Realizability models and the axiom of choice

joint work with J-L. Krivine (IRIF- University Paris 7)

Laura Fontanella

I2M - Université Aix-Marseille

September 15, 2018

What is Realizability?

Realizability aims at extracting the computational content of mathematical proofs.

A short history

Kleene 1945

Correspondence between formulas of Heyting arithmetic and (sets of indexes of) recursive functions.

Curry Howard 1958

Isomorphism between proofs in intuitionistic logic and simply typed lambda-terms.

Griffin 1990

Correspondence between classical logic and lambda-terms plus control operators.

Krivine 2000-2004

Realisability models of set theory (ZF+DC)

The Axiom of Choice

Open problem: can we realize the Axiom of Choice?

Krivine 2004

Realizability models of Dependent Choice

F. + Krivine 2018 (work in progress)

Realizability model of $ZF + \forall\alpha\ AC_\alpha$.

The Axiom of Choice

Open problem: can we realize the Axiom of Choice?

Krivine 2004

Realizability models of Dependent Choice

F. + Krivine 2018 (work in progress)

Realizability model of $ZF + \forall\alpha\ AC_\alpha$.

The Axiom of Choice

Open problem: can we realize the Axiom of Choice?

Krivine 2004

Realizability models of Dependent Choice

F. + Krivine 2018 (work in progress)

Realizability model of $ZF + \forall\alpha\ AC_\alpha$.

Variants of the Axiom of Choice

		AC						
		$\downarrow$						
		FI	$\rightarrow$	FB				
		$\downarrow$	$\swarrow$		$\searrow$			
CB*	$\leftarrow$	**PP**				WU	$\overset{\rightarrow}{\not\leftarrow}$	AC_ω^-
$\downarrow$		$\downarrow$	$\searrow$		$\nearrow$			$\uparrow\ \not\downarrow$
WPP	$\rightarrow$	PPIdm	$\overset{\rightarrow}{\not\leftarrow}$	$\forall\alpha\ AC_\alpha$	$\overset{\rightarrow}{\not\leftarrow}$	DC	$\overset{\rightarrow}{\not\leftarrow}$	AC_ω
					$\not\searrow$			$\not\uparrow\ \downarrow$
				$\not\swarrow$		Idm	$\overset{\rightarrow}{\not\leftarrow}$	$\aleph_0$-TC
					$\swarrow$	$\not\uparrow$	$\not\searrow$	$\not\uparrow\ \not\downarrow$
				PW		LCR	$\not\rightarrow$	AC_ω^-

Krivine's realizability models

Forcing in 1 slide

Forcing (Cohen 1962)

Technique for defining models of set theory and proving relative consistency and independence results.

- Fix a (complete) boolean algebra $\mathbb{B}$.
- We assign to each formula φ of ZF a value $||\varphi||$ in $\mathbb{B}$. If φ is 'definitely true' we give it value $\mathbb{1}$; if it is 'definitely false' we give it value 0, otherwise we assign it some intermediate value in $\mathbb{B}$ in a way that 'respects the logic'.
- Then we look at the formulas that have value $\mathbb{1}$, they form a coherent theory. Thus there is a model that satisfy those formulas, the *forcing model*.

Forcing in 1 slide

Forcing (Cohen 1962)

Technique for defining models of set theory and proving relative consistency and independence results.

- Fix a (complete) boolean algebra $\mathbb{B}$.
- We assign to each formula φ of ZF a value $||\varphi||$ in $\mathbb{B}$. If φ is 'definitely true' we give it value $\mathbb{1}$; if it is 'definitely false' we give it value 0, otherwise we assign it some intermediate value in $\mathbb{B}$ in a way that 'respects the logic'.
- Then we look at the formulas that have value $\mathbb{1}$, they form a coherent theory. Thus there is a model that satisfy those formulas, the *forcing model*.

Forcing in 1 slide

Forcing (Cohen 1962)

Technique for defining models of set theory and proving relative consistency and independence results.

- Fix a (complete) boolean algebra $\mathbb{B}$.
- We assign to each formula φ of ZF a value $||\varphi||$ in $\mathbb{B}$. If φ is 'definitely true' we give it value $\mathbb{1}$; if it is 'definitely false' we give it value 0, otherwise we assign it some intermediate value in $\mathbb{B}$ in a way that 'respects the logic'.
- Then we look at the formulas that have value $\mathbb{1}$, they form a coherent theory. Thus there is a model that satisfy those formulas, the *forcing model*.

Forcing vs. realizability

Forcing	**Realizability**
$\mathbb{B}$: set of conditions (Boolean algebra)	Λ: the 'programs' ; Π : the 'stacks'
$\|\|\varphi\|\| \in \mathbb{B}$	$\|\varphi\| \subseteq \Lambda$; $(\!\|\varphi\|\!) \subseteq \Pi$
$\mathbb{1}$ maximal condition	$I,\ K,\ W,\ C,\ B,\ cc \in \Lambda$ 'instructions'
$\{\mathbb{1}\}$	$\Lambda^* \subseteq \Lambda$: the 'trustful' programs. Contains the instructions
$\wedge$ $\vee$	( ) 'application' ; . 'push' ; $\star$ 'process' k_π 'continuation'
$\leq$ partial order on $\mathbb{B} \setminus \{0\}$	$\succ$ preorder on $\Lambda \star \Pi$
$\perp \subseteq \mathbb{B} \times \mathbb{B}$	$\perp \subseteq \Lambda \star \Pi$ final segment
V 'ground model' $V^{\mathbb{P}}$ the Boolean-valued model	$\mathcal{M}$ 'ground model' $\mathcal{N}$ 'realizability model'
$V^{\mathbb{P}} \models \varphi$ if $\|\|\varphi\|\| = \mathbb{1}$ $\mathbb{1} \Vdash \varphi$ reads "$\mathbb{1}$ forces φ"	$\mathcal{N} \models \varphi$ if $\exists \theta \in \Lambda^* \ (\theta \in \|\varphi\|)$ $\theta \Vdash \varphi$ reads "θ realizes φ"

Krivine's machine

$\succ$ is the least preorder on $\Lambda \star \Pi$ such that for all $\xi, \eta, \zeta \in \Lambda$ and $\pi, \sigma \in \Pi$,

- $\xi(\eta) \star \pi \succ \xi \star \eta . \pi$
- $I \star \xi . \pi \succ \xi \star \pi$
- $K \star \xi . \eta . \pi \succ \xi \star \pi$
- $E \star \xi . \eta . \pi \succ \xi(\eta) \star \pi$
- $W \star \xi . \eta . \pi \succ \xi \star \eta . \eta . \pi$
- $C \star \xi . \eta . \zeta . \pi \succ \xi \star \zeta . \eta . \pi$
- $B \star \xi . \eta . \zeta . \pi \succ \xi(\eta(\zeta)) \star \pi$
- $cc \star \xi . \pi \succ \xi \star k_\pi . \pi$
- $k_\pi \star \xi . \sigma \succ \xi \star \pi$

Non extensional set theory ZF_ε

$\mathcal{L} = \{\varepsilon\ , \in, \subseteq\}$.
$x \simeq y$ is the formula $x \subseteq y \wedge y \subseteq x$

- Extensionality: $\forall x \forall y (x \in y \iff \exists z\ \varepsilon\ y (x \simeq z))$;
  $\forall x \forall y (x \subseteq y \iff \forall z\ \varepsilon\ x (z \in y))$
- Foundation:
  $\forall x_1 ... \forall x_n \forall a (\forall x (\forall y\ \varepsilon\ x F[y, x_1, ..., x_n] \Rightarrow F[x, x_1, ..., x_n]) \Rightarrow F[a, x_1, ..., x_n])$
- Pairing: $\forall a \forall b \exists x (a\ \varepsilon\ x \wedge b\ \varepsilon\ x)$
- Union: $\forall a \exists b \forall x\ \varepsilon\ a \forall y\ \varepsilon\ x (y\ \varepsilon\ b)$
- Powerset: $\forall a \exists b \forall x \exists y\ \varepsilon\ b \forall z (z\ \varepsilon\ y \iff (z\ \varepsilon\ a \wedge z\ \varepsilon\ x))$
- Replacement: $\forall x_1 ... \forall x_n \forall a \exists b \forall x\ \varepsilon\ a (\exists y F[x, y, x_1, ..., x_n] \Rightarrow (\exists y\ \varepsilon\ b F[x, y, x_1 ... x_n]))$
- Infinity $\forall x_1 ... x_n \forall a \exists b [a\ \varepsilon\ b \wedge \forall x\ \varepsilon\ b (\exists y F[x, y, x_1, ..., x_n] \Rightarrow \exists y\ \varepsilon\ b F[x, y, x_1, ..., x_n])]$

ZF_ε is a conservative extension of ZF.

The realizability relation

We define the two truth values $|\varphi| \subseteq \Lambda$ and $(\!|\varphi|\!) \subseteq \Pi$.

$\xi \in |\varphi| \iff \forall \pi \in (\!|\varphi|\!)(\xi \star \pi \in \perp\!\!\!\perp)$

$\xi \Vdash \varphi$ means $\xi \in |\varphi|$

- $(\!|\top|\!) = \emptyset$, $(\!|\perp|\!) = \Pi$,
- $(\!|a \not\varepsilon b|\!) = \{\pi \in \Pi;\ (a, \pi) \in b\}$
- $(\!|a \subseteq b|\!) = \{\xi \,.\, \pi;\ \exists c (c, \pi) \in a \text{ and } \xi \Vdash c \notin b\}$
- $(\!|a \notin b|\!) = \{\xi \,.\, \xi' \,.\, \pi;\ \exists c (c, \pi) \in b \text{ and } \xi \Vdash a \subseteq c \text{ and } \xi' \Vdash c \subseteq a\}$
- $(\!|\varphi \Rightarrow \psi|\!) = \{\xi \,.\, \pi;\ \xi \Vdash \varphi \text{ and } \pi \in (\!|\psi|\!)\}$
- $(\!|\forall x \varphi|\!) = \bigcup_a (\!|\varphi[a/x]|\!)$

Adequacy lemma

Adequacy lemma

Let $A_1, ..., A_n, A$ be closed formulas of ZF_ε and suppose $x_1 : A_1, ..., x_n : A_n \vdash t : A$.
If $\xi_1 \Vdash A_1, ...\xi_n \Vdash A_n$, then $t[\xi_1/x_1, ..., \xi_n/x_n] \Vdash A$.

Corollary

If $\vdash t : A$, then $t \Vdash A$

Theorem

The axioms of ZF_ε are realized.

Non extensional choice

Non extensional functions

$$\varepsilon - Func(f) \equiv \forall x, y, y' \ ((x, y)\varepsilon\, f \wedge (x, y')\varepsilon\, f \Rightarrow y = y')$$

Non extensional Axiom of Choice (NEAC)

$$\forall z \exists f \ (f \subseteq z \wedge \varepsilon - Func(f) \wedge \forall x, y \exists y' \ ((x, y)\varepsilon\, z \Rightarrow (x, y')\varepsilon\, f))$$

Krivine 2004

Realizability models of DC (using NEAC and the 'unicity' of natural numbers).

F. + Krivine 2018

Realizability model of $\forall\alpha\ AC_\alpha$ (using NEAC and the 'unicity' of ordinals in the model).

Non extensional choice

Non extensional functions

$\varepsilon - Func(f) \equiv \forall x, y, y' \; ((x,y)\varepsilon\, f \wedge (x,y')\varepsilon\, f \Rightarrow y = y')$

Non extensional Axiom of Choice (NEAC)

$$\forall z \exists f \; (f \subseteq z \wedge \varepsilon - Func(f) \wedge \forall x, y \exists y' \; ((x,y)\varepsilon\, z \Rightarrow (x,y')\varepsilon\, f))$$

Krivine 2004

Realizability models of DC (using NEAC and the ‘unicity’ of natural numbers).

F. + Krivine 2018

Realizability model of $\forall\alpha\ AC_\alpha$ (using NEAC and the ‘unicity’ of ordinals in the model).

Non extensional choice

Non extensional functions

$\varepsilon - Func(f) \equiv \forall x, y, y' \; ((x,y)\varepsilon\, f \wedge (x,y')\varepsilon\, f \Rightarrow y = y')$

Non extensional Axiom of Choice (NEAC)

$$\forall z \exists f \; (f \subseteq z \wedge \varepsilon - Func(f) \wedge \forall x, y \exists y' \; ((x,y)\varepsilon\, z \Rightarrow (x,y')\varepsilon\, f))$$

Krivine 2004

Realizability models of DC (using NEAC and the ‘unicity’ of natural numbers).

F. + Krivine 2018

Realizability model of $\forall\alpha\; AC_\alpha$ (using NEAC and the ‘unicity’ of ordinals in the model).

Thank you