Compactness for abelian groups, graphs, topological spaces, trees and others

joint work with M. Magidor and Y. Hayut

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Compactness: Given some structure (e.g. a set of ordinals, a group, a topological space etc.) if every substructure of smaller cardinality satisfies a certain property, then the whole structure satisfies the same property.

Example of compactness: König's lemma

Compactness and Reflection

Reflection: Given some structure $\mathcal{S}$, if the structure satisfies some property $\mathcal{P}$, then there is a substructure $\mathcal{S}'$ of smaller cardinality with the same property.

When we have compactness for some property, then we have reflection for the negation of the property, and vice versa.

Shelah's compactness theorem for singular cardinals

Shelah's compactness theorem for singular cardinals (Shelah 1975)

If κ is **singular**, then

- Given an abelian group *G* of size κ, if every subgroup of size $< \kappa$ is free abelian, then *G* is free abelian.
- Given a graph *G* of size κ. If every subgraph of *G* of size $< \kappa$ has coloring number $\leq \gamma < \kappa$, then *G* has coloring number $\leq \gamma$.
- Given *A* a family of κ many countable sets, if every subfamily of size $< \kappa$ has a transversal, then *A* has a transversal.
- (Magidor 2015) Given *X* a topological space locally of cardinality $< \kappa$, if *X* is $< \kappa$-collectionwise Hausdorff, then *X* is collectionwise Hausdorff

When do we have compactness for structures of cardinality a **regular** cardinal?

Compactness for regular cardinals

- **V=L** implies the failure of compactness properties for regular cardinals (e.g. if $V = L$ than no regular cardinal κ except weakly compacts has the tree property, i.e. the generalisation of König's lemma to κ)
- **Large cardinals** imply compactness properties for regular cardinals (e.g. if κ is weakly compact all the compactness properties above hold for structures of size κ)

Lemma

Let κ be a measurable cardinal and let G be an abelian group of size κ. If G is $<\kappa$-free abelian, then G is free abelian.

Proof.

W.l.o.g. we can assume that $G = (\kappa, +)$. Fix an elementary embedding $j : V \to M$ with critical point κ. By elementarity

$$M \models j(G) = (j(\kappa), j(+)) \text{ is abelian and every subgroup of size } < j(\kappa) \text{ is free abelian}$$

Let H be the subgroup of $j(G)$ generated by κ, then H is isomorphic to G ($j \restriction G$ is the isomorphism). H is free abelian, hence G is also free abelian. □

Large cardinals and compactness for abelian groups

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Large cardinals and compactness for graphs

Lemma

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Proof.

W.l.o.g. we can assume that $G = (\kappa, E)$. Fix an elementary embedding $j : V \to M$ with critical point κ. By elementarity

$$M \models \text{every subgraph of } j(G) = (j(\kappa), j(E)) \text{ of size} < j(\kappa) \text{ has coloring number} \leq j(\gamma) = \gamma$$

Let H be the subgraph of $j(G)$ generated by κ, then by elementarity G is isomorphic to H.H has coloring number $\leq \gamma$, hence G has also coloring number $\leq \gamma$. □

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W.l.o.g. we can assume that $X = (\kappa, \tau)$. Fix an elementary embedding $j : V \to M$ with critical point κ. By elementarity

$$M \models j(X) = (j(\kappa), j(\tau)) \text{ is a topological space } < j(\kappa)\text{-collectionwise Hausdorff}$$

Let H be the subspace of $j(X)$ generated by κ, then by elementarity X is isomorphic to H.Let C be a closed discrete set in X, then $C^* := j''C$ is a closed discrete set in H, hence in $j(X)$. Since $j(X)$ is $< j(\kappa)$-collectionwise Hausdorff and C^* has size $\leq \kappa < j(\kappa)$, we have a separation $\{U_x{}^*\}_{x \in C^*}$ for C^*.Taking $U_x := j^{-1''} U_x$ for every x, we get a separation for C. □

Large cardinals and compactness for being collectionwise Hausdorff

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When do we have compactness for structures of cardinality a **small regular** cardinal?

Delta-reflection

Definition (Magidor, Shelah '94)

For $\kappa < \lambda$, $\Delta_{\kappa,\lambda}$ is the following statement:
given a stationary set $S \subseteq E^{\lambda}_{<\kappa}$ and an algebra $\mathcal{A}$ on λ with $< \kappa$ operations, there exists a subalgebra $\mathcal{A}'$ of $\mathcal{A}$ such that the order type of $\mathcal{A}'$ is a regular cardinal $< \kappa$ and

$$S \cap \mathcal{A}' \text{ is stationary in } sup(\mathcal{A}')$$

We say that λ has the Delta-reflection if $\Delta_{\kappa,\lambda}$ holds for every $\kappa < \lambda$.

Applications of Delta-reflection

Applications (Magidor, Shelah '94)

Suppose that κ has the Delta-reflection, then

- Given an abelian group *G* of size κ, if every subgroup of size $< \kappa$ is free abelian, then *G* is free abelian.
- Given a graph *G* of size κ. If every subgraph of *G* of size $< \kappa$ has coloring number $\leq \gamma < \kappa$, then *G* has coloring number $\leq \gamma$.
- Given *A* a family of κ many countable sets, if every subfamily of size $< \kappa$ has a transversal, then *A* has a transversal.
- Given *X* a topological space locally of cardinality $< \kappa$, if *X* is $< \kappa$-collectionwise Hausdorff, then *X* is collectionwise Hausdorff

Consistency of the Delta-reflection at $\aleph_{\omega^2+1}$

If κ is weakly compact, then κ has the Delta-reflection.

Theorem (Magidor, Shelah '94)

Cons$(\exists(\kappa_n)_{n<\omega}\text{supercompact cardinals}) \to$ *Cons*$(\Delta_{\aleph_{\omega^2},\aleph_{\omega^2+1}})$

Moreover, $\aleph_{\omega^2+1}$ is the smallest regular cardinal that can have the Delta-reflection.

Theorem (Magidor, Shelah '94)

Cons$(\exists(\kappa_n)_{n<\omega}\text{supercompact cardinals}) \to$
Cons$(\kappa \text{ is the first cardinal fixed point and for every regular } \lambda > \kappa,\ \Delta^-_{\kappa,\lambda})$

Consistency of the Delta-reflection at $\aleph_{\omega^2+1}$

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The tree property

The tree property is the generalization of König's lemma to uncountable cardinals.

Definition

A regular cardinal κ has the tree property (denoted *TP*(κ)) if for every tree of height κ with levels all of size $< \kappa$ there is a branch of size κ

Theorem

An inaccessible cardinal is weakly compact if and only if it has the tree property.

Delta-reflection and the tree property

Theorem

(F. , Magidor 2015)
$Cons(\exists(\kappa_n)_{n<\omega} \textit{supercompact cardinals}) \rightarrow Cons(\Delta_{\aleph_{\omega^2},\aleph_{\omega^2+1}} + TP(\aleph_{\omega^2+1}))$

Theorem

(F. , Magidor 2015) The Delta-reflection at $\aleph_{\omega^2+1}$ does not imply the tree property at $\aleph_{\omega^2+1}$. (i.e. $Cons(\exists(\kappa_n)_{n<\omega} \textit{supercompact cardinals}) \rightarrow Cons(\Delta_{\aleph_{\omega^2},\aleph_{\omega^2+1}} + \neg TP(\aleph_{\omega^2+1}))$)

Square principle

(Jensen) Square $\square_\kappa$:

There exists a sequence $\langle C_\alpha;\ \alpha \in Lim(\kappa^+)\rangle$ such that

1. every $C_\alpha \subseteq \alpha$ is a club;
2. $o.t.(C_\alpha) \leq \kappa$
3. $\beta \in \mathrm{Lim}(C_\alpha)$ implies $C_\beta = C_\alpha \cap \beta$;

Square principle

Square is an anti-compactness principle

- (Solovay) $\square_\kappa$ implies the failure of the reflection of stationary subsets of κ^+ (hence it implies the failure of the Delta-reflection at κ^+).
- (Todorčević ?) $\square_\kappa$ implies the failure of the tree property at κ^+
- (Rinot) If $\square_\kappa$ holds and $2^\kappa = \kappa^+$, then there is a graph *G* of size κ^+ such that every subgraph of size $< \kappa^+$ is countably chromatic while *G* has chromatic number κ.

Todorčević square

(Todorčević) $\square(\kappa)$:

There exists a sequence $\langle C_\alpha;\ \alpha \in Lim(\kappa)\rangle$ such that

1. every $C_\alpha \subseteq \alpha$ is a club;
2. $\beta \in \mathrm{Lim}(C_\alpha)$ implies $C_\beta = C_\alpha \cap \beta$;
3. there are no threads for the sequence, i.e. there is no club $C \subset \kappa$ such that $\beta \in \mathrm{Lim}(C)$ implies $C_\beta = C \cap \beta$;

Fact: $\square_\kappa$ implies $\square(\kappa^+)$

Todorčević square

$\square(\kappa)$ is an anti-compactness principle

- (Todorčević) $\square(\kappa)$ implies the failure of the tree property at κ
- (Veličković) $\square(\kappa)$ implies the existence of two stationary subsets of E^{κ}_{ω} that do not reflect simultaneously (i.e. there is no α such that both reflect to α).

Theorem (F. , Hayut)

$Cons(\exists(\kappa_n)_{n<\omega}\text{supercompact cardinals}) \rightarrow Cons(\Delta_{\aleph_{\omega^2},\aleph_{\omega^2+1}} + \square(\aleph_{\omega^2+1}))$

- in particular the Delta-reflection does not imply the tree property at $\aleph_{\omega^2+1}$ (another proof of F. , Magidor 2015).

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$Cons(\exists(\kappa_n)_{n<\omega}\text{supercompact cardinals}) \rightarrow Cons(\Delta_{\aleph_{\omega^2},\aleph_{\omega^2+1}} + \square(\aleph_{\omega^2+1}))$

- in particular the Delta-reflection does not imply the tree property at $\aleph_{\omega^2+1}$ (another proof of F. , Magidor 2015).

Some open questions

- What other compactness properties are incompatible with $\square(\kappa)$? Does $\square(\kappa)$ implies the failure of compactness for being countably chromatic?
- For which cardinals is it consistent to have compactness for being countably chromatic?
- Is there a natural principle of reflection for small cardinals that implies every interesting compactness property?

Merci