Large Properties for Small Cardinals

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Definition

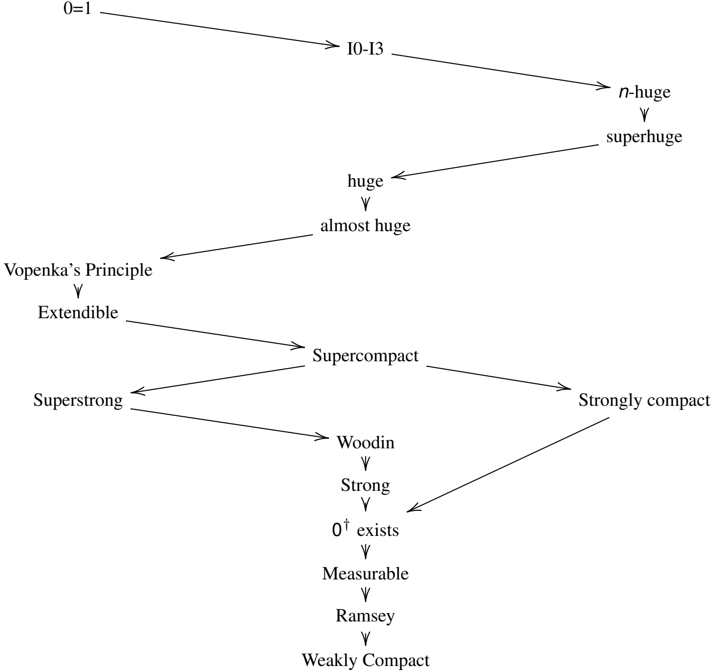
A cardinal κ is *inaccessible* if it is regular and strong limit (i.e. for every $\lambda < \kappa$, $2^{\lambda} < \kappa$).

Theorem

If there exists an inaccessible cardinal κ*, then* $\langle V_{\kappa}, \in \rangle \models ZFC$.

Corollary

The existence of an inaccessible cardinal is not provable in ZFC.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n-huge
superhuge
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almost huge
Vopenka's Principle
Extendible
Supercompact
Superstrong
Strongly compact
Woodin
Strong
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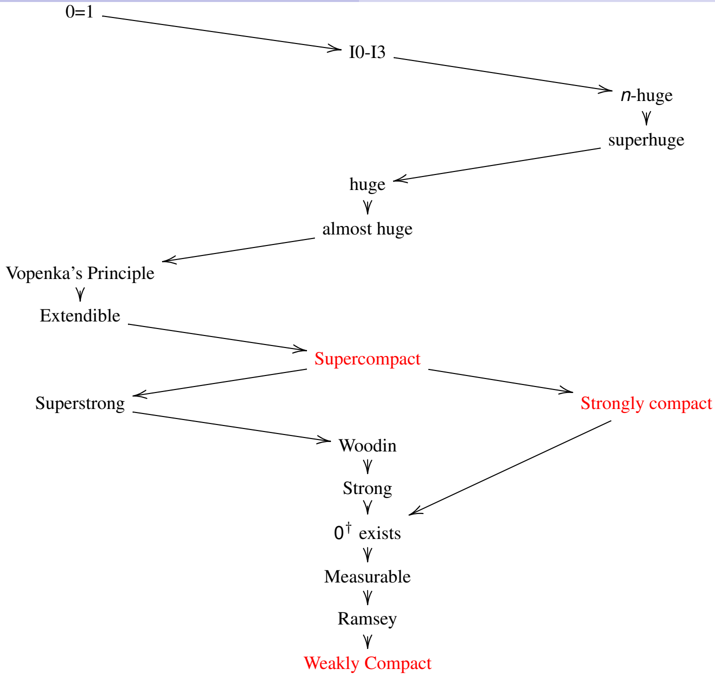
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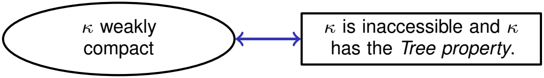
The Combinatorial Essence of Large Cardinals



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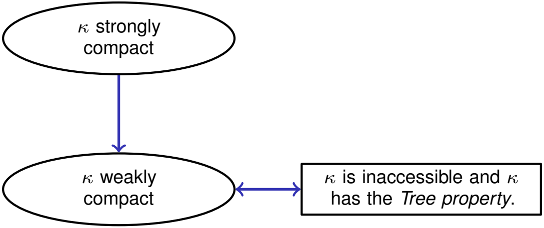

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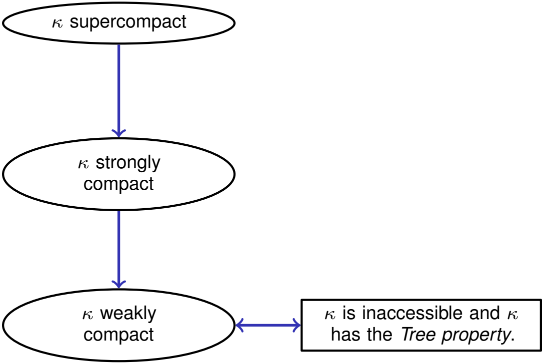

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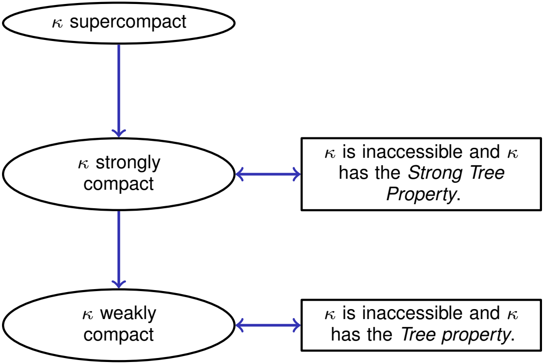

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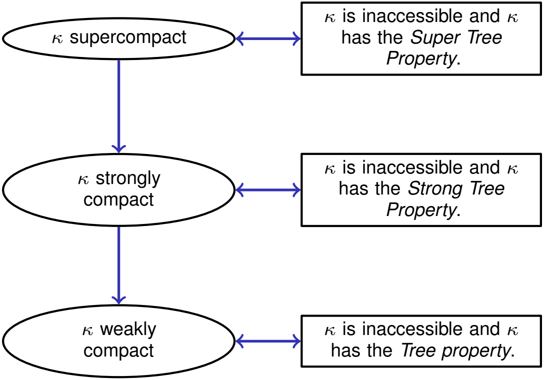

The Tree Property

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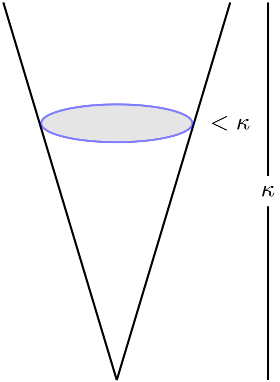

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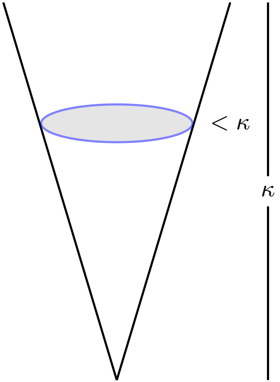
A regular cardinal κ satisfies the *tree property* if, and only if, every κ-tree has a cofinal branch.



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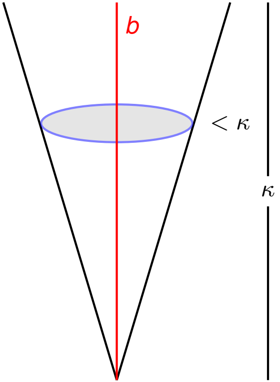

The Tree Property

Remark

Every κ-tree T can be seen as a subset of $2^{<\kappa}$;
for every $\alpha < \kappa$, $Lev_\alpha(T) = \{f \in T;\ dom(f) = \alpha\}$;
a cofinal branch is a function $b : \kappa \to 2$ such that $b \restriction \alpha \in Lev_\alpha(T)$, for all $\alpha < \kappa$.

- (König's Lemma) $\aleph_0$ satisfies the tree property;
- (Aronszajn) $\aleph_1$ does not satisfy the tree property;
- (Mitchell) for every $n \geq 2$, it is consistent that $\aleph_n$ satisfies the tree property.

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The Strong Tree Property

Definition

Let $\lambda \geq \kappa$ and $Fn(\kappa, \lambda) := \{f : X \to 2;\ X \in [\lambda]^{<\kappa}\}$. A (κ, λ)-*tree* is a subset $F \subseteq Fn(\kappa, \lambda)$ such that:

1. for all $f \in F$, if $X \subseteq dom(f)$, then $f \upharpoonright X \in F$;
2. for all $X \in [\lambda]^{<\kappa}$, $Lev_X(F) := \{f \in F;\ dom(f) = X\} \neq \emptyset$ and has size $< \kappa$.

Definition

A *cofinal branch* for a (κ, λ)-tree F is a function $b : \lambda \to 2$ such that $b \upharpoonright X \in Lev_X(F)$, for all $X \in [\lambda]^{<\kappa}$

Definition

κ (regular) satisfies the *Strong Tree Property* if for all $\lambda \geq \kappa$, every (κ, λ)-tree has a cofinal branch.

The Super Tree Property

Definition

Let F be a (κ, λ)-tree. A sequence $D := \langle d_X;\ X \in [\lambda]^{<\kappa} \rangle$ is an *F-level sequence* if $d_X \in Lev_X(F)$, for all $X \in [\lambda]^{<\kappa}$.

Definition

Let F be a (κ, λ)-tree and $D := \langle d_X;\ X \in [\lambda]^{<\kappa} \rangle$ an F-level sequence. An *ineffable branch* for D is a function $b : \lambda \to 2$ such that

$$\{X \in [\lambda]^{<\kappa};\ b \restriction X = d_X\}$$

is stationary.

Definition

κ (regular) satisfies the *Super Tree Property* if, for all $\lambda \geq \kappa$ and for all (κ, λ)-tree F, every F-level sequence has an ineffable branch.

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Definition

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- What regular cardinals can satisfy the Strong or the Super Tree Properties?

Strong Tree Properties for Small Cardinals

Weiss

Let $n \geq 2$, if $Cons(ZFC + \exists\kappa \text{ supercompact })$, then
$Cons(ZFC + \aleph_n \text{ has the Super Tree Property })$.

Fontanella

If $Cons(ZFC + \exists\langle\kappa_n\rangle_{n<\omega} \text{ supercompact cardinals})$, then
$Cons(ZFC + \forall n \geq 2,\ \aleph_n \text{ has the Super Tree Property })$.

Is it possible to construct a model where every regular cardinal larger than $\aleph_1$ satisfies the super tree property?

- (Neeman) The tree property can consistently hold up to $\aleph_{\omega+1}$.
- (Friedman, Halilovic) The tree property can consistently hold at $\aleph_{\omega+2}$.

The Super Tree Property for two successive cardinals

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use $\mathbb{M}(\aleph_n, \kappa)$.

κ inacc.
$+$ super tree property

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$\kappa = \aleph_{n+2}$
$+$ super tree property

The Super Tree Property for two successive cardinals

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For every natural number $n \geq 2$, if $Cons(ZFC + \exists\kappa, \lambda \text{ supercompact })$, then $Cons(ZFC + \aleph_n \text{ and } \aleph_{n+1} \text{ have the Super Tree Property })$.

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λ
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$\lambda = \aleph_3$
$+$ super tree property

$\mathbb{M}(\aleph_0, \kappa) * \mathbb{M}(\aleph_1, \lambda)$

$\kappa = \aleph_2$
$+$ super tree property ?

Cummings and Foreman's Iteration

$\langle \kappa_n \rangle_{n<\omega}$ supercompact cardinals and $\langle L_n \rangle_{n<\omega}$ Laver functions. $\mathbb{R}_\omega$ is an interation of length ω. At stage n, force with

$$\mathbb{Q}_n := \mathbb{R}(\aleph_n^{V_{n-1}}, \kappa_n, V_{n-2}, V_{n-1}, L_n).$$

The poset $\mathbb{Q}_n$

1. makes $\kappa_n = \aleph_{n+2}$ while preserving the super tree property at κ_n;
2. anticipates a fragment of the tail of the iteration Tail_n

Then take the inverse limite.

Fontanella

If $Cons(ZFC + \exists\langle\kappa_n\rangle_{n<\omega}$ supercompact cardinals$)$, then $Cons(ZFC + \forall n \geq 2,\ \aleph_n$ has the Super Tree Property $)$.

We prove that $V[G_\omega] \models \aleph_2$ has the super tree property.

Proof.

In $V[G_\omega]$, fix F an $(\aleph_2, \mu)$-tree and $D := \{d_X;\ X \in [\mu]^{<\aleph_2}\}$ an F- level sequence. In that model $\kappa_0 = \aleph_2$.

Fix an elementary embedding $j : V \to N$ such that:

- $cr(j) = \kappa_0$, $j(\kappa_0) > \sigma$ and ${}^{\sigma}N \subseteq N$, for σ large enough;
- $j(L_0)(\kappa_0)$ is a name for a fragment of $Tail_0$

Step 1 : lift j to an elementary embedding $j : V[G_\omega] \to N[H^*]$
(use the fact that $j(\mathbb{Q}_0) \upharpoonright \kappa_0 + 1 = \mathbb{Q}_0 * j(L)(\kappa_0)$);
Step 2 : find an ineffable branch b for D in $N[H^*]$
($j(F)$ is an $(j(\kappa_0), j(\mu))$-tree and $j[\mu] \in [j(\mu)]^{<j(\kappa_0)}$, the value of $j(d)_{j[\mu]}$ provides an ineffable branch);
Step 3 : prove that $b \in V[G_\omega]$; (use two preservation theorems). □

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Strong Tree Properties at Successors of Singular Cardinals

Question

Can we prove the consistency of the strong or the super tree property at $\aleph_{\omega+1}$?

Fontanella (based on Magidor, Shelah)

If κ is a singular limit of supercompact cardinals, then κ^+ satisfies the Strong Tree Property.

Sinapova (based on Magidor, Shelah)

Assume that $\langle \kappa_n \rangle_{n<\omega}$ is an increasing sequence of supercompact cardinals, then there is a model of ZFC where $\aleph_{\omega+1}$ has the Tree Property.

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