Le problème de la justification des axiomes mathématiques et les grands cardinaux en théorie des ensembles

Laura Fontanella

Kurt Gödel Research Center for Mathematical Logic, University of Vienna
http://www.logique.jussieu.fr/ fontanella

28/02/14

Oxford English Dictionary

«An axiom is a self-evident proposition requiring no formal demonstration to prove its truth, but received and assented as soon as mentioned»

Structural axioms

Axioms defining some kind of structure (e.g. groups, rings, vector spaces, topological spaces, Hilbert spaces, etc.).

Foundational axioms

« Either laws of valid reasoning that are supposed to apply to *all* parts of mathematics, or axioms for such fundamental concepts as number, set and function that underlie *all* mathematical concepts. » (Feferman 1999)

What is self-evident?

In 1883 Cantor introduced a "law of thought"

«... fundamental, rich in consequences, and particularly marvellous for its general validity... It is always possible to bring any well-defined set into the form of a well-ordered set.»

Can we trust our intuition of what is self-evident?

An example: the idea that *every property determines a set.*

Russell's paradox

Let A be the set of all sets that do not belong to themselves. Does A belong to itself?

$$A \in A \iff A \notin A$$

Intrinsic justifications

reasons for believing in their obviousness or self evidence

Extrinsic justifications

pragmatic, heuristic justifications, stated in terms of their consequences, or inter theoretic connections, or explanatory power, for example. (It is these extrinsic justifications that often mimic the techniques of natural science)

(Maddy 1988)

Are Zermelo-Fraenkel axioms supported by intrinsic reasons?

The Axiom of Extensionality

Two sets with the same elements are equal.

An extensional notion of set is preferable because it is simpler, clearer and more convenient and because it can simulate intentional notions when needed. (Fraenkel, Bar-Hillel and Levy 1973)

Are Zermelo-Fraenkel axioms supported by intrinsic reasons?

The Axiom of Foundation

Every non empty set contains an element that is disjoint from it.

Zermelo first introduced (in 1906) a weak form of the Axiom of Foundation, namely $A \notin A$, to block Russel's paradox. The stronger versions was suggested only later in 1917 by Mirimanoff and was accepted and included in Zermelo-Fraenkel axioms because...

> «... no field of set their or mathematics is in any general *need* of sets which are not well-founded» (Fraenkel, Bar-Hillel and Levy 1973)

An intrinsic motivation...

The Iterative Conception

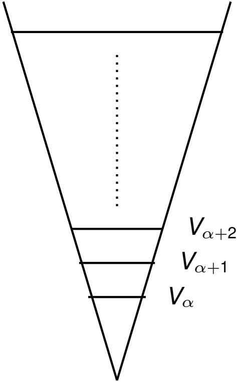
Due to Zermelo (1930). Sets are conceived as being *built up* from below in stages starting from the empty set at the lowest stage.

The Von Neuman Universe

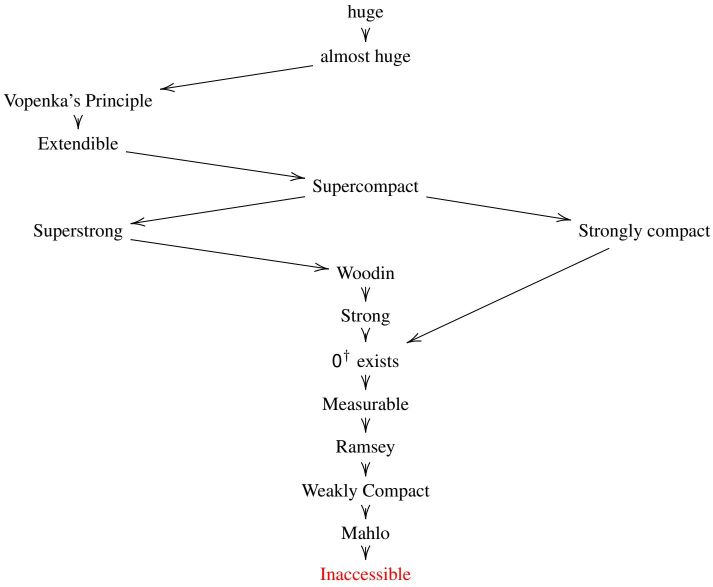
Von Neuman Hierarchy

$$V_0 := \emptyset;$$
$$V_{\alpha+1} := \mathcal{P}(V_\alpha);$$
if α is a limit ordinal,
$$V_\alpha := \bigcup_{\beta<\alpha} V_\beta.$$

$$V := \bigcup_{\alpha \in Ord} V_\alpha$$



The Hierarchy of Large Cardinals



Inaccessible cardinals

Definition

A cardinal κ is weakly inaccessible if it satisfies

- for every cardinal $\gamma < \kappa$, $\gamma^+ < \kappa$;
- for every sequence $\langle \kappa_i \rangle_{i<\gamma}$ of ordinals less than κ with $\gamma < \kappa$, the supremum $\sup_{i<\gamma} \kappa_i < \kappa$.

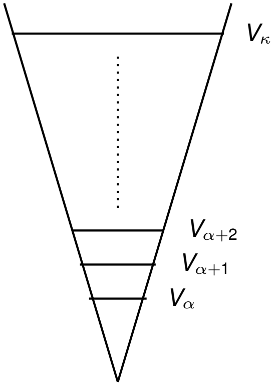
Definition

A cardinal κ is (strongly) inaccessible if it satisfies

- for every cardinal $\gamma < \kappa$, $2^\gamma < \kappa$;
- for every sequence $\langle \kappa_i \rangle_{i<\gamma}$ of ordinals less than κ with $\gamma < \kappa$, the supremum $\sup_{i<\gamma} \kappa_i < \kappa$.

Large cardinals yield a model of *ZFC*

If κ is an inaccessible cardinal, then V_κ is a model of *ZFC*.



Some intrinsic motivations or rules of thumbs in favour of inaccessible cardinals.

Whimsical Identity

$\aleph_0$ could be defined as the inaccessible cardinal. But this identity would seem "accidental" [like the identity between "human" and "featherless biped"]. The set-theoretic universe should be rich enough to rule out such accidental identities. (Kanamori and Magidor 1978 and Martin)

We will see that Whimsical identity may produce hill effects.

Uniformity

The cumulative hierarchy is very rich, so when a certain property occurs at some stage of the hierarchy, it must occur at some higher stage or the universe would lose in complexity.

We will see that Whimsical identity may produce hill effects.

Inexhaustibility

The universe of sets is too complex to be exhausted by any handful of operations, in particular by power set and replacement, the two given by the axioms of Zermelo and Fraenkel. Thus there must be an ordinal number after all the ordinals generated by replacement and power set. This is an inaccessible.

The Maximize rule

«Intrinsic necessity depends on the concept of iterative model. In a general way, hypotheses which purport to enrich the content of power sets... or to introduce more ordinals conform to the intuitive model. *We believe that the collection of all ordinals is very 'long' and each power set (of an infinite set) is very 'thick'. Hence, any axioms to such effects are in accordance with our intuitive concept*» (Wang 1974)

Reflection

The universe of sets is so complex that it cannot be completely described. Therefore, anything true of the universe must already be true of some initial segment of the universe. [Any property holding in V reflects at some V_α] In particular, since V is closed by power set and replacement, there must be some V_α which is also closed by power set and reflection.

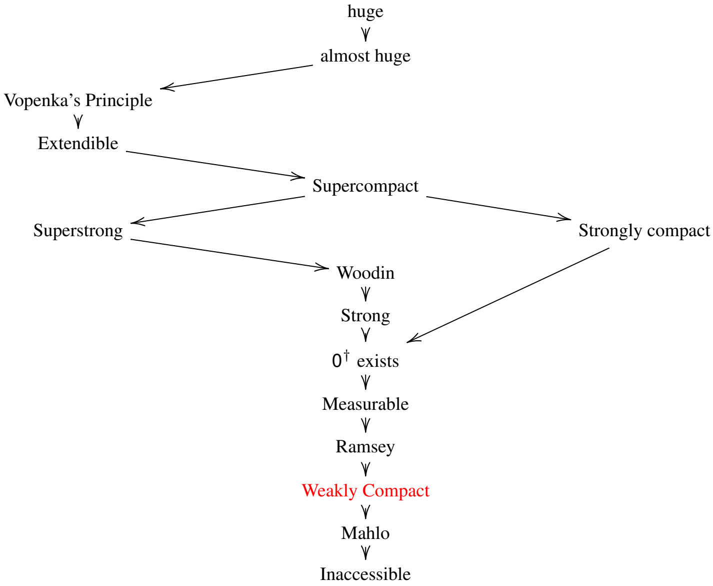
For any natural number n, one can prove from ZFC a reflection principle which says that given any ordinal α, there is an ordinal $\beta > \alpha$ such V_β satisfies all first order sentences of set theory which are true for V and contain fewer than n quantifiers.

Cantorian finitism

The sequence of natural numbers continues to produce interesting complexities arbitrarily far out, so the sequence of transfinite ordinals should do the same.

Objection: while the sequence of natural numbers does continue to produce arbitrarily large prime numbers, it doesn't produce adjacent primes after 2 and 3, or even primes after 2.

The Hierarchy of Large Cardinals



Weakly compact cardinals

Weakly compact cardinals

A cardinal κ is weakly compact if every partition of the pairs of κ into two groups has an homogeneous set of size κ

Thus this is a generalisation of Ramsey' theorem.

Ramsey's theorem holds at $\aleph_0$ so the existence of weakly compact cardinals can be justified by the *uniformity* or *the whimsical identity* arguments.

An ill-effect

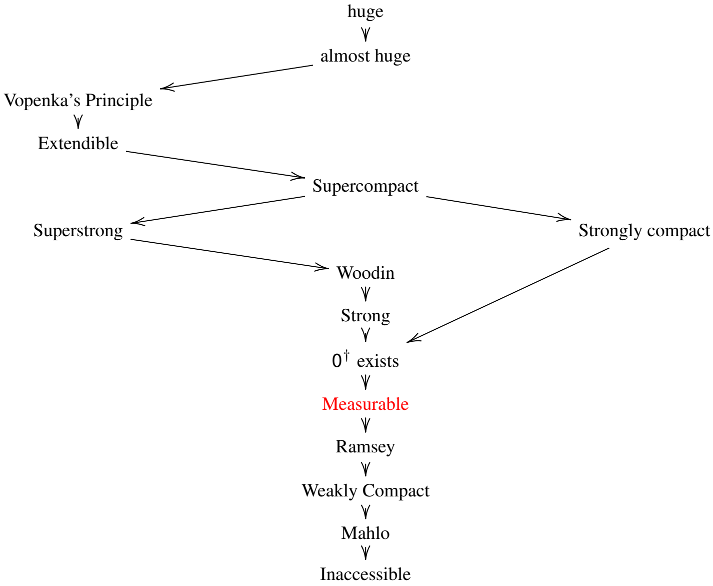
The proof of Ramsey's theorem also gives a homogeneous set for partitions of n-tuples of integers into m groups, but this property cannot be consistently generalised to an uncountable cardinal.

Weak compactness is equivalent to a weak compactness theorem for the language $L_{\kappa,\kappa}$, and to a certain tree property generalising König's lemma etc.

Diversity

«It turned out that weak compactness has many diverse characterisations, which is good evidence for the naturalness and efficacy of the concept. » (Kanamori and Magidor 1978)

The Hierarchy of Large Cardinals



Measurable cardinals

Measurable cardinals

A measure on a cardinal κ is a division of its subsets into large and small in such a way that κ is large, $\emptyset$ and singletons are small, intersections of fewer than κ large sets are remain large, complements of large sets are small, complements of small sets are large.

$\aleph_0$ is measurable. Hence we can appeal to *uniformity*, *whimsical identity*. Also the *Maximize* rule applies.

Theorem

(Scott) Suppose there is a measurable cardinal, then $V \neq L$.

Gödel proved the consistency of *AC* and *CH* through the *Constructible Universe L* which is defined like the V_α except at successor stages where we take $L_{\alpha+1} := Def(L_\alpha)$ where $Def(L_\alpha)$ denotes the set of all definable subsets of L_α.

Theorem

(Gödel 1938) $(L, \in)$ is a model of $AC + CH$

Theorem

(Gödel 1938) If *ZF* is consistent, then $ZF + V = L$ is consistent

Gödel believed *CH* to be *false*.

Gödel 1947

«[In order to determine the value of the continuum, we will need new axioms.] The simplest of these assert the existence of inaccessible numbers. This axiom, roughly speaking, means nothing else but that the totality of sets obtainable by exclusive use of the process of formation of sets expressed in the other axioms forms again a set (and therefore a new basis for a further application of these process). »

but he continued...

Gödel 1947

«there is a little hope of solving [the continuum problem] by means of those axioms of infinity which can be set up on the basis of principles known today ... probably there exist other axioms based on hitherto unknown principles ... which a more profound understanding of the concepts underlying logic and mathematics would enable us to recognise as implied by these concepts»

Small cardinals are consistent with $V = L$. Measurable cardinals are not, so they became viable possibility to settle *CH*.

Contrary to Gödel's hope, the continuum hypothesis is *still* unsolved by these axioms. Since it has been shown to be independent of all the axioms considered so far, assuming their consistency.

All these axioms are preserved under most forcing extensions, and thus, they can be shown to be relatively consistent with both CH and $\neg CH$.

Reinhardt cardinals

A cardinal κ is measurable iff there exists a transitive class M and an elementary embedding $j : V \to M$ such that κ is the critical point (i.e. $j(\kappa) > \kappa$ and $j \restriction \kappa$ is the identity).

Reinhardt cardinal

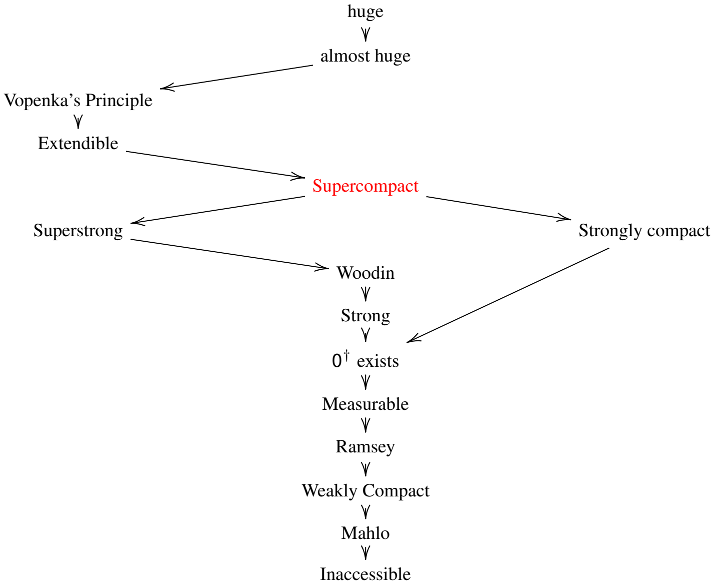
A cardinal κ is Reinhardt if there exists an elementary embedding $j : V \to V$ with critical point κ

Theorem

(Kunen) There are no Reinhardt cardinals.

The proof makes essential use of the axiom of choice. The problem is open without this assumption.

The Hierarchy of Large Cardinals



"Large" Large Cardinals

Definition

A cardinal κ is supercompact if and only if, for every θ, there exists an elementary embedding $j : V \to M$ of V into an inner model M with critical point κ such that $j(\kappa) > \theta$ and M is closed by θ-sequences.

Intrinsic reasons: *Generalization*? *Maximize*?

Tarski (and Gödel)

«We see at the moment no cogent intuitive reasons which could induce us to believe in the existence of ["large" large cardinals], or which at least would make it very plausible that the hypothesis stating the existence of such cardinals is consistent with familiar axioms of set theory. »

Gödel

«However [the new axioms] are supported by rater strong argument from analogy ... [Finally, we may look for axioms which are] so abundant in their verifiable consequences... that quite irrespective of their intrinsic necessity they would have to be assumed in the same sense as any well established physical theory»

The "linearity" of large cardinals.

Kanamori and Magidor 1978

As our edifice grew, we saw how one by one the large cardinals fell into place in a linear hierarchy. This is especially remarkable in view of the ostensibly disparate ideas that motivate their formulation. As remarked by H. Friedman, this hierarchical aspect of the theory of large cardinals is somewhat a mystery.

The Axiom of Determinacy

AD

The axiom of determinacy (abbreviated as AD) states that certain two-person games of length ω with perfect information in which both players choose natural numbers is determined; that is, one of the two players has a winning strategy.

By a theorem of Woodin, ZF+AD is equiconsistent with ZFC + there exists infinitely many Woodin cardinals.

AD is inconsistent with *AC*.

The axiom of determinacy implies that all subsets of the real numbers are Lebesgue measurable, have the property of Baire, and the perfect set property. The last implies a weak form of the continuum hypothesis (namely, that every uncountable set of reals has the same cardinality as the full set of reals).
Moreover, if to the hypothesis of an infinite set of Woodin cardinals is added the existence of a measurable cardinal larger than all of them, a very strong theory of Lebesgue measurable sets of reals emerges, as it is then provable that the axiom of determinacy is true in L(R), and therefore that every set of real numbers in L(R) is determined.

Inner Model Theory

The Inner Model Program seeks generalisations of *L* (inner models i.e. transitive subclasses of V or of a generic extension of V) for large cardinal axioms.

«The large cardinals axioms for which there is an inner model theory are consistent.» (Woodin)

The inner models are defined layer by layer working up through the hierarchy of large cardinal axioms. Each layer provides the foundation for the next, and L is the first layer. In constructing the inner model for a specific large cardinal axiom, one obtains an exhaustive analysis of all weaker large cardinal axioms.

One important use of inner models is the proof of consistency results. If it can be shown that every model of a large cardinal axiom A has an inner model satisfying a large cardinal axiom B, then if A is consistent, B must also be consistent. It is one of the tools used to rank axioms by consistency strength.

The Universe View

Behind these intrinsic (or extrinsic) motivations for large cardinals

1. There is a *unique* absolute background concept of set, *the* universe of sets. The goal of set theorists is to find *the* theory of sets that appropriately describe this universe.
2. we have an *intuition* of this universe of sets (it is rich, complex etc.)
3. every set theoretic assertion has a *definite truth value*. Axioms are (or should be) *true*.

Intrinsic and extrinsic motivations often get confused.

(Zermelo 1908, about the axiom of choice) « That this axiom, even though it was never formulated in textbook style, has frequently been used, and successfully at that, in the most diverse fields of mathematics ... is an indisputable fact. Such an extensive use of a principle can be explained only by its *self-evidence*, which of course, must not be confused with its provability. No matter if this self-evidence is to a certain degree subjective – it is surely a necessary source of mathematical principles.

Are extrinsic motivations good criterions for *truth*?

$0 = 1$ has many wonderful consequences :)

Even a platonist might refuse 2 or might be convinced that these aren't the "right" intuitions of the universe of sets.

Even a platonist might refuse 1 and 3 (ex. Hamkins' multiverse view)

A different (non-platonistic) view

Ant theory (axiomatic system) is respectable unless (until) it leads us to a contradiction.

- 1 (hence 2) is refused. There are different respectable/interesting concepts of sets, different respectable/interesting theories of sets.
- 3 is refused. None of these theories of sets are "true", nor they are the "right ones".
- each theory gives us an interesting framework (ex. non euclidean geometries)

Axioms are neither true nor false, but we can bet that we will never discovery their inconsistency.

Woodin's prediction

«There will be no discovery ever of an inconsistency in [the theory "ZFC+ there exists infinitely many Woodin cardinals"]» (Woodin)

Theories with better *explanatory power* are preferable. *Fruitful* theories are more interesting.

Despite Godel's hope, large cardinals did not solve the continuum problem.

Applications of large large cardinals only occurred as consistency results.

Merci