REFLECTION PRINCIPLES AND LARGE CARDINALS

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1. Lecture one

A reflection principle: a statement establishing for some kind of structure (e.g. a set of ordinals, a tree, a topological space ...) and a given property, that if the structure satisfies the property, then there is a substructure of smaller cardinality that satisfies the same property.

A compactness principle: a statement establishing for some kind of structure (e.g. a set of ordinals, a tree, a topological space ...) and a given property, that if every substructure of smaller cardinality satisfies the property, then the whole structure satisfies the property.

Reflection is the dual of compactness: if we have reflection for some property, than we have compactness for the negation of the property, and vice-versa.

Example:

- König's Lemma: if T is a tree of height ω whose levels are all finite, then T has an infinite branch.

(see the Appendix for the definition of "tree" in set theory)

König's Lemma is a compactness result for the property of having a branch cofinal through the tree (i.e. of the same order type as the height of the tree); it can be seen as a reflection principle for the property of not having a branch cofinal through the height of the tree.

1.1. Generalized compactness.

We recall Compactness Theorem.

Theorem 1.1. *(Compactness theorem) Let Γ be a first order theory, then Γ is satisfiable if and only if it is finitely satisfiable (i.e., every finite subset of Γ has a model).*

Let κ, λ be two infinite cardinals. $\mathscr{L}_{\kappa,\lambda}$ is the logic that allows conjunctions and disjunctions of less than κ many formulas, and allows quantifications over less than λ many variables.

$\mathscr{L}_{\omega,\omega}$ corresponds to the standard first order logic.

Definition 1.2. *A regular cardinal κ is a* strongly compact *cardinal, if for every theory Γ in $\mathscr{L}_{\kappa,\kappa}$, Γ is satisfiable if and only if it is $< \kappa$-satisfiable (i.e., every set of less than κ many sentences of Γ has a model).*

Definition 1.3. *A regular cardinal κ is a* weakly compact *cardinal, if for every theory Γ in $\mathscr{L}_{\kappa,\kappa}$ with at most κ non logical symbols, Γ is satisfiable if and only if it is $<\kappa$-satisfiable (i.e., every set of less than κ many sentences of Γ has a model).*

Stronger versions of compactness: For a given logic $\mathscr{L}$, we say that $\mathscr{L}$ is μ-compact if every theory of $\mathscr{L}$ is satisfiable if and only if it is $<\mu$-satisfiable.

- (Magidor) there exists an *extendible* cardinal if and only if there is a cardinal μ such that second order logic is μ-compact.
- (Stavi) Vopěnka's principle holds if and only if for every logic $\mathscr{L}$ there is a cardinal μ such that $\mathscr{L}$ is μ-compact.

1.2. The tree property.

Tree property: A cardinal κ has the tree property if every tree of height κ with levels all of size less than κ, has a *cofinal* branch (i.e., a branch of size κ).

Fact 1.4. *If κ has the tree property, then κ is regular.*

Proof. Otherwise, $\kappa = \lim_{i<\gamma} \kappa_i$ where each $\kappa_i < \kappa$ and $\gamma < \kappa$. For each $i < \gamma$ let $f_i : \kappa_i \to 2$ be such that $f_i(\alpha) = i$ for every α. Let T be $\{f_i \upharpoonright \alpha;\ \alpha < \kappa_i,\ i \in I\}$, then T is a tree of height κ, every α-level has size at most $\gamma < \kappa$, yet, T does not have a cofinal branch. □

Proposition 1.5. *If κ is a weakly compact cardinal, then κ has the tree property.*

Proof. Let $(T, <)$ be a tree of height κ with levels all of size less than κ. For each $\alpha < \kappa$, let $\{t_i^\alpha; i < \gamma_\alpha\}$ enumerate the α-th level. Fix for each $t \in T$, a propositional symbol P_t. Then $\Gamma := \{\bigvee_{i<\gamma_\alpha} P_{t_i^\alpha};\ \alpha < \kappa\} \cup \{\neg(P_s \wedge P_t);\ s, t \in T \text{ are } <\text{-incomparable}\}$ is $<\kappa$-satisfiable (because $ht(T) = \kappa$). Let $\mathscr{M}$ be a model of Γ, then $\{t \in T; M \models\ P_t\}$ is a cofinal branch. □

Theorem 1.6. *(Erdös and Tarski) An inaccessible cardinal is weakly compact if and only if it satisfies the tree property.*

Proposition 1.7. *The following hold.*

- *(Aronszajn) ω_1 does not have the tree property.*
- *(Specker) If CH holds, then ω_2 does not have the tree property.*
- *(Kurepa) if $2^{<\kappa} = \kappa$, then κ^+ does not have the tree property.*
- *(Mitchell)* $\mathrm{Cons}(\exists\kappa$ *weakly compact* $) \to \mathrm{Cons}(\omega_2$ *has the tree property*$)$.
- *(Mitchell) For every regular τ,* $\mathrm{Cons}(\exists\kappa$ *weakly compact* $) \to \mathrm{Cons}(\tau^{++}$ *has the tree property*$)$.

Open questions:

- Is it possible to build a model where every regular cardinal has the tree property?
- Is there a model of the tree property at $\aleph_{\omega+1}$ where SCH fails at $\aleph_\omega$?

Exercice: prove that if κ is weakly compact, then κ is (strongly) inaccessible.

2. Lecture two

2.1. **Elementary embeddings.**

Given two structures M, N in a language $\mathscr{L}$, a function $j : M \to N$ is an *elementary embedding* if j is injective and for every formula $\varphi(x_1, \ldots, x_n)$ of $\mathscr{L}$ and every $a_1, \ldots, a_n \in M$,

$$M \models \varphi(a_1, \ldots, a_n) \iff N \models \varphi(j(a_1), \ldots j(a_n))$$

Suppose M and N contains all the ordinals, then the *critical point* of an elementary embedding $j : M \to N$, denoted $cr(j)$ is the least ordinal α that is moved by j (if such an ordinal exists).

We are going to consider elementary embeddings $j : V \to M$ where M is an inner model, i.e. M is a transitive $\in$-model of ZF that contains all the ordinals. Transitive means that for every $x \in M$, $x \subseteq M$.

Remark 2.1. *If $j : V \to M$ is an elementary embedding where $M \subseteq V$ is an inner model and j has critical point κ, then $j(\kappa) > \kappa$. Because j is injective.*

Definition 2.2. *$\kappa > \omega$ is a measurable cardinal if and only if there is an elementary embedding $j : V \to M$ where M is an inner model of ZFC and κ is the critical point of j.*

Proposition 2.3. *$\kappa > \omega$ is a strongly compact cardinal if and only if for every λ there is an elementary embedding $j : V \to M$ where M is an inner model of ZFC, κ is the critical point of j, $j(\kappa) > \lambda$ and every set $X \subseteq M$ of size at most λ is covered by a set $Y \in M$ such that $M \models |Y| < j(\kappa)$.*

Definition 2.4. *$\kappa > \omega$ is a* supercompact cardinal*, if for every λ there is an elementary embedding $j : V \to M$ where M is an inner model of ZFC, κ is the critical point of j, $j(\kappa) > \lambda$ and every set $X \subseteq M$ of size at most λ belongs to M.*

Definition 2.5. *$\kappa > \omega$ is a* huge*, if for every λ there is an elementary embedding $j : V \to M$ where M is an inner model of ZFC, κ is the critical point of j, and every set $X \subseteq M$ of size at most $j(\kappa)$ belongs to M.*

Reflection for the chromatic number. Given a graph G, the *chromatic number* of G, denoted $\chi(G)$ is the least ν such that there is a colouring of the vertices of G in ν many colours such that any two adjacents vertices have distinct colours.

Proposition 2.6. *Let κ be a measurable cardinal and G a graph of size κ, let $\mu < \kappa$ and let $\chi(G) > \mu$, then there is an induced subgraph H of size $< \kappa$ such that $\chi(H) > \mu$.*

Proof. W.l.o.g. G has underlying set κ. Let $j : V \to M$ be an elementary embedding with critical point κ, then $j(G)$ is a graph with underlying set $j(\kappa)$. The subgraph induced on κ is G and, since $M \subseteq V$, it is clear that in M we have $\chi(G) > \mu$. So M thinks that $j(G)$ has a subgraph of size $\kappa < j(\kappa)$ that has chromatic number $> \mu$. Moreover $j(\mu) = \mu$ because $cr(j) = \kappa$, so by elementarity, G has a subgraph of size $< \kappa$ with chromatic number $> \mu$. □

Reflection for transversals. A *transversal* is a one-to-one choice function.

Proposition 2.7. *Let κ be a measurable cardinal and let $\mathcal{F}$ be a family of countable sets with $|\mathcal{F}| = \kappa$. If $\mathcal{F}$ has no transversals, then there is $\mathcal{F}' \subseteq \mathcal{F}$ of size $< \kappa$ that has no transversals.*

Proof. Let $j : V \to M$ be an elementary embedding with critical point κ, then $j(F)$ is a family of countable sets with $|j(F)| = j(\kappa)$. $j''F$ is a subfamily of $j(F)$ of size $< \kappa$. Since F has no transversal, $j''F$ has no transversal. So $j(F)$ has a subfamily of size $< j(\kappa)$ with no transversals. By elementarity F has a subfamily of size $< \kappa$ with no transversals. □

Exercice: Using elementary embeddings, prove that if κ is measurable, then κ has the tree property.

2.2. Rado's conjecture.

Definition 2.8. *Given a tree T of height ν^+, we say that T is* special *if there exists a function $f : T \to \nu$ such that for every two nodes $x < y$, one has $f(x) \neq f(y)$.*

In particular, a special tree has no cofinal branch.

Rado's conjecture RC: any tree T of height ω_1 is special if and only if all subtrees of cardinality $\aleph_1$ are specials.

Theorem 2.9. *(Todorčević)* Cons($\exists\kappa$ *strongly compact*) $\to$ Cons(RC).

Proposition 2.10. *Suppose that κ is a supercompact cardinal and T is a of height ω_1. If every subtree of T of size $< \kappa$ is special, then T is special.*

Proof. Let λ be the size of T. Fix a λ-supercompact embedding with critical point κ. W.l.o.g. T has underlying set λ. By elementarity $j(T)$ is a tree with underlying set $j(\lambda)$ and every subtree of $j(T)$ of size $< j(\kappa)$ is special. Let $T^* := (j''\lambda, <_{j(T)})$, then $T^* \in M$ by the closure of M. It is easy to see that T^* is isomorphic to T (the isomorphism is the embedding j). It follows that the height of T^* is the same as T, namely ω_1. Moreover $|T^*| = \lambda < j(\kappa)$ so T^* is special so there is a specializing function $f^* : T^* \to \omega$. We define $f : T \to \omega$ by letting $f(\alpha) := j^{-1}f^*j(\alpha)$, then f witnesses that T is special. □

To prove the consistency of Rado's Conjecture one forces with $Coll(\omega_1, < \kappa)$ so κ becomes $\aleph_2$. Then one shows that the conclusion of this proposition holds also in the generic extension, thus every non special tree of height ω_1 has a non-special subtree of size $< \kappa = \omega_2$, namely of size $\aleph_1$.

Difficult open question Is RC equiconsistent with a strongly compact cardinal?

- RC is independent from CH;
- RC implies $2^{\aleph_0} \leq \aleph_2$;
- RC implies $\theta^{\aleph_0} = \theta$ for every regular cardinal θ;
- RC implies SCH.

Theorem 2.11. *(Solovay) If κ is strongly compact, then SCH holds* above κ.

RC has the strength of a strongly compact cardinal and it is a sort of property of strong compactness for $\aleph_2$, so by analogy it is not surprising that RC implies SCH "above $\aleph_2$" thus everywhere.

Open question: Does the strong tree property at $\aleph_2$ implies SCH?

A tree witnessing the failure of the tree property at some cardinal κ is called a κ-Aronszajn tree.

Theorem 2.12. *(Torres-Perez, Todorčević) Assume RC. Then CH is equivalent to the existence of a special $\aleph_2$-Aronszajn tree.*

3. Lecture three

3.1. Stationary sets.

We recall the definition of club and stationary set.

Let κ be a limit ordinal of uncountable cofinality. A *club* is a subset $C \subseteq \kappa$ such that

- C is closed (i.e., for every sequence $\langle \beta_i \rangle_{i<\gamma}$ of ordinals in C such that $\gamma < \kappa$, $\lim_{i<\gamma} \beta_i \in C$);
- C is unbounded (i.e., for every $\beta < \kappa$ there is $\beta' \in C$ such that $\beta' > \beta$).

$S \subseteq \kappa$ is *stationary* if for every club $C \subseteq \kappa$, $S \cap C \neq \emptyset$.

Example: For κ regular,

- for every $\gamma < \kappa$, $\kappa \setminus \gamma$ is a club of κ.
- for $\lambda < \kappa$, we let $E^\kappa_\lambda := \{\alpha < \kappa;\ \mathrm{cof}(\alpha) = \lambda\}$. E^κ_λ is a stationary subset of κ.

Fact 3.1. *(exercice) The intersection of less than κ club subsets of κ is a club.*

Theorem 3.2. *(Fodor) If S is a stationary subset of κ and f is a function on κ such that for every $\alpha \in S \setminus \{0\}$, $f(\alpha) < \alpha$, then f is constant on a stationary subset of S.*

3.2. Reflection of stationary sets.

Reflection of stationary sets: the reflection of stationary sets holds at a cardinal κ if for every stationary set $S \subseteq \kappa$, there exists $\alpha < \kappa$ of uncountable cofinality, such that $S \cap \alpha$ is stationary (we say that S "reflects" at α).

Applications: For a cardinal κ the following are equiconsistent:

- every stationary subset of κ reflects;
- every κ-free abelian group is κ^+-free.

Fact 3.3. *If κ is the successor of a regular cardinal λ, then there is a stationary subset of κ that does not reflect.*

Proof. The set $E^\kappa_\lambda := \{\alpha < \kappa;\ \mathrm{cof}(\alpha) = \lambda\}$ is stationary, but it does not reflect. Indeed, for every $\alpha < \lambda^+$ we can always find a club $C_\alpha \subseteq \alpha$ of points of cofinality $< \lambda$. □

Proposition 3.4. *If κ is a weakly compact cardinal, then every stationary subset of κ reflects.*

We prove the reflection of stationary sets from a measurable cardinal. Let $S \subseteq \kappa$ be a stationary subset. Let $j : V \to M$ be an elementary embedding witnessing the fact that κ is measurable. $j(S)$ is a stationary subset of $j(\kappa)$. $S = j(S) \cap \kappa$, thus $j(S)$ reflects to $\kappa < j(\kappa)$. So M thinks that $j(\kappa)$ reflects to some ordinal below $< j(\kappa)$, thus by elementarity S reflects to some ordinal $\alpha < \kappa$.

Proposition 3.5. *If κ is a supercompact cardinal, then for every $\lambda > \kappa$, every stationary subset of $E^{\lambda}_{<\kappa}$ reflects to some point $\alpha < \lambda$ of cofinality $< \kappa$.*

Proof. Let $S \subseteq E^{\lambda}_{<\kappa}$ be a stationary subset of λ. Let $j : V \to M$ be a λ-supercompact embedding with critical point κ (see the characterisation of strong compactness). By elementarity $j(S)$ is a stationary subset of $E^{j(\lambda)}_{<\kappa}$ (the points of $j(S)$ remain of cofinality $< \kappa$ because κ is the critical point). Let $\gamma := \sup j'' \lambda$. Then $\text{cof}^M(\gamma) < j(\kappa)$ and $\gamma < j(\lambda)$. It is easy to see that $j(S) \cap \gamma = j'' S$ and, since S is stationary, so is $j(S) \cap \gamma$. Thus M thinks that $j(S)$ reflects to some ordinal of cofinality $< j(\kappa)$. By elementarity, S reflects to an ordinal of cofinality $< \kappa$. □

Exercice: show that κ strongly compact is enough for this result.

Theorem 3.6. *(Magidor '62)*
Cons($\exists \langle \kappa_n \rangle_{n<\omega}$ *supercompact cardinals*) $\to$ Cons(*every stationary subset of* $\aleph_{\omega+1}$ *reflects*).

Open questions: Is there a model where $\aleph_{\omega+1}$ satisfies both the tree property and the reflection of stationary sets?

3.3. Delta-reflection.

Definition 3.7. *(Magidor Shelah) Given two cardinals $\kappa < \lambda$, $\Delta_{\kappa,\lambda}$ is the statement that for every cardinal $\mu < \kappa$, for every stationary set $S \subseteq E^{\lambda}_{<\kappa} := \{\alpha < \lambda;\ \text{cof}(\alpha) < \kappa\}$ and for every algebra A on λ with μ operations, there exists a subalgebra A' of order type a regular cardinal $\eta < \kappa$ such that $S \cap A'$ is stationary in* $\sup(A')$.

We say that λ has the *Delta-reflection* if $\Delta_{\kappa,\lambda}$ holds for every $\kappa < \lambda$.

The Delta-reflection at some cardinal κ has many applications. Suppose that κ has the Delta-reflection, then

(a) κ satisfies the reflection of stationary sets;
(b) every κ-free abelian group of size κ is free;
(c) If G is a graph of size κ and every subgraph of G of smaller cardinality has coloring number[1] $\lambda < \kappa$ then G has coloring number λ;
(d) If F is a family of κ many countable sets, and if every subfamily F' of F of smaller cardinality has a transversal (i.e. a one-to-one choice function on F'), then the whole family has a transversal;
(e) If X is a topological space locally of cardinality $< \kappa$ and X is κ-collectionwise Hausdorff (i.e. every closed discrete subset of X of size $< \kappa$ is separated), then X is collectionwise Hausdorff.

C is separated if there is a family of mutually disjoint open sets $\{U_y;\ y \in C\}$ such that for all y, $C \cap U_y = \{y\}$.

It should be pointed out that all these compactness phenomenons can be demonstrated in ZFC for κ singular, they all follow from *Shelah's compactness theorem for singular cardinals.* Thus the

[1] A graph has coloring number λ if there is a well order on the graph such that every element is connected to less than λ many elements preceding it in this well order

Theorem 3.8. *If κ is a weakly compact cardinal, then κ has the Delta-reflection.*

Theorem 3.9. *(Magidor , Shelah '94)*
$\mathrm{Cons}(\exists\langle\kappa_n\rangle_{n<\omega}$ *supercompact cardinals*$) \to \mathrm{Cons}(\aleph_{\omega^2+1}$ *has the Delta-reflection*$)$.

Fact 3.10. *(Magidor, Shelah, Ekler et al.) No regular cardinal below $\aleph_{\omega^2+1}$ can have the Delta-reflection.*

It is important to point out the fact that, unlike the tree property, the Delta-reflection is compatible with GCH: in fact, in Magidor and Shelah's model, GCH holds.

Theorem 3.11. *(F. , Magidor '15)*
- *the Delta-reflection does not imply the tree property (one can build a model where $\aleph_{\omega^2+1}$ has the Delta-reflection but not the tree property);*
- *Delta-reflection and tree property are compatible at $\aleph_{\omega^2+1}$ (both properties hold at $\aleph_{\omega^2+1}$ in Magidor and Shelah's model).*

In particular $\aleph_{\omega^2+1}$ can have simultaneously the Delta-reflection and the tree property.

Magidor and Shelah also proved that under the same assumptions one can build a model where, for the first cardinal fixed point κ, the properties $(a), ..., (e)$ above hold *with no limitation on the size of the structure*, i.e. for instance, every κ-free abelian group (of any size) is free.

Open question: Does Delta-reflection implies compactness for the chromatic number?

4. Lecture four

4.1. Clubs of $\mathcal{P}_\kappa(\lambda)$.

Let κ be a regular cardinal, and let $\lambda \geq \kappa$, we denote by $\mathcal{P}_\kappa(\lambda)$ the set $\{x \subseteq \lambda;\ |x| < \kappa\}$ and we denote by $[\lambda]^\kappa$ the set $\{x \subseteq \lambda;\ |x| = \kappa\}$.

Definition 4.1. *A set $C \subseteq \mathcal{P}_\kappa(\lambda)$ is a* club *if the following hold:*
1. *C is closed (i.e., for any chain $x_0 \subseteq x_1 \subseteq \ldots \subseteq x_\zeta \subseteq \ldots$, $\zeta < \alpha$ of sets in C, with $\alpha < \kappa$, the union $\bigcup_{\zeta<\alpha} x_\zeta$ is in X);*
2. *C is unbounded (i.e., for every $x \in \mathcal{P}_\kappa(\lambda)$ there is $y \supseteq x$ such that $x \in C$).*

Definition 4.2. *A set $S \subseteq \mathcal{P}_\kappa(\lambda)$ is stationary if $S \cap C \neq \emptyset$ for every club $C \subseteq \mathcal{P}_\kappa(\lambda)$.*

Exercice 4.3. *The intersection of less than κ club subsets of $\mathcal{P}_\kappa(\lambda)$ is a club of $\mathcal{P}_\kappa(\lambda)$.*

Theorem 4.4. *(Fodor-Jech) If S is a stationary subset of $\mathcal{P}_\kappa(\lambda)$ and f is a function on $\mathcal{P}_\kappa(\lambda)$ such that for every $x \in S \setminus \{\emptyset\}$, $f(x) \in x$, then f is constant on a stationary subset of S.*

Fact 4.5. *(Menas) For every club $C \subseteq \mathcal{P}_\kappa(\lambda)$, there is a function $f : [\lambda]^{<\omega} \to \mathcal{P}_\kappa(\lambda)$, such that if $\mathrm{Cl}_f := \{x \in \mathcal{P}_\kappa(\lambda);\ \forall e \subseteq x\ (f(e) \subseteq x)\}$, then $\mathrm{Cl}_f \subseteq C$*

Proof. By induction on $|e|$, we find for each $e \in [\lambda]^{<\omega}$ an infinite set $f(e) \in C$ such that $e \subseteq f(e)$ in such a way that $f(e') \subseteq f(e)$ whenever $e' \subseteq e$. So f is defined. To see that $\mathrm{Cl}_f \subseteq C$, let x be a closure point of f. Then, $x = \bigcup\{f(e);\ e \in [x]^{<\omega}\}$. This is the union of a family of less than κ many elements of C, so $x \in C$. □

4.2. Reflection principle.

If S is a stationary subset of $[\lambda]^{\aleph_0}$, and $X \in [\lambda]^{\aleph_1}$, then we say that S *reflects* at X if $S \cap [X]^{\aleph_0}$ is stationary in $[X]^{\aleph_0}$.

The following principle was introduced by Foreman, Magidor and Shelah.

(Weak) Reflection principle: For every regular $\lambda \geq \aleph_2$, every stationary set $S \subseteq [\lambda]^{\aleph_0}$ reflects at some $X \in [\lambda]^{\aleph_1}$ such that $X \supseteq \omega_1$.

We denote by $RP(\lambda)$ the reflection principle at λ, namely for stationary subsets of $[\lambda]^{\aleph_0}$.

Attention: the condition $X \supseteq \omega_1$ is important (for the following proposition).

Proposition 4.6. *(Feng, Jech) Let $\lambda \geq \aleph_2$ be a regular cardinal, then the following are equivalent:*

(1) *The reflection principle holds at λ;*
(2) *for every stationary subset $S \subseteq [\lambda]^{\aleph_0}$, the set $\{X \in [\lambda]^{\aleph_1};\ S$ reflects at $X\}$ is stationary.*

Proof.
$(\Leftarrow)$: it is enough to observe that $\{X \in [\lambda]^{\aleph_1};\ X \supseteq \omega_1\}$ is a club.

$(\Rightarrow)$: Let $S \subseteq [\lambda]^{\aleph_0}$ be a stationary set. Suppose by contradiction that there is a club $C \subseteq [\lambda]^{\aleph_1}$ such that S does not reflect in any set of C. Let $g : [\lambda]^{<\omega} \to \mathcal{P}_{\aleph_2}(\lambda)$, such that $\mathrm{Cl}_g \subseteq C$. Cl_g is a club in $[\lambda]^{\aleph_0}$, so the set $S \cap \mathrm{Cl}_g$ is stationary, thus by the reflection principle, it reflects to some set $X \supseteq \omega_1$ of size $\aleph_1$. In particular S reflects on X, so X does not belong to C.

Fix a bijection $f : \omega_1 \to X$. Observe that $\{y \in [X]^{\aleph_0};\ f^{-1}[y] \subseteq y\}$ is a club in $[X]^{\aleph_0}$ (here we use $\omega_1 \subseteq X$, otherwise this set is empty), hence $T := \{y \in S \cap \mathrm{Cl}_g \cap [X]^{\aleph_0};\ f^{-1}[y] \subseteq y\}$ is stationary in $[X]^{\aleph_0}$. It follows that $\{f^{-1}[y];\ y \in T\}$ is stationary in $[\omega_1]^{\aleph_0}$, moreover $f[f^{-1}[y]] = y \supseteq f^{-1}[y]$, therefore the set

$$T^* := \{a \in [\omega_1]^{\aleph_0};\ a \subseteq f[a] \text{ and } f[a] \in S \cap \mathrm{Cl}_g\}$$

is stationary in $[\omega_1]^{\aleph_0}$.

Now we prove that X is closed by g, thus $X \in \mathrm{Cl}_g$ contradicting $X \notin C$. Let $e \in [X]^{<\omega}$, then $f^{-1}[e] \in [\omega_1]^{<\omega}$ so there is $a \in T^*$ such that $a \supseteq f^{-1}[e]$. By definition of T^*, we have $a = f^{-1}[y]$ for some $y \in T \subseteq S \cap \mathrm{Cl}_g$. Observe that $e = f[f^{-1}[e]] \subseteq f[a] = y$, thus $g[e] \subseteq y$ because y is closed by g. Since $y \subseteq f[\omega_1] = X$, we proved $g[e] \subseteq X$ as required. □

For $\lambda = \aleph_2$ the assumption $X \supseteq \omega_1$ can be dropped.

Theorem 4.7. *(Shelah, Todorčević) The reflection principle at $\aleph_2$ implies that the continuum is at most ω_2.*

Proof. For each uncountable $\alpha < \omega_2$, let $C_\alpha \subseteq [\alpha]^{\aleph_0}$ be a club of cardinality $\aleph_1$, and let $D = \bigcup_{\omega_1 \leq \alpha < \omega_2} C_\alpha$. D must contain a club: otherwise $S := D \setminus [\omega_2]^{\aleph_0}$ is stationary, so by $RP(\aleph_2)$, using the proposition above, we can find an ordinal $\aleph_1 \leq \alpha < \omega_2$ such that $S \cap \alpha$ is stationary on α (indeed, ω_2 is a club of $[\omega_2]^{\aleph_0}$), a contradiction. We have $|D| = \aleph_2$ (because it contains a club). However, a result of Baumgartner and Taylor shows that every club of $[\omega_2]^{\aleph_0}$ has size $\aleph_2^{\aleph_0}$, hence $\aleph_2^{\aleph_0} = \aleph_2$, in particular the continuum is at most $\aleph_2$. □

Theorem 4.8. *If the existence of a supercompact cardinal is consistent with ZFC, then the reflection principle is also consistent.*

Proposition 4.9. *The reflection principle at $\aleph_2$ is equiconsistent with the existence of a weakly compact cardinal.*

Theorem 4.10. *The reflection principle implies the singular cardinal hypothesis.*

4.3. **The strong reflection principle.** Todorčević formulated a stronger version of the reflection principle, called *Strong reflection principle.*

Strong reflection principle, SRP: for every κ, every $S \subseteq [\kappa]^\omega$ and for every regular $\theta > \kappa$ there is an increasing continuous $\in$-chain $\{N_\alpha;\ \alpha < \omega_1\}$ of countable elementary models of H_θ (with N_0 containing a predefined element of H_θ) such that for all $\alpha < \omega_1$, $N_\alpha \cap \kappa \in S$ if and only if there exists a countable elementary submodel M of H_θ such that $N_\alpha \subseteq M$, $M \cap \omega_1 = N_\alpha \cap \omega_1$ and $M \cap \kappa \in S$.

Definition 4.11. *A set $S \subseteq [\lambda]^\omega$ is projective stationary if for every stationary set $T \subseteq \omega_1$, the set $\{X \in S;\ X \cap \omega_1 \in T\}$ is stationary.*

Equivalently, for every club $C \subseteq [\lambda]^\omega$, the projection $(S \cap C) \restriction \omega_1$ contains a club.

Strong reflection principle is equivalent to: for every $\lambda \geq \aleph_2$, if $S \subseteq [H_\lambda]^\omega$ is projective stationary, then there exists an elementary chain $\langle M_\alpha;\ \alpha < \omega_1\rangle$ of countable models such that $M_\alpha \in S$ for all α.

Theorem 4.12. *(Woodin) SRP implies that the continuum is $\aleph_2$.*

5. Lecture five

General Chang conjecture:

Definition 5.1. *Given a countable first order language $\mathscr{L}$ with a distinguished unary predicate R, a structure $\mathscr{M}$ in $\mathscr{L}$ is said to be a (λ, κ)-structure if the underlying set of $\mathscr{M}$ has size λ and $R^{\mathscr{M}}$ is a set of size κ.*

The general form of Chang Conjecture, denoted $(\lambda_1, \kappa_1) \twoheadrightarrow (\lambda_0, \kappa_0)$, states that every (λ_1, κ_1)-structure has a (λ_0, κ_0) elementary substructure.

This is a two-cardinal version of Löwenheim-Skolem theorem.

Fact 5.2. *Let κ be a huge cardinal and let $j : V \to M$ witnessing its hugeness, let $\lambda = j(\kappa)$ then $(\lambda, \kappa) \twoheadrightarrow (\kappa, < \kappa)$.*

Proof. Let $\mathscr{N} = (N, R)$ be a (λ, κ)-structure. We can assume that $N = \lambda$ and $R = \kappa$. Let $j : V \to M$ be an elementary embedding witnessing the fact that κ is huge and with $j(\kappa) = \lambda$. Then $j(\mathscr{N}) = (j(N), j(R))$ is a $(j(\lambda), j(\kappa))$-structure. Let $N^* := j''\lambda$ and $R^* := j''\ R = \kappa$ then (N^*, R^*) is a (λ, κ)-structure. Moreover $\lambda = j(\kappa)$ and $\kappa < j(\kappa)$ so it is in fact a $(j(\kappa), < j(\kappa))$-structure. (N^*, R^*) is an elementary substrucure of $j(\mathscr{N})$, so M thinks that $j(\mathscr{N})$ has an elementary substructure of order type $(j(\kappa, < j(\kappa)))$. By elementarity there exists in V an elementary $(\kappa, < \kappa)$-substructure for $\mathscr{N}$. □

The usual **Chang's Conjecture, CC** is $(\omega_2, \omega_1) \twoheadrightarrow (\omega_1, \omega)$

Fact 5.3. *(Todorčević) RC implies CC.*

Theorem 5.4. *(Silver, Donder)* Cons($\exists\kappa$ ω_1*-Erdös cardinal*) $\to$ Cons(CC).

(The existence of an ω_1-Erdös cardinal is a weaker axiom than the existence of a measurable cardinal).

Theorem 5.5. *(Laver)* Cons($\exists\kappa$ *huge*) $\to$ Cons($(\omega_3, \omega_2) \twoheadrightarrow (\omega_2, \omega_1)$).

Theorem 5.6. *for* $n \geq 1$, Cons($\exists\kappa$ *huge*) $\to$ Cons($(\omega_{n+2}, \omega_{n+1}) \twoheadrightarrow (\omega_{n+1}, \omega_n)$).

Theorem 5.7. *(Levinski, Magidor, Shelah)* Cons($\exists\kappa$ *huge*) $\to$ Cons($(\aleph_{\omega+1}, \aleph_\omega) \twoheadrightarrow (\omega_1, \omega_0)$).

Open question:

- For $n \geq 1$, is $(\aleph_{\omega_{n+1}}, \aleph_{\omega_n}) \twoheadrightarrow (\omega_n, \omega_{n-1})$) consistent?
- Is it possible to prove the result of Levinski, Magidor and Shelah from a supercompact cardinal?

Strong Chang Conjecture, CC^*: there are arbitrarily large uncountable regular cardinals θ such that for every well ordering $<$ of H_θ and every countable elementary submodel $M \prec (H_\theta, \in, <)$ and every ordinal $\eta < \omega_2$, there exists an elementary countable submodel M^* such that

(1) $M \subseteq M^*$ and $M \cap \omega_1 = M^* \cap \omega_1$
(2) $(M^* \cap \omega_2) \setminus \eta \neq \emptyset$.

Theorem 5.8. *(Todorčević)* CC^* *implies* $2^{\aleph_0} \leq \omega_2$.

Theorem 5.9. *(Torres-Perez, Wu)* $CC^* + \neg CH$ *implies* $TP(\aleph_2)$

5.1. **Square principles.**

Square principle was introduced by Jensen.
Square principle: $\square_\kappa$ is the statement that there exists a sequence $\langle C_\alpha;\ \alpha \in \mathrm{Lim}(\kappa^+)\rangle$ such that

(1) every $C_\alpha \subseteq \alpha$ is a club;
(2) $\beta \in \mathrm{Lim}(C_\alpha)$ implies $C_\beta = C_\alpha \cap \beta$;
(3) $o.t.(C_\alpha) \leq \kappa$

Fact 5.10. *(3) can be replaced by the following: if* $\mathrm{cof}(\alpha) < \kappa$, *then* $o.t.(C_\alpha) < \kappa$.

Proof. Fix a club $C \subseteq \kappa$ with $o.t.(C) = \mathrm{cof}(\kappa)$, then replace C_α by $\{\beta \in C_\alpha;\ o.t.(C_\alpha \cap \beta) \in C\}$ whenever $o.t.(C_\alpha) \in Lim(C) \cup \{\kappa\}$. $\square$

Theorem 5.11. *Let* $\kappa > \omega$, *then* $\square_\kappa$ *implies that there is a stationary subset of* κ^+ *that does not reflect.*

Proof. Let $\langle C_\alpha;\ \alpha \in \mathrm{Lim}(\kappa^+)\rangle$ be a square sequence. For every $\alpha < \kappa^+$ limit ordinal, let $f(\alpha) = o.t.(C_\alpha)$. The function f is a regressive function on $\kappa^+ - (\kappa + 1)$, so by Fodor theorem there exists $T \subseteq \kappa^+$ stationary such that $f \restriction T$ is constant. We show that T does not reflect. For every $\alpha < \kappa^+$ of uncountable cofinality and every $\beta \in \mathrm{acc}(C_\alpha)$, we have $f(\beta) = o.t.(C_\beta) = o.t.(C_\alpha \cap \beta) < o.t.(C_\alpha)$. Then $f \restriction \mathrm{acc}(C_\alpha)$ is injective, so $|T \cap \mathrm{acc}(C_\alpha)| \leq 1$. In particular $T \cap \alpha$ is not stationary in α (because $\mathrm{acc}(C_\alpha)$ is a club in α), thus T does not reflect. $\square$

Corollary 5.12. *If κ is strongly compact, then $\square_\mu$ fails for every $\mu \geq \kappa$.*

$\square_\kappa$ has a generalisation, which is due to Schimmerling.

$\square_{\kappa,\lambda}$: there exists a sequence $\langle \mathcal{C}_\alpha;\ \alpha \in \mathrm{Lim}(\kappa^+)\rangle$ such that

(1) for every $C \in \mathcal{C}_\alpha$, $C \subseteq \alpha$ is a club of order type $\leq \kappa$;
(2) $0 < |\mathcal{C}_\alpha| \leq \lambda$
(3) for every $C \in \mathcal{C}_\alpha$, if $\beta \in \mathrm{Lim}(C)$, then $C \cap \beta \in \mathcal{C}_\beta$;

Remark 5.13. *We note that the* silly square *principle $\square_{\mu,\mu^+}$ is always true, since we may just fix D_α club in α for every $\alpha < \mu^+$ and let $C_\beta = \{D_\alpha \cap \beta : \beta \in Lim(D_\alpha) \cup \{\alpha\}\}$.*

$\square_\kappa$ corresponds to $\square_{\kappa,1}$.

Weak square: $\square^*_\kappa$ is the principle $\square_{\kappa,\kappa}$

Theorem 5.14. *(Jensen) $\square^*_\mu$ is equivalent to the existence of a special μ^+-Aronszajn tree*

Even the weak square is incompatible with strongly compact cardinals.

Theorem 5.15. *(Shelah) If κ is a strongly compact cardinal then $\square^*_\mu$ fails for every singular cardinal μ such that* $\mathrm{cof}(\mu) < \kappa < \mu$

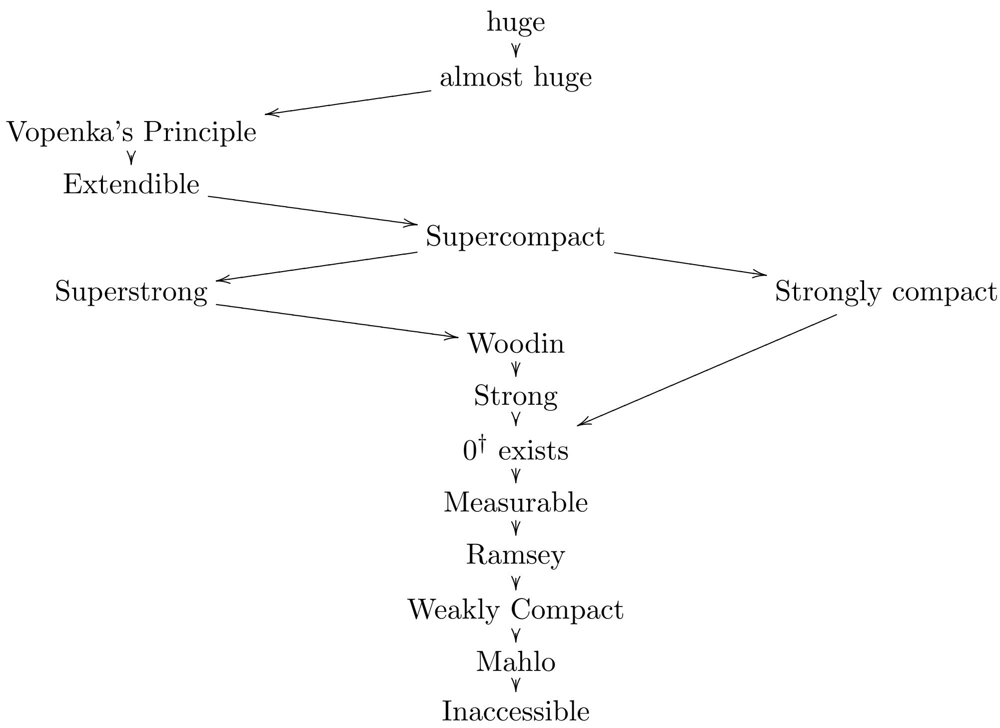
Theorem 5.16.

- *(Todorcevic) RC implies the failure of $\square_\kappa$ for all uncountable κ.*
- *(Torres-Perez, Todorcevic) RC implies the failure of $\square^*_\kappa$ for all singular κ of countable cofinality*
- *(Torres-Perez, Todorcevic) Assume RC, then CH is equivalent to $\square^*_{\omega_1}$ (we already metioned this results)*

Theorem 5.17. *(Todorcevic) CC implies the failure of $\square_{\omega_1}$.*

Hayut and my self proved that the Delta-reflection at κ^+ is compatible with another square principle denoted $\square(\kappa^+)$ that was introduced by Todorcevic and is weaker than $\square_\kappa$ (The Delta-reflection implies the failure of $\square_\kappa$ because it implies the reflection of stationary sets).

6. Appendix



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