Large Properties at Small Cardinals

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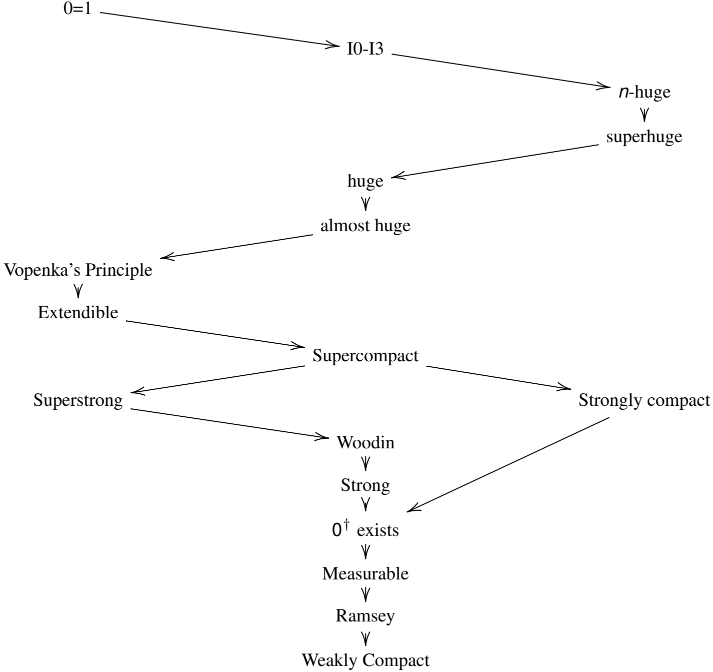
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superhuge
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Vopenka's Principle
Extendible
Supercompact
Superstrong
Strongly compact
Woodin
Strong
0† exists
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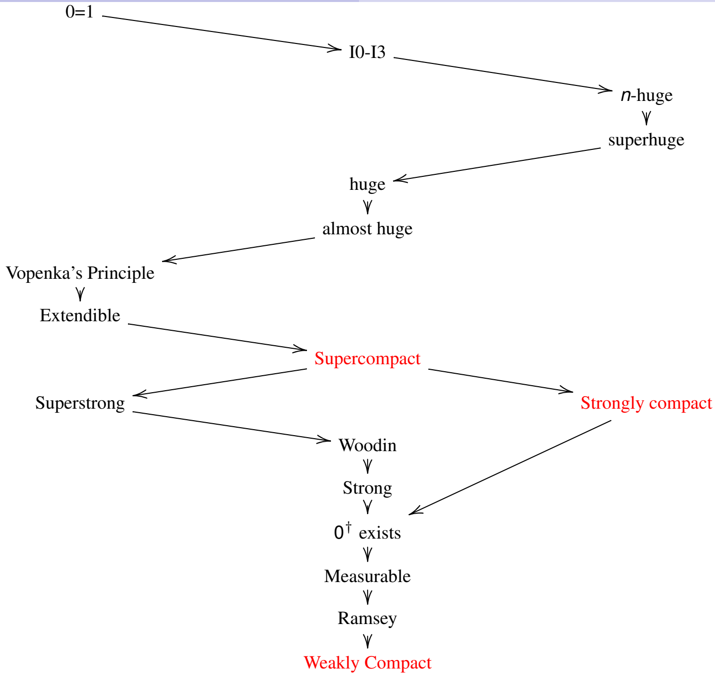
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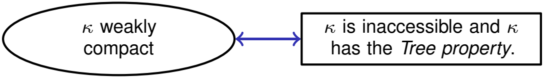
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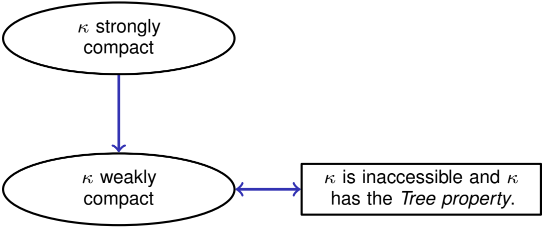
κ is inaccessible and κ has the *Tree property*.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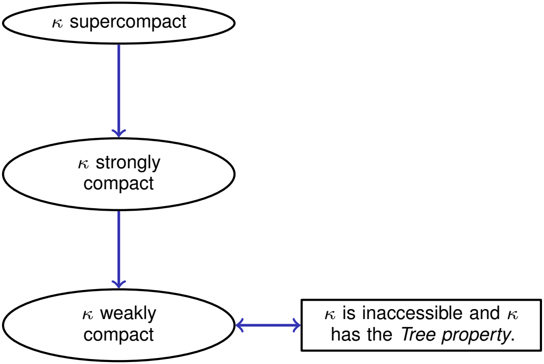
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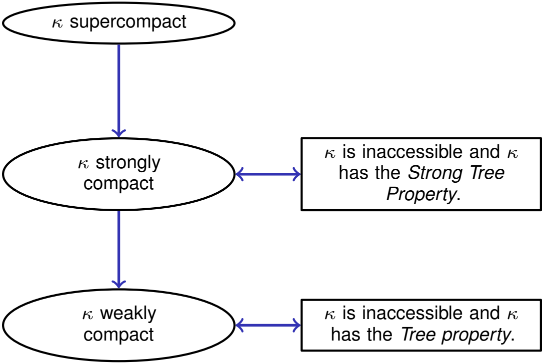
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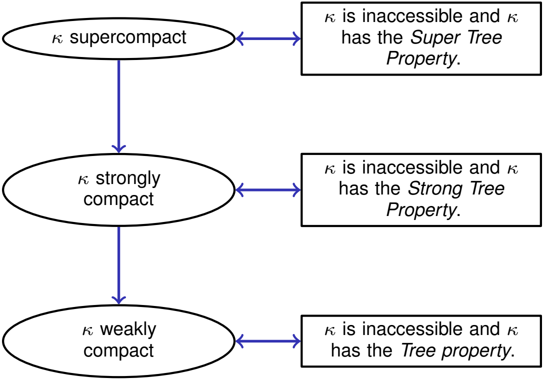
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- What cardinals can satisfy those properties?
- How can we use them?
- Can we find similar characterizations of other large cardinals?

The Tree Property

Let κ be a regular cardinal.

Definition

- A κ-tree is a tree of height κ with levels all of size less than κ.
- we say that κ satisfies the *tree property* if every κ-tree has a cofinal branch.

Theorem

- (König's Lemma) $\aleph_0$ satisfies the tree property;
- (Aronszajn) $\aleph_1$ does not satisfy the tree property;
- (Mitchell) for every $n \geq 2$ if *Cons*($ZFC + \exists\kappa$ weakly compact) then *Cons*($ZFC + \aleph_n$ has the tree property).

The Strong Tree Property

Definition

Let $\lambda \geq \kappa$, a (κ, λ)-tree is a subset $F \subseteq \{f : X \to 2;\ X \in [\lambda]^{<\kappa}\}$ such that:

1. for all $f \in F$, if $X \subseteq dom(f)$, then $f \upharpoonright X \in F$;
2. for all $X \in [\lambda]^{<\kappa}$, $Lev_X(F) := \{f \in F;\ dom(f) = X\} \neq \emptyset$ and has size $< \kappa$.

Definition

A cofinal branch for a (κ, λ)-tree F is a function $b : \lambda \to 2$ such that $b \upharpoonright X \in Lev_X(F)$, for all $X \in [\lambda]^{<\kappa}$

Definition

κ (regular) satisfies the Strong Tree Property if for all $\lambda \geq \kappa$, every (κ, λ)-tree has a cofinal branch.

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The Super Tree Property

Definition

Let F be a (κ, λ)-tree. A sequence $D := \langle d_X;\ X \in [\lambda]^{<\kappa} \rangle$ is an *F-level sequence* if $d_X \in Lev_X(F)$, for all $X \in [\lambda]^{<\kappa}$.

Definition

Let F be a (κ, λ)-tree and $D := \langle d_X;\ X \in [\lambda]^{<\kappa} \rangle$ an F-level sequence. An *ineffable branch* for D is a function $b : \lambda \to 2$ such that

$$\{X \in [\lambda]^{<\kappa};\ b \upharpoonright X = d_X\}$$

is stationary.

Definition

κ satisfies the *Super Tree Property* if, for all $\lambda \geq \kappa$ and for all (κ, λ)-tree F, every F-level sequence has an ineffable branch.

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The Tree Property at Small Cardinals

Mitchell

Let $n \geq 2$, if $Cons(ZFC + \exists\kappa \text{ weakly compact})$, then
$Cons(ZFC + \aleph_n \text{ has the Tree Property })$.

Abraham

If $Cons(ZFC + \exists\kappa < \lambda \text{ such that } \kappa \text{ is supercompact and } \lambda \text{ is weakly compact})$, then
$Cons(ZFC + \aleph_2 \text{ and } \aleph_3 \text{ have the Tree Property })$.

Cummings, Foreman

If $Cons(ZFC + \exists\langle\kappa_n\rangle_{n<\omega} \text{ supercompact cardinals})$, then
$Cons(ZFC + \forall n \geq 2\ (\aleph_n \text{ has the tree property}))$.

The Tree Property at Small Cardinals

Magidor, Shelah +Sinapova

If $Cons(ZFC + \exists\langle\kappa_n\rangle_{n<\omega}$ supercompact cardinals), then
$Cons(ZFC + \aleph_{\omega+1}$ has the tree property)).

Neeman

If $Cons(ZFC + \exists\langle\kappa_n\rangle_{n<\omega}$ supercompact cardinals), then
$Cons(ZFC +$ every regular cardinal $\leq \aleph_{\omega+1}$ has the tree property)).

Friedman, Halilović

If $Cons(ZFC + \exists\langle\kappa$weakly compact hypermeasurable), then
$Cons(ZFC + \aleph_{\omega+2}$ has the tree property)).

Open Question

Is it possible to construct a model where all regular cardinals above $\aleph_1$ simultaneously satisfy the tree property?

Strong Tree Properties at Small Cardinals

Weiss

Let $n \geq 2$, if $Cons(ZFC + \exists\kappa \text{ supercompact })$, then
$Cons(ZFC + \aleph_n \text{ has the Super Tree Property })$.

Fontanella

If $Cons(ZFC + \exists\kappa, \lambda \text{ supercompact cardinals })$, then
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Mitchell's forcing

Mitchell's forcing

Let $\tau < \kappa$ regulars. $(p, q) \in \mathbb{M}(\tau, \kappa)$ iff

1. $p \in \mathbb{P}(\tau, \kappa)$ i.e. $p \in Add(\tau, \kappa)$ and for all $\alpha \in dom(p)$, α is a succ. ordinal
2. $q :]\tau, \kappa[\to V$ of size $\leq \tau$ s.t. every $\alpha \in dom(q)$ is a succ. cardinal and

$$\Vdash_{\mathbb{P}(\tau,\kappa)\restriction\alpha} \; q(\alpha) \in Add(\tau^+)$$

$(p, q) \leq (p', q')$ iff $p \leq p'$, $dom(q') \subseteq dom(q)$, and for every $\alpha \in dom(q')$

$$p \restriction \alpha \Vdash q(\alpha) \leq q'(\alpha).$$

Lemma

$\mathbb{M}(\tau, \kappa)$ is a projection of $\mathbb{P}(\tau, \kappa) \times \mathbb{Q}(\tau, \kappa)$ where $\mathbb{Q}(\tau, \kappa)$ is a τ^+-closed forcing.

Weiss

Let $n \geq 2$, if $Cons(ZFC + \exists\kappa \text{ supercompact })$, then $Cons(ZFC + \aleph_n \text{ has the Super Tree Property })$.

Preserving Branches

The Sunflower Property

Let θ be reg. and $\mathbb{P} \subseteq Add(\theta, ...)$. $\mathbb{P}$ has the θ-sunflower property if for every $\langle p_X;\ X \in [\mu]^{<\theta} \rangle$ with $\mu \geq \theta$ there is

$I \subseteq [\mu]^{<\theta}$ cofinal and $q \in \mathbb{P}$ s.t. for every $X, Y \in I$ there is $Z \in I$ s.t $X, Y \subseteq Z$ and

$$p_X \cap p_Z = q = p_Y \cap p_Z$$

Preservation Lemma 1

Let θ be reg. and let F be a (θ, μ)-tree with $\mu \geq \theta$. Assume $\mathbb{P}$ is a forcing with the θ-sunflower property, then $\mathbb{P}$ does not add cofinal branches to F.

Preservation Lemma 2

Let θ be reg. and F a (θ, μ)-tree with $\mu \geq \theta$. Assume $\mathbb{Q}$ is a η^+-closed forcing with $\eta < \theta \leq 2^\eta$ then $\mathbb{Q}$ does not add cofinal branches to F.

The Super Tree Property at Small Cardinals

Weiss

Let $n \geq 2$, if $Cons(ZFC + \exists\kappa \text{ supercompact })$, then $Cons(ZFC + \aleph_n \text{ has the Super Tree Property })$.

use $\mathbb{M}(\aleph_n, \kappa)$.

κ inacc.
$+$ super tree property

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use $\mathbb{M}(\aleph_n, \kappa)$.

$\kappa = \aleph_{n+2}$
$+$ super tree property

The Super Tree Property at Small Cardinals

Fontanella

For every natural number $n \geq 2$, if *Cons*($ZFC + \exists\kappa, \lambda$ supercompact), then *Cons*($ZFC + \aleph_n$ and $\aleph_{n+1}$ have the Super Tree Property).

λ inacc.
$+$ super tree property

κ inacc.
$+$ super tree property

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For every natural number $n \geq 2$, if $Cons(ZFC + \exists\kappa, \lambda \text{ supercompact })$, then $Cons(ZFC + \aleph_n \text{ and } \aleph_{n+1} \text{ have the Super Tree Property })$.

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$+$ super tree property

$\mathbb{M}(\aleph_0, \kappa)$

$\kappa = \aleph_2$
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For every natural number $n \geq 2$, if $Cons(ZFC + \exists\kappa, \lambda \text{ supercompact })$, then $Cons(ZFC + \aleph_n \text{ and } \aleph_{n+1} \text{ have the Super Tree Property })$.

$\lambda = \aleph_3$
$+$ super tree property

$$\mathbb{M}(\aleph_0, \kappa) * \mathbb{M}(\aleph_1, \lambda)$$

$\kappa = \aleph_2$
$+$ super tree property ?

The Main Forcing

(Cummings-Foreman based on Abraham)

Let $V \subseteq W$ models, in V we let $\mathbb{P} := Add(\tau, \kappa)$ and assume $W \models \tau < \kappa$ reg. ,κ inacc. $\mathbb{P}$ is τ^+-c.c. and τ-distrib. $L \in W$ is a function $L : \kappa \to V_\kappa$ we define in W the poset

$$\mathbb{R}(\tau, \kappa, V, W, L).$$

As usual $\mathbb{R} \upharpoonright 0$ is the trivial forcing. For $\beta \leq \kappa$, $(p, q, f) \in \mathbb{R} \upharpoonright \beta$ iff

1. $p \in \mathbb{P} \upharpoonright \beta = Add(\tau, \beta)^V$;
2. q is a partial fnc. on β of size $\leq \tau$ s.t. for every $\alpha \in dom(q)$, α is a succ. ord., $q(\alpha) \in W^{\mathbb{P}\upharpoonright\alpha}$ and $\Vdash^W_{\mathbb{P}\upharpoonright\alpha} q(\alpha) \in Add(\tau^+)$
3. f is a partial fnc. on β of size $\leq \tau$, for every $\alpha \in dom(f)$, α is a limit ord., $f(\alpha) \in W^{\mathbb{R}\upharpoonright\alpha}$ and

$$\Vdash^W_{\mathbb{R}\upharpoonright\alpha} f(\alpha) \in L(\alpha) \text{ and } L(\alpha) \text{ is a } \tau^+\text{-directed closed forcing}$$

$(p', q', f') \leq (p, q, f)$ iff $p' \leq p$, $p' \upharpoonright \alpha \Vdash^W_{\mathbb{P}\upharpoonright\alpha} q'(\alpha) \leq q(\alpha)$ and $(p', q', f') \upharpoonright \alpha \Vdash^W_{\mathbb{R}\upharpoonright\alpha} f'(\alpha) \leq f(\alpha)$.

The Main Forcing

Intuition

the forcing $\mathbb{R}(\tau, \kappa, V, W)$

- makes κ become τ^{++} while preserving the super tree property at κ,
- anticipates a fragment of the rest of the iteration that we are going to define (using L).

Lemma

$\mathbb{R}(\tau, \kappa, V, W)$ is a projection of $\mathbb{P}(\tau, \kappa)^V \times \mathbb{U}(\tau, \kappa, V, W)$ where $\mathbb{U}(\tau, \kappa, V, W)$ is a τ^+-closed forcing.

Cummings and Foreman's Iteration

Cummings-Foreman's Iteration

$\langle \kappa_n \rangle_{n<\omega}$ supercompact cardinals and $\langle L_n \rangle_{n<\omega}$ Laver functions. $\mathbb{R}_\omega$ is the inverse limit of $\langle \mathbb{R}_n \rangle_{n<\omega}$.

1. $\mathbb{R}_1$ is $\mathbb{Q}_0 := \mathbb{R}(\aleph_0, \kappa_0, V, V, L_0)$;
2. $\mathbb{R}_{n+1} := \mathbb{Q}_0 * \ldots * \dot{\mathbb{Q}}_n$ where $\dot{\mathbb{Q}}_n$ is an $\mathbb{R}_n$-name for

$$\mathbb{R}(\kappa_{n-2}, \kappa_n, V[\dot{G}_{n-1}], V[\dot{G}_n], L_n^*)$$

Cummings and Foreman's Iteration

So at stage $n+1$, we force with
a poset $\mathbb{Q}_n$ that

1. makes $\kappa_n = \aleph_{n+2}$ while preserving the super tree property at κ_n;
2. anticipates a fragment of the tail of the iteration $Tail_{n+1}$ (using L_n);

Fontanella

If $Cons(ZFC + \exists\langle\kappa_n\rangle_{n<\omega}$ supercompact cardinals$)$, then $Cons(ZFC + \forall n \geq 2,\ \aleph_n$ has the Super Tree Property $)$.

We prove that $V[G_\omega] \models \aleph_2$ has the super tree property.

Proof.

In $V[G_\omega]$, fix F an $(\aleph_2, \mu)$-tree and $D := \langle d_X;\ X \in [\mu]^{<\aleph_2}\rangle$ an F- level sequence. In that model $\kappa_0 = \aleph_2$.

Fix an elementary embedding $j : V \to N$ such that:

- $cr(j) = \kappa_0$, $j(\kappa_0) > \sigma$ and ${}^{\sigma}N \subseteq N$, for σ large enough;
- $j(L_0)(\kappa_0)$ is an $\mathbb{R}_1$-name ($\mathbb{Q}_0$-name) for a fragment of $Tail_1$

Step 1 : lift j to an elementary embedding $j : V[G_\omega] \to N[H^*]$ (use the fact that $j(\mathbb{Q}_0) \upharpoonright \kappa_0 + 1 = \mathbb{Q}_0 * j(L_0)(\kappa_0)$);

Step 2 : find an ineffable branch b for D in $N[H^*]$ ($j(F)$ is an $(j(\kappa_0), j(\mu))$-tree and $j[\mu] \in [j(\mu)]^{<j(\kappa_0)}$, the value of $j(d)_{j[\mu]}$ provides an ineffable branch);

Step 3 : prove that $b \in V[G_\omega]$; (use the two preservation lemmas). □

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If $Cons(ZFC + \exists\langle\kappa_n\rangle_{n<\omega}$ supercompact cardinals), then $Cons(ZFC + \forall n \geq 2,\ \aleph_n$ has the Super Tree Property).

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The Strong Tree Property at $\aleph_{\omega+1}$

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If $Cons(ZFC + \exists\langle\kappa_n\rangle_{n<\omega}$ supercompact cardinals$)$, then $Cons(ZFC + \aleph_{\omega+1}$ has the Strong Tree Property$)$.

Magidor, Shelah

If κ is a singular limit of strongly compact cardinals, then κ^+ satisfies the Tree Property.

Key Lemma

If κ is a singular limit of strongly compact cardinals, then κ^+ satisfies the Strong Tree Property.

A Partition Property for Strongly Compact Cardinals

For $S \subseteq [\lambda]^{<\mu}$ cofinal

$$[[\, S \,]]^2 \text{ the set of all pairs } (X, Y) \in S \times S \text{ such that } X \subseteq Y.$$

Definition

κ reg. and $\nu \geq \kappa$, the principle $\varphi(\kappa, \nu^+)$ establishes that f.e. $\lambda \geq \nu^+$ and $S \subseteq [\lambda]^{<\nu^+}$ stationary, every $c : [[\, S \,]]^2 \to \gamma$ with $\gamma < \kappa$ has a quasi homogenous set H of color i which is also stationary, i.e.

$$\text{for every } X, Y \in H \text{ there is } W \supseteq X, Y \text{ in } H \text{ such that } c(X, W) = i = c(Y, W).$$

Theorem

Let κ be a strongly compact cardinal, then $\varphi(\kappa, \nu^+)$ holds for every $\nu \geq \kappa$.

Lemma

If ν is a singular limit of regular cardinals $\langle\kappa_i\rangle_{i<cof(\nu)}$ that satisfy the property $\varphi(\kappa_i,\nu^+)$, then ν^+ has the strong tree property.

Assume $\nu = \lim_{n<\omega}\kappa_n$ with every κ_n satisfying $\varphi(\kappa_n,\nu^+)$. Let F be a (ν^+,μ)-tree. For every $X \in [\mu]^{<\nu^+}$, we assume that $Lev_X(F) = \{f_i^X;\ i < |Lev_X(F)|\}$.

Step 1

There exists $n < \omega$ and a stationary set $S \subseteq [\mu]^{<\nu^+}$, such that for all $X, Y \in S$, there are $\zeta, \eta < \kappa_n$ with $f_\zeta^X \restriction (X \cap Y) = f_\eta^Y \restriction (X \cap Y)$.

proof

Given a function $f \in Lev_X$, we write $\# f = i$ for $i < \nu$, when $f = f_i^X$. Define $c : [[\ [\mu]^{<\nu^+}]]^2 \to \omega$ by $c(X,Y) = \min\{i;\ \#(f_0^Y \restriction X) < \kappa_i\}$. Since $\varphi(\kappa_0,\nu^+)$ holds, there is $n < \omega$ and a stationary quasi homogenous set $S \subseteq [\mu]^{<\nu^+}$ of color n. Then, for every $X, Y \in S$, there is $Z \supseteq X, Y$ in S such that $c(X,Z) = n = c(Y,Z)$. This means that $\#(f_0^Z \restriction X),\ \#(f_0^Z \restriction Y) < \kappa_n$. So, let $\zeta, \eta < \kappa_n$ be such that $f_0^Z \restriction X = f_\zeta^X$ and $f_0^Z \restriction Y = f_\eta^Y$, then $f_\zeta^X \restriction (X \cap Y) = f_0^Z \restriction (X \cap Y) = f_\eta^Y \restriction (X \cap Y)$, as required.

Let n and S be like in the Spine Lemma, we prove the following fact.

Step 2

There is a cofinal $S' \subseteq S$ and an ordinal $\zeta < \kappa_n$ such that for all $X, Y \in S'$, we have $f_\zeta^X \restriction (X \cap Y) = f_\zeta^Y \restriction (X \cap Y)$ (the set S' is even stationary).

proof

For every $(X, Y) \in [[\ S\ ]]^2$, let $\bar{c}(X, Y)$ be the minimum couple $(\zeta, \eta) \in \kappa_n \times \kappa_n$, in the lexicografical order, such that $f_\eta^Y \restriction X = f_\zeta^X$; the function is well defined by definition of n and S. We use $\varphi(\kappa_{n+1}, \nu^+)$ with the function $\bar{c} : [[\ S\ ]]^2 \to \kappa_n \times \kappa_n$. Hence, there exists a quasi homogenous set S' of color $(\zeta, \eta) \in \kappa_n \times \kappa_n$. It follows that for every $X, Y \in S'$, there is $Z \supseteq X, Y$ in S' such that $\bar{c}(X, Z) = (\zeta, \eta) = \bar{c}(Y, Z)$. This means that $f_\eta^Z \restriction X = f_\zeta^X$ and $f_\eta^Z \restriction Y = f_\zeta^Y$, hence $f_\zeta^X \restriction (X \cap Y) = f_\eta^Z \restriction (X \cap Y) = f_\zeta^Y \restriction (X \cap Y)$.

Set $b = \bigcup_{X \in S'} f_\zeta^X$, by the previous lemma b is a function; b is a cofinal branch for F.

Corollary

Let ν be a singular limit of strongly compact cardinals, then ν^+ has the strong tree property.

Fontanella

If $Cons(ZFC + \exists\langle\kappa_n\rangle_{n<\omega}$ supercompact cardinals), then $Cons(ZFC + \aleph_{\omega+1}$ has the Strong Tree Property).

We mimic the proof of the previous lemma. Start with $\langle\kappa_n\rangle_{n<\omega}$ indestructible supercompact. Force to make κ_n the n-th succ. of κ_n and in the resulting model V prove there is $\mu < \kappa_0$ of cof. ω such that $\mathbb{L}(\mu) := Add(\omega, \mu) \times Add(\mu^+, < \kappa_0)$ works. By contrad. there is no such μ. For every μ... $\dot{F}_\mu$ is a counterexample (an $\mathbb{L}(\mu)$-name for a (ν^+, λ_μ)-tree). For every (a, b, μ) with μ... and $(a, b) \in \mathbb{L}(\mu)$ and for every $X, Y \in [\lambda]^{<\nu^+}$ and $\zeta, \eta < \nu$ define

$$(X, \zeta)\ S_i\ (Y, \eta) \iff (a, b) \Vdash (X, \zeta) <_{\dot{F}_\mu} (Y, \eta).$$

Step 1: there is $n < \omega$ and D cofinal such that $\{\ S_i \restriction D \times \kappa_n\ \}_i$ is a spine (using the supercompactness of κ_0) Step 2: find a cofinal branch in V (using the supercompactness of κ_{n+1}).

Strong Tree Properties and the Singular Cardinal Hypothesis

The Singular Cardinal Hypothesis *SCH*

If κ is a singular strong limit cardinal, then $2^\kappa = \kappa^+$.

Solovay

If κ is a strongly compact cardinal, then the singular cardinal hypothesis holds *above* κ.

Question

- Does the strong tree property for a regular cardinal κ entails the singular cardinal hypothesis above κ?
- Does the strong tree property at $\aleph_2$ entails the singular cardinal hypothesis?

Thank you for your attention.

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