

How colorings reduce when colors increase

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STRENGTH OF A THEOREM

Some theorems are more **effective** than others.

Theorem (Intermediate value theorem)

For every continuous function f over $[a, b]$ and every $y \in [f(a), f(b)]$, there is some $x \in [a, b]$ such that $f(x) = y$.

Theorem (König's lemma)

Every infinite, finitely branching tree has an infinite path.

STRENGTH OF A THEOREM

Provability strength

- ▶ Reverse mathematics
- ▶ Intuitionistic reverse mathematics

Computational strength

- ▶ Computable reducibility
- ▶ Uniform reducibility

Provability approach

REVERSE MATHEMATICS

Goal

Determine which axioms are required to prove **ordinary** theorems in reverse mathematics.

- ▶ Simpler proofs
- ▶ More insights

Subsystems of second-order arithmetic.

BASE THEORY RCA_0

- ▶ Basic Peano axioms
- ▶ Σ_1^0 induction scheme

$$(\varphi(0) \wedge \forall n.(\varphi(n) \rightarrow \varphi(n+1))) \rightarrow \forall n.\varphi(n)$$

where $\varphi(n)$ is any Σ_1^0 formula of L_2

- ▶ Δ_1^0 comprehension scheme

$$\forall n(\varphi(n) \leftrightarrow \psi(n)) \rightarrow \exists X.\forall n.(x \in X \leftrightarrow \varphi(n))$$

where $\varphi(n)$ is any Σ_1^0 formula of L_2 in which X does not occur freely and $\psi(n)$ is any Π_1^0 formula of L_2 .

HOW TO THINK ABOUT RCA_0 ?

RCA_0 captures **computable** mathematics

RCA_0 has model $\mathcal{M} = \{\omega, S, <, +, \cdot\}$ where

- ▶ ω are the standard integers
- ▶ $S = \{X \in 2^\omega : X \text{ is computable}\}$ is the second-order part

Computational approach

THEOREMS AS PROBLEMS

Many theorems \mathbf{P} are of the form

$$(\forall X)[\Phi(X) \rightarrow (\exists Y)\Psi(X, Y)]$$

where Φ and Ψ are arithmetic formulas.

We may think of \mathbf{P} as a class of **problems**.

- ▶ An X such that $\Phi(X)$ holds is an **instance**.
- ▶ A Y such that $\Psi(X, Y)$ holds is a **solution** to X .

THEOREMS AS PROBLEMS

Examples:

- ▶ (König's lemma)
Every **infinite, finitely branching tree** has an **infinite path**.
- ▶ (Ramsey's theorem)
Every **k -coloring** has an **infinite monochromatic subset**.
- ▶ (The atomic model theorem)
Every **complete atomic theory** has an **atomic model**.
- ▶ ...

COMPUTABLE REDUCIBILITY

Definition (Computable reducibility)

A theorem P is *computably reducible* to a theorem Q if every P -instance I computes a Q -instance J such that for every solution X to J , $X \oplus I$ computes a solution to I .

Intuition:

If $P \leq_c Q$ then solving Q is *harder* than solving P .

PROVABILITY VS COMPUTATIONAL APPROACH

If we forget induction,

$$P \leq_c Q$$

can be seen as

$$\text{RCA}_0 \vdash Q \rightarrow P$$

where only one application of Q is allowed.

Ramsey's theorem

RAMSEY'S THEORY

Every **sufficiently large** collection of objects has an **arbitrarily large** sub-collection whose objects satisfy some **structural properties**.

RAMSEY'S THEOREM

Definition

Given a coloring $f : [\mathbb{N}]^n \rightarrow k$, a set H is *f-homogeneous* if there exists a color $i < k$ such that $f([H]^n) = i$.

Definition (Ramsey's theorem)

Every coloring $f : [\mathbb{N}]^n \rightarrow k$ has an infinite *f-homogeneous* set.

RAMSEY'S THEOREM

Over k -tuples

RT k
 n

Using n colors

RAMSEY'S THEOREM

Fix the number of colors n .

RAMSEY'S THEOREM FOR k -TUPLES

Theorem (Jockusch, 1972)

Every computable coloring $f : [\mathbb{N}]^k \rightarrow n$ has a Π_k^0 infinite f -homogeneous set.

Theorem (Jockusch, 1972)

For every $k \geq 3$, there is a computable coloring $f : [\mathbb{N}]^k \rightarrow n$ such that every infinite f -homogeneous set computes $\emptyset^{(k-2)}$.

RAMSEY'S THEOREM FOR k -TUPLES

Theorem (Simpson, 2009)

For each $k_1, k_2 \geq 3$, $\text{RCA}_0 \vdash \text{RT}_n^{k_1} \leftrightarrow \text{RT}_n^{k_2}$.

What about RT_n^2 ?

RAMSEY'S THEOREM FOR PAIRS

Theorem (Seetapun, 1995)

For every computable coloring $f : [\mathbb{N}]^2 \rightarrow n$ and every non-computable set C , there is an infinite f -homogeneous set $H \not\leq_T C$.

Corollary

RT_n^2 does not imply RT_n^3 over RCA_0 .

HOW MANY APPLICATIONS?

When $3 \leq k_1 < k_2$, the proof of

$$\text{RCA}_0 \vdash \text{RT}_n^{k_1} \rightarrow \text{RT}_n^{k_2}$$

involves multiple applications of $\text{RT}_n^{k_1}$.

How many applications of $\text{RT}_n^{k_1}$ are necessary?

HOW MANY APPLICATIONS?

Theorem (Jockusch, 1972)

For every $k \geq 2$, there is a computable coloring $f : [\mathbb{N}]^k \rightarrow n$ with no Σ_k^0 infinite f -homogeneous set.

Corollary

For every $k \geq 2$, $\text{RT}_n^k \not\leq_c \text{RT}_n^{k+1}$.

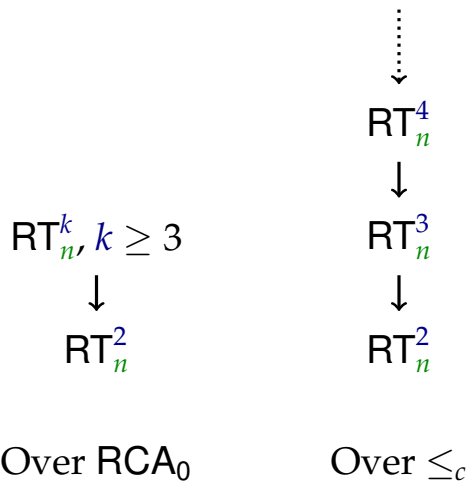
At least 2 applications of RT_n^k are necessary to prove RT_n^{k+1} .

HOW MANY APPLICATIONS?

Theorem (Cholak, Jockusch, Slaman, 2001)

For every $k \geq 2$, every set $P \gg \emptyset^{(k-1)}$, and every computable coloring $f : [\mathbb{N}]^k \rightarrow n$, there is an infinite f -homogeneous set H such that $H' \leq_T P$.

- ▶ At most 3 applications of RT_n^3 are necessary to prove RT_n^4
- ▶ Exactly 2 applications of RT_n^k are necessary to prove RT_n^{k+1} whenever $k \geq 4$.

SUMMARY FOR A FIXED n 

RAMSEY'S THEOREM

Fix the size of tuples k .

RAMSEY'S THEOREM

Theorem (Folklore)

For every $n, m \geq 2$, $\text{RCA}_0 \vdash \text{RT}_n^k \leftrightarrow \text{RT}_m^k$

Proof for $m = n^2$.

- ▶ Take a coloring $f : [\mathbb{N}]^k \rightarrow n^2$
- ▶ Define $g : [\mathbb{N}]^k \rightarrow n$ by merging colors by blocks of size n
- ▶ Apply RT_n^k to g to obtain H such that $|f([H]^2)| \leq n$.
- ▶ Apply again RT_n^k to f restricted to H .

□

HOW MANY APPLICATIONS?

Theorem (Patey)

Fix some $n > m \geq 2$ and n sets B_0, \dots, B_{n-1} whose complements are hyperimmune. For every m -partition $A_0 \cup \dots \cup A_{m-1} = \mathbb{N}$, there exists an infinite subset H of some A_i and a pair $j_0 < j_1 < n$ such that every infinite H -computable set intersects both B_{j_0} and B_{j_1} .

HOW MANY APPLICATIONS?

Theorem

For every $n > m \geq 2$, $\text{RT}_n^2 \not\leq_c \text{RT}_m^2$.

Proof (Part I).

- ▶ Define a Δ_2^0 partition $B_0 \cup \dots \cup B_{n-1} = \mathbb{N}$ such that the \bar{B} 's are **hyperimmune**.
- ▶ Consider its Δ_2^0 approximation function as a computable instance of RT_n^2 .

□

HOW MANY APPLICATIONS?

Theorem

For every $n > m \geq 2$, $\text{RT}_n^2 \not\leq_c \text{RT}_m^2$.

Proof (Part II).

- ▶ Fix computable instance $f : [\mathbb{N}]^2 \rightarrow m$ of RT_m^2 .
- ▶ Construct a **p-cohesive** set C such that the \bar{B}' 's are hyperimmune relative to C .
- ▶ Define $\tilde{f} : \mathbb{N} \rightarrow m$ by $\tilde{f}(x) = \lim_{s \in C} f(x, s)$
- ▶ Apply previous theorem to obtain an infinite \tilde{f} -homogeneous set H such that $H \oplus C$ does not compute an infinite set homogeneous for the B' 's.

□

HOW MANY APPLICATIONS?

Theorem (Patey)

For every $n > m \geq 2$, $\text{RT}_n^k \not\leq_c \text{RT}_m^k$.

Proof.

By induction over $k \geq 2$ using **prehomogeneous** sets. □

SUMMARY FOR A FIXED k

$$RT_n^k, n \geq 2$$

Over RCA_0

$$\begin{array}{c} \vdots \\ \downarrow \\ RT_4^k \\ \downarrow \\ RT_3^k \\ \downarrow \\ RT_2^k \end{array}$$

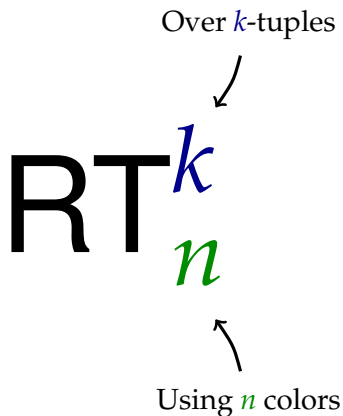
Over \leq_c

RAMSEY'S THEOREM

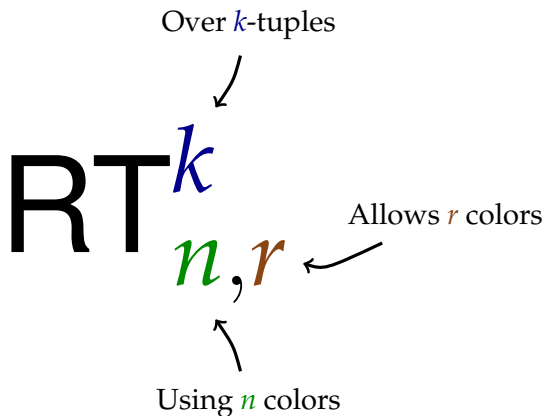
Over k -tuples

RT^k_n

Using n colors



RAMSEY'S THEOREM



THIN SET THEOREM

 TS_n^k $RT_{n,n-1}^k$

ALLOWING MORE COLORS

Theorem (Wang, 2014)

Fix some k and some *sufficiently large* n . For every computable instance f of TS_n^k and every non-computable set C , there is an infinite solution to f which does not compute C .

Corollary

For every k and sufficiently large n , TS_n^k does not imply RT_2^3 over RCA_0 .

ALLOWING MORE COLORS

Theorem (Dorais, Dzhafarov, Hirst, Mileti, Shafer, 2015)

$$\text{RCA}_0 \vdash \text{TS}_{n^s}^{ks+1} \rightarrow \text{TS}_n^{k+1}$$

Theorem (Dorais, Dzhafarov, Hirst, Mileti, Shafer, 2015)

$$\text{RCA}_0 \vdash \text{TS}_{2^k}^{k+2} \rightarrow \text{TS}_2^3$$

ALLOWING MORE COLORS

Theorem (Patey)

For every $n \geq 2$, TS_{n+1}^2 does not imply TS_n^2 over RCA_0 .

Theorem (Patey)

Fix some $m \geq 2$. For every k and sufficiently large n , TS_n^k does not imply TS_m^2 over RCA_0 .

SUMMARY FOR $k = 2$ RT_2^2  TS_3^2  TS_4^2 Over RCA_0

CONCLUSION

- ▶ Computable reducibility gives a more fine-grained analysis than reverse mathematics.
- ▶ Ramsey's theorem is not robust for computable reducibility.
- ▶ Changing the number of allowed colors has a great impact on the strength of Ramsey's theorem.

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QUESTIONS

Thank you for listening!