Effectiveness and aperiodicity of subshifts

Sebastián Barbieri
LIP, ENS de Lyon
Based on works with Nathalie Aubrun, Mathieu Sablik and Stephan Thomassé

Journées GT Calculabilités 2015

Outline

1. Background
2. Effectiveness in groups
3. Aperiodicity

G-Subshifts

- G is a (finitely generated) group.
- $\mathcal{A}$ is a finite alphabet.
- $\mathcal{A}^G$ is the set of functions from G to $\mathcal{A}$
- $\sigma : G \times \mathcal{A}^G \to \mathcal{A}^G$ is the left shift action given by :

$$\sigma_g(x)_h = x_{g^{-1}h}$$

G-Subshifts

- G is a (finitely generated) group.
- $\mathcal{A}$ is a finite alphabet.
- $\mathcal{A}^G$ is the set of functions from G to $\mathcal{A}$
- $\sigma : G \times \mathcal{A}^G \to \mathcal{A}^G$ is the left shift action given by :

$$\sigma_g(x)_h = x_{g^{-1}h}$$

Definition : G-subshift

$X \subset \mathcal{A}^G$ is a *G-subshift* if it invariant under the action of σ and closed for the product topology on $\mathcal{A}^G$.

G-Subshifts

- G is a (finitely generated) group.
- $\mathcal{A}$ is a finite alphabet.
- $\mathcal{A}^G$ is the set of functions from G to $\mathcal{A}$
- $\sigma : G \times \mathcal{A}^G \to \mathcal{A}^G$ is the left shift action given by :

$$\sigma_g(x)_h = x_{g^{-1}h}$$

Definition : G-subshift

$X \subset \mathcal{A}^G$ is a *G-subshift* if it invariant under the action of σ and closed for the product topology on $\mathcal{A}^G$.

Equivalently, X is a G-subshift if it can be defined by a set of forbidden patterns : $\exists \mathcal{F} \subset \bigcup_{F \subset G, |F| < \infty} \mathcal{A}^F$ such that

$$X = X_{\mathcal{F}} := \{x \in \mathcal{A}^G \mid \forall P \in \mathcal{F} : P \not\sqsubset x\}$$

$\mathbb{Z}$-Subshift examples

Example : full shift. Let $\mathcal{A} = \{0, 1\}$ and $\mathcal{F} = \emptyset$. Then $X_{\mathcal{F}} = \mathcal{A}^{\mathbb{Z}}$ is the set of all bi-infinite words.

$\mathbb{Z}$-Subshift examples

Example : full shift. Let $\mathcal{A} = \{0,1\}$ and $\mathcal{F} = \emptyset$. Then $X_{\mathcal{F}} = \mathcal{A}^{\mathbb{Z}}$ is the set of all bi-infinite words.

Example : Fibonacci shift. Let $\mathcal{A} = \{0,1\}$ and $\mathcal{F} = \{11\}$. Then $X_{\mathcal{F}}$ is the set of all bi-infinite words which have no pairs of consecutive 1's.

$$x = \dots 010100010100100100100 \cdots \in X_{\mathcal{F}}$$

$\mathbb{Z}$-Subshift examples

Example : full shift. Let $\mathcal{A} = \{0,1\}$ and $\mathcal{F} = \emptyset$. Then $X_{\mathcal{F}} = \mathcal{A}^{\mathbb{Z}}$ is the set of all bi-infinite words.

Example : Fibonacci shift. Let $\mathcal{A} = \{0,1\}$ and $\mathcal{F} = \{11\}$. Then $X_{\mathcal{F}}$ is the set of all bi-infinite words which have no pairs of consecutive 1's.

$$x = \dots 0101000101001001001 00 \cdots \in X_{\mathcal{F}}$$

Example : one-or-less subshift

$$X_{\leq 1} := \{x \in \{0,1\}^{\mathbb{Z}} \mid |\{n \in \mathbb{Z} : x_n = 1\}| \leq 1\}.$$

Is a $\mathbb{Z}$-subshift as it is defined by the set $\mathcal{F} = \{10^n1 | n \in \mathbb{N}_0\}$.

$$x = \dots 0000000000010000000000 \cdots \in X_{\leq 1}$$

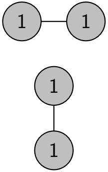
Example in $\mathbb{Z}^2$: Fibonacci shift

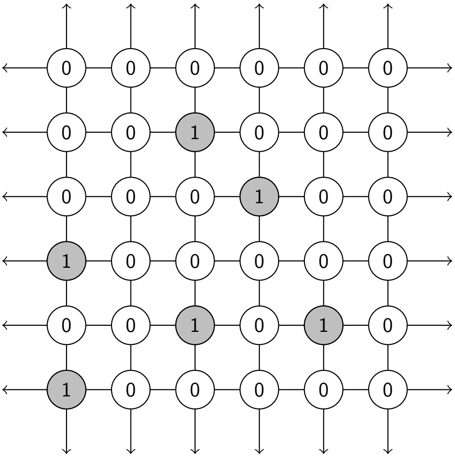
Example : Fibonacci shift. X_{fib} is the set of assignments of $\mathbb{Z}^2$ to $\{0, 1\}$ such that there are no two adjacent ones.

Example in $\mathbb{Z}^2$: Fibonacci shift

Example : Fibonacci shift. X_{fib} is the set of assignments of $\mathbb{Z}^2$ to $\{0,1\}$ such that there are no two adjacent ones.

$\mathcal{F} =$

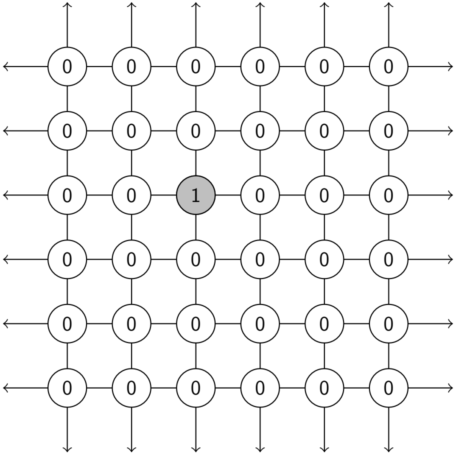




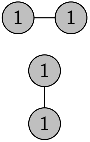
Example : one-or-less subshift

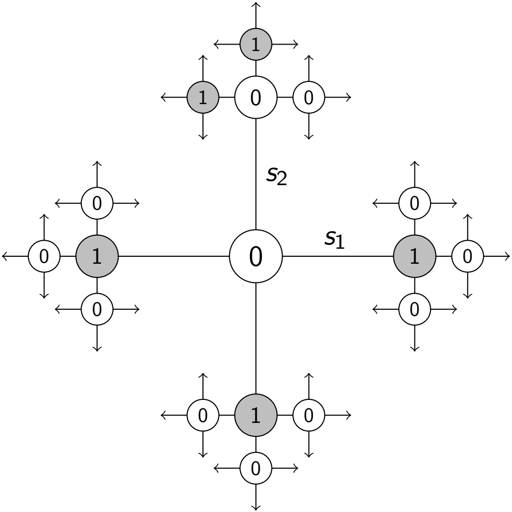
Example : one-or-less subshift.

$$X_{\leq 1} := \{x \in \{0,1\}^{\mathbb{Z}^d} \mid |\{z \in \mathbb{Z}^d : x_z = 1\}| \leq 1\}.$$



Example : S-Fibonacci shift for $G = F_2$

$\mathcal{F} =$ 



Interesting classes

G-SFTs

A G-subshift X is said to be of finite type (G-SFT) if there exists a finite set of patterns $\mathcal{F}$ such that $X = X_{\mathcal{F}}$.

Interesting classes

G-SFTs

A G-subshift X is said to be of finite type (G-SFT) if there exists a finite set of patterns $\mathcal{F}$ such that $X = X_{\mathcal{F}}$.

Example : S-Fibonacci shift. For every group G generated by a finite set S the S-Fibonacci shift is a G-SFT.

Interesting classes

G-SFTs

A G-subshift X is said to be of finite type (G-SFT) if there exists a finite set of patterns $\mathcal{F}$ such that $X = X_{\mathcal{F}}$.

Example : S-Fibonacci shift. For every group G generated by a finite set S the S-Fibonacci shift is a G-SFT.

Sofic G-subshifts

A G-subshift Y over $\mathcal{A}$ is said to be a sofic G-subshift if there exists a G-SFT X and a surjective cellular automaton $\phi : X \to Y$. That is, we have a G-SFT where we allow to delete some information.

Example : $X_{\leq 1}$ is a sofic G-subshift if G is $\mathbb{Z}^d$ or a finitely generated free group F_k.

Interesting classes

Remark : These classes are interesting from a computational perspective because they can be defined with a finite amount of data. How far can we take this idea ?

Interesting classes

Remark : These classes are interesting from a computational perspective because they can be defined with a finite amount of data. How far can we take this idea ?

Definition : Effectiveness in $\mathbb{Z}$

A $\mathbb{Z}$-subshift $X \subset \mathcal{A}^{\mathbb{Z}}$ is said to be effective if there is a recognizable set $\mathcal{F} \subset \mathcal{A}^*$ such that $X = X_{\mathcal{F}}$.

Interesting classes

Remark : These classes are interesting from a computational perspective because they can be defined with a finite amount of data. How far can we take this idea ?

Definition : Effectiveness in $\mathbb{Z}$

A $\mathbb{Z}$-subshift $X \subset \mathcal{A}^{\mathbb{Z}}$ is said to be effective if there is a recognizable set $\mathcal{F} \subset \mathcal{A}^*$ such that $X = X_{\mathcal{F}}$.

Question : How can the idea of effectiveness be translated into general groups ?

Outline

First approach : $\mathbb{Z}$-effectiveness

Let G be a finitely generated group and $S \subset G$ a finite generator.

First approach : $\mathbb{Z}$-effectiveness

Let G be a finitely generated group and $S \subset G$ a finite generator.

Definition : $\mathbb{Z}$-effectiveness

A G-subshift $X \subset \mathcal{A}^G$ is *$\mathbb{Z}$-effective* if there is a Turing machine which enumerates a set of pattern codings such that the set of consistent pattern codings defines a set $\mathcal{F}$ such that $X = X_{\mathcal{F}}$.

First approach : $\mathbb{Z}$-effectiveness

Let G be a finitely generated group and $S \subset G$ a finite generator.

Definition : $\mathbb{Z}$-effectiveness

A G-subshift $X \subset \mathcal{A}^G$ is *$\mathbb{Z}$-effective* if there is a Turing machine which enumerates a set of pattern codings such that the set of consistent pattern codings defines a set $\mathcal{F}$ such that $X = X_{\mathcal{F}}$.

Question : Is it always possible to recognize if a pattern coding is inconsistent ?

Example : the Baumslag-Solitar group $BS(1,2)$

Consider the group $BS(1,2) = \langle a, b \mid ab = ba^2 \rangle$.

Example : the Baumslag-Solitar group $BS(1,2)$

Consider the group $BS(1,2) = \langle a, b \mid ab = ba^2 \rangle$.

- The pattern coding

$$\begin{array}{ccc} (\epsilon, 0) & (b, 1) & (a, 1) \\ (ab, 0) & (ba^2, 0) & (ba, 1) \end{array}$$

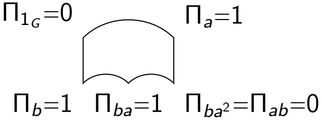
Example : the Baumslag-Solitar group $BS(1,2)$

Consider the group $BS(1,2) = \langle a, b \mid ab = ba^2 \rangle$.

- The pattern coding

$$\begin{array}{ccc} (\epsilon, 0) & (b, 1) & (a, 1) \\ (ab, 0) & (ba^2, 0) & (ba, 1) \end{array}$$

is *consistent* and defines the pattern



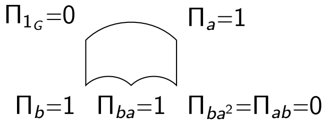
Example : the Baumslag-Solitar group $BS(1,2)$

Consider the group $BS(1,2) = \langle a, b \mid ab = ba^2 \rangle$.

- The pattern coding

$$\begin{array}{ccc} (\epsilon, 0) & (b, 1) & (a, 1) \\ (ab, 0) & (ba^2, 0) & (ba, 1) \end{array}$$

  is *consistent* and defines the pattern



- The pattern coding

$$\begin{array}{ccc} (\epsilon, 0) & (a^2, 1) & (bab^{-1}a, 1) \\ (a, 1) & (ba, 1) & (abab^{-1}, 0) \end{array}$$

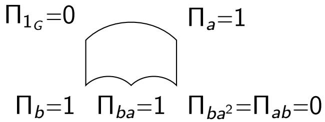
Example : the Baumslag-Solitar group $BS(1,2)$

Consider the group $BS(1,2) = \langle a, b \mid ab = ba^2 \rangle$.

- The pattern coding

$$
\begin{matrix}
(\epsilon, 0) & (b, 1) & (a, 1) \\
(ab, 0) & (ba^2, 0) & (ba, 1)
\end{matrix}
$$

is *consistent* and defines the pattern



- The pattern coding

$$
\begin{matrix}
(\epsilon, 0) & (a^2, 1) & (bab^{-1}a, 1) \\
(a, 1) & (ba, 1) & (abab^{-1}, 0)
\end{matrix}
$$

is *inconsistent* since $abab^{-1}$ and $bab^{-1}a$ represent the same element.

$$abab^{-1} = ba^3b^{-1} = ba(b^{-1}b)a^2b^{-1} = bab^{-1}abb^{-1} = bab^{-1}a$$

Limitations of $\mathbb{Z}$-effectiveness

Definition : Word problem

Let $S \subset G$ be a finite generator of G. The word problem of G asks whether two words on $S \cup S^{-1}$ are equivalent in G. Formally :

$$WP(G) = \left\{ w \in \left(S \cup S^{-1} \right)^* \mid w =_G 1_G \right\}.$$

Limitations of $\mathbb{Z}$-effectiveness

Definition : Word problem

Let $S \subset G$ be a finite generator of G. The word problem of G asks whether two words on $S \cup S^{-1}$ are equivalent in G. Formally :

$$WP(G) = \left\{ w \in \left(S \cup S^{-1} \right)^{*} \mid w =_G 1_G \right\}.$$

Example : Decidable word problem. The word problem for $\mathbb{Z}^2 \simeq \langle a, b | ab = ba \rangle$ is :

$$WP(\mathbb{Z}^2) = \{ w \in \{a, b, a^{-1}, b^{-1}\}^* \mid |w|_a = |w|_{a^{-1}} \wedge |w|_b = |w|_{b^{-1}} \}$$

Limitations of $\mathbb{Z}$-effectiveness

Definition : Word problem

Let $S \subset G$ be a finite generator of G. The word problem of G asks whether two words on $S \cup S^{-1}$ are equivalent in G. Formally :

$$WP(G) = \left\{ w \in \left(S \cup S^{-1} \right)^* \mid w =_G 1_G \right\}.$$

Example : Decidable word problem. The word problem for $\mathbb{Z}^2 \simeq \langle a, b | ab = ba \rangle$ is :

$$WP(\mathbb{Z}^2) = \{ w \in \{a, b, a^{-1}, b^{-1}\}^* \mid |w|_a = |w|_{a^{-1}} \wedge |w|_b = |w|_{b^{-1}} \}$$

Example : Undecidable word problem. If $f : \mathbb{N} \to \{0, 1\}$ is non-computable the group $G = \langle a, b, c, d \mid ab^n = c^n d, n \in f^{-1}(1) \rangle$ has undecidable word problem.

Limitations of $\mathbb{Z}$-effectiveness

Remark : If G is not recursively presented, It is not possible to recognize whether a pattern coding is consistent !

Limitations of $\mathbb{Z}$-effectiveness

Remark : If G is not recursively presented, It is not possible to recognize whether a pattern coding is consistent !

Remark : Even if G is finitely presented, there are simple subshifts which are not $\mathbb{Z}$-effective !

Limitations of $\mathbb{Z}$-effectiveness

Remark : If G is not recursively presented, It is not possible to recognize whether a pattern coding is consistent !

Remark : Even if G is finitely presented, there are simple subshifts which are not $\mathbb{Z}$-effective !

Remark [Theorem : Novikov(55), Boone(58)]

There are finitely presented groups with undecidable word problem !

Theorem

For a recursively presented group the one-or-less subshift :

$$X_{\leq 1} := \{x \in \{0,1\}^G \mid |\{g \in G : x_g = 1\}| \leq 1\}.$$

is not $\mathbb{Z}$-effective if $WP(G)$ is undecidable.

G-effectiveness

Another approach : Don't codify anything !

G-effectiveness

Another approach : Don't codify anything !

Definition : G-machine

A G-machine is a Turing machine whose tape has been replaced by the group G. The transition function is $\delta : Q \times \Sigma \to Q \times \Sigma \times (S \cup S^{-1} \cup \{1_G\})$ where S is a finite set of generators of G.

G-effectiveness

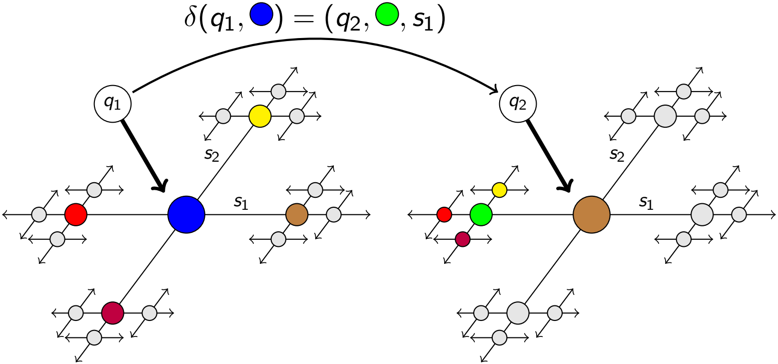
Another approach : Don't codify anything !

Definition : G-machine

A G-machine is a Turing machine whose tape has been replaced by the group G. The transition function is $\delta : Q \times \Sigma \to Q \times \Sigma \times (S \cup S^{-1} \cup \{1_G\})$ where S is a finite set of generators of G.

Remark : Computation is over patterns instead of words.

Example : Transition in a F_2-machine



G-effectiveness

Definition :

- A set of patterns $\mathcal{P}$ is said to be recognizable if there is a G-machine which accepts if and only if $P \in \mathcal{P}$.
- A set of patterns $\mathcal{P}$ is said to be decidable if there is a G-machine which accepts if $P \in \mathcal{P}$ and rejects otherwise.

G-effectiveness

Definition :

- A set of patterns $\mathcal{P}$ is said to be recognizable if there is a G-machine which accepts if and only if $P \in \mathcal{P}$.
- A set of patterns $\mathcal{P}$ is said to be decidable if there is a G-machine which accepts if $P \in \mathcal{P}$ and rejects otherwise.

G-effectiveness

A G-subshift $X \subset \mathcal{A}^G$ is G-effective if there exists a set of forbidden patterns $\mathcal{F}$ such that $X = X_{\mathcal{F}}$ and $\mathcal{F}$ is G-recognizable.

What can we say about G-effectiveness ?

Remark : The one-or-less subshift $X_{\leq 1}$ is G-effective for every finitely generated group G.

What can we say about G-effectiveness ?

Remark : The one-or-less subshift $X_{\leq 1}$ is G-effective for every finitely generated group G.

Theorem

Let G be an infinite, finitely generated group, then every $\mathbb{Z}$-effective subshift is G-effective.

What can we say about G-effectiveness ?

Remark : The one-or-less subshift $X_{\leq 1}$ is G-effective for every finitely generated group G.

Theorem

Let G be an infinite, finitely generated group, then every $\mathbb{Z}$-effective subshift is G-effective.

Theorem

Let G be finitely generated group with decidable word problem then every G-effective subshift is $\mathbb{Z}$-effective.

Some results about G-effectiveness ?

Theorem

A subshift is G-effective if and only if it satisfies the conditions of $\mathbb{Z}$-effectiveness with a Turing machine which has access to an oracle of $WP(G)$.

Some results about G-effectiveness ?

Theorem

A subshift is G-effective if and only if it satisfies the conditions of $\mathbb{Z}$-effectiveness with a Turing machine which has access to an oracle of $WP(G)$.

- We have also shown that the class of G-effective subshifts contains every G-SFT, every sofic and every $\mathbb{Z}$-effective G-subshift.

Outline

Aperiodicity in a subshift

Definition : Strongly aperiodic

A G-subshift X is said to be *strongly aperiodic* if

$$\forall x \in X,\ stab_{\sigma}(x) := \{g \in G \mid gx = x\} = \{1_G\}$$

Aperiodicity in a subshift

Definition : Strongly aperiodic

A G-subshift X is said to be *strongly aperiodic* if

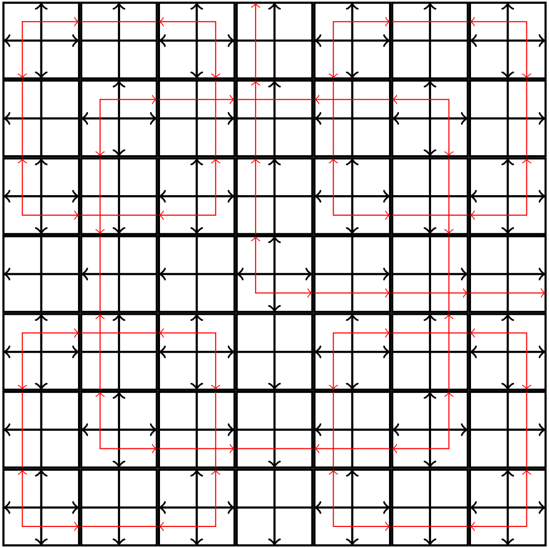
$$\forall x \in X,\ stab_\sigma(x) := \{g \in G \mid gx = x\} = \{1_G\}$$

Example in $G = \mathbb{Z}$. Let $\mathcal{A} = \{0, 1, 2\}$ and $\mathcal{F} = \{ww \mid w \in \mathcal{A}^*\}$. Then $X_\mathcal{F}$ is strongly aperiodic.

Some known facts

- $\mathbb{Z}$-SFTs are never strongly aperiodic.
- There are strongly aperiodic $\mathbb{Z}^2$-SFTs. (1964 Berger, 1971 Robinson, 1996 Kari)
- There are weakly aperiodic SFTs in Baumslag Solitar groups (2013 Aubrun-Kari)
- There are strongly aperiodic SFTs in the Heisenberg group (2014 Sahin-Schraudner)
- The existence of a strongly aperiodic G-SFT implies G is one ended (2014 Cohen)
- A finitely presented group which admits a strongly aperiodic SFT has decidable word problem (2015 Jeandel)

The Robinson tiling



Our result

Theorem :

For every infinite and finitely generated group G there exists a strongly aperiodic G-effective subshift.

Our result

Theorem :

For every infinite and finitely generated group G there exists a strongly aperiodic G-effective subshift.

Corollary :

For a recursively presented group, there exists a $\mathbb{Z}$-effective strongly aperiodic subshift if and only if $WP(G)$ is decidable.

An ingredient for the proof

Definition

Let (X, d) be a metric space. We say $F \subset G$ is *r-covering* if for each $x \in G$ there is $y \in F$ such that $d(x, y) \leq r$. We say F is *s-separating* if for each $x \neq y \in F$ then $d(x, y) > s$

An ingredient for the proof

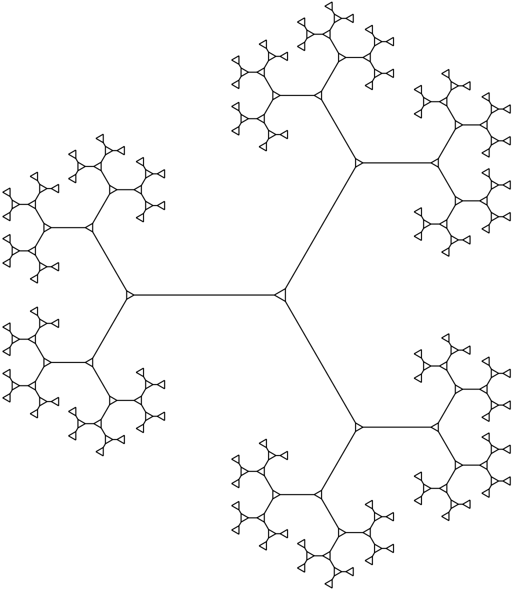
Definition

Let (X, d) be a metric space. We say $F \subset G$ is *r-covering* if for each $x \in G$ there is $y \in F$ such that $d(x, y) \leq r$. We say F is *s-separating* if for each $x \neq y \in F$ then $d(x, y) > s$

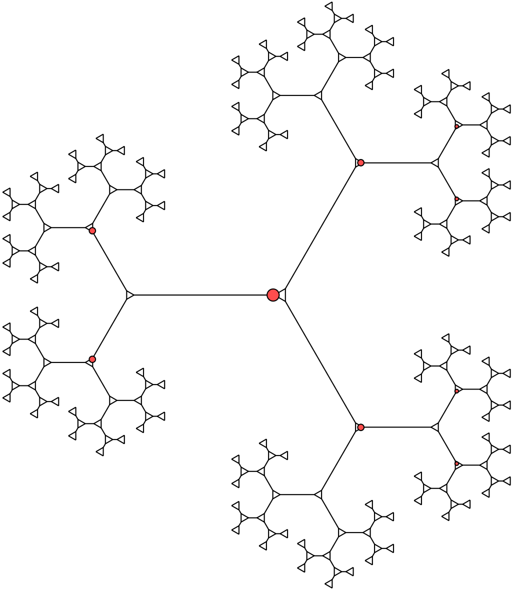
Proposition

If X is countable, then for any $r \in \mathbb{R}$ there exists $Y \subset X$ such that Y is both r-separating and r-covering.

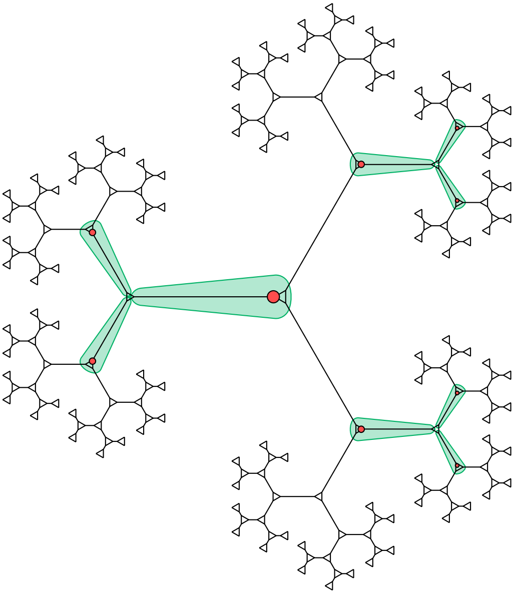
Example : 2-covering and 2-separating set in PSL($\mathbb{Z}$, 2)



Example : 2-covering and 2-separating set in $PSL(\mathbb{Z}, 2)$



Example : 2-covering and 2-separating set in $PSL(\mathbb{Z}, 2)$



Proof

- First we create a layer with a hierarchical structure.
- $Y \subset (S \cup S^{-1} \cup \{1_G\})^G$
- The points $y \in Y$ codify forests with a property :

Proof

- First we create a layer with a hierarchical structure.
- $Y \subset (S \cup S^{-1} \cup \{1_G\})^G$
- The points $y \in Y$ codify forests with a property :

Property

for every $n \in \mathbb{N}$, G can be partitioned in sets $(\mathcal{C}_i)_{i \in I}$ such that $\exists g_i \in \mathcal{C}_i$ such that

$$B(g_i, n) \subset \mathcal{C}_i \subset B(g_i, 5^n)$$

And for each $\mathcal{C}_i$ there is either a single $h \in \mathcal{C}_i$ with $x_h = 1_G$ and for every other $g \in \mathcal{C}_i$ then $gx_g \in \mathcal{C}_i$ or $\forall g \in \mathcal{C}_i$ $x_g \neq 1_G$ and there is a single $h \in \mathcal{C}_i$ such that $hx_h \notin \mathcal{C}_i$.

Proof

- First we create a layer with a hierarchical structure.
- $Y \subset (S \cup S^{-1} \cup \{1_G\})^G$
- The points $y \in Y$ codify forests with a property :

Property

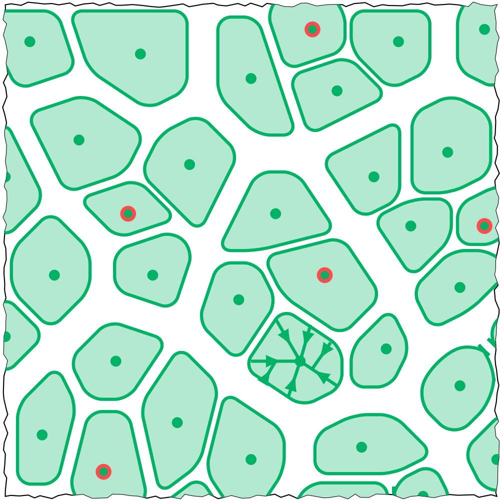
for every $n \in \mathbb{N}$, G can be partitioned in sets $(\mathcal{C}_i)_{i \in I}$ such that $\exists g_i \in \mathcal{C}_i$ such that

$$B(g_i, n) \subset \mathcal{C}_i \subset B(g_i, 5^n)$$

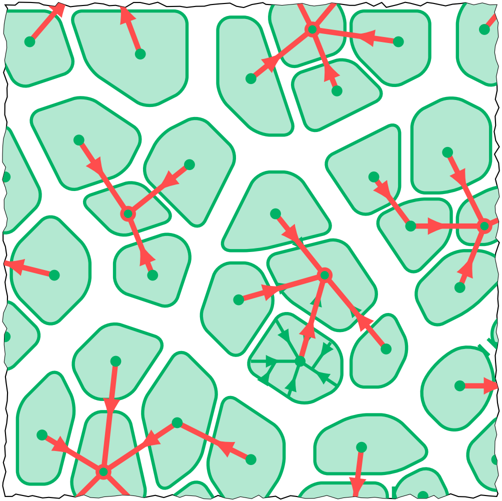
And for each $\mathcal{C}_i$ there is either a single $h \in \mathcal{C}_i$ with $x_h = 1_G$ and for every other $g \in \mathcal{C}_i$ then $gx_g \in \mathcal{C}_i$ or $\forall g \in \mathcal{C}_i$ $x_g \neq 1_G$ and there is a single $h \in \mathcal{C}_i$ such that $hx_h \notin \mathcal{C}_i$.

Remark : This property can be easily verified with a TM with access to $WP(G)$.

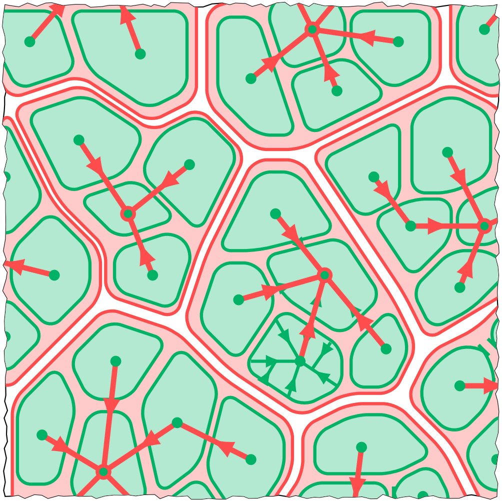
Cluster structure



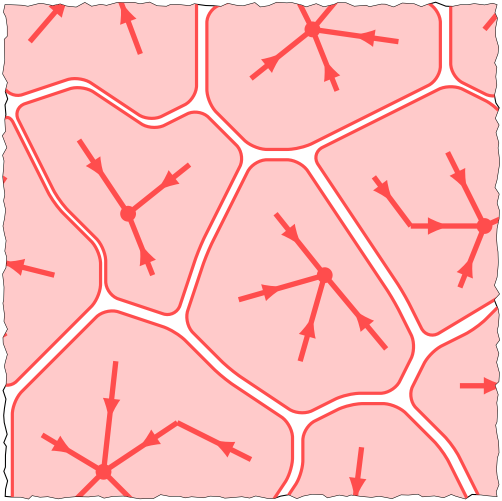
Cluster structure



Cluster structure



Cluster structure



Second layer

Remark : We are not done yet !

Example : $G = \mathbb{Z}$.

$$y = \ldots, +1, +1, +1, +1, +1, +1, \cdots \in Y$$

Second layer

Remark : We are not done yet !

Example : $G = \mathbb{Z}$.

$$y = \ldots, +1, +1, +1, +1, +1, +1, \cdots \in Y$$

Consider an infinite word $\mathcal{W}$ without squares, such as the one produced by $\phi : \{0,1,2\} \to \{0,1,2\}^*$ given by :

$$\phi(k) = \begin{cases} 01210, & \text{if } k = 0 \\ 12021, & \text{if } k = 1 \\ 20102, & \text{if } k = 2 \end{cases}$$

Second layer

Remark : We are not done yet !

Example : $G = \mathbb{Z}$.

$$y = \ldots, +1, +1, +1, +1, +1, +1, \cdots \in Y$$

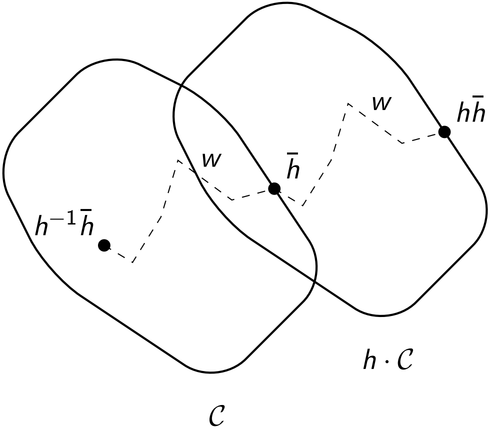
Consider an infinite word $\mathcal{W}$ without squares, such as the one produced by $\phi : \{0,1,2\} \to \{0,1,2\}^*$ given by :

$$\phi(k) = \begin{cases} 01210, & \text{if } k = 0 \\ 12021, & \text{if } k = 1 \\ 20102, & \text{if } k = 2 \end{cases}$$

We consider $X \subset ((S \cup S^{-1} \cup \{1_G\}) \times \{0,1,2\})^G$ such that for $x \in X$ then $\pi_1(x) \in Y$ and every path in $\pi_1(x)$ contains a subword of $\mathcal{W}$ in the second layer.

Final argument

The existence of $h \neq 1_G$ such that $h \in stab_\sigma(x)$ creates a square word.



Corollary :

For a recursively presented group, there exists a $\mathbb{Z}$-effective strongly aperiodic subshift if and only if $WP(G)$ is decidable.

Corollary :

For a recursively presented group, there exists a $\mathbb{Z}$-effective strongly aperiodic subshift if and only if $WP(G)$ is decidable.

Proof : As $WP(G)$ is decidable, every G-effective subshift is $\mathbb{Z}$-effective and thus our construction shows the existence. Jeandel's result gives the other direction.

Current work

- Use simulation theorems with our construction to produce strongly aperiodic SFTs in some classes of groups.

Current work

- Use simulation theorems with our construction to produce strongly aperiodic SFTs in some classes of groups.
- Apply the idea of clusters to generate entropies in amenable groups.

Merci beaucoup pour votre attention !

Avez-vous des questions ?