Large Properties at Small Cardinals

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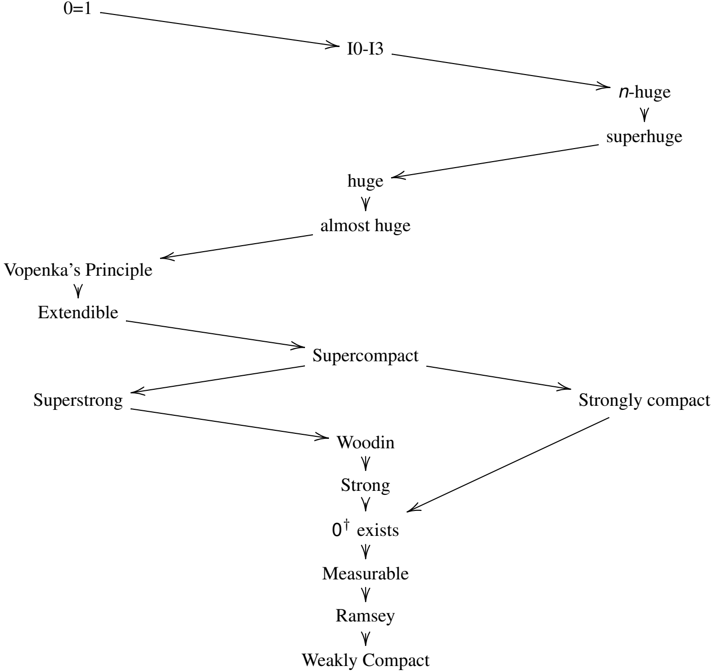
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Superstrong
Strongly compact
Woodin
Strong
$0^\dagger$ exists
Measurable
Ramsey
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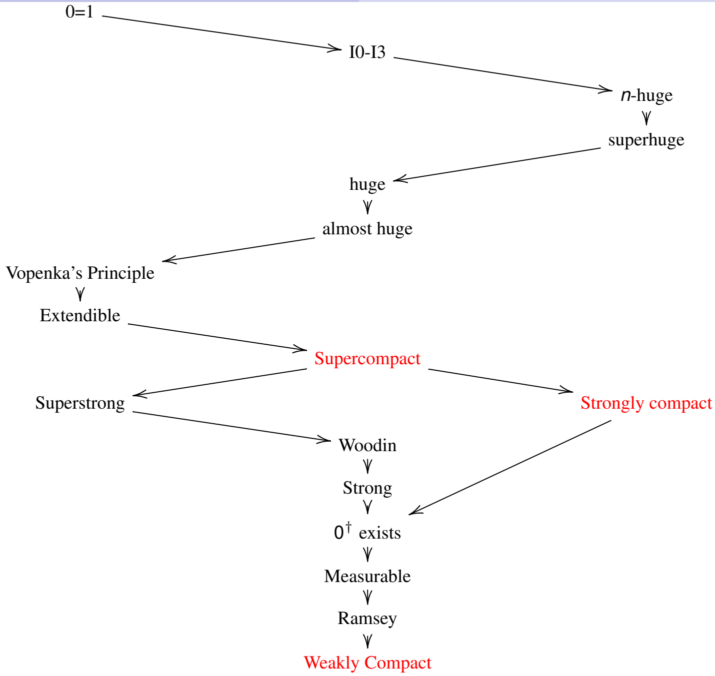
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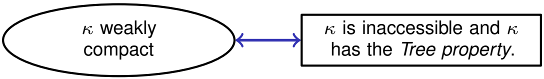
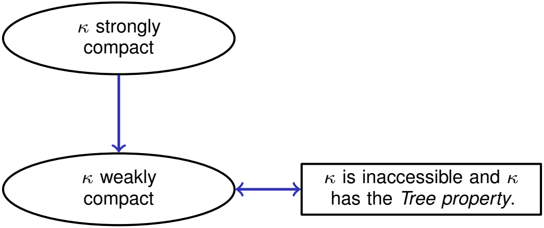

κ weakly compact
κ is inaccessible and κ has the *Tree property*.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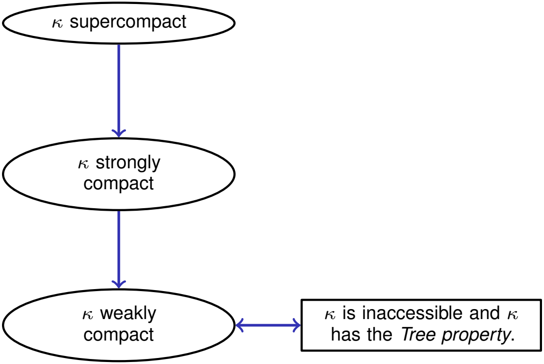
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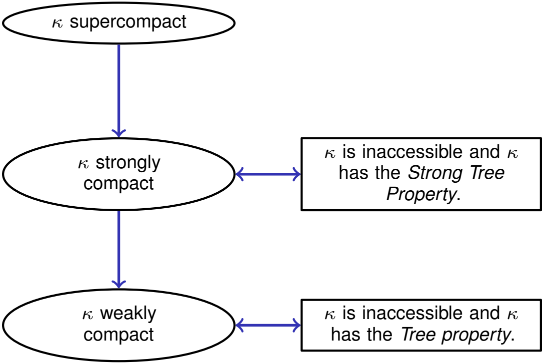
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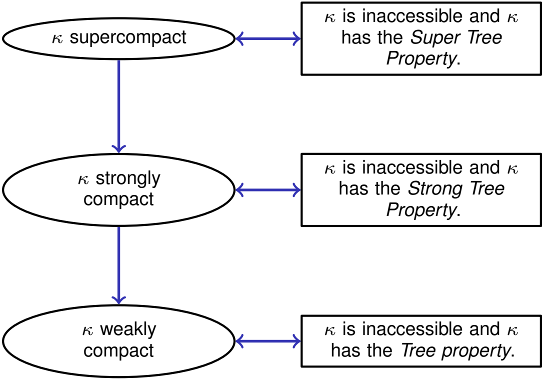
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The Tree Property

Let κ be a regular cardinal.

Definition

- A κ-tree is a tree of height κ with levels all of size less than κ.
- we say that κ satisfies the *tree property* if every κ-tree has a cofinal branch.

Theorem

- (König's Lemma) $\aleph_0$ satisfies the tree property;
- (Aronszajn) $\aleph_1$ does not satisfy the tree property;
- (Mitchell) for every $n \geq 2$ if *Cons*(*ZFC* $+ \exists\kappa$ weakly compact) then *Cons*(*ZFC* $+ \aleph_n$ has the tree property).

The Strong Tree Property

Definition

Let $\lambda \geq \kappa$ and $Fn(\kappa, \lambda) := \{f : X \to 2;\ X \in [\lambda]^{<\kappa}\}$. A (κ, λ)-tree is a subset $F \subseteq Fn(\kappa, \lambda)$ such that:

1. for all $f \in F$, if $X \subseteq dom(f)$, then $f \upharpoonright X \in F$;
2. for all $X \in [\lambda]^{<\kappa}$, $Lev_X(F) := \{f \in F;\ dom(f) = X\} \neq \emptyset$ and has size $< \kappa$.

Definition

A cofinal branch for a (κ, λ)-tree F is a function $b : \lambda \to 2$ such that $b \upharpoonright X \in Lev_X(F)$, for all $X \in [\lambda]^{<\kappa}$

Definition

κ (regular) satisfies the Strong Tree Property if for all $\lambda \geq \kappa$, every (κ, λ)-tree has a cofinal branch.

The Super Tree Property

Definition

Let F be a (κ, λ)-tree. A sequence $D := \langle d_X;\ X \in [\lambda]^{<\kappa} \rangle$ is an F-level sequence if $d_X \in Lev_X(F)$, for all $X \in [\lambda]^{<\kappa}$.

Definition

Let F be a (κ, λ)-tree and $D := \langle d_X;\ X \in [\lambda]^{<\kappa} \rangle$ an F-level sequence. An ineffable branch for D is a function $b : \lambda \to 2$ such that

$$\{X \in [\lambda]^{<\kappa};\ b \restriction X = d_X\}$$

is stationary.

Definition

κ satisfies the Super Tree Property if, for all $\lambda \geq \kappa$ and for all (κ, λ)-tree F, every F-level sequence has an ineffable branch.

The Super Tree Property

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Let F be a (κ, λ)-tree. A sequence $D := \langle d_X;\ X \in [\lambda]^{<\kappa} \rangle$ is an F-level sequence if $d_X \in Lev_X(F)$, for all $X \in [\lambda]^{<\kappa}$.

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Definition

κ satisfies the Super Tree Property if, for all $\lambda \geq \kappa$ and for all (κ, λ)-tree F, every F-level sequence has an ineffable branch.

Strong Tree Properties at Small Cardinals

Weiss

Let $n \geq 2$, if *Cons*($ZFC + \exists\kappa$ supercompact), then
Cons($ZFC + \aleph_n$ has the Super Tree Property).

Fontanella

If *Cons*($ZFC + \exists\kappa, \lambda$ supercompact cardinals), then
Cons($ZFC + \aleph_2$ and $\aleph_3$ have the Super Tree Property).

Fontanella

If *Cons*($ZFC + \exists\langle\kappa_n\rangle_{n<\omega}$ supercompact cardinals), then
Cons($ZFC + \forall n \geq 2,\ \aleph_n$ has the Super Tree Property).

Fontanella

If *Cons*($ZFC + \exists\langle\kappa_n\rangle_{n<\omega}$ supercompact cardinals), then
Cons($ZFC + \aleph_{\omega+1}$ has the Strong Tree Property).

The Super Tree Property at Small Cardinals

Weiss

Let $n \geq 2$, if *Cons*(*ZFC* $+ \exists\kappa$ supercompact), then *Cons*(*ZFC* $+ \aleph_n$ has the Super Tree Property).

use $\mathbb{M}(\aleph_n, \kappa)$.

κ inacc.
$+$ super tree property

The Super Tree Property at Small Cardinals

Weiss

Let $n \geq 2$, if $Cons(ZFC + \exists\kappa \text{ supercompact })$, then $Cons(ZFC + \aleph_n \text{ has the Super Tree Property })$.

use $\mathbb{M}(\aleph_n, \kappa)$.

$\kappa = \aleph_{n+2}$

$+$ super tree property

The Super Tree Property at Small Cardinals

Fontanella

For every natural number $n \geq 2$, if $Cons(ZFC + \exists \kappa, \lambda \text{ supercompact })$, then $Cons(ZFC + \aleph_n \text{ and } \aleph_{n+1} \text{ have the Super Tree Property })$.

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$\mathbb{M}(\aleph_0, \kappa)$

$\kappa = \aleph_2$
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For every natural number $n \geq 2$, if *Cons*(*ZFC* $+ \exists\kappa, \lambda$ supercompact), then *Cons*(*ZFC* $+ \aleph_n$ and $\aleph_{n+1}$ have the Super Tree Property).

$\lambda = \aleph_3$ + super tree property

$\kappa = \aleph_2$ + super tree property ?

$$\mathbb{M}(\aleph_0, \kappa) * \mathbb{M}(\aleph_1, \lambda)$$

Cummings and Foreman's Iteration

$\langle \kappa_n \rangle_{n<\omega}$ supercompact cardinals and $\langle L_n \rangle_{n<\omega}$ Laver functions. $\mathbb{R}_\omega$ is an interation of length ω. At stage $n+1$, force with
a poset $\mathbb{Q}_n$ that

1. makes $\kappa_n = \aleph_{n+2}$ while preserving the super tree property at κ_n;
2. anticipates a fragment of the tail of the iteration $Tail_{n+1}$

Then take the inverse limite.

Fontanella

If $Cons(ZFC + \exists\langle\kappa_n\rangle_{n<\omega}$ supercompact cardinals), then $Cons(ZFC + \forall n \geq 2,\ \aleph_n$ has the Super Tree Property).

We prove that $V[G_\omega] \models \aleph_2$ has the super tree property.

Proof.

In $V[G_\omega]$, fix F an $(\aleph_2, \mu)$-tree and $D := \langle d_X;\ X \in [\mu]^{<\aleph_2}\rangle$ an F- level sequence. In that model $\kappa_0 = \aleph_2$.

Fix an elementary embedding $j : V \to N$ such that:

- $cr(j) = \kappa_0$, $j(\kappa_0) > \sigma$ and ${}^{\sigma}N \subseteq N$, for σ large enough;
- $j(L_0)(\kappa_0)$ is an $\mathbb{R}_1$-name ($\mathbb{Q}_0$-name) for a fragment of $Tail_1$

Step 1 : lift j to an elementary embedding $j : V[G_\omega] \to N[H^*]$ (use the fact that $j(\mathbb{Q}_0) \restriction \kappa_0 + 1 = \mathbb{Q}_0 * j(L_0)(\kappa_0)$);

Step 2 : find an ineffable branch b for D in $N[H^*]$ ($j(F)$ is an $(j(\kappa_0), j(\mu))$-tree and $j[\mu] \in [j(\mu)]^{<j(\kappa_0)}$, the value of $j(d)_{j[\mu]}$ provides an ineffable branch);

Step 3 : prove that $b \in V[G_\omega]$; (use two preservation lemmas). □

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Step 3 : prove that $b \in V[G_\omega]$; (use two preservation lemmas). □

Preserving Branches

Preservation Lemma 1

Let θ be reg. and F a (θ, μ)-tree with $\mu \geq \theta$. Assume $\mathbb{Q}$ is a η^+-closed forcing with $\eta < \theta \leq 2^\eta$ then $\mathbb{Q}$ does not add cofinal branches to F.

The Sunflower Property

Let θ be reg. and $\mathbb{P} \subseteq Add(\theta, ...)$. $\mathbb{P}$ has the θ-sunflower property if for every $\langle p_X;\ X \in [\mu]^{<\theta} \rangle$ with $\mu \geq \theta$ there is

$$I \subseteq [\mu]^{<\theta} \text{ and } q \in \mathbb{P} \text{ s.t. for every } X, Y \in I \text{ there is } Z \in I \text{ s.t } X, Y \subseteq Z \text{ and}$$

$$p_X \cap p_Z = q = p_Y \cap p_Z$$

Preservation Lemma 2

Let θ be reg. and let F be a (θ, μ)-tree with $\mu \geq \theta$. Assume $\mathbb{P}$ is a forcing with the θ-sunflower property, then $\mathbb{P}$ does not add cofinal branches to F.

The Strong Tree Property at $\aleph_{\omega+1}$

Fontanella

If *Cons*(*ZFC* $+ \exists\langle\kappa_n\rangle_{n<\omega}$ supercompact cardinals), then *Cons*(*ZFC* $+ \aleph_{\omega+1}$ has the Strong Tree Property).

Magidor, Shelah

If κ is a singular limit of strongly compact cardinals, then κ^+ satisfies the Tree Property.

Key Lemma

If κ is a singular limit of strongly compact cardinals, then κ^+ satisfies the Strong Tree Property.

A Partition Property for Strongly Compact Cardinals

For $S \subseteq [\lambda]^{<\mu}$ cofinal

$$[[\, S \,]]^2 \text{ the set of all pairs } (X, Y) \in S \times S \text{ such that } X \subseteq Y.$$

Definition

κ reg. and $\nu \geq \kappa$, the principle $\varphi(\kappa, \nu^+)$ establishes that f.e. $\lambda \geq \nu^+$ and $S \subseteq [\lambda]^{<\nu^+}$ stationary, every $c : [[\, S \,]]^2 \to \gamma$ with $\gamma < \kappa$ has a quasi homogenous set H of color i which is also stationary, i.e.

$$\text{for every } X, Y \in H \text{ there is } W \supseteq X, Y \text{ in } H \text{ such that } c(X, W) = i = c(Y, W).$$

Theorem

Let κ be a strongly compact cardinal, then $\varphi(\kappa, \nu^+)$ holds for every $\nu \geq \kappa$.

Lemma

If $\nu = \lim_{n<\omega} \kappa_n$ and $\varphi(\kappa_n, \nu^+)$, then ν^+ has the Strong Tree Property.

Thank you for your attention.

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