From Large Cardinals to Large Combinatorial Properties

Laura Fontanella

Kurt Gödel Research Center for Mathematical Logic, University of Vienna
http://www.logique.jussieu.fr/ fontanella

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Inaccessible cardinals

Definition

A cardinal κ is weakly inaccessible if it satisfies

- for every cardinal $\gamma < \kappa$, $\gamma^+ < \kappa$;
- for every sequence $\langle \kappa_i \rangle_{i<\gamma}$ of ordinals less than κ with $\gamma < \kappa$, the supremum $\sup_{i<\gamma} \kappa_i < \kappa$.

Definition

A cardinal κ is (strongly) inaccessible if it satisfies

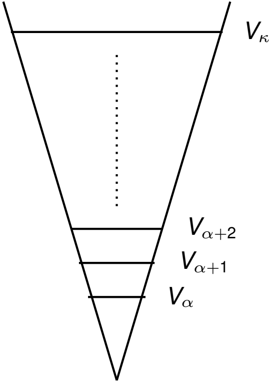
- for every cardinal $\gamma < \kappa$, $2^\gamma < \kappa$;
- for every sequence $\langle \kappa_i \rangle_{i<\gamma}$ of ordinals less than κ with $\gamma < \kappa$, the supremum $\sup_{i<\gamma} \kappa_i < \kappa$.

Inaccessible cardinals imply *ZFC* is consistent

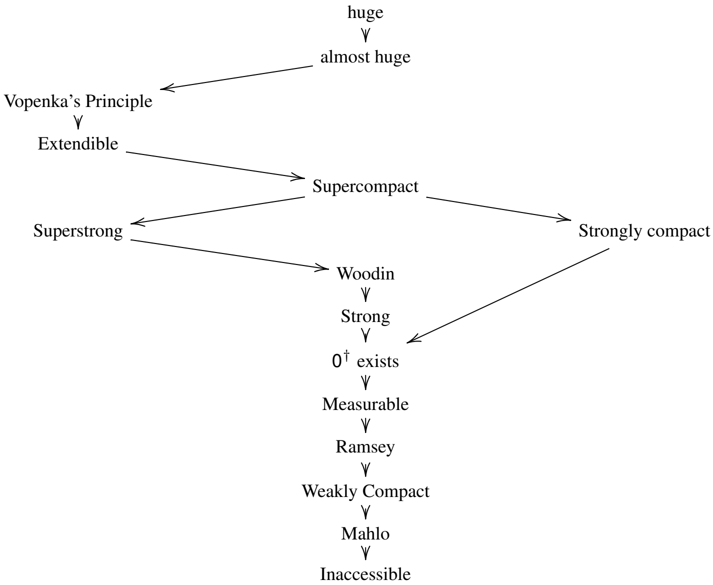
Von Neuman Hierarchy

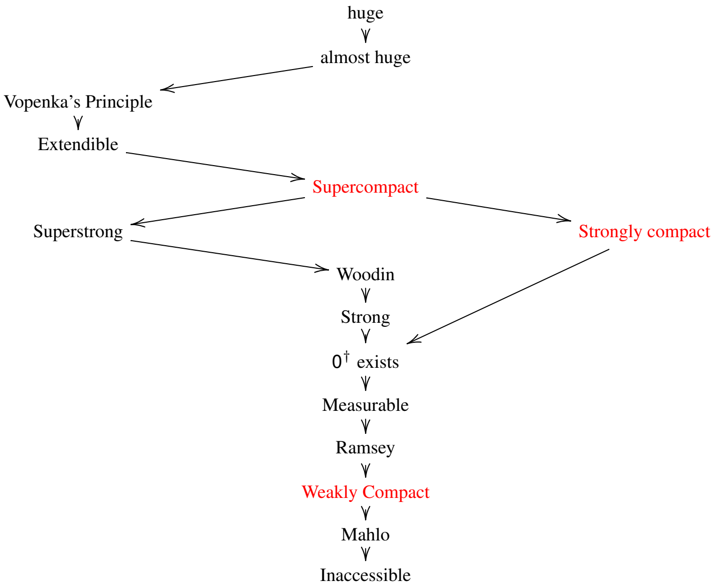
$$V_0 := \emptyset;$$
$$V_{\alpha+1} := \mathcal{P}(V_\alpha);$$
if α is a limit ordinal,
$$V_\alpha := \bigcup_{\beta<\alpha} V_\beta.$$

$V := \bigcup_{\alpha \in Ord} V_\alpha$ is the set-theoretic universe.



The hierarchy of large cardinals




huge
almost huge
Vopenka's Principle
Extendible
Supercompact
Superstrong
Strongly compact
Woodin
Strong
$0^\dagger$ exists
Measurable
Ramsey
Weakly Compact
Mahlo
Inaccessible

Weakly compact, Strongly compact and Supercompact cardinals

Definition

Given a cardinal κ,

- κ is weakly compact if any collection of sentences of the the infinitary language $L_{\kappa,\kappa}$ using at most κ non-logical symbols, if κ-satisfiable, is satisfiable;
- κ is strongly compact if any collection of sentences of the the infinitary language $L_{\kappa,\kappa}$, if κ-satisfiable, is satisfiable;

Proposition

A cardinal κ is strongly compact if and only if, for every θ, there exists an elementary embedding $j : V \to M$ of V into an inner model M with critical point κ such that for every $X \subseteq M$ of size $\leq \theta$, there exists $Y \in M$ such that $X \subseteq Y$ and $M \models |Y| < j(\kappa)$.

Definition

A cardinal κ is supercompact if and only if, for every θ, there exists an elementary embedding $j : V \to M$ of V into an inner model M with critical point κ such that $j(\kappa) > \theta$ and M is closed by θ-sequences.

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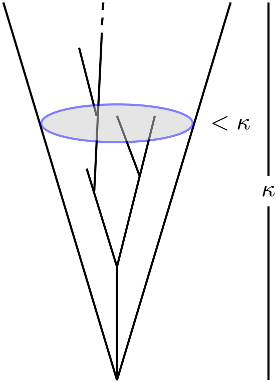
The Tree Property

A κ-tree, for a regular κ, is a tree of height κ and levels of size $< \kappa$.



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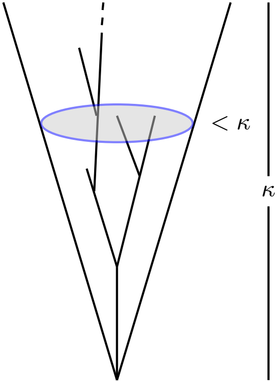
A κ-tree, for a regular κ, is a tree of height κ and levels of size $< \kappa$.

A regular cardinal κ satisfies the tree property if, and only if, every κ-tree has a cofinal branch.



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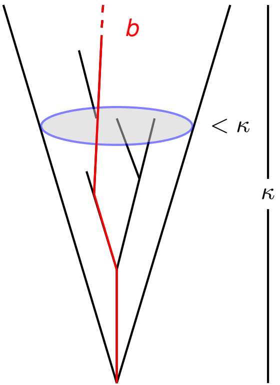

The Tree Property

Let κ be a regular cardinal.

Theorem

- (König's Lemma 1936) $\aleph_0$ satisfies the tree property;
- (Aronszajn 1934) $\aleph_1$ does not satisfy the tree property;
- (Specker 1949) If $\tau^{<\tau} = \tau$, then the tree property fails at τ^+;
- (Mitchell 1972) If *Cons*(*ZFC* $+ \exists\kappa \text{ weakly compact}$), then for every regular τ such that $\tau^{<\tau} = \tau$, we have *Cons*(*ZFC* $+ \tau^{++} \text{ has the tree property}$).

Erdös & Tarski 1961

κ is weakly compact iff it is inaccessible and it satisfies the tree property.

Jech 1973, Di Prisco & Zwicker 1980, Donder & Weiss 2010

κ is strongly compact iff it is inaccessible and it satisfies the strong tree property.

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The Strong Tree Property

Definition

Let $\lambda \geq \kappa$, a (κ, λ)-tree is a subset $F \subseteq \{f : X \to 2;\ X \in [\lambda]^{<\kappa}\}$ such that:

1. for all $f \in F$, if $X \subseteq dom(f)$, then $f \restriction X \in F$;
2. for all $X \in [\lambda]^{<\kappa}$, $Lev_X(F) := \{f \in F;\ dom(f) = X\} \neq \emptyset$ and has size $< \kappa$.

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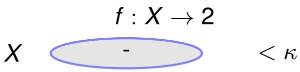

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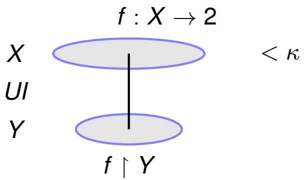

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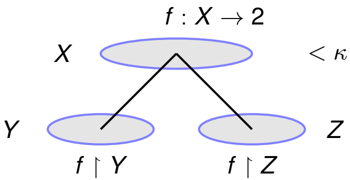

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Definition

A cofinal branch for a (κ, λ)-tree F is a function $b : \lambda \to 2$ such that $b \restriction X \in Lev_X(F)$, for all $X \in [\lambda]^{<\kappa}$

Definition

κ (regular) satisfies the Strong Tree Property if for all $\lambda \geq \kappa$, every (κ, λ)-tree has a cofinal branch.

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The Super Tree Property

Definition

Let F be a (κ, λ)-tree. A sequence $D := \langle d_X;\ X \in [\lambda]^{<\kappa} \rangle$ is an *F-level sequence* if $d_X \in Lev_X(F)$, for all $X \in [\lambda]^{<\kappa}$.

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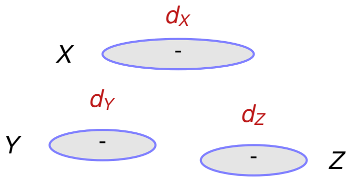

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Definition

Let F be a (κ, λ)-tree and $D := \langle d_X;\ X \in [\lambda]^{<\kappa} \rangle$ an F-level sequence. An ineffable branch for D is a cofinal branch $b : \lambda \to 2$ such that

$$\{X \in [\lambda]^{<\kappa};\ b \upharpoonright X = d_X\}$$

is stationary.

Definition

κ satisfies the Super Tree Property if, for all $\lambda \geq \kappa$ and for all (κ, λ)-tree F, every F-level sequence has an ineffable branch.

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Fontanella 2012

If $Cons(ZFC + \exists \langle \kappa_n \rangle_{n<\omega} \text{ supercompact cardinals})$, then $Cons(ZFC + \forall n \geq 2,\ \aleph_n \text{ has the Super Tree Property })$.

Fontanella 2012

If $Cons(ZFC + \exists \langle \kappa_n \rangle_{n<\omega} \text{ supercompact cardinals})$, then $Cons(ZFC + \aleph_{\omega+1} \text{ has the Strong Tree Property })$.

The Super Tree Property at Small Cardinals

Weiss

Let $n \geq 2$, if $Cons(ZFC + \exists\kappa \text{ supercompact })$, then $Cons(ZFC + \aleph_n \text{ has the Super Tree Property })$.

The Super Tree Property at Small Cardinals

Weiss

Let $n \geq 2$, if *Cons*($ZFC + \exists\kappa$ supercompact), then *Cons*($ZFC + \aleph_n$ has the Super Tree Property).

use $\mathbb{M}(\aleph_n, \kappa)$.

$\kappa = \aleph_{n+2}$
$+$ super tree property

The Super Tree Property at Small Cardinals

Fontanella - Main Theorem 1

If $Cons(ZFC + \exists\langle\kappa_n\rangle_{n<\omega}$ supercompact cardinals), then $Cons(ZFC + \forall n \geq 2,\ \aleph_n$ has the Super Tree Property).

λ inaccessible
+ super tree property

κ inaccessible
+ super tree property

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λ inaccessible
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$\mathbb{M}(\aleph_0, \kappa)$

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$\lambda = \aleph_3$ + super tree property

$\kappa = \aleph_2$ + super tree property ?

$$\mathbb{M}(\aleph_0, \kappa) * \mathbb{M}(\aleph_1, \lambda)$$

Cummings and Foreman's Iteration

$\langle \kappa_n \rangle_{n<\omega}$ supercompact cardinals

At stage $n+1$, we force with $\mathbb{Q}_n$.

1. $\mathbb{Q}_n$ makes $\kappa_n = \aleph_{n+2}$ while preserving the super tree property at κ_n;
2. $\mathbb{Q}_n$ anticipates the tail of the iteration $Tail_{n+1}$ (using the Laver function L_n).

κ_n inaccessible
+ super tree property

κ_1 inaccessible
+ super tree property

κ_0 inaccessible
+ super tree property

κ_n inaccessible
$+$ super tree property

$\mathbb{Q}_0$

κ_1 inaccessible
$+$ super tree property

κ_0 $= \aleph_2$
$+$ super tree property

κ_n inaccessible
$+$ super tree property

$\mathbb{Q}_0 * \dot{\mathbb{Q}}_1$

$\kappa_1 = \aleph_3$
$+$ super tree property

$\kappa_0 = \aleph_2$
$+$ super tree property

$\kappa_n = \aleph_{n+2}$ + super tree property

$\mathbb{Q}_0 * \dot{\mathbb{Q}}_1 * \cdots * \dot{\mathbb{Q}}_n * \cdots$

$\kappa_1 = \aleph_3$ + super tree property

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The Strong Tree Property at $\aleph_{\omega+1}$

Fontanella - Main Theorem 2

If $Cons(ZFC + \exists\langle\kappa_n\rangle_{n<\omega} \text{ supercompact cardinals})$, then $Cons(ZFC + \aleph_{\omega+1} \text{ has the Strong Tree Property})$.

Magidor & Shelah 1996

If ν is a singular limit of strongly compact cardinals, then ν^+ satisfies the Tree Property.

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ν^+ strong tree property

ν

κ_n strongly compact

κ_1 strongly compact

κ_0 strongly compact

ν^+ strong tree property

ν

κ_n strongly compact

$$\mathbb{L}_\mu := \mathit{Coll}(\omega, \mu) \times \mathit{Coll}(\mu^+, < \kappa_0) \times \prod_{n<\omega} \mathit{Coll}(\kappa_n, < \kappa_{n+1})$$

κ_1 strongly compact

κ_0 strongly compact

μ $\mathit{cof}(\mu) = \omega$

ω

$\nu^+ = \aleph_{\omega+1}$ strong tree property

$\nu = \aleph_\omega$

$\kappa_n = \aleph_{n+2}$

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$\kappa_1 = \aleph_3$

$\kappa_0 = \aleph_2$

μ $\mathit{cof}(\mu) = \omega$

ω

From large cardinals to large combinatorial properties

- Can every regular cardinal satisfy the tree property, strong tree property or the super tree property?
- What are the consequences of the "strong compactnesss" or "supercompactness" of these small cardinals?
- Can we characterize all large cardinals in terms of combinatorial properties?

Thank you.