Axioms as definitions: revisiting Hilbert

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What is an axiom in mathematics?

Self evidence, intrinsic motivations

"an axiom is a self-evident proposition requiring no formal demonstration to prove its truth, but received and assented as soon as mentioned" (Oxford English Dictionary).

Problems:

- self-evidence is subjective
- Axiom of Choice, Axiom of Infinity were not immediately received and assented
- even less controversial axioms of ZF are not strictly speaking obvious Ex: the Axiom of Foundation $A \notin A$ vs. 'every set is well-founded'

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Extrinsic motivations

Axioms can (also) be justified by their “success”, “fruitfulness”, or by "pragmatic or inter-theoretic motivations" (Maddy).

Remark: different incompatible theories can be successful
Ex: V=L vs. Axiom of measurable cardinals

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A different approach: Axioms as definitions of a concept.

"*In my opinion, a concept can be fixed logically only by its relations to other concepts. These relations formulated in certain statements, I call axioms, thus arriving at the view that axioms (perhaps together with propositions assigning names to concepts) are the definitions of the concepts*". (Hilbert, letter to Frege 22 Sept. 1900)

The Frege-Hilbert controversy

“ *[...] it is surely obvious that every theory is only a scaffolding or schema of concepts together with their necessary relations to one another, and that the basic elements can be thought in any way one likes. If in speaking of my points I think of some system of things, e.g. the system: love, law, chimney-sweep... and then assume all my axioms as relations between these things, then my propositions, e.g. Pythagoras’ theorem, are also valid for these things.* ” (Hilbert 29 Dic. 1899)

Axioms as definitions: matters of truth and existence

- Hilbert: consistency is a sufficient condition for truth and existence
- Shapiro (*Ante rem* Structuralism): satisfaction rather than consistency, and also categoricity
- *In rebus* Structuralism: the structure exists only inasmuch as some concrete system of things instantiates the structure, but no system is the *unique* intended interpretation of the theory, all concrete systems would do equally well. The real meaning of a sentence φ in a theory T is:
  Every concrete system that satisfies T, also satisfies φ.
  or, *Any possible system that satisfies T, also satisfies φ.*
- Arealism: mathematics is not a body of truths.

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Existential quantifiers in set theory

$$ZF \vdash \nexists y \forall x (x \in y \iff Ord(x))$$

- ‘the class of ordinals does not exist’ (absent in the literature)
- ‘the class of ordinals is not a set’
- ‘there is no set that contains all the ordinals (and nothing more)’

Rephrasing the axioms of ZFC

- (Extensionality) Two *sets* are equal if they contain the same *sets* as elements.
- (Pairing) Given two sets a and b, the pair $\{a, b\}$ (i.e. the collection containing exactly a and b as elements) **is a set**;
- (Separation) Given a formula $\varphi(x, \vec{p})$ with *sets* parameters $\vec{p}$, and given a *set* a, the collection of all x in a that satisfy $\varphi(x, \vec{p})$ **is a set**;
- (Union) Given a *set* a, the union of a (i.e. the collection of all *sets* that belong to some element of a) **is a set**;
- (Power Set) Given a *set* a, the collection of all subsets of a **is a set**;
- (Replacement) Given a function $f(x)$ (defined with set parameters) and a *set* a, the collection of all $f(x)$ with $x \in a$ **is a set**;
- (Foundation) A collection of *sets* that does not have an $\in$-minimal element **is not a set**;
- (Infinity) Consider all the collections *of sets* that contain the empty set and are closed by the operation $x \mapsto x \cup \{x\}$, at least one of such collections **is a set**;
- (Choice) For every family of nonempty sets *which is itself a set*, the image of the choice function **is a set**.

Existential quantifiers in set theory- continue

The existential quantifiers in the axioms of *ZFC* do not establish the actual existence of something. Instead, they are ways of singling out collections which are already given. The existential quantifiers *select* among such collections, those that are worth the title of 'set'.

Set theory is concerned with a domain of individuals, which we shall call simply 'objects' and among which are the 'sets'. (Zermelo 1908)

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When we infer '∃' of a certain collection, we make the collection available for the operations definable from the other axioms. Thus being a set means being available for the other axioms. It follows that the meaning of 'set' depends on the theory. In this sense, the whole theory ZF (or ZFC) is a **definition** of 'set' through the 'membership relation' (so, more precisely, it's a definition of the concepts of 'set' and 'membership').

"*[...] to give a definition of a point in three lines is to my mind an impossibility, for only the whole structure of axioms yields a complete definition and hence every new axiom changes the concept.*"(Hilbert 29 Dic. 1899)

The meaning of 'set' changes if we change the theory. So being a set in the sense of ZF is different than in ZFC, or ZFC+V=L etc.

The foundational role of set theory

If many theories of sets or concepts of sets are legitimate, then should we consider this multitude of set theories as the object of study of set theorists, just as we consider that the multitude of geometries are the object of study of geometers?

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A possible criterion: maximize the *expressive power* of the concept defined.

Some concepts of sets may be more “expressive” than others: you can express ‘set in the sense of V=L’ in the theory of large cardinals (by simply saying that the object in question is in L), the converse is not true.

(Steel) “The language of set theory as used by the believer in V = L can certainly be translated into the language of set theory as used by the believer in measurable cardinals, via the translation $\varphi \mapsto \varphi^L$. There is no translation in the other direction.”

Thank you