Reflection and anti-reflection at the successor of a singular cardinal

joint work with Yair Hayut

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Abstract

We present the following result:

Theorem (F. , Hayut)

$Cons(\exists(\kappa_n)_{n<\omega}\text{supercompact cardinals}) \rightarrow Cons(\Delta_{\aleph_{\omega^2},\aleph_{\omega^2+1}} + \square(\aleph_{\omega^2+1}))$

- $\Delta_{\aleph_{\omega^2},\aleph_{\omega^2+1}}$ is a strong version of the reflection of stationary sets at $\aleph_{\omega^2+1}$.
- $\square(\kappa^+)$ is a weak version of $\square_\kappa$.

Reflection and compactness

Reflection: Given some structure (e.g. a set of ordinals, a group, a topological space etc.), if the structure satisfies a certain property, then there is a substructure of smaller cardinality with the same property.

Reflection of stationary sets

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Let κ be a regular cardinal,
Refl(κ): for every stationary subset S of κ, there exists $\alpha < \kappa$ of uncountable cofinality such that

$$S \cap \alpha \text{ is a stationary subset of } \alpha.$$

Applications: *Refl*(κ) is equiconsistent with "every κ-free abelian group is κ^+-free"

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Reflection of stationary sets

- If κ is weakly compact, then *Refl*(κ) holds.
- *Refl*(κ^+) fails if κ is a regular cardinal.

Theorem (Magidor '82)

Cons$(\exists(\kappa_n)_{n<\omega}\text{supercompact cardinals}) \rightarrow$ *Cons*(*Refl*$(\aleph_{\omega+1})$)

Delta-reflection

Definition (Magidor, Shelah '94)

For $\kappa < \lambda$, $\Delta_{\kappa,\lambda}$ is the following statement:
given a stationary set $S \subseteq E^{\lambda}_{<\kappa}$ and an algebra $\mathcal{A}$ on λ with $< \kappa$ operations, there exists a subalgebra $\mathcal{A}'$ of $\mathcal{A}$ such that the order type of $\mathcal{A}'$ is a regular cardinal $< \kappa$ and

$$S \cap \mathcal{A}' \text{ is stationary in } sup(\mathcal{A}')$$

We say that λ has the Delta-reflection if $\Delta_{\kappa,\lambda}$ holds for every $\kappa < \lambda$.

Applications of Delta-reflection

Applications (Magidor, Shelah)

Suppose that κ has the Delta-reflection, then

- *Refl*(κ) holds
- every almost free abelian group of size κ is free.
- Given a graph *G* of size κ. If every subgraph of *G* of size $< \kappa$ has coloring number $\gamma < \kappa$, then *G* has coloring number γ.
- Given *A* a family of κ sets all of size $< \kappa$, if every subfamily of size $< \kappa$ has a transversal, then *A* has a transversal.
- Given *X* a topological space locally of cardinality $< \kappa$, if *X* is $< \kappa$-collectionwise Hausdorff, then *X* is collectionwise Hausdorff

Consistency of the Delta-reflection at $\aleph_{\omega^2+1}$

Theorem (Magidor, Shelah '94)

$\mathit{Cons}(\exists(\kappa_n)_{n<\omega}\text{supercompact cardinals}) \rightarrow \mathit{Cons}(\Delta_{\aleph_{\omega^2},\aleph_{\omega^2+1}})$

Moreover, $\aleph_{\omega^2+1}$ is the smallest regular cardinal that can have the Delta-reflection.

Square principle

(Jensen) Square $\square_\kappa$:

There exists a sequence $\langle C_\alpha;\ \alpha \in Lim(\kappa^+)\rangle$ such that

1. every $C_\alpha \subseteq \alpha$ is a club;
2. $\beta \in \mathrm{Lim}(C_\alpha)$ implies $C_\beta = C_\alpha \cap \beta$;
3. $o.t.(C_\alpha) \leq \kappa$

Square principle

Square is an anti-reflection principle

- (Solovay) $\square_\kappa$ implies $\neg$*Refl*(κ^+) (in particular it implies the failure of the Delta-relfection at κ^+).
- (Solovay) if κ is strongly compact, then $\square_\mu$ fails for every $\mu \geq \kappa$.

Todorčević square

(Todorčević) $\square(\kappa)$:

There exists a sequence $\langle C_\alpha;\ \alpha \in Lim(\kappa)\rangle$ such that

1. every $C_\alpha \subseteq \alpha$ is a club;
2. $\beta \in \mathrm{Lim}(C_\alpha)$ implies $C_\beta = C_\alpha \cap \beta$;
3. there are no threads for the sequence, i.e. there is no club $C \subset \kappa$ such that $\beta \in \mathrm{Lim}(C)$ implies $C_\beta = C \cap \beta$;

Fact: $\square_\kappa$ implies $\square(\kappa^+)$

Todorčević square

$\square(\kappa)$ is an anti-reflection principle

- (Veličković) $\square(\kappa)$ implies the existence of two stationary subsets of E^{κ}_{ω} that do not reflect simultaneously (i.e. there is no α such that both reflect to α).
- (Rinot) $\square(\kappa)$ implies that every stationary subset of κ can be split into κ many disjoint stationary parts that do not reflect simultaneously
- (Todorčević) $\square(\kappa)$ implies the failure of the tree property at κ
- (Solovay, Veličković) if κ is strongly compact, then $\square(\mu)$ fails for every $\mu \geq \kappa$.

We present the following result:

Theorem (F. , Hayut)

$Cons(\exists(\kappa_n)_{n<\omega}\text{supercompact cardinals}) \to Cons(\Delta_{\aleph_{\omega^2},\aleph_{\omega^2+1}} + \Box(\aleph_{\omega^2+1}))$

- The Delta-reflection at κ^+ is incompatible even with the weak square $\Box^*_\kappa$, so in a way this result is optimal.
- The Delta-reflection implies the failure of the approachability property, so in particular $\Box(\aleph_{\omega^2+1})$ does not imply the approachability property at $\aleph_{\omega^2}$
- $\Box(\kappa^+)$ implies the failure of the tree property at κ^+, so in particular the Delta-reflection does not imply the tree property at $\aleph_{\omega^2+1}$ (see also F. , Magidor).

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- $\square(\kappa^+)$ implies the failure of the tree property at κ^+, so in particular the Delta-reflection does not imply the tree property at $\aleph_{\omega^2+1}$ (see also F. , Magidor).

Delta reflection at the successor of a singular cardinal

Theorem (Solovay)

Suppose $\lambda = lim_{n<\omega}\kappa_n$ is a limit of supercompact cardinals, then λ^+ has the Delta-reflection.

Proof.

Let S and A be a stationary set and an algebra as in the statement of the Delta-reflection. Let $n < \omega$ large enough so that $S \subseteq E^{\lambda^+}_{<\kappa_n}$ and A has $< \kappa_n$ many operations. Fix a λ^+-supercompact embedding $j : V \to M$ with critical point κ_n. Let B be the subalgebra of $j(A)$ generated by $j'' \lambda^+$. Then by the closure of M, we have $B \in M$. Moreover the domain of B is precisely $j''\lambda^+$, thus the order type of B is $\lambda^+ < j(\kappa)$. We have $j(S) \cap B = j'' S$, hence this is stationary in $sup(j'' \lambda^+)$. It follows that $M \models \exists X$ subalgebra of $j(A)$ of order type $< j(\kappa)$ such that $j(S) \cap X$ is stationary in $\sup(X)$. By elementarity there exists a subalgebra X of A of order type $< \kappa$ such that S reflects on $sup(X)$. □

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Delta reflection at $\aleph_{\omega^2+1}$

Theorem (Magidor, Shelah '94)

$Cons(\exists(\kappa_n)_{n<\omega}\text{supercompact cardinals}) \to Cons(\Delta_{\aleph_{\omega^2},\aleph_{\omega^2+1}})$

Use a forcing $\mathbb{P}$ similar to diagonal Prikry forcing. The conditions have the following form

$$p = \langle \alpha_0, g_0, f_0, ..., \alpha_{n-1}, g_{n-1}, f_{n-1}, A_n, g_n, F_n... \rangle$$

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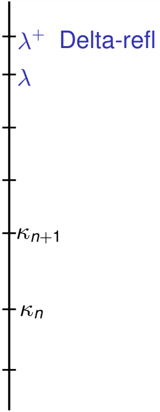

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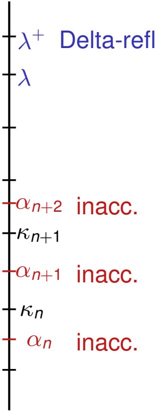

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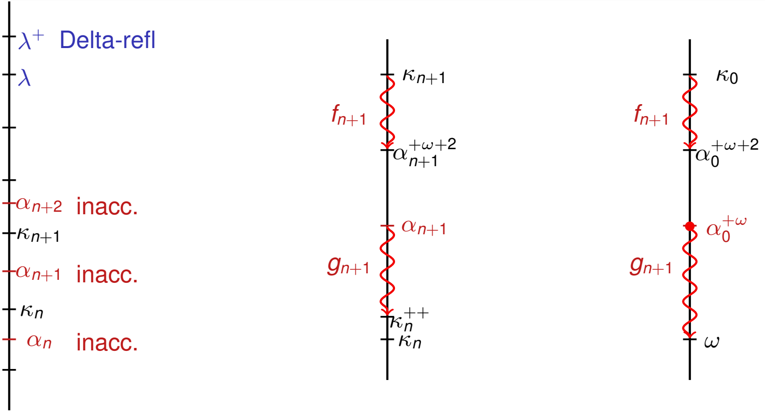

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Delta-reflection and square

We want both the Delta-reflection at $\aleph_{\omega^2+1}$ and $\square(\aleph_{\omega^2+1})$.

Problem: if $\square(\lambda^+)$ holds, then there are no λ^+-supercompact cardinals.

An attempted solution: Force with

- $\mathbb{S}$: forces a $\square(\lambda^+)$-sequence $\mathcal{S}$
- $\mathbb{T}$: adds a thread to $\mathcal{S}$

Then $\mathbb{S} * \mathbb{T}$ contains a λ^+-directed closed dense subset, thus

$$V^{\mathbb{S}*\mathbb{T}} \models \text{ each } \kappa_n \text{ is supercompact}$$

Forcing with $\mathbb{P}$, we have

$$V^{(\mathbb{S}*\mathbb{T})\times\mathbb{P}} \models \Delta_{\aleph_{\omega^2},\aleph_{\omega^2+1}}$$

Finally, we need a preservation lemma that shows that $\mathbb{T}$ can be removed, i.e. if the Delta-relection holds after $\mathbb{T}$, then it already held before. Thus

$$V^{\mathbb{S}\times\mathbb{P}} \models \Delta_{\aleph_{\omega^2},\aleph_{\omega^2+1}} + \square(\aleph_{\omega^2+1})$$

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New problem: $\mathbb{T}$ destroys stationary sets, so it may destroy stationary sets that do not reflect in $V^{\mathbb{S}*\mathbb{P}}$, thus the preservation lemma cannot be proven.

New solution: we do some preparation, namely we define an iteration $\mathbb{R}$ that preventively destroy all the stationary sets in $V^{\mathbb{S}\times\mathbb{P}}$ that would be destroyed by $\mathbb{T}$.

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Factorising $\mathbb{P}$

$$\mathbb{C}_n := \prod_{m \geq n} Coll(\kappa_m^{++}, < \kappa_{m+1})$$

For $c, c' \in \mathbb{C}_0$, let

- $c \sim c' \iff \exists n \forall m \geq n \; c(m) = c'(m)$
- $c \leq^* c' \iff \exists n \forall m \geq n \; c(m) \leq c'(m)$

$$\mathbb{C}_{fin} := (\mathbb{C}_0 / \sim, \leq^*)$$

$\mathbb{P}$ can be factorised like this

$$\mathbb{P} \equiv \mathbb{C}_{fin} * \mathbb{P}^*$$

The preparation

In $V^{\mathbb{C}_{fin} \times \mathbb{S}}$ we define $\mathbb{R}$ such that if E is a stationary set in $V^{(\mathbb{C}_{fin} \times \mathbb{S}) * \mathbb{R}}$, then $V^{(\mathbb{C}_{fin} \times \mathbb{S}) * \mathbb{R}} \models$ "$1_{\mathbb{T}} \Vdash E$ is stationary".

For every $n < \omega$, $(\mathbb{C}_n \times \mathbb{S}) * \mathbb{R} * \mathbb{T}$ contains a κ_n^+-directed closed dense subsets, thus

$$V^{(\mathbb{C}_n \times \mathbb{S}) * \mathbb{R} * \mathbb{T}} \models \kappa_n \text{ is supercompact}$$

In this model fix a normal ultrafilter on $\mathcal{P}_{\kappa_n}(\lambda^+)$, it has a projection to a normal ultrafilter U_n on κ_n, U_n is already in V. From $\{U_n\}_{n<\omega}$ define $\mathbb{P}$ in V.

The final model is

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In this model fix a normal ultrafilter on $\mathcal{P}_{\kappa_n}(\lambda^+)$, it has a projection to a normal ultrafilter U_n on κ_n, U_n is already in V. From $\{U_n\}_{n<\omega}$ define $\mathbb{P}$ in V.

The final model is

$$V^{(\mathbb{C}_{fin}\times\mathbb{S})*(\mathbb{R}\times\mathbb{P}^*)}$$

The preparation

In $V^{\mathbb{C}_{fin} \times \mathbb{S}}$ we define $\mathbb{R}$ such that if E is a stationary set in $V^{(\mathbb{C}_{fin} \times \mathbb{S}) * \mathbb{R}}$, then $V^{(\mathbb{C}_{fin} \times \mathbb{S}) * \mathbb{R}} \models$ "$1_{\mathbb{T}} \Vdash E$ is stationary".

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The idea of the proof

Part 1:
$V^{\mathbb{S}} \models \square(\lambda^+)$

A forcing $\mathbb{B}$ does not add a thread to a $\square(\lambda^+)$-sequence if $\mathbb{B} \times \mathbb{B}$ does not change the cofinality of λ^+.

$\mathbb{C}_{fin}, \mathbb{R}$ and $\mathbb{P}^*$ satisfy this requirement, thus

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Define in $V^{(\mathbb{C}_{fin}\times\mathbb{S})*\mathbb{R}}$ "fake versions" S^* of $\dot{S}$ and A^* of $\dot{A}$. By the preparation $\mathbb{R}$, there exists a generic G_T for $\mathbb{T}$ such that

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