Enumeration Degrees in Algebra

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Plan

Starting point : Groups

Theorem (Boone 57, Novikov 55)

There exists f.p. groups with undecidable word problems.

Theorem (ess. Kuznetsov 58)

f.p. simple groups have decidable word problem.

Starting point : Symbolic Dynamics

Theorem (Robinson 71)

There exists subshifts of finite type with an uncomputable language

Theorem (Ballier-J. 08, Hochman 09)

Minimal subshifts of finite type have a computable language.

Goal of the talk

Is there a common framework ?

Yes: originally observed by Kuznetsov 55, see also Mal'tsev 61.

What can we say if the structure is not finitely/recursively presented ?

This talk

Key ingredient - Enumeration reducibility

Most results involve *enumeration reducibility*

Definition (Informal)

$A \leq_e B$ if we can enumerate A from any enumeration of B

Definition (Informal)

$A \leq_e^f B$ if we can enumerate A from any enumeration of B via f

$\leq_e$ is a preorder, smallest degree is exactly the r.e. sets.

Plan

Quasivariety

Let I be a infinite computable countable set.

Definition

Let $\mathcal{S} = (F_n, a_n)_{n\in\mathbb{N}}$ be a computably enumerable set of pairs of finite subsets of I and elements of I.
The quasivariety defined by $\mathcal{S}$ is the set of all $x \in \{0,1\}^I$ s.t.

$$\forall n, \bigwedge_{i\in F_n} x_i = 1 \implies x_{a_n} = 1$$

The quasivariety defined by $\mathcal{S}$ is the set of all $X \subseteq I$ s.t.

$$\forall n, \bigwedge_{i\in F_n} i \in X \implies a_n \in X$$

Note: quasivarieties are Π^0_1 sets.

Example : Subshifts

Definition

Let A be a finite alphabet.
A subshift S is a closed shift-invariant subset of $A^{\mathbb{Z}}$.
Equivalently there exists a set $\mathcal{F}$ s.t. S is exactly the set of biinfinite words that do not contain any element of $\mathcal{F}$.

The set of all words over $\{a, b\}^{\mathbb{Z}}$ with at most two letters b is a subshift.
The set of all words over $\{a, b\}^{\mathbb{Z}}$ with no cubes is a subshift.

The set of all subshifts of $A^{\mathbb{Z}}$ is a quasivariety.

Example : Subshifts

A subshift S is entirely characterized by the set of words L that cannot appear in S.
$L \subseteq A^\star = I$ is such an set iff it is factorial and extensible:

- For any $a \in A$, if $w \in L$ then $aw \in L$.
- For any $a \in A$, if $w \in L$ then $wa \in L$.
- If $aw \in L$ for all a, then $w \in L$.
- If $wa \in L$ for all a, then $w \in L$.

Thus subshifts, seen as languages, form a quasivariety.

- The largest point correspond to $L = A^\star$ and to $S = \emptyset$
- The smallest point correspond to $L = \emptyset$ and to $S = A^{\mathbb{Z}}$

Example: Groups

The set of all f.g. groups with k generators is a quasivariety.

A group G is f.g. with k generators iff there is a epimorphism from $\mathbb{F}_k$ (the free group with k generators) into G
Such a group can be seen as a normal subgroup of $\mathbb{F}_k$.

Example: Groups

A set $S \subseteq \mathbb{F}_k = I$ is a normal subgroup of $\mathbb{F}_k$ iff:

- If (nothing) then $1 \in S$
- If $g \in S$ then $g^{-1} \in S$
- If $g \in S, h \in S$ then $gh \in S$
- For any h, if $g \in S$ then $hgh^{-1} \in S$

The quasivariety we obtain is called the space of marked groups and was introduced by Grigorchuk (85).

Other examples

- Ideals in a computable (commutative) ring
- Quasivarieties from universal algebra
  - I will be the set of equations of the form $a = b$ where a and b are terms.
- Universal theories in a finite language.
  - I will be the set of all atomic formulas

Some properties and notations

Quasivariety will be denoted V. $X \in V$ will be called a *point*

V is a complete semi-lattice.

- V contains a maximal element, the set I itself.
- V contains a minimal element. More generally, for any set $Y \subseteq I$, there exists a smallest element X of V containing Y.

Plan

Presentations

Definition

A presentation of $X \in V$ is any subset R of X s.t. X is the smallest point of V containing R.
X is finitely (recursively) presented if it admits a finite (recursively enumerable) presentation.

Finitely presented subshifts correspond to subshifts of finite type.
Recursively presented subshifts correspond to effectively closed subshifts.
Finitely (recursively) presented groups correspond to finitely (recursively) presented groups

Presentations

Proposition

Let V be a quasivariety If Y is a presentation of X, then $X \leq_e Y$.
Consequently, X is the smallest (for $\leq_e$) presentation of X.

Proposition (Uniform Version)

There exists f s.t. Y is a presentation of X iff $X \leq_e Y$ via f.

In particular a recursively presented point is recursively enumerable (seen as a subset of I)
Proof is essentially the completeness theorem.

Presentation - Converse

Proposition

Let $Y \subseteq X$.
Then $X \leq_e Y$ iff $\exists$ a quasivariety where Y is a presentation of X.

Maximal elements

Definition (Maximal points)

A point $X \in V$ is maximal if $X \subseteq Y$, $Y \in V$ implies $Y = I$ or $Y = X$.

In the language of groups, this correspond to *simple* groups.
In the language of symbolic dynamics, this correspond to *minimal* subshifts.

Main Theorem

Theorem

Let V be a quasivariety and X maximal. Then $\overline{X} \leq_e X$.

$\overline{X} = I \setminus X$ is the complement of X

Generalizes Kuznetsov theorem, dropping any computability assumption on X.

Idea

Y is a presentation of X iff $X \leq_e^f Y$ (first proposition).

$x \in \overline{X}$ iff $I \leq_e^f X \cup \{x\}$

Let $a \notin X$ be fixed.

$x \in \overline{X}$ iff from any enumeration of $X \cup \{x\}$ we will enumerate a

$x \in \overline{X}$ iff from any enumeration of X we will enumerate x

Generalizations

corollary

Let V be a quasivariety. If X is maximal and recursively presented, then X is computable.

Exactly the theorems in the introduction

Theorem

Let V be a quasivariety s.t. V is finitely presented. If X maximal, then $\overline{X} \leq_e X$ uniformly.

Generalizations

Same conclusions under the following hypothesis:

- There exists a computably enumerable family $\mathcal{Y}$ of finitely presented points so that $X \subsetneq Y, Y \in V$ implies $Y \in \mathcal{Y}$
- There exists a computably enumerable set $S \subseteq \overline{X}$ s.t. $X \subsetneq Y, Y \in V$ implies $Y \cap S \neq \emptyset$

First one is satisfied for *just infinite groups*, the second one for *isolated* and *finitely discriminable* groups.

Converse

Theorem

Let $X \subseteq I$. If $\overline{X} \leq_e X$, then there exists a quasivariety V s.t. X is maximal.

Hence $\leq_e$ captures exactly what we want.

Plan

Question

All theorems are of the form

If something, then $\overline{X} \leq_e X$.

There are examples where the converse is not true.

What information is $\overline{X}$ lacking to enumerate X ?

Maximality function

Definition

$A \leq_e B$ if there exists an enumeration of finite sets $f(a, n)$ s.t.

$$x \in A \iff \exists n, f(a, n) \subseteq B$$

Definition

The enumeration function of $\overline{X} \leq_e^f X$ is the function defined on $\overline{X}$ by:

$$g(x) = \min\{n | f(x, n) \subseteq X\}$$

Seems ad-hoc, but in all examples, by uniformity, f does not depend on X

Maximality function

Theorem

g is computable from X
X is computable from $\overline{X}$ and any total function that bounds g.

Applications

For subshifts, g can be replaced by the *quasiperiodicity* function:

Definition (Quasiperiodicity function)

$g(n) = k$ if any pattern of size n that appear in X can be found in any pattern of size k that appear in X.

Theorem

For a minimal subshift S, the quasiperiodicity function can be computed from the forbidden language of S. From the quasiperiodicity function, and the language of S, one can enumerate the forbidden language of S.

Applications: Groups

Definition

Let a be a fixed generator.
$g(n) = k$ if for all words $w \neq 1$ in G of size n, a can be written as the product of at most k conjugates of w (each conjugate having size at most k)

Theorem

For a simple group G , the simplicity function can be computed from the word problem of G From the simplicity function and the complement of the word problem, one can enumerate the word problem of G

Conclusions

- Unification of existing results in a new context.
- Results for algebraic structures that are not recursive
- Lot of new questions coming from analogy between different algebraic structures

Prospective Theorems

Prospective Theorem [J.-Vanier 2015]

A subshift has a computable language iff it is the subaction of a minimal subshift, which it iself a subaction of a SFT.

Prospective Theorem [Guillon-Zinoviadis 2018]

A subshift has a computable language iff it is the subaction of a minimal SFT.

Prospective Theorems

Prospective Theorem [Vanier 2018]

Let A, B s.t. $\overline{A} \oplus B \equiv_e A$.
Then there exists a minimal subshift for which the language of forbidden words has enumeration degree A and the quasiperiodicity function has Turing degree B.

Prospective Theorem

Prospective Theorem [J. 2017]

For any f.g. group G, there exists a f.g. simple group H with the same Turing degree and a computable simplicity function.

Prospective Theorem [J.-PhD Student 2019]

T is finitely axiomatisable relative to T' iff T is enumeration reducible to T'.