The ramified analytic hierarchy in second-order arithmetic

Félix Castro

December 17, 2024

1. The ramified analytic hierarchy (RAH)
   - Introduction: constructible universe and ramified analytic hierarchy
   - Presentation of (classical) second-order arithmetic ($\mathbf{PA2}$)
   - The axiom scheme of collection
   - A glimpse at intuitionistic logic
   - Well-preorders
   - The ramified analytic hierarchy
2. Equality in finite type arithmetic

Popularization and constructible universe

Rough analogy.

Foundations of mathematics	Playgrounds for mathematicians
Expressive enough	Entertaining
Consistent	Safe

Example of foundations: set theory $(\mathbf{ZF})$

Popularization and constructible universe

Rough analogy.

Foundations of mathematics	*Playgrounds for mathematicians*
Expressive enough	*Entertaining*
Consistent	*Safe*

Example of foundations: set theory ($\mathbf{ZF}$)

Gödel defined a proper class L (the constructible universe) [21]

- L is an inner model of $\mathbf{ZF}$
- $\mathrm{L} \vDash \mathbf{AC} + \mathrm{GCH}$
- $\mathrm{L} \vDash \mathrm{V} = \mathrm{L}$ ("all the sets are constructible")

Theorem (Gödel). $\mathbf{ZF} + \mathrm{V} = \mathrm{L} \vdash \phi$ *implies* $\mathbf{ZF} \vdash \phi^{\mathrm{L}}$

[21] K. Gödel. The consistency of the axiom of choice and the generalized continuum hypothesis ('38)

Popularization and constructible universe

Rough analogy.

Foundations of mathematics	*Playgrounds for mathematicians*
Expressive enough	*Entertaining*
Consistent	*Safe*

Example of foundations: set theory ($\mathbf{ZF}$)

Gödel defined a proper class L (the constructible universe) [21]

- L is an inner model of $\mathbf{ZF}$
- $\mathrm{L} \vDash \mathbf{AC} + \mathrm{GCH}$
- $\mathrm{L} \vDash \mathrm{V} = \mathrm{L}$ ("all the sets are constructible")

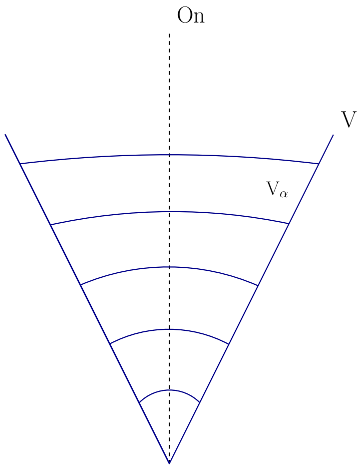
Theorem (Gödel). $\mathbf{ZF} + \mathrm{V} = \mathrm{L} \vdash \phi$ *implies* $\mathbf{ZF} \vdash \phi^{\mathrm{L}}$

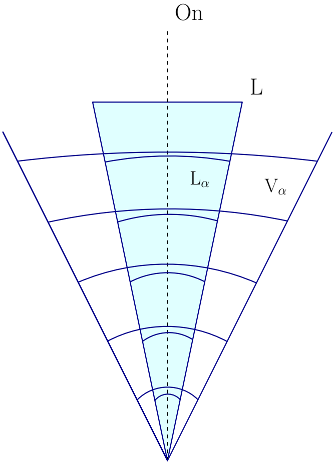
Analogy (sequel).

results of relative consistency	*adding new toys is safe*

[21] K. Gödel. The consistency of the axiom of choice and the generalized continuum hypothesis ('38)

Construction of L





- $L_\alpha = \bigcup_{\beta<\alpha} \mathbf{Def}(L_\beta)$
- $\mathbf{Def}(X) \subseteq \mathscr{P}(X)$
- $\mathbf{Def}(X) \triangleq \{Y \subseteq X \mid Y$ definable from $X\}$
- $L = \bigcup_{\alpha \in \mathrm{On}} L_\alpha$

Ramified analytic hierarchy

Kleene defined the ramified analytic hierarchy (RAH) [26]

- $\mathrm{RAH}_\alpha = \bigcup_{\beta<\alpha} \mathbf{Def^2}(\mathrm{RAH}_\beta)$
    - For $X \subseteq \mathscr{P}(\omega)$:

      $\mathbf{Def^2}(X) \subseteq \mathscr{P}(\omega)$

      $\mathbf{Def^2}(X) \triangleq$ subsets of ω definable in 2^{nd}-order logic in the structure (ω, X)

[26] S. C. Kleene. Quantification of number-theoretic functions ('60)

Ramified analytic hierarchy

Kleene defined the ramified analytic hierarchy (RAH) [26]

- $\mathrm{RAH}_\alpha = \bigcup_{\beta<\alpha} \mathbf{Def^2}(\mathrm{RAH}_\beta)$
  - For $X \subseteq \mathscr{P}(\omega)$:
    $\mathbf{Def^2}(X) \subseteq \mathscr{P}(\omega)$
    $\mathbf{Def^2}(X) \triangleq$ subsets of ω definable in 2nd-order logic in the structure (ω, X)
- $\mathrm{RAH} = \bigcup_{\alpha\in\mathrm{On}} \mathrm{RAH}_\alpha$
  - RAH is a model of $\mathbf{PA2}$
  - $\mathrm{RAH} \vDash \mathbf{AC}$
  - $\mathrm{RAH} \vDash \mathrm{V} = \mathrm{RAH}$

[26] S. C. Kleene. Quantification of number-theoretic functions ('60)

Ramified analytic hierarchy

Kleene defined the ramified analytic hierarchy (RAH) [26]

- $\mathrm{RAH}_\alpha = \bigcup_{\beta<\alpha} \mathbf{Def^2}(\mathrm{RAH}_\beta)$
  - For $X \subseteq \mathscr{P}(\omega)$:
    $\mathbf{Def^2}(X) \subseteq \mathscr{P}(\omega)$
    $\mathbf{Def^2}(X) \triangleq$ subsets of ω definable in 2^{nd}-order logic in the structure (ω, X)
- $\mathrm{RAH} = \bigcup_{\alpha\in\mathrm{On}} \mathrm{RAH}_\alpha$
  - RAH is a model of $\mathbf{PA2}$
  - $\mathrm{RAH} \vDash \mathbf{AC}$
  - $\mathrm{RAH} \vDash \mathrm{V} = \mathrm{RAH}$
- Proofs done in $\mathbf{ZF}$
  - recursion-theoretic and model-theoretic arguments + ordinals [10]

[26] S. C. Kleene. Quantification of number-theoretic functions ('60)

[10] R. Boyd, G. Hensel, H. Putnam. A recursion-theoretic characterization of the ramified analytical hierarchy ('69)

RAH in 2^{nd}-order arithmetic and related frameworks

- Study of RAH in 3^{rd}-order arithmetic with choice (A_3) [51]
  - $\mathrm{RAH} \models \mathrm{A}_3$ and $\mathrm{RAH} \models V = \mathrm{RAH}$
    - Interpretation between set theory and arithmetic (Zbierski,[2])
    - Correspondence in levels between L and RAH (Boolos,[2])

[2] K. R. Apt, W. Marek. Second order arithmetic and related topics ('74)

[51] Z. Vetulani. Ramified analysis and the minimal β-models of higher order arithmetics ('84)

RAH in 2^{nd}-order arithmetic and related frameworks

- Study of RAH in 3^{rd}-order arithmetic with choice (A_3) [51]
  - $\mathrm{RAH} \models \mathrm{A}_3$ and $\mathrm{RAH} \models V = \mathrm{RAH}$
    - Interpretation between set theory and arithmetic (Zbierski,[2])
    - Correspondence in levels between L and RAH (Boolos,[2])
- Study of RAH in 2^{nd}-order arithmetic ($\mathbf{PA2}$) [15]
  - $\mathrm{RAH} \models \mathbf{PA2}$ and $\mathrm{RAH} \models V = \mathrm{RAH}$
    - Use a fact about well-orders that does not seem to be "easily" provable

[2] K. R. Apt, W. Marek. Second order arithmetic and related topics ('74)

[51] Z. Vetulani. Ramified analysis and the minimal β-models of higher order arithmetics ('84)

[15] L. Colson, S. Grigorieff. Syntactical truth predicates for second order arithmetic ('01)

RAH in 2^{nd}-order arithmetic and related frameworks

- Study of RAH in 3^{rd}-order arithmetic with choice (A_3) [51]
  - $\mathrm{RAH} \vDash \mathrm{A}_3$ and $\mathrm{RAH} \vDash V = \mathrm{RAH}$
    - Interpretation between set theory and arithmetic (Zbierski,[2])
    - Correspondence in levels between L and RAH (Boolos,[2])
- Study of RAH in 2^{nd}-order arithmetic $(\mathbf{PA2})$ [15]
  - $\mathrm{RAH} \vDash \mathbf{PA2}$ and $\mathrm{RAH} \vDash V = \mathrm{RAH}$
    - Use a fact about well-orders that does not seem to be "easily" provable
    - Their proof of this fact is flawed

[2] K. R. Apt, W. Marek. Second order arithmetic and related topics ('74)

[51] Z. Vetulani. Ramified analysis and the minimal β-models of higher order arithmetics ('84)

[15] L. Colson, S. Grigorieff. Syntactical truth predicates for second order arithmetic ('01)

Goal and contributions

- Carve the path to a computational study of constructibility
  - Constructibility in System $\mathrm{F} \rightsquigarrow \mathrm{RAH}$
    - Impredicative quantification relativized to RAH
    - **Goal:** design a type system with a notion of extensional choice
  - What is very interesting is that we hit a snag!

Goal and contributions

- Carve the path to a computational study of constructibility
    - Constructibility in System $\mathrm{F} \rightsquigarrow \mathrm{RAH}$
        - Impredicative quantification relativized to RAH
        - **Goal:** design a type system with a notion of extensional choice
    - What is very interesting is that we hit a snag!
- Contributions of Part 1

Goal and contributions

- Carve the path to a computational study of constructibility
  - Constructibility in System $\mathrm{F} \rightsquigarrow \mathrm{RAH}$
    - Impredicative quantification relativized to RAH
    - **Goal:** design a type system with a notion of extensional choice
  - What is very interesting is that we hit a snag!
- Contributions of Part 1
  - Chapter 1: a new arithmetic over pure binary trees

Goal and contributions

- Carve the path to a computational study of constructibility
  - Constructibility in System $\mathrm{F} \rightsquigarrow \mathrm{RAH}$
    - Impredicative quantification relativized to RAH
    - **Goal:** design a type system with a notion of extensional choice
  - What is very interesting is that we hit a snag!
- Contributions of Part 1
  - Chapter 1: a new arithmetic over pure binary trees
  - Chapter 2: the axiom scheme of collection in 2^{nd}-order logic

Goal and contributions

- Carve the path to a computational study of constructibility
  - Constructibility in System $\mathrm{F} \rightsquigarrow \mathrm{RAH}$
    - Impredicative quantification relativized to RAH
    - **Goal:** design a type system with a notion of extensional choice
  - What is very interesting is that we hit a snag!
- Contributions of Part 1
  - Chapter 1: a new arithmetic over pure binary trees
  - Chapter 2: the axiom scheme of collection in 2^{nd}-order logic
  - Chapter 3: study of this scheme in the realm of intuitionistic logic

Goal and contributions

- Carve the path to a computational study of constructibility
  - Constructibility in System $\mathrm{F} \rightsquigarrow \mathrm{RAH}$
    - Impredicative quantification relativized to RAH
    - **Goal:** design a type system with a notion of extensional choice
  - What is very interesting is that we hit a snag!
- Contributions of Part 1
  - Chapter 1: a new arithmetic over pure binary trees
  - Chapter 2: the axiom scheme of collection in 2^{nd}-order logic
  - Chapter 3: study of this scheme in the realm of intuitionistic logic
  - Chapter 4: well-preorders in 2^{nd}-order logic

Goal and contributions

- Carve the path to a computational study of constructibility
  - Constructibility in System $\mathrm{F} \rightsquigarrow \mathrm{RAH}$
    - Impredicative quantification relativized to RAH
    - **Goal:** design a type system with a notion of extensional choice
  - What is very interesting is that we hit a snag!
- Contributions of Part 1
  - Chapter 1: a new arithmetic over pure binary trees
  - Chapter 2: the axiom scheme of collection in 2^{nd}-order logic
  - Chapter 3: study of this scheme in the realm of intuitionistic logic
  - Chapter 4: well-preorders in 2^{nd}-order logic
  - Chapter 5: RAH studied in $\mathbf{PA2}^{-} + \mathbf{Coll}$

What is arithmetic ?

- Is arithmetic about integers?

What is arithmetic ?

- Is arithmetic about integers?
  - Not only! It is about combinatorics
  - Example: Gödel's incompleteness theorem

What is arithmetic ?

- Is arithmetic about integers?
    - Not only! It is about combinatorics
    - Example: Gödel's incompleteness theorem
- Linear structures are not suited for encodings

What is arithmetic ?

- Is arithmetic about integers?
  - Not only! It is about combinatorics
  - Example: Gödel's incompleteness theorem
- Linear structures are not suited for encodings
- Arithmetic over the type of pure binary trees

```
Inductive 𝔹 :=
| 0 : 𝔹
| ⟨_,_⟩ : 𝔹 → 𝔹 → 𝔹
```

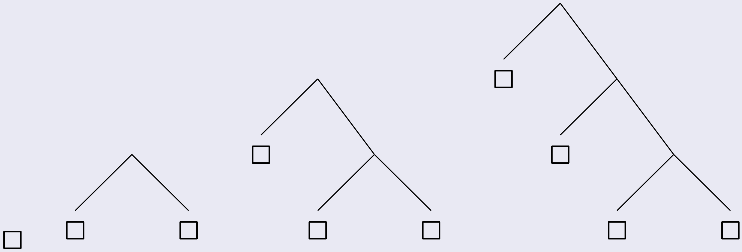
Individuals of $\mathbf{PA2}$: the type of pure binary trees

Codes of primitive recursive functions	f, g	$::=$	$\mathbf{0} \mid \mathrm{id} \mid \mathrm{fst} \mid \mathrm{snd}$ $\mid (f \circ g) \mid \langle f, g \rangle \mid [f \vert g]$
Terms	t, u	$::=$	$x \mid 0 \mid \langle t, u \rangle \mid f(t)$

Individuals of **PA2**: the type of pure binary trees

Codes of primitive recursive functions	f, g	$::=$	$\mathbf{0} \mid \mathrm{id} \mid \mathrm{fst} \mid \mathrm{snd}$ $\mid (f \circ g) \mid \langle f, g \rangle \mid [f \mid g]$
Terms	t, u	$::=$	$x \mid 0 \mid \langle t, u \rangle \mid f(t)$

Examples of pure binary trees

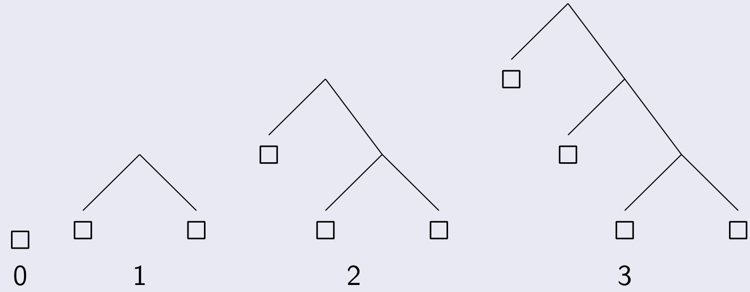


0 $\quad \langle 0, 0 \rangle$ $\quad \langle 0, \langle 0, 0 \rangle \rangle$ $\quad \langle 0, \langle 0, \langle 0, 0 \rangle \rangle \rangle$

Individuals of **PA2**: the type of pure binary trees

Codes of primitive recursive functions	f, g	$::=$	$\mathbf{0} \mid \mathrm{id} \mid \mathrm{fst} \mid \mathrm{snd}$ $\mid (f \circ g) \mid \langle f, g \rangle \mid [f \mid g]$
Terms	t, u	$::=$	$x \mid 0 \mid \color{red}{\langle t, u \rangle} \mid f(t)$

Encoding of integers as right combs



Individuals of $\mathbf{PA2}$: the type of pure binary trees

Codes of primitive recursive functions	f, g	$::=$	$\mathbf{0} \mid \mathrm{id} \mid \mathrm{fst} \mid \mathrm{snd}$ $\mid (f \circ g) \mid \langle f, g\rangle \mid [f\mid g]$
Terms	t, u	$::=$	$x \mid 0 \mid \langle t, u\rangle \mid f(t)$

Axioms of injectivity and non confusion:

- $\langle x, y\rangle = \langle x', y'\rangle \Rightarrow (x = x' \wedge y = y')$
- $\langle x, y\rangle \neq 0$

Individuals of **PA2**: the type of pure binary trees

Codes of primitive recursive functions	f, g	$::=$	$\mathbf{0} \mid \mathrm{id} \mid \mathrm{fst} \mid \mathrm{snd}$ $\mid (f \circ g) \mid \langle f, g \rangle \mid [f \vert g]$
Terms	t, u	$::=$	$x \mid 0 \mid \langle t, u \rangle \mid f(t)$

Axioms of injectivity and non confusion:

- $\langle x, y \rangle = \langle x', y' \rangle \Rightarrow (x = x' \wedge y = y')$
- $\langle x, y \rangle \neq 0$

Axioms of computation:

- $\mathbf{0}(x) = 0$
- $\mathrm{id}(x) = x$
- $\mathrm{fst}(0) = 0$
- $\mathrm{fst}(x, y) = x$
- $[\mathrm{f}|\mathrm{g}](0) = 0$
- $(f \circ g)(x) = f(g(x))$
- $\langle f, g \rangle(x) = \langle f(x), g(x) \rangle$
- $\mathrm{snd}(0) = 0$
- $\mathrm{snd}(x, y) = y$
- $[\mathrm{f}|\mathrm{g}](x, 0) = f(x)$
- $[\mathrm{f}|\mathrm{g}](\mathrm{x}, \langle \mathrm{y}, \mathrm{z} \rangle) = \mathrm{g}(\mathrm{x}, \mathrm{y}, \mathrm{z}, [\mathrm{f}|\mathrm{g}](\mathrm{x}, \mathrm{y}), [\mathrm{f}|\mathrm{g}](\mathrm{x}, \mathrm{z}))$

Formulas of **PA2**

Formulas ϕ, ψ ::= $\bot \mid t = u \mid t \in X$
$\mid \phi \Rightarrow \psi \mid \forall x \phi \mid \forall X \phi$

Sets E ::= $\{x \mid \phi\}$

[47] S. G. Simpson. Subsystems of second order arithmetic ('99)

Formulas of **PA2**

$$\begin{array}{llll} \text{Formulas} & \phi, \psi & ::= & \bot \mid t = u \mid t \in X \\ & & & \mid \phi \Rightarrow \psi \mid \forall x \phi \mid \forall X \phi \end{array}$$

$$\begin{array}{llll} \text{Sets} & E & ::= & \{x \mid \phi\} \end{array}$$

Example: the sets $\mathbb{B}$ and $\mathbb{N}$

$$\mathbb{B} \triangleq \{x \mid \forall X (0 \in X \Rightarrow \forall y \forall z (y \in X \Rightarrow z \in X \Rightarrow \langle y, z \rangle \in X) \Rightarrow x \in X)\}$$

$$\mathbb{N} \triangleq \{x \mid x \in \mathbb{B} \wedge \texttt{checknat}(x) = 1\}$$

where:

- $\texttt{checknat} \triangleq [\langle \mathbf{0}, \mathbf{0} \rangle | [\mathrm{id} | \mathbf{0}] \circ \langle \mathrm{snd} \circ \mathrm{snd} \circ \mathrm{snd} \circ \mathrm{snd}, \mathrm{fst} \circ \mathrm{snd} \rangle] \circ \langle \mathbf{0}, \mathrm{id} \rangle$
- $\phi \wedge \psi \triangleq (\phi \Rightarrow \psi \Rightarrow \bot) \Rightarrow \bot$

[47] S. G. Simpson. Subsystems of second order arithmetic ('99)

Formulas of **PA2**

Formulas	ϕ, ψ	$::=$	$\bot \mid t = u \mid t \in X$
			$\mid \phi \Rightarrow \psi \mid \forall x \phi \mid \forall X \phi$
Sets	E	$::=$	$\{x \mid \phi\}$

Axiom of induction:

- $0 \in X \Rightarrow \forall x \forall y \big(x \in X \Rightarrow y \in X \Rightarrow \langle x, y\rangle \in X\big) \Rightarrow \forall x (x \in X)$

Axiom scheme of comprehension:

- $\exists X \forall x \big(x \in X \Leftrightarrow \phi\big)$ (i.e. $\exists X\, X = \{x \mid \phi\}$)

where $X \notin FV(\phi)$

[47] S. G. Simpson. Subsystems of second order arithmetic ('99)

Slice of a set

Motivation: Encoding functions from the individuals to the reals

Slice of a set

Motivation: Encoding functions from the individuals to the reals

Definition

The slice of a set E at the individual x is defined as the set

$$E[x] \quad \triangleq \quad \{y \mid \langle x, y \rangle \in E\}$$

Slice of a set

Motivation: Encoding functions from the individuals to the reals

Definition

The slice of a set E at the individual x is defined as the set

$$E[x] \quad \triangleq \quad \{y \mid \langle x, y \rangle \in E\}$$

Solution: A set E encodes the function $x \mapsto E[x]$

Slice of a set

Motivation: Encoding functions from the individuals to the reals

Definition

The slice of a set E at the individual x is defined as the set

$$E[x] \quad \triangleq \quad \{y \mid \langle x, y\rangle \in E\}$$

Solution: A set E encodes the function $x \mapsto E[x]$

Example: $M \triangleq \{\langle n, m\rangle \mid n \in \mathbb{N} \land (\exists k \in \mathbb{N})\, m = k \times n\}$ encodes the function

$$\begin{cases} \mathbb{N} & \rightarrow & \mathscr{P}(\omega) \\ n & \mapsto & M[n] = n\mathbb{N} \end{cases}$$

Slice of a set

Motivation: Encoding functions from the individuals to the reals

Definition

The slice of a set E at the individual x is defined as the set

$$E[x] \quad \triangleq \quad \{y \mid \langle x, y\rangle \in E\}$$

Solution: A set E encodes the function $x \mapsto E[x]$
Example: $M \triangleq \{\langle n, m\rangle \mid n \in \mathbb{N} \wedge (\exists k \in \mathbb{N})\, m = k \times n\}$ encodes the function

$$\begin{cases} \mathbb{N} & \rightarrow & \mathscr{P}(\omega) \\ n & \mapsto & M[n] = n\mathbb{N} \end{cases}$$

Notation. $\mathbf{S}(E)$ *is the image of* $x \mapsto E[x]$ *i.e.* $\mathbf{S}(E) = \{E[x] \mid x \in \iota\}$

1. The ramified analytic hierarchy (RAH)
   - Introduction: constructible universe and ramified analytic hierarchy
   - Presentation of (classical) second-order arithmetic ($\mathbf{PA2}$)
   - The axiom scheme of collection
   - A glimpse at intuitionistic logic
   - Well-preorders
   - The ramified analytic hierarchy
2. Equality in finite type arithmetic

Axiom of choice ($\mathbf{AC}_\iota$) and axiom of collection ($\mathbf{Coll}$)

Definition. $\mathbf{AC}_\iota : \forall x \exists Y \phi(x, Y) \Rightarrow \exists Z \forall x \phi(x, Z[x])$

Intuition: Extracts a function $x \mapsto Z[x]$ from $\phi(x, Y)$

Axiom of choice ($\mathbf{AC}_\iota$) and axiom of collection ($\mathbf{Coll}$)

Definition. $\mathbf{AC}_\iota : \forall x \exists Y \phi(x,Y) \Rightarrow \exists Z \forall x \phi(x, Z[x])$

Intuition: Extracts a function $x \mapsto Z[x]$ from $\phi(x,Y)$

Definition. $\mathbf{Coll} : \forall x \exists Y \phi(x,Y) \Rightarrow \exists Z \forall x \exists y \phi(x, Z[y])$

Intuition: All individuals have an image by $\phi(x,Y)$ in $\mathbf{S}(Z)$

Remark: Called blurred countable choice in Dominik Kirst's work

Axiom of choice ($\mathbf{AC}_\iota$) and axiom of collection ($\mathbf{Coll}$)

Definition. $\mathbf{AC}_\iota : \forall x \exists Y \phi(x, Y) \Rightarrow \exists Z \forall x \phi(x, Z[x])$

Intuition: Extracts a function $x \mapsto Z[x]$ from $\phi(x, Y)$

Definition. $\mathbf{Coll} : \forall x \exists Y \phi(x, Y) \Rightarrow \exists Z \forall x \exists y \phi(x, Z[y])$

Intuition: All individuals have an image by $\phi(x, Y)$ in $\mathbf{S}(Z)$
Remark: Called blurred countable choice in Dominik Kirst's work

Remark. $\mathbf{PA2} \vdash \mathbf{Coll} \Leftrightarrow \mathbf{AC}_\iota$

Axiom of choice ($\mathbf{AC}_\iota$) and axiom of collection ($\mathbf{Coll}$)

Definition. $\mathbf{AC}_\iota : \forall x \exists Y \phi(x, Y) \Rightarrow \exists Z \forall x \phi(x, Z[x])$

Intuition: Extracts a function $x \mapsto Z[x]$ from $\phi(x, Y)$

Definition. $\mathbf{Coll} : \forall x \exists Y \phi(x, Y) \Rightarrow \exists Z \forall x \exists y \phi(x, Z[y])$

Intuition: All individuals have an image by $\phi(x, Y)$ in $\mathbf{S}(Z)$
Remark: Called blurred countable choice in Dominik Kirst's work

Remark. $\mathbf{PA2} \vdash \mathbf{Coll} \Leftrightarrow \mathbf{AC}_\iota$

Proof of $\Rightarrow$.

1. *From* $\mathbf{Coll}$ *and* $\forall x \exists Y \phi(x, Y)$, *exists* Z *s.t.* $\forall x \exists y \phi(x, Z[y])$

Axiom of choice ($\mathbf{AC}_\iota$) and axiom of collection ($\mathbf{Coll}$)

Definition. $\mathbf{AC}_\iota : \forall x \exists Y \phi(x,Y) \Rightarrow \exists Z \forall x \phi(x, Z[x])$

Intuition: Extracts a function $x \mapsto Z[x]$ from $\phi(x,Y)$

Definition. $\mathbf{Coll} : \forall x \exists Y \phi(x,Y) \Rightarrow \exists Z \forall x \exists y \phi(x, Z[y])$

Intuition: All individuals have an image by $\phi(x,Y)$ in $\mathbf{S}(Z)$
Remark: Called blurred countable choice in Dominik Kirst's work

Remark. $\mathbf{PA2} \vdash \mathbf{Coll} \Leftrightarrow \mathbf{AC}_\iota$

Proof of $\Rightarrow$.

1. *From* $\mathbf{Coll}$ *and* $\forall x \exists Y \phi(x,Y)$, *exists* Z *s.t.* $\forall x \exists y \phi(x, Z[y])$
2. *Define* Z_0 *by* $Z_0[x] = Z[y_x]$ *where* y_x *is the smallest* y *s.t.* $\phi(x, Z[y])$

Axiom of choice ($\mathbf{AC}_\iota$) and axiom of collection ($\mathbf{Coll}$)

Definition. $\mathbf{AC}_\iota : \forall x \exists Y \phi(x,Y) \Rightarrow \exists Z \forall x \phi(x, Z[x])$

Intuition: Extracts a function $x \mapsto Z[x]$ from $\phi(x,Y)$

Definition. $\mathbf{Coll} : \forall x \exists Y \phi(x,Y) \Rightarrow \exists Z \forall x \exists y \phi(x, Z[y])$

Intuition: All individuals have an image by $\phi(x,Y)$ in $\mathbf{S}(Z)$
Remark: Called blurred countable choice in Dominik Kirst's work

Remark. $\mathbf{PA2} \vdash \mathbf{Coll} \Leftrightarrow \mathbf{AC}_\iota$

Proof of $\Rightarrow$.

1. *From* $\mathbf{Coll}$ *and* $\forall x \exists Y \phi(x,Y)$, *exists* Z *s.t.* $\forall x \exists y \phi(x, Z[y])$
2. *Define* Z_0 *by* $Z_0[x] = Z[y_x]$ *where* y_x *is the smallest* y *s.t.* $\phi(x, Z[y])$

Remark: The proof uses induction and classical logic

Giving up on induction

- Computational content of 2^{nd}-order arithmetic: classical realizability
  - proofs $\rightsquigarrow$ programs and formulas $\rightsquigarrow$ sets of programs
  - induction is not directly realized
  - sound interpretation of 2^{nd}-order arithmetic without induction ($\mathbf{PA2}^-$)

Giving up on induction

- Computational content of 2nd-order arithmetic: classical realizability
  - proofs $\rightsquigarrow$ programs and formulas $\rightsquigarrow$ sets of programs
  - induction is not directly realized
  - sound interpretation of 2nd-order arithmetic without induction ($\mathbf{PA2}^-$)
- The process of relativization: $(\forall x\phi)^{\mathbb{B}} \triangleq \forall x(x \in \mathbb{B} \Rightarrow \phi^{\mathbb{B}})$

Prop.2.2.3.2. $\mathbf{PA2} \vdash \phi$ *implies* $\mathbf{PA2}^- \vdash \phi^{\mathbb{B}}$

Giving up on induction

- Computational content of 2^{nd}-order arithmetic: classical realizability
  - proofs $\rightsquigarrow$ programs and formulas $\rightsquigarrow$ sets of programs
  - induction is not directly realized
  - sound interpretation of 2^{nd}-order arithmetic without induction ($\mathbf{PA2}^-$)
- The process of relativization: $(\forall x \phi)^{\mathbb{B}} \triangleq \forall x (x \in \mathbb{B} \Rightarrow \phi^{\mathbb{B}})$

Prop.2.2.3.2. $\mathbf{PA2} \vdash \phi$ *implies* $\mathbf{PA2}^- \vdash \phi^{\mathbb{B}}$

- $\mathbf{PA2}^-$ will be our playground!

Giving up on induction

- Computational content of 2^{nd}-order arithmetic: classical realizability
  - proofs $\rightsquigarrow$ programs and formulas $\rightsquigarrow$ sets of programs
  - induction is not directly realized
  - sound interpretation of 2^{nd}-order arithmetic without induction ($\mathbf{PA2}^-$)
- The process of relativization: $(\forall x \phi)^{\mathbb{B}} \triangleq \forall x (x \in \mathbb{B} \Rightarrow \phi^{\mathbb{B}})$

Prop.2.2.3.2. $\mathbf{PA2} \vdash \phi$ *implies* $\mathbf{PA2}^- \vdash \phi^{\mathbb{B}}$

- $\mathbf{PA2}^-$ will be our playground!

Advantages of $\mathbf{PA2}^-$

- *direct computational interpretation*
- *induction retrieved through relativization*
- *a wider sets of individuals*

Recap and open problem

Recap

$\mathbf{Coll} : \forall x \exists Y \phi(x, Y) \Rightarrow \exists Z \forall x \exists y \phi(x, Z[y])$

$\mathbf{AC}_\iota : \forall x \exists Y \phi(x, Y) \Rightarrow \exists Z \forall x \phi(x, Z[x])$

PA2

- with induction
- $\mathbf{PA2} \vdash \phi$
- $\mathbf{PA2} \vdash \mathbf{AC}_\iota \Rightarrow \mathbf{Coll}$
- $\mathbf{PA2} \vdash \mathbf{Coll} \Rightarrow \mathbf{AC}_\iota$

PA2$^-$

- without induction
- $\mathbf{PA2}^- \vdash \phi^{\mathbb{B}}$
- $\mathbf{PA2}^- \vdash \mathbf{AC}_\iota \Rightarrow \mathbf{Coll}$
- O.P. $\mathbf{PA2}^- \vdash \mathbf{Coll} \Rightarrow \mathbf{AC}_\iota$?

Recap and open problem

Recap

$\mathbf{Coll} : \forall x \exists Y \phi(x, Y) \Rightarrow \exists Z \forall x \exists y \phi(x, Z[y])$
$\mathbf{AC}_\iota : \forall x \exists Y \phi(x, Y) \Rightarrow \exists Z \forall x \phi(x, Z[x])$

PA2

- with induction
- $\mathbf{PA2} \vdash \phi$
- $\mathbf{PA2} \vdash \mathbf{AC}_\iota \Rightarrow \mathbf{Coll}$
- $\mathbf{PA2} \vdash \mathbf{Coll} \Rightarrow \mathbf{AC}_\iota$

PA2⁻

- without induction
- $\mathbf{PA2}^- \vdash \phi^{\mathbb{B}}$
- $\mathbf{PA2}^- \vdash \mathbf{AC}_\iota \Rightarrow \mathbf{Coll}$
- O.P. $\mathbf{PA2}^- \vdash \mathbf{Coll} \Rightarrow \mathbf{AC}_\iota$?

Open problem 1. Is $\mathbf{Coll}$ *weaker than* $\mathbf{AC}_\iota$ *in* $\mathbf{PA2}^-$*?*

Recap and open problem

Recap

$\mathbf{Coll}: \forall x \exists Y \phi(x, Y) \Rightarrow \exists Z \forall x \exists y \phi(x, Z[y])$
$\mathbf{AC}_\iota: \forall x \exists Y \phi(x, Y) \Rightarrow \exists Z \forall x \phi(x, Z[x])$

PA2

- with induction
- $\mathbf{PA2} \vdash \phi$
- $\mathbf{PA2} \vdash \mathbf{AC}_\iota \Rightarrow \mathbf{Coll}$
- $\mathbf{PA2} \vdash \mathbf{Coll} \Rightarrow \mathbf{AC}_\iota$

$\mathbf{PA2}^-$

- without induction
- $\mathbf{PA2}^- \vdash \phi^{\mathbb{B}}$
- $\mathbf{PA2}^- \vdash \mathbf{AC}_\iota \Rightarrow \mathbf{Coll}$
- O.P. $\mathbf{PA2}^- \vdash \mathbf{Coll} \Rightarrow \mathbf{AC}_\iota$?

Open problem 1. Is **Coll** *weaker than* $\mathbf{AC}_\iota$ *in* $\mathbf{PA2}^-$*?*

Conjecture: Yes!

Negative translation

- $\mathbf{HA2}^-$ is the intuitionistic version of $\mathbf{PA2}^-$
- Negative translation again by relativization
  - 1st-order relat. $\mathbf{G}_{\mathbb{B}}(\forall x \phi) \triangleq \forall x (x \in \mathbb{B} \Rightarrow \mathbf{G}_{\mathbb{B}}(\phi))$
  - 2nd-order relat. $\mathbf{G}_{\mathbb{B}}(\forall X \phi) \triangleq \forall X (\mathrm{Sbl}(X) \Rightarrow \mathbf{G}_{\mathbb{B}}(\phi))$
  - Stable predicates: $\mathrm{Sbl}(X) \triangleq (\forall x \in \mathbb{B})(\neg\neg x \in X \Rightarrow x \in X)$

Prop. 3.2.1.2. $\mathbf{PA2} \vdash \phi$ *implies* $\mathbf{HA2}^- \vdash \mathbf{G}_{\mathbb{B}}(\phi)$

Intuition: Negative translation done by relativization to the stable predicates

The schemes $\mathbf{Coll}_{\mathbf{Dom}}$ and $\mathbf{AC}_{\mathbf{Dom}}$

Definition. $\mathbf{Coll}_{\mathbf{Dom}} : \forall D((\forall x \in D)\exists Y \phi(x, Y) \Rightarrow \exists Z(\forall x \in D)\exists y \phi(x, Z[y]))$

Definition. $\mathbf{Coll_{Dom}} : \forall D((\forall x \in D)\exists Y \phi(x, Y) \Rightarrow \exists Z(\forall x \in D)\exists y \phi(x, Z[y]))$

Remark: $\mathbf{PA2^-} \vdash \mathbf{Coll} \Leftrightarrow \mathbf{Coll_{Dom}}$

The schemes $\mathbf{Coll}_{\mathbf{Dom}}$ and $\mathbf{AC}_{\mathbf{Dom}}$

Definition. $\mathbf{Coll}_{\mathbf{Dom}} : \forall D((\forall x \in D)\exists Y \phi(x, Y) \Rightarrow \exists Z(\forall x \in D)\exists y \phi(x, Z[y]))$

Remark: $\mathbf{PA2}^{-} \vdash \mathbf{Coll} \Leftrightarrow \mathbf{Coll}_{\mathbf{Dom}}$

Theorem 3.2.2.1. $\mathbf{Coll}_{\mathbf{Dom}}$ *implies its negative translation in* $\mathbf{HA2}^{-}$

The schemes $\mathbf{Coll}_{\mathbf{Dom}}$ and $\mathbf{AC}_{\mathbf{Dom}}$

Definition. $\mathbf{Coll}_{\mathbf{Dom}} : \forall D((\forall x \in D)\exists Y \phi(x, Y) \Rightarrow \exists Z(\forall x \in D)\exists y \phi(x, Z[y]))$

Remark: $\mathbf{PA2}^- \vdash \mathbf{Coll} \Leftrightarrow \mathbf{Coll}_{\mathbf{Dom}}$

Theorem 3.2.2.1. $\mathbf{Coll}_{\mathbf{Dom}}$ *implies its negative translation in* $\mathbf{HA2}^-$

Intuition: **Coll** lacks a "tiny bit" of classical logic in its formulation

The schemes $\mathbf{Coll_{Dom}}$ and $\mathbf{AC_{Dom}}$

Definition. $\mathbf{Coll_{Dom}} : \forall D((\forall x \in D)\exists Y \phi(x, Y) \Rightarrow \exists Z(\forall x \in D)\exists y \phi(x, Z[y]))$

Remark: $\mathbf{PA2^-} \vdash \mathbf{Coll} \Leftrightarrow \mathbf{Coll_{Dom}}$

Theorem 3.2.2.1. $\mathbf{Coll_{Dom}}$ *implies its negative translation in* $\mathbf{HA2^-}$

Intuition: **Coll** lacks a “tiny bit” of classical logic in its formulation

Definition. $\mathbf{AC_{Dom}} : \forall D((\forall x \in D)\exists Y \phi(x, Y) \Rightarrow \exists Z(\forall x \in D)\phi(x, Z[x]))$

Remark: $\mathbf{PA2^-} \vdash \mathbf{AC} \Leftrightarrow \mathbf{AC_{Dom}}$

Theorem 3.2.2.2. $\mathbf{AC_{Dom}}$ *implies its negative translation in* $\mathbf{HA2^-}$

Definition. $\mathbf{Coll_{Dom}} : \forall D((\forall x \in D)\exists Y \phi(x, Y) \Rightarrow \exists Z(\forall x \in D)\exists y \phi(x, Z[y]))$

Remark: $\mathbf{PA2^-} \vdash \mathbf{Coll} \Leftrightarrow \mathbf{Coll_{Dom}}$

Theorem 3.2.2.1. $\mathbf{Coll_{Dom}}$ *implies its negative translation in* $\mathbf{HA2^-}$

Intuition: **Coll** lacks a "tiny bit" of classical logic in its formulation

Definition. $\mathbf{AC_{Dom}} : \forall D((\forall x \in D)\exists Y \phi(x, Y) \Rightarrow \exists Z(\forall x \in D)\phi(x, Z[x]))$

Remark: $\mathbf{PA2^-} \vdash \mathbf{AC} \Leftrightarrow \mathbf{AC_{Dom}}$

Theorem 3.2.2.2. $\mathbf{AC_{Dom}}$ *implies its negative translation in* $\mathbf{HA2^-}$

Remark: $\mathbf{HA2^-} \vdash \mathbf{AC_{Dom}} \Rightarrow \mathbf{Coll_{Dom}}$

The schemes $\mathbf{Coll_{Dom}}$ and $\mathbf{AC_{Dom}}$

Definition. $\mathbf{Coll_{Dom}} : \forall D((\forall x \in D)\exists Y \phi(x, Y) \Rightarrow \exists Z(\forall x \in D)\exists y \phi(x, Z[y]))$

Remark: $\mathbf{PA2^-} \vdash \mathbf{Coll} \Leftrightarrow \mathbf{Coll_{Dom}}$

Theorem 3.2.2.1. $\mathbf{Coll_{Dom}}$ *implies its negative translation in* $\mathbf{HA2^-}$

Intuition: $\mathbf{Coll}$ lacks a "tiny bit" of classical logic in its formulation

Definition. $\mathbf{AC_{Dom}} : \forall D((\forall x \in D)\exists Y \phi(x, Y) \Rightarrow \exists Z(\forall x \in D)\phi(x, Z[x]))$

Remark: $\mathbf{PA2^-} \vdash \mathbf{AC} \Leftrightarrow \mathbf{AC_{Dom}}$

Theorem 3.2.2.2. $\mathbf{AC_{Dom}}$ *implies its negative translation in* $\mathbf{HA2^-}$

Remark: $\mathbf{HA2^-} \vdash \mathbf{AC_{Dom}} \Rightarrow \mathbf{Coll_{Dom}}$

Theorem 3.3.4.3. $\mathbf{Coll_{Dom}}$ *does not imply* $\mathbf{AC_{Dom}}$ *in* $\mathbf{HA2^-}$

It uses... intuitionistic realizability

Intuitionistic realizability in a nutshell

Closed formulas with parameters.

x	$\rightsquigarrow$	$\mathbf{B}$
X	$\rightsquigarrow$	$\mathbf{B} \to \mathscr{P}(\Lambda)$

Definition

Interpretation of formulas as sets of closed λ-terms (Λ):

$$
\begin{array}{rcl}
|t \in \dot{P}| & \triangleq & \{M \mid \exists N \in P(\llbracket t \rrbracket), M \rightsquigarrow N\} \\
|\forall x \phi| & \triangleq & \bigcap_{b \in \mathbf{B}} |\phi[x := \dot{b}]| \\
|\exists x \phi| & \triangleq & \bigcup_{b \in \mathbf{B}} |\phi[x := \dot{b}]|
\end{array}
\qquad
\begin{array}{rcl}
|\phi \Rightarrow \psi| & \triangleq & \{M \mid \forall N \in |\phi|, MN \in |\psi|\} \\
|\forall X \phi| & \triangleq & \bigcap_{P : \mathbf{B} \to \mathscr{P}(\Lambda)} |\phi[X := \dot{P}]| \\
|\exists X \phi| & \triangleq & \bigcup_{P : \mathbf{B} \to \mathscr{P}(\Lambda)} |\phi[X := \dot{P}]|
\end{array}
$$

Intuitionistic realizability in a nutshell

Closed formulas with parameters.

x	$\rightsquigarrow$	$\mathbf{B}$
X	$\rightsquigarrow$	$\mathbf{B} \to \mathscr{P}(\Lambda)$

Definition

Interpretation of formulas as sets of closed λ-terms (Λ):

$$
\begin{array}{rcl}
|t \in \dot{P}| & \triangleq & \{M \mid \exists N \in P(\llbracket t \rrbracket), M \rightsquigarrow N\} \\
|\forall x \phi| & \triangleq & \bigcap_{b \in \mathbf{B}} |\phi[x := \dot{b}]| \\
|\exists x \phi| & \triangleq & \bigcup_{b \in \mathbf{B}} |\phi[x := \dot{b}]|
\end{array}
\qquad
\begin{array}{rcl}
|\phi \Rightarrow \psi| & \triangleq & \{M \mid \forall N \in |\phi|, MN \in |\psi|\} \\
|\forall X \phi| & \triangleq & \bigcap_{P : \mathbf{B} \to \mathscr{P}(\Lambda)} |\phi[X := \dot{P}]| \\
|\exists X \phi| & \triangleq & \bigcup_{P : \mathbf{B} \to \mathscr{P}(\Lambda)} |\phi[X := \dot{P}]|
\end{array}
$$

Theorem 3.3.2.1 (Soundness). $\mathbf{HA2}^{-} \vdash \phi$ *implies* $|\phi| \neq \emptyset$

In the next slides

1. $\mathbf{Coll}_{\mathbf{Dom}}$ is realized
2. $\mathbf{AC}_{\mathbf{Dom}}$ is not realized

$\mathbf{Coll}_{\mathbf{Dom}}$ is realized

Reminder. $\mathbf{Coll}_{\mathbf{Dom}} : \forall D((\forall x \in D)\exists Y \phi(x, Y) \Rightarrow \exists Z(\forall x \in D)\exists y \phi(x, Z[y]))$

Theorem 3.3.3.1. All instances of $\mathbf{Coll}_{\mathbf{Dom}}$ *are realized by* $\lambda\xi.\xi$

$\mathbf{Coll}_{\mathbf{Dom}}$ is realized

Reminder. $\mathbf{Coll}_{\mathbf{Dom}} : \forall D((\forall x \in D)\exists Y \phi(x, Y) \Rightarrow \exists Z(\forall x \in D)\exists y \phi(x, Z[y]))$

Theorem 3.3.3.1. All instances of $\mathbf{Coll}_{\mathbf{Dom}}$ *are realized by* $\lambda\xi.\xi$

Idea of proof.

- *Fixing a domain* P_D*, show*

$$|(\forall x \in \dot{P}_D)\exists Y \phi(x, Y)| \subseteq |\exists Z(\forall x \in \dot{P}_D)\exists y \phi(x, Z[y])|$$

$\mathbf{Coll}_{\mathbf{Dom}}$ is realized

Reminder. $\mathbf{Coll}_{\mathbf{Dom}} : \forall D((\forall x \in D)\exists Y \phi(x, Y) \Rightarrow \exists Z(\forall x \in D)\exists y \phi(x, Z[y]))$

Theorem 3.3.3.1. All instances of $\mathbf{Coll}_{\mathbf{Dom}}$ *are realized by* $\lambda\xi.\xi$

Idea of proof.

- *Fixing a domain* P_D*, show*

$$|(\forall x \in \dot{P_D})\exists Y \phi(x, Y)| \subseteq |\exists Z(\forall x \in \dot{P_D})\exists y \phi(x, Z[y])|$$

- *Using an external axiom of choice, define* P_Z *such that*

$$|(\forall x \in \dot{P_D})\exists Y \phi(x, Y)| \subseteq |(\forall x \in \dot{P_D})\exists y \phi(x, \dot{P_Z}[y])|$$

$\mathbf{Coll_{Dom}}$ is realized

Reminder. $\mathbf{Coll_{Dom}} : \forall D((\forall x \in D)\exists Y \phi(x, Y) \Rightarrow \exists Z(\forall x \in D)\exists y \phi(x, Z[y]))$

Theorem 3.3.3.1. All instances of $\mathbf{Coll_{Dom}}$ *are realized by* $\lambda\xi.\xi$

Idea of proof.

- *Fixing a domain* P_D*, show*

$$|(\forall x \in \dot{P_D})\exists Y \phi(x, Y)| \subseteq |\exists Z(\forall x \in \dot{P_D})\exists y \phi(x, Z[y])|$$

- *Using an external axiom of choice, define* P_Z *such that*

$$|(\forall x \in \dot{P_D})\exists Y \phi(x, Y)| \subseteq |(\forall x \in \dot{P_D})\exists y \phi(x, \dot{P_Z}[y])|$$

Intuition:

- The set $Z[y]$ needs to depend on x and on a proof of $x \in D$
- The variable y is used to encode an individual x and a realizer of $x \in D$

$\mathbf{AC_{Dom}}$ is not realized

Reminder. $\mathbf{AC_{Dom}} : \forall D((\forall x \in D)\exists Y \phi(x, Y) \Rightarrow \exists Z(\forall x \in D)\phi(x, Z[x]))$

Theorem 3.3.4.2. $\mathbf{AC_{Dom}}$ *is not realized*

$\mathbf{AC_{Dom}}$ is not realized

Reminder. $\mathbf{AC_{Dom}} : \forall D((\forall x \in D)\exists Y \phi(x, Y) \Rightarrow \exists Z(\forall x \in D)\phi(x, Z[x]))$

Theorem 3.3.4.2. $\mathbf{AC_{Dom}}$ *is not realized*

Idea of proof. *Find an instance of* $\mathbf{AC_{Dom}}$ *and a domain such that*

- *the premise is realized*
- *the conclusion is not (contradiction with eq. theory of* λ*-calculus)*

$\mathbf{AC_{Dom}}$ is not realized

Reminder. $\mathbf{AC_{Dom}} : \forall D((\forall x \in D)\exists Y \phi(x, Y) \Rightarrow \exists Z(\forall x \in D)\phi(x, Z[x]))$

Theorem 3.3.4.2. $\mathbf{AC_{Dom}}$ *is not realized*

Idea of proof. *Find an instance of* $\mathbf{AC_{Dom}}$ *and a domain such that*

- *the premise is realized*
- *the conclusion is not (contradiction with eq. theory of* λ*-calculus)*

Intuition:

- The set $Z[x]$ needs to depend on x and on a proof of $x \in D$
- The proof of $x \in D$ is proof relevant in intuitionistic realizability

Recap

$\mathbf{Coll}: \forall x \exists Y \phi(x,Y) \Rightarrow \exists Z \forall x \exists y \phi(x, Z[y])$
$\mathbf{Coll_{Dom}}: \forall D((\forall x \in D)\exists Y \phi(x,Y) \Rightarrow \exists Z(\forall x \in D)\exists y \phi(x, Z[y]))$
$\mathbf{AC}_\iota: \forall x \exists Y \phi(x,Y) \Rightarrow \exists Z \forall x \phi(x, Z[x])$
$\mathbf{AC_{Dom}}: \forall D((\forall x \in D)\exists Y \phi(x,Y) \Rightarrow \exists Z(\forall x \in D)\phi(x, Z[x]))$

PA2

- classical logic
- with induction
- $\mathbf{PA2} \vdash \phi$
- $\mathbf{AC}_\iota \Rightarrow \mathbf{Coll}$
- $\mathbf{Coll} \Rightarrow \mathbf{AC}_\iota$
- $\mathbf{Coll} \Leftrightarrow \mathbf{Coll_{Dom}}$
- $\mathbf{AC} \Leftrightarrow \mathbf{AC_{Dom}}$

PA2⁻

- classical logic
- without induction
- $\mathbf{PA2}^- \vdash \phi^{\mathbb{B}}$
- $\mathbf{AC}_\iota \Rightarrow \mathbf{Coll}$
- O.P. $\mathbf{Coll} \Rightarrow \mathbf{AC}_\iota$?
- $\mathbf{Coll} \Leftrightarrow \mathbf{Coll_{Dom}}$
- $\mathbf{AC} \Leftrightarrow \mathbf{AC_{Dom}}$

HA2⁻

- intuitionistic logic
- without induction
- $\mathbf{HA2}^- \vdash \mathbf{G}_{\mathbb{B}}(\phi)$
- $\mathbf{AC_{Dom}} \Rightarrow \mathbf{Coll_{Dom}}$
- $\mathbf{Coll_{Dom}} \not\Rightarrow \mathbf{AC_{Dom}}$
- $\mathbf{Coll_{Dom}} \Rightarrow \mathbf{G}(\mathbf{Coll_{Dom}})$
- $\mathbf{AC_{Dom}} \Rightarrow \mathbf{G}(\mathbf{AC_{Dom}})$

Well-orders or well-preorders in $\mathbf{PA2}^{-}$?

Well-order. *Order s.t. all nonempty subsets of its field have a least element*

Well-orders or well-preorders in $\mathbf{PA2}^-$?

Well-order. *Order s.t. all nonempty subsets of its field have a least element*

Problem: Define the supremum of a family of well-orders?

- Let $\mathscr{U}$ s.t. for all $x, \mathscr{U}[x]$ is a well-order
- Define $\sup(\mathscr{U})$?

Well-orders or well-preorders in $\mathbf{PA2}^{-}$?

Well-order. *Order s.t. all nonempty subsets of its field have a least element*

Problem: Define the supremum of a family of well-orders?

- Let $\mathscr{U}$ s.t. for all $x, \mathscr{U}[x]$ is a well-order
- Define $\sup(\mathscr{U})$? Solution: Use induction...

Well-orders or well-preorders in $\mathbf{PA2}^-$?

Well-order. *Order s.t. all nonempty subsets of its field have a least element*

Problem: Define the supremum of a family of well-orders?

- Let $\mathscr{U}$ s.t. for all $x, \mathscr{U}[x]$ is a well-order
- Define $\sup(\mathscr{U})$? Solution: Use induction...

Preorder. *Drop the antisymmetry from an order*

Well-preorder. *Preorder s.t. all nonempty subsets of its field have a least element*

Thm.4.2.2.1. *The sup of a family of well-preorders can be defined in* $\mathbf{PA2}^-$

Well-orders or well-preorders in $\mathbf{PA2}^-$?

Well-order. Order s.t. all nonempty subsets of its field have a least element

Problem: Define the supremum of a family of well-orders?

- Let $\mathscr{U}$ s.t. for all $x, \mathscr{U}[x]$ is a well-order
- Define $\sup(\mathscr{U})$? Solution: Use induction...

Preorder. Drop the antisymmetry from an order

Well-preorder. Preorder s.t. all nonempty subsets of its field have a least element

Thm.4.2.2.1. The sup of a family of well-preorders can be defined in $\mathbf{PA2}^-$

Idea of proof.

Well-orders or well-preorders in $\mathbf{PA2}^-$?

Well-order. *Order s.t. all nonempty subsets of its field have a least element*

Problem: Define the supremum of a family of well-orders?

- Let $\mathscr{U}$ s.t. for all $x, \mathscr{U}[x]$ is a well-order
- Define $\sup(\mathscr{U})$? Solution: Use induction...

Preorder. *Drop the antisymmetry from an order*

Well-preorder. *Preorder s.t. all nonempty subsets of its field have a least element*

Thm.4.2.2.1. *The sup of a family of well-preorders can be defined in* $\mathbf{PA2}^-$

Idea of proof.

1. *Use the well-preorder* $\preccurlyeq_{\mathrm{WPO}}$ *on the class of* WPO

Well-orders or well-preorders in $\mathbf{PA2}^-$?

Well-order. *Order s.t. all nonempty subsets of its field have a least element*

Problem: Define the supremum of a family of well-orders?

- Let $\mathscr{U}$ s.t. for all $x, \mathscr{U}[x]$ is a well-order
- Define $\sup(\mathscr{U})$? Solution: Use induction...

Preorder. *Drop the antisymmetry from an order*

Well-preorder. *Preorder s.t. all nonempty subsets of its field have a least element*

Thm.4.2.2.1. *The sup of a family of well-preorders can be defined in* $\mathbf{PA2}^-$

Idea of proof.

1. *Use the well-preorder* $\preccurlyeq_{\mathrm{WPO}}$ *on the class of* WPO
2. *Compare the initial segments of the well-preorders in* $\mathbf{S}(\mathscr{U})$

Well-orders or well-preorders in $\mathbf{PA2}^-$?

Well-order. *Order s.t. all nonempty subsets of its field have a least element*

Problem: Define the supremum of a family of well-orders?

- Let $\mathscr{U}$ s.t. for all $x, \mathscr{U}[x]$ is a well-order
- Define $\sup(\mathscr{U})$? Solution: Use induction...

Preorder. *Drop the antisymmetry from an order*

Well-preorder. *Preorder s.t. all nonempty subsets of its field have a least element*

Thm.4.2.2.1. *The sup of a family of well-preorders can be defined in $\mathbf{PA2}^-$*

Idea of proof.

1. *Use the well-preorder $\preccurlyeq_{\mathrm{WPO}}$ on the class of* WPO
2. *Compare the initial segments of the well-preorders in* $\mathbf{S}(\mathscr{U})$
3. *Formally:* $\sup(\mathscr{U}) \triangleq \{\langle\langle x, x'\rangle, \langle y, y'\rangle\rangle \mid \mathscr{U}[x]_{<x'} \preccurlyeq_{\mathrm{WPO}} \mathscr{U}[y]_{<y'}\}$

Functional relation: a formula $\mathscr{F}(X,Y)$ s.t. $\forall X \exists! Y \mathscr{F}(X,Y)$

Thm.4.2.3.1. Iterating $\mathscr{F}$ over a w.p.o. gives a unique output (in $\mathbf{PA2}^-$)

Transfinite recursion and iteration over ω

Functional relation: a formula $\mathscr{F}(X,Y)$ s.t. $\forall X \exists ! Y \mathscr{F}(X,Y)$

Thm.4.2.3.1. Iterating $\mathscr{F}$ over a w.p.o. gives a unique output (in $\mathbf{PA2}^-$)

Generalization: $\mathscr{F}$ func. up to $\cong_{\mathrm{WPO}}$ i.e. $(\mathscr{F}(\alpha,\beta) \wedge \mathscr{F}(\alpha,\gamma)) \Rightarrow \beta \cong_{\mathrm{WPO}} \gamma$

Transfinite recursion and iteration over ω

Functional relation: a formula $\mathscr{F}(X,Y)$ s.t. $\forall X \exists! Y \mathscr{F}(X,Y)$

Thm.4.2.3.1. Iterating $\mathscr{F}$ over a w.p.o. gives a unique output (in $\mathbf{PA2}^-$)

Generalization: $\mathscr{F}$ func. up to $\cong_{\mathrm{WPO}}$ i.e. $(\mathscr{F}(\alpha,\beta) \wedge \mathscr{F}(\alpha,\gamma)) \Rightarrow \beta \cong_{\mathrm{WPO}} \gamma$

Thm.4.2.4.1. Iterating $\mathscr{F}$ over ω gives a unique output up to $\cong_{\mathrm{WPO}}$ (in $\mathbf{PA2}^- + \mathbf{Coll}$)

Transfinite recursion and iteration over ω

Functional relation: a formula $\mathscr{F}(X,Y)$ s.t. $\forall X \exists! Y \mathscr{F}(X,Y)$

Thm.4.2.3.1. Iterating $\mathscr{F}$ over a w.p.o. gives a unique output (in $\mathbf{PA2}^-$)

Generalization: $\mathscr{F}$ func. up to $\cong_{\mathrm{WPO}}$ i.e. $(\mathscr{F}(\alpha,\beta) \wedge \mathscr{F}(\alpha,\gamma)) \Rightarrow \beta \cong_{\mathrm{WPO}} \gamma$

Thm.4.2.4.1. Iterating $\mathscr{F}$ over ω gives a unique output up to $\cong_{\mathrm{WPO}}$ (in $\mathbf{PA2}^- + \mathbf{Coll}$)

Idea of proof.

Transfinite recursion and iteration over ω

Functional relation: a formula $\mathscr{F}(X,Y)$ s.t. $\forall X \exists! Y \mathscr{F}(X,Y)$

Thm.4.2.3.1. Iterating $\mathscr{F}$ over a w.p.o. gives a unique output (in $\mathbf{PA2}^-$)

Generalization: $\mathscr{F}$ func. up to $\cong_{\text{WPO}}$ i.e. $(\mathscr{F}(\alpha,\beta) \wedge \mathscr{F}(\alpha,\gamma)) \Rightarrow \beta \cong_{\text{WPO}} \gamma$

Thm.4.2.4.1. Iterating $\mathscr{F}$ over ω gives a unique output up to $\cong_{\text{WPO}}$ (in $\mathbf{PA2}^- + \mathbf{Coll}$)

Idea of proof.

- *The axiom of collection shows the existence of a bound for the finite iterations of $\mathscr{F}$: "collecting all the iterations"*

Transfinite recursion and iteration over ω

Functional relation: a formula $\mathscr{F}(X,Y)$ s.t. $\forall X \exists! Y \mathscr{F}(X,Y)$

Thm.4.2.3.1. Iterating $\mathscr{F}$ over a w.p.o. gives a unique output (in $\mathbf{PA2}^-$)

Generalization: $\mathscr{F}$ func. up to $\cong_{\mathrm{WPO}}$ i.e. $(\mathscr{F}(\alpha,\beta) \wedge \mathscr{F}(\alpha,\gamma)) \Rightarrow \beta \cong_{\mathrm{WPO}} \gamma$

Thm.4.2.4.1. Iterating $\mathscr{F}$ over ω gives a unique output up to $\cong_{\mathrm{WPO}}$ (in $\mathbf{PA2}^- + \mathbf{Coll}$)

Idea of proof.

- *The axiom of collection shows the existence of a bound for the finite iterations of $\mathscr{F}$: "collecting all the iterations"*
- *Take the supremum of all the w.p.o. in this bound*

1. The ramified analytic hierarchy (RAH)
   - Introduction: constructible universe and ramified analytic hierarchy
   - Presentation of (classical) second-order arithmetic (**PA2**)
   - The axiom scheme of collection
   - A glimpse at intuitionistic logic
   - Well-preorders
   - The ramified analytic hierarchy
2. Equality in finite type arithmetic

Steps to the construction and the analysis of RAH

- Step 1. Internalization of (Tarski's) truth
  - $\mathscr{U} \rightsquigarrow$ structure where individuals range over $\mathbb{B}$ and reals over $\mathbf{S}(\mathscr{U})$
  - Thm.5.1.2.2. $\mathbf{PA2}^- \vdash \phi^{\mathbf{S}(\mathscr{U})} \Leftrightarrow \mathrm{Sat}(\mathscr{U}, \ulcorner\phi\urcorner)$

Steps to the construction and the analysis of RAH

- Step 1.Internalization of (Tarski's) truth
  - $\mathscr{U} \rightsquigarrow$ structure where individuals range over $\mathbb{B}$ and reals over $\mathbf{S}(\mathscr{U})$
  - Thm.5.1.2.2. $\mathbf{PA2}^- \vdash \phi^{\mathbf{S}(\mathscr{U})} \Leftrightarrow \mathrm{Sat}(\mathscr{U}, \ulcorner\phi\urcorner)$
- Step 2. Operator $\mathbf{Def^2} : \mathrm{Fam} \to \mathrm{Fam}$
  - $\mathbf{Def^2}(\mathscr{U})[\hat{x}.\ulcorner\phi\urcorner] = \{x \mid \phi^{\mathbf{S}(\mathscr{U})}\}$

Steps to the construction and the analysis of RAH

- Step 1.Internalization of (Tarski's) truth
  - $\mathscr{U} \rightsquigarrow$ structure where individuals range over $\mathbb{B}$ and reals over $\mathbf{S}(\mathscr{U})$
  - Thm.5.1.2.2. $\mathbf{PA2}^- \vdash \phi^{\mathbf{S}(\mathscr{U})} \Leftrightarrow \mathrm{Sat}(\mathscr{U}, \ulcorner\phi\urcorner)$
- Step 2. Operator $\mathbf{Def^2} : \mathrm{Fam} \to \mathrm{Fam}$
  - $\mathbf{Def^2}(\mathscr{U})[\hat{x}.\ulcorner\phi\urcorner] = \{x \mid \phi^{\mathbf{S}(\mathscr{U})}\}$
- Step 3. Iteration of $\mathbf{Def^2}$ along w.p.o.$\rightsquigarrow (\mathrm{RAH}_\alpha)_{\alpha \in \mathrm{WPO}}$
  - $\mathbf{S}(\mathrm{RAH}_{\alpha+1}) \triangleq$ sets definable from $\mathbf{S}(\mathrm{RAH}_\alpha)$

Steps to the construction and the analysis of RAH

- Step 1.Internalization of (Tarski's) truth
  - $\mathscr{U} \rightsquigarrow$ structure where individuals range over $\mathbb{B}$ and reals over $\mathbf{S}(\mathscr{U})$
  - Thm.5.1.2.2. $\mathbf{PA2}^- \vdash \phi^{\mathbf{S}(\mathscr{U})} \Leftrightarrow \mathrm{Sat}(\mathscr{U}, \ulcorner\phi\urcorner)$
- Step 2. Operator $\mathbf{Def^2} : \mathrm{Fam} \to \mathrm{Fam}$
  - $\mathbf{Def^2}(\mathscr{U})[\hat{x}.\ulcorner\phi\urcorner] = \{x \mid \phi^{\mathbf{S}(\mathscr{U})}\}$
- Step 3. Iteration of $\mathbf{Def^2}$ along w.p.o.$\rightsquigarrow (\mathrm{RAH}_\alpha)_{\alpha\in\mathrm{WPO}}$
  - $\mathbf{S}(\mathrm{RAH}_{\alpha+1}) \triangleq$ sets definable from $\mathbf{S}(\mathrm{RAH}_\alpha)$
- Step 4. $\mathbf{S}(\mathrm{RAH}) \triangleq \bigcup_{\alpha\in\mathrm{WPO}} \mathbf{S}(\mathrm{RAH}_\alpha)$ can be well-ordered
  - sets compared through the level they appear at

Steps to the construction and the analysis of RAH

- Step 1.Internalization of (Tarski's) truth
  - $\mathscr{U} \rightsquigarrow$ structure where individuals range over $\mathbb{B}$ and reals over $\mathbf{S}(\mathscr{U})$
  - Thm.5.1.2.2. $\mathbf{PA2}^- \vdash \phi^{\mathbf{S}(\mathscr{U})} \Leftrightarrow \mathrm{Sat}(\mathscr{U}, \ulcorner\phi\urcorner)$
- Step 2. Operator $\mathbf{Def^2} : \mathrm{Fam} \to \mathrm{Fam}$
  - $\mathbf{Def^2}(\mathscr{U})[\hat{x}.\ulcorner\phi\urcorner] = \{x \mid \phi^{\mathbf{S}(\mathscr{U})}\}$
- Step 3. Iteration of $\mathbf{Def^2}$ along w.p.o.$\rightsquigarrow (\mathrm{RAH}_\alpha)_{\alpha \in \mathrm{WPO}}$
  - $\mathbf{S}(\mathrm{RAH}_{\alpha+1}) \triangleq$ sets definable from $\mathbf{S}(\mathrm{RAH}_\alpha)$
- Step 4. $\mathbf{S}(\mathrm{RAH}) \triangleq \bigcup_{\alpha \in \mathrm{WPO}} \mathbf{S}(\mathrm{RAH}_\alpha)$ can be well-ordered
  - sets compared through the level they appear at
- Step 5. The principle of reflection (in $\mathbf{PA2}^- + \mathbf{Coll}$)
  - Thm.5.4.1.1. $\mathbf{PA2}^- + \mathbf{Coll} \vdash \forall \alpha \exists \beta \succcurlyeq_{\mathrm{WPO}} \alpha \forall x (\phi^{\mathbf{S}(\mathrm{RAH})}(x) \Leftrightarrow \phi^{\mathbf{S}(\mathrm{RAH}_\beta)}(x))$
  - Open problem 3. Is $\mathbf{Coll}$ necessary?

Steps to the construction and the analysis of RAH

- Step 1.Internalization of (Tarski's) truth
  - $\mathscr{U} \rightsquigarrow$ structure where individuals range over $\mathbb{B}$ and reals over $\mathbf{S}(\mathscr{U})$
  - Thm.5.1.2.2. $\mathbf{PA2}^{-} \vdash \phi^{\mathbf{S}(\mathscr{U})} \Leftrightarrow \mathrm{Sat}(\mathscr{U}, \ulcorner\phi\urcorner)$
- Step 2. Operator $\mathbf{Def^2} : \mathrm{Fam} \to \mathrm{Fam}$
  - $\mathbf{Def^2}(\mathscr{U})[\hat{x}.\ulcorner\phi\urcorner] = \{x \mid \phi^{\mathbf{S}(\mathscr{U})}\}$
- Step 3. Iteration of $\mathbf{Def^2}$ along w.p.o.$\rightsquigarrow (\mathrm{RAH}_\alpha)_{\alpha\in\mathrm{WPO}}$
  - $\mathbf{S}(\mathrm{RAH}_{\alpha+1}) \triangleq$ sets definable from $\mathbf{S}(\mathrm{RAH}_\alpha)$
- Step 4. $\mathbf{S}(\mathrm{RAH}) \triangleq \bigcup_{\alpha\in\mathrm{WPO}} \mathbf{S}(\mathrm{RAH}_\alpha)$ can be well-ordered
  - sets compared through the level they appear at
- Step 5. The principle of reflection (in $\mathbf{PA2}^{-} + \mathbf{Coll}$)
  - Thm.5.4.1.1. $\mathbf{PA2}^{-} + \mathbf{Coll} \vdash \forall\alpha\exists\beta \succcurlyeq_{\mathrm{WPO}} \alpha\forall x(\phi^{\mathbf{S}(\mathrm{RAH})}(x) \Leftrightarrow \phi^{\mathbf{S}(\mathrm{RAH}_\beta)}(x))$
  - Open problem 3. Is $\mathbf{Coll}$ necessary?
- Step 6. $\mathbf{S}(\mathrm{RAH})$ models the axiom of comprehension
  - $\mathbf{S}(\mathrm{RAH})$ is a well-ordered model of $\mathbf{PA2}$ (in $\mathbf{PA2}^{-} + \mathbf{Coll}$)

$\boldsymbol{\lambda}\mathbf{PA2}_{\mathbf{AC}} \longrightarrow \boldsymbol{\lambda}\mathbf{PA2}^{-}_{\mathbf{Coll}}$

Type system for **PA2** with an operator of extensional choice	Type system for $\mathbf{PA2}^{-} + \mathbf{Coll}$
$\Gamma \vdash M : \phi$	$\Gamma^{\mathbf{S}(\mathrm{RAH})} \vdash M^{*} : \phi^{\mathbf{S}(\mathrm{RAH})}$
unknown computational content	Krivine realizability

Future work. *Extract a computational content for* $\boldsymbol{\lambda}\mathbf{PA2}_{\mathbf{AC}}$

What is $\mathbf{HA}^\omega$?

$\mathbf{HA}^\omega$ is

- a first-order many sorted (intuitionistic) theory
- a formal theory to speak about functionals
- a conservative extension of Heyting Arithmetic

The theory $\mathbf{HA}^\omega$

Formulas of $\mathbf{HA}^\omega$.

$$\begin{array}{rcl}
\sigma, \tau & ::= & \mathbf{N} \mid \sigma \to \tau \\
t, u & ::= & x^\sigma \mid \lambda x^\sigma . t \mid (t\,u) \\
 & & \mid 0 \mid \mathrm{s}\,t \mid \mathrm{Rec}^\sigma\, t\, u\, v \\
\Phi, \Psi & ::= & t =_{\mathbf{N}} u \mid \bot \\
 & & \mid \Phi \Rightarrow \Psi \mid \Phi \wedge \Psi \\
 & & \mid \forall x^\sigma . \Phi \mid \exists x^\sigma . \Phi
\end{array}$$

Axioms of $\mathbf{HA}^\omega$.

- *the axioms of equality* *(roughly an axiomatization of $\cong$)*
- $x =_{\mathbf{N}} y \Rightarrow \Phi[z := x] \Rightarrow \Phi[z := y]$ *(indiscernibility of identicals)*
- $\neg(\mathrm{s}\,x) =_{\mathbf{N}} 0$ *(axiom of non-confusion)*
- $\Phi[z := 0] \Rightarrow (\forall x^{\mathbf{N}}\, \Phi[z := x] \Rightarrow \Phi[z := \mathrm{s}\,x]) \Rightarrow \forall x^{\mathbf{N}}\, \Phi[z := x]$ *(induction)*

Extensions of $\mathbf{HA}^\omega$

$$\begin{tikzcd}
& \mathbf{HA}^\omega \arrow[d, "\text{Extending equality}"] & \\
& \mathbf{N\text{-}HA}^\omega \arrow[dl, "\text{ext}"'] \arrow[dr, "\text{dec}"] & \\
\mathbf{E\text{-}HA}^\omega & & \mathbf{I\text{-}HA}^\omega
\end{tikzcd}$$

- $\mathbf{N\text{-}HA}^\omega$: adding equality at all level of sorts
- $\mathbf{E\text{-}HA}^\omega$: adding extensional equality at all level of sorts
- $\mathbf{I\text{-}HA}^\omega$: adding decidable equality at all level of sorts
  - $\mathsf{eqdec}_\sigma : \sigma \to \sigma \to \mathbf{N}$ such that $\forall x^\sigma \forall y^\sigma (x =_\sigma y \Leftrightarrow \mathsf{eqdec}_\sigma\, x\, y =_{\mathbf{N}} 0)$

[49] A. S. Troelstra. Metamathematical Investigation of Intuitionistic Arithmetic and Analysis ('73)

Two models of $\mathbf{HA}^\omega$

Plain model. $\mathbf{M}$ *is defined in* $\mathbf{ZF}$ *by*

$$\begin{array}{rcl} \mathbf{M}_\mathbf{N} & \triangleq & \mathbb{N} \\ \mathbf{M}_{\sigma\to\tau} & \triangleq & \mathbf{M}_\tau^{\mathbf{M}_\sigma} \end{array}$$

Terms of System T *are interpreted as functions*

Remark: $\mathbf{M} \vDash \mathbf{E\text{-}HA}^\omega$

Model of Hereditary Recursive Operations. $\mathbf{HRO}$ *is defined by*

$$\begin{array}{rcl} \mathbf{HRO}_\mathbf{N} & \triangleq & \mathbb{N} \\ \mathbf{HRO}_{\sigma\to\tau} & \triangleq & \{e \in \mathbb{N} \mid \forall n \in \mathbf{HRO}_\sigma \ \{e\}(n) \downarrow\in \mathbf{HRO}_\tau\} \end{array}$$

Terms of System T *are interpreted as indexes of recursive functions*

Remark: $\mathbf{HRO} \vDash \mathbf{I\text{-}HA}^\omega$

Contribution, future work and on-going work

- Contribution: Translation by parametricity from $\mathbf{E\text{-}HA}^\omega$ to $\mathbf{HA}^\omega$[1,11]
  - Thm.6.3.2.1. If $\Delta;\Gamma \vdash_e M : \Phi$ then $\Delta^1, \Delta^2; \Delta^{pm}, \Gamma^{pm} \vdash M^{pm} : \Phi^{pm}$
- Future work: Computational analysis of this translation
- On-going work: Translation characterizing the model $\mathbf{HRO}$
  - Implement a quoting instruction in $\mathbf{HA}^\omega$[44]

[1] T. Altenkirch, S. Boulier, A. Kaposi, N. Tabareau. Setoid type theory – a syntactic translation ('19)

[11] F.Castro. An interpretation of $\mathbf{E\text{-}HA}^\omega$ inside $\mathbf{HA}^\omega$ ('23)

[44] P.-M. Pédrot. Upon this quote I will build my Church Thesis ('24)

End of the talk

Thank you all for listening