Is typed realizability only predicative?

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Is typed realizability only (categorically) predicative?

No

Typed realizability models for HOL

[Paulin-Mohring. Extracting F_ω's programs from proofs in the calculus of constructions, 89']

Yes

Impredicativity $\rightsquigarrow$ untypedness

[Lietz, Streicher. Impredicativity entails untypedness, 02']

Question: Triposes from typed realizability?

Background

Def. *A typed partial combinatory algebra (tpca) is the data of*

- *a set of types* $\mathtt{Ty}$ *equipped with binary operations* $\to$ *and* $\times$
- *a set* $|A|$ *of realizers for each types* $A \in \mathtt{Ty}$
- *partial application functions* $._{A,B} : |A \to B| \times |A| \to |B|$
- *realizers* $\mathtt{k}_{A,B}, \mathtt{s}_{A,B,C}, \mathtt{pair}_{A,B}, \mathtt{fst}_{A,B}, \mathtt{snd}_{A,B}$

Ex. $\mathtt{Ty} \triangleq \mathscr{P}(\mathbb{N})$ *with* $A \to B \triangleq \{n \mid \forall m \in |A|, \{n\}(m) \downarrow \in |B|\}$ *and* $|A| = A$
Rk. $\big(\mathbb{N}, \{\_\}(\_)\big)$ *is* $\mathbf{K_1}$

Def. $\mathscr{T}$ *is essentially untyped if*

there exists a type U *such that all* $|A|$ *are partial retracts of* $|U|$ *i.e.*

$$\exists (U : \mathtt{Ty}) \forall (A : \mathtt{Ty}) \exists (s_A : |A \to U|) \exists (r_A : |U \to A|) \forall (a : |A|)\, r_A(s_A(a)) = a$$

Ex. $\mathscr{P}(\mathbb{N})$ *has no universal types*

Pseudofunctor from tpca

Fix $\mathscr{T}$ a tcpa and define

$$\begin{array}{rccl} \mathbb{P}: & \mathbf{Set}^{\mathrm{op}} & \to & \mathbf{pHA} \\ & I & \mapsto & \mathbb{P}(I) \triangleq (\bigsqcup_{A:\mathrm{Ty}} \mathscr{P}(|A|)^I, \leq_I) \quad \text{“family of fixed type”} \end{array}$$

For $p : \mathscr{P}(|A|)^I, q : \mathscr{P}(|B|)^I$, $p \leq_I q \quad \triangleq \quad p(i) \to q(i)$ is uniformly realized:

$$p \leq_I q \quad \triangleq \quad \exists (t : |A \to B|)(\forall i \in I)\ t \Vdash p(i) \to q(i)$$

Theorem

If $\mathbb{P}$ is a tripos then $\mathscr{T}$ is essentially untyped.

Proof.

The type of the generic predicate is universal. □

Tripos from (expressive enough) tpca. *Erase the type structure [1]*

Ex. For the tpca $\mathscr{P}(\mathbb{N})$, it gives the effective tripos.
Other solution. Consider

$$\begin{array}{rccl} \mathbb{P}: & \mathbf{Set}^{\mathrm{op}} & \to & \mathbf{pHA} \\ & I & \mapsto & \mathbb{P}(I) \triangleq \left(\left(\bigsqcup_{A:\mathrm{Ty}} \mathscr{P}(|A|)\right)^I, \leq_I\right) \quad \text{“type varying along the family”} \end{array}$$

Question: How to define $p \leq q$?

Idea: introduce external subtypings and cast operators
$$p \leq q \triangleq \exists (C : \mathrm{Ty}) \exists (t : |C|)(\forall i \in I) \exists (\rho : C \lhd\!: A \to B).\, \rho\langle t\rangle \Vdash p(i) \to q(i)$$

Problem: need arbitrary intersection types
If $f : I \to \{*\}$ and $p : \mathbb{P}(I)$ with $p(i) = |A_i|$, then type of $(\forall_f p)(*)$? $(= \bigcap_{i \in I} A_i$?)

Ex: For the tpca $\mathscr{P}(\mathbb{N})$, define $A \lhd\!: B \triangleq |A| \subseteq |B|$ and $\rho\langle t\rangle \triangleq t$.
It gives a tripos (related to Grayson modified realizability tripos).

[1] Cohen, Grunfeld, Kirst, Miquey. Syntactic Effectful Realizability in Higher-Order Logic,25'

Line of research

Question: Generic way to build triposes from (some) tpcas?

Our approach: Use evidenced frame to highlight the needed subtypings

An evidenced frame is the data of a triple $(\Phi, E, \cdot \xrightarrow{\cdot} \cdot)$ *where*

- Φ *is a set of propositions*
- E *is a set of evidences*
- $(\cdot \xrightarrow{\cdot} \cdot) \subseteq \Phi \times E \times \Phi$ *is the entailment relation*

along with extra properties.

Road toward an EF from a tpca

Ingredients

Propositions: types from an impredicative universe ($\mathtt{Set}$*)*
quantify over subsets of propositions: $\forall(X : \mathtt{Set} \to \mathtt{Prop}).A(X) : \mathtt{Set}$

Evidences: (typed) terms from $\mathtt{Set}$

Entailment relation: use cast operators $\rho : A \lhd: B$
simulates Curry-style polymorphism: quantifiers as intersection types
deals with type mismatch: composition of evidences needs to be total

Problem: choice principle on the subtypings
needed for the introduction rule of the universal arrow

Description of the construction (formalized in Rocq)

Meta-theory

CoC *with an impredicative (proof relevant) universe, Σ-types, an inductive type of subtypings and a choice principle.* Question: *Consistency?*

Propositions

A proposition is a type $A : \mathtt{Set}$ with a predicate $A \to \mathtt{Prop}$:

$$\Phi \triangleq \Sigma(A : \mathtt{Set}). A \to \mathtt{Prop}$$

If $\phi : \Phi$, write $[\![\phi]\!]^{\mathbf{T}}$ for its type and $[\![\phi]\!]^{\mathbf{S}}$ for its predicate.

Evidences

An evidence is a term of a type A (in $\mathtt{Set}$):

$$\mathrm{E} \triangleq \Sigma(A : \mathtt{Set}). A$$

If $e : \mathrm{E}$, write $\mathtt{Tm}(e)$ for the element and $\mathtt{Ty}(e)$ for its type.

Subtyping relation

$$\frac{A : \mathtt{Set}}{A \lhd: A} \qquad \frac{A \lhd: B \quad B \lhd: C}{A \lhd: C}$$

$$\frac{P : \kappa \to \mathtt{Set} \quad A : \kappa}{(\Pi(X:\kappa).PX) \lhd: PA}\ (\kappa \in \{\mathtt{Set}, \Phi, \Phi \to \mathtt{Prop}\}) \qquad \frac{A : \mathtt{Prop} \quad x : A \quad B : \mathtt{Set}}{(A \to B) \lhd: B}$$

$$\frac{A, B, C : \mathtt{Set} \quad A \lhd: B}{((A \to B) \to C) \lhd: C}$$

$$\frac{P : \Phi \to \mathtt{Prop} \quad A, B : \mathtt{Set} \quad C : \Phi \quad F : \Pi(X:\Phi).PX \to A \lhd: [\![C]\!]^{\mathbf{T}} \to [\![X]\!]^{\mathbf{T}}}{\big((\Pi(X:\Phi).PX \to A \to [\![C]\!]^{\mathbf{T}} \to [\![X]\!]^{\mathbf{T}}) \to B\big) \lhd: B}$$

Elimination principle. $\rho\langle a\rangle : B$ *if* $\rho : A \lhd: B$ *and* $a : A$

Entailement

$$A \xrightarrow{e} B \quad \triangleq \quad \exists\big(\rho : \mathtt{Ty}(e) \lhd: [\![A]\!]^{\mathbf{T}} \to [\![B]\!]^{\mathbf{T}}\big).\Pi(a : [\![A]\!]^{\mathbf{T}}).[\![A]\!]^{\mathbf{S}}(a) \to [\![B]\!]^{\mathbf{S}}\big(\rho\langle \mathtt{Tm}(e)\rangle(a)\big)$$

Choice principle

$$\big(\Pi(X:\Phi).PX \to \exists\big(\rho : A \lhd: [\![B]\!]^{\mathbf{T}} \to [\![X]\!]^{\mathbf{T}}\big).RX\rho\big) \to$$
$$\exists\big(F : \Pi(X:\Phi).P\,X \to A \lhd: [\![B]\!]^{\mathbf{T}} \to [\![X]\!]^{\mathbf{T}}\big).\Pi(X:\Phi)(p:PX).RX(F\,X\,p)$$

Recall

$$\frac{P:\Phi \to \texttt{Prop} \quad A,B:\texttt{Set} \quad C:\Phi \quad F:\Pi(X:\Phi).PX \to A \lhd: [\![C]\!]^{\mathbf{T}} \to [\![X]\!]^{\mathbf{T}}}{\big((\Pi(X:\Phi).PX \to A \to [\![C]\!]^{\mathbf{T}} \to [\![X]\!]^{\mathbf{T}}) \to B\big) \lhd: B}$$

Theorem (in Rocq + `impredicative-set` + AC)

$(\Phi, \mathrm{E}, \cdot \dot{\to} \cdot)$ *is an evidenced frame.*

Conclusion

To-go

- set-based triposes from typed realizability $\rightsquigarrow$ erase types
- or add subtypings on top of the type system

Future works

- Consistency of the meta-theory?
- Refine the construction (remove the use of the choice principle)
- EF from internal complete category in topos-theoretic setting

Question: When a category of assemblies over a tpca can be recontructed as the category of $\neg\neg$-separated objects for some topos? [1]

Idea: If (up to subtypings) it implements arbitrary intersection types

[1] Lietz, Streicher. Impredicativity entails untypedness,02'