Compactness and reflection at small cardinals

joint work with Menachem Magidor

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Compactness and reflection

Reflection: Given some structure (e.g. a group, a topological space etc.), if the structure satisfies a certain property, then there is a substructure of smaller cardinality with the same property.

Compactness: Given some structure if every substructure of smaller cardinality satisfies a certain property, then the whole structure satisfies the same property.

Compactness and large cardinals

Definition

A cardinal κ is strongly compact if and only if $\mathcal{L}_{\kappa,\kappa}$ satisfies the compactness theorem, i.e. every collection of $\mathcal{L}_{\kappa,\kappa}$-sentences is satisfiable if it is $<\kappa$-satisfiable.

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A cardinal κ is weakly compact if and only if $\mathcal{L}_{\kappa,\kappa}$ satisfies the *weak* compactness theorem, i.e. every collection of $\mathcal{L}_{\kappa,\kappa}$-sentences *with at most* κ *non logical symbols* is satisfiable if it is $<\kappa$-satisfiable.

The tree property

Definition

Let κ be a regular cardinal, we say that κ has the tree property if every tree of height κ with levels all of size less than κ, has a cofinal branch (i.e. a branch of size κ).
$TP(\kappa)$ means that κ has the tree property.

Assuming the consistency of large cardinals, the following are consistent:

- (Mitchell 1972) the tree property holds at every $\aleph_n$ with $n \geq 2$ (at any double successor of a regular cardinal)
- (Magidor, Shelah 1996) the tree property holds at $\aleph_{\omega+1}$ (at any successor of singular cardinal)

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The stationary reflection

Definition

Let κ be a regular cardinal and let $S \subseteq \kappa$ be a stationary set, we say that *S* reflects if there exists $\alpha < \kappa$ of uncountable cofinality such that $S \cap \alpha$ is a stationary subset of α. *Refl*(κ) is the statement that every stationary subset of κ reflects.

Refl(κ^+) fails if κ is a regular cardinal.

Theorem (Magidor 1982)

If the existence of infinitely many supercompact cardinals is consistent, then *Refl*$(\aleph_{\omega+1})$ is consistent.

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Reflection of stationary sets and the tree property at $\aleph_{\omega^2+1}$

Theorem (F. , Magidor 2014)

If the existence of infinitely many supercompact cardinals is consistent, then $\mathit{Refl}(\aleph_{\omega^2+1}) + \mathit{TP}(\aleph_{\omega^2+1})$ is consistent.

Definition

For $\kappa < \lambda$, $\Delta_{\kappa,\lambda}$ is the following statement:
given a stationary set $S \subseteq E^{\lambda}_{<\kappa}$ and an algebra $\mathcal{A}$ on λ with $< \kappa$ operations, there exists a subalgebra $\mathcal{A}'$ of $\mathcal{A}$ such that the order type of $\mathcal{A}'$ is a regular cardinal $< \kappa$ and $S \cap \mathcal{A}'$ is stationary in the *sup*$(\mathcal{A}')$

Let κ be singular. Δ_{κ,κ^+} implies *Refl*(κ^+).

Δ_{κ,κ^+} implies

- every almost free abelian group of size κ^+ is free.
- If G is a graph of size κ^+ and every subgraph of G of size $< \kappa$ has coloring number $\gamma < \kappa$, then G has coloring number γ.
- Given A a family of κ^+ sets all of size $< \kappa$, if every subfamily of size $< \kappa$ has a transversal, then A has a transversal.

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A strong principle of reflection at $\aleph_{\omega^2+1}$

Theorem (Magidor, Shelah 1994)

If the existence of infinitely many supercompact cardinals is consistent, then $\Delta_{\aleph_{\omega^2},\aleph_{\omega^2+1}}$ is consistent.

$\aleph_{\omega^2+1}$ is the smallest regular cardinal that can consistently satisfy the "Delta reflection".

Delta-reflection and the tree property at $\aleph_{\omega^2+1}$

Theorem 1 (F. , Magidor 2014)

If the existence of infinitely many supercompact cardinals is consistent, then $\Delta_{\aleph_{\omega^2},\aleph_{\omega^2+1}} + TP(\aleph_{\omega^2+1})$ is consistent.

Corollary

If the existence of infinitely many supercompact cardinals is consistent, then $Refl(\aleph_{\omega^2+1}) + TP(\aleph_{\omega^2+1})$ is consistent.

Theorem 2 (F. , Magidor 2014)

The Delta-relfection does not imply the tree property. If the existence of infinitely many supercompact cardinals is consistent, then $\Delta_{\aleph_{\omega^2},\aleph_{\omega^2+1}} + \neg TP(\aleph_{\omega^2+1})$ is consistent.

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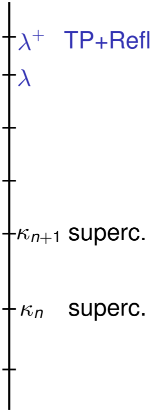

Theorem 1: Delta reflection and the tree property at the successor of a singular cardinal

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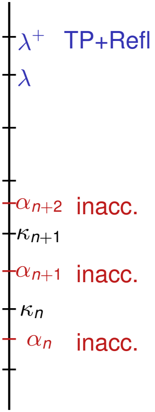

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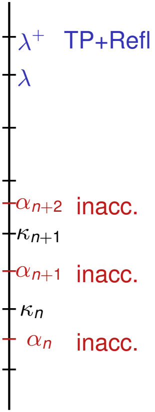

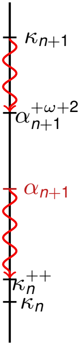

Let $\langle \kappa_n \rangle_{n<\omega}$ be indestructible supercompact cardinals, let $\lambda := lim_{n<\omega} \kappa_n$.

$$\mathbb{S}_n := \prod_{m \geq n} Col(\kappa_m^{++}, < \kappa_{m+1})$$

$V^{\mathbb{S}_n} \models \kappa_n$ is supercompact.

Let $\dot{W}_n$ be an $\mathbb{S}_n$-name for a normal ultrafilter on $\mathcal{P}_{\kappa_n}(\lambda^+)$.
Let U_n be the projection of this ultrafilter to κ_n, we can assume U_n is in V.

Theorem 2: Independence of the tree property from the Δ-reflection

Theorem 2 (F. , Magidor 2014)

Assuming the consistency of infinitely many supercompact cardinals, there is a model of *ZFC* in which $\Delta_{\aleph_{\omega^2}, \aleph_{\omega^2+1}}$ holds but $TP(\aleph_{\omega^2+1})$ fails.

$\langle \kappa_n \rangle_{n<\omega}$ and λ are as before. We force with

$$\mathbb{R} * \mathbb{P}$$

where:

- $\mathbb{R}$ the forcing that adds a Suslin tree $\dot{T}$ at λ^+. (The conditions of $\mathbb{R}$ are homogeneous trees $t \subset {}^{<\lambda^+}2$ of successor height)
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Thank you