Lowness of the piegeonhole principle

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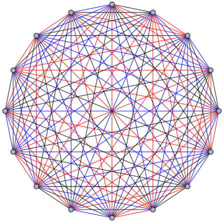


Section 1

Ramsey Theory

Motivation

It all started with this guy...

Theorem (Ramsey's theorem)

Let $n \geqslant 1$. For each coloration of $[\omega]^n$ in a finite number of color, there exists a set $X \in [\omega]^\omega$ such that each element of $[X]^n$ has the same color (X is said to be monochromatic).

Motivation

Ramsey Theory

A general question

Suppose we have some mathematical structure that is then cut into finitely many pieces. How big must the original structure be in order to ensure that at least one of the pieces has a given interesting property ?

Examples :

1. Van der Waerden's theorem
2. Hindman's theorem
3. ...

Motivation

Example (Van der Waerden's theorem)

For any given c and n, there is a number $w(c,n)$, such that if $w(c,n)$ consecutive numbers are colored with c different colors, then it must contain an arithmetic progression of length n whose elements all have the same color.

We know that :

$$w(c,n) \leqslant 2^{2^{c^{2^{2^{n+9}}}}}$$

Example (Hindmam's theorem)

If we color the natural numbers with finitely many colors, there must exists a monochromatic infinite set closed by finite sums.

Partition regularity

Theorems in Ramsey theory often assert, in their stronger form, that certain classes are *partition regular* :

Definition (Partition regularity)

A *partition regular* class is a collection of sets $\mathcal{L} \subseteq 2^\omega$ such that :

1. $\mathcal{L}$ is not empty
2. If $X \in \mathcal{L}$ and $Y_0 \cup \cdots \cup Y_k \supseteq X$, then there is $i \leqslant k$ such that $Y_i \in \mathcal{L}$

Partition regularity

The following classes are partition regular :

Classical combinatorial results :

1. The class of infinite sets
2. The class of sets with positive upper density
3. The class of sets X s.t. $\sum_{n \in X} \frac{1}{n} = \infty$
4. The class of sets containing arbitrarily long arithmetic progressions (Van der Waerden's theorem)
5. The class of sets containing an infinite set closed by finite sum (Hindman's theorem)

... and *new* type of results involving computability :

1. Given X non-computable, the class of sets containing an infinite set which does not compute X (Dzhafarov and Jockusch)

Seetapun's theorem

Theorem (Dzhafarov and Jockusch)

Given X non-computable, Given $A^0 \cup A^1 = \omega$, there exists $G \in [A^0]^\omega \cup [A^1]^\omega$ such that G does not compute X.

This theorem comes from Reverse mathematics :

What is the computational strength of Ramsey's theorem ?

that is, given a computable coloring of say $[\omega]^2$, must all monochromatic sets have a specific computational power ?

Theorem (Seetapun)

For any non-computable set X and any computable coloring of $[\omega]^2$, there is an infinite monochromatic set which does not compute X.

Theorem (Jockusch)

There exists a computable coloring of $[\omega]^3$, every solution of which computes $\emptyset'$.

Modern approach of Seetapun's theorem

Modern approach of Seetapun's theorem (Cholak, Jockusch, Slaman) :

Definition

A set C is $\{R_n\}_{n\in\omega}$-*cohesive* if $C \subseteq^* R_n$ or $C \subseteq^* \overline{R_n}$ for every n.

Definition

A coloring $c : \omega^2 \to \{0,1\}$ is *stable* if $\forall x \ \lim_{y\in\omega} c(x,y)$ exists.

1. Given a computable coloring $c : \omega^2 \to \{0,1\}$, let $R_n = \{y \ : \ c(n,y) = 0\}$. Let C be $\{R_n\}_{n\in\omega}$-cohesive. Then c restricted to C is stable.
2. Let c be a stable coloring. Let A_c be the $\Delta^0_2(c)$ set defined as $A_c(x) = \lim_y c(x,y)$. An infinite subset of A_c or of $\overline{A_c}$ can be used to compute a solution to c.

$\to$ Find a cohesive set C (cohesive for the recursive sets) which does not compute X and use Dzhafarov and Jockusch relative to C with $A_{c\upharpoonright C}$.

Background of RT^2_2 vs SRT^2_2

Definition

RT^2_2 : Any coloring $c : \omega^2 \to \{0,1\}$ admits an infinite homogeneous set.

The key idea of Cholak, Jockusch and Slaman is to split RT^2_2 into simpler principles (original motivation was to find a low_2 solution to RT^2_2) :

Definition

COH : For any sequence of sets $\{R_n\}_{n\in\omega}$ there is an $\{R_n\}_{n\in\omega}$-cohesive set.

Definition

SRT^2_2 : Any stable coloring admits a monochromatic set.

$\leftrightarrow$ (over RCA_0)

D^2_2 : For any Δ^0_2 set A, there is a set $X \in [A]^\omega \cup [\overline{A}]^\omega$.

We have that RT^2_2 is equivalent to $\mathrm{SRT}^2_2 + \mathrm{COH}$ over RCA_0.

The question

Theorem (Cholak, Jockusch and Slaman)

$\mathrm{RT}^2_2 \leftrightarrow_{\mathrm{RCA}_0} \mathrm{STR}^2_2 + \mathrm{COH}.$

Theorem (Hirschfeldt, Jockusch, Kjoss-Hanssen, Lempp and Slaman)

RT^2_2 *is strictly stronger than COH over* RCA_0.

Question

Do we have that RT^2_2 *is strictly stronger than* SRT^2_2 *over* RCA_0 *?*

$\leftrightarrow$

Do we have that SRT^2_2 *implies* COH *over* RCA_0 *?*

Theorem (Chong, Slaman, Yang)

RT^2_2 *is strictly stronger than* SRT^2_2 *over* RCA_0.

The question

Theorem (Chong, Slaman, Yang)

SRT^2_2 *does not imply* COH *over* RCA_0.

Proposition (Folklore)

If X computes a p-cohesive set (a set which is cohesive for primitive recursive sets), then X cannot be of low degree (with $X' \leqslant_T \emptyset'$).

The separation is done by building a non-standard models of $\mathrm{SRT}^2_2 + \mathrm{RCA}_0$ containing only sets which are low within the model. The model has to be non-standard by the following :

Theorem (Downey, Hirschfeldt, Lempp and Solomon)

There is a Δ^0_2 set A with no infinite low set in it or in its complement.

The proof of DHLS uses Σ^0_2-induction.

The new question

Question

Do we have that SRT^2_2 *implies* COH *over* RCA_0 *in* ω*-models ?*

$\leftrightarrow$

Is every ω*-models of* $D^2_2 \oplus RCA_0$ *also a model of* COH *?*

Question

Let A *be a* Δ^0_2 *set. Is there an infinite subset* G *of* A *or of the complement of* A*, such that* G *computes no p-cohesive set ?*

Question

What about any set A*, not necessarily* Δ^0_2 *?*

Section 2

Splitting ω in two

The question

What can we encode inside every infinite subsets of both two halves of ω ?

A splitting :

Such that :

1. **Each infinite subset** of the blue part has some comp. power
2. **Each infinite subset** of the red part has some comp. power

Answer : Not much...

A precision

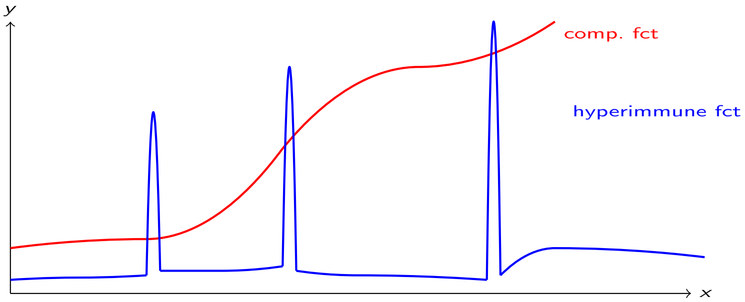
$\boxed{\text{What if we drop the complement thing ?}}$

Consider any set X. Then we can encode X into every infinite subset of a set A the following way : We let A be all the integers which correspond to an encoding of the prefixes of X (using some computable bijection between 2^ω and ω).

Encoding Hyperimmunity

Definition (Hyperimmunity)

A set X is of *hyperimmune degree* if X computes a function $f : \omega \to \omega$, which is not dominated by any computable function.



Theorem

There exists a covering $A^0 \cup A^1 \supseteq \omega$, such that every $X \in [A^0]^\omega \cup [A^1]^\omega$ is of hyperimmune degree.

Encoding Hyperimmunity

Theorem

There exists a covering $A^0 \cup A^1 \supseteq \omega$, such that every $X \in [A^0]^\omega \cup [A^1]^\omega$ is of hyperimmune degree.

We split ω by alternating larger and larger blocks of consecutive integers in A^0 and A^1.

For X infinite subset of A^0 or A^1, the hyperimmune function is given by $f(n)$ to be the n-th number which appears in X.

Encoding DNC

Definition (Diagonally non-computable degree)

A set X is of *DNC degree* (diagonally non-computable) if X computes a function $f : \omega \to \omega$, such that $f(n) \neq \Phi_n(n)$ for every n.

Theorem

The following are equivalent for a set X :

1. *X is of DNC degree.*
2. *X computes a function which on input n can output a string of Kolmogorov complexity greater than n.*
3. *X computes an infinite subset of a Martin-Löf random set.*

Encoding DNC

Definition (Informal definition of Kolmorogov complexity)

We say $K(\sigma) \geqslant n$ if the size of the smallest program which outputs σ is at least n.

Definition (Informal definition of Martin Löf randomness)

We say X is Martin Löf random is the Kolmogorov complexity of each of its prefix σ is greater than $|\sigma|$.

Theorem

X is DNC iff X computes an infinite subset of a Martin-Löf random set.

$$
\begin{array}{ll}
 & 00101110101001101100110100101101011001010101010\ldots \\
\rightarrow & 00001000000000100000000000000100011000000000010\ldots \\
\rightarrow & 11111111011111111011111101111111110111101111\ldots
\end{array}
$$

Cone avoidance

Theorem [Dzhafarov and Jockusch]

Let $X \subseteq \omega$ be non-computable. For every covering $A^0 \cup A^1 \supseteq \omega$, we have some $G \in [A^0]^\omega \cup [A^1]^\omega$ such that $G \not\geq_T X$.

The proof uses computable Mathias Forcing, where conditions are elements $\langle \sigma_0, \sigma_1, Y \rangle$ with

1. $\sigma_0 \subseteq A^0$ and $\sigma_1 \subseteq A^1$
2. $Y \cap A^0$ and $Y \cap A^1$ are both infinite.
3. Y does not compute X

We have that $\langle \sigma_0, \sigma_1, Y \rangle$ extends $\langle \tau_0, \tau_1, Z \rangle$ if

1. σ_0 extends τ_0 and σ_1 extends τ_1
2. $\sigma_0 - \tau_0 \subseteq Z$ and $\sigma_1 - \tau_1 \subseteq Z$
3. $Y \subseteq Z$

The forcing yields two generics $G^0 = \sigma_0^0 \preceq \sigma_0^1 \preceq \sigma_0^2 \preceq \dots$ and $G^1 = \sigma_1^0 \preceq \sigma_1^1 \preceq \sigma_1^2 \preceq \dots$. One of them is guarantied not to compute X, but we don't know which one in advance...

PA degrees

Definition

A set X is of P.A. degree if X computes a complete and consistent extension of Peano arithmetic.

Theorem

The following are equivalent :

1. X *is of P.A. degree.*
2. X *is diagonally non-computable with a* $\{0,1\}$*-valued function.*
3. X *computes an infinite path in any non-empty* Π^0_1 *class.*

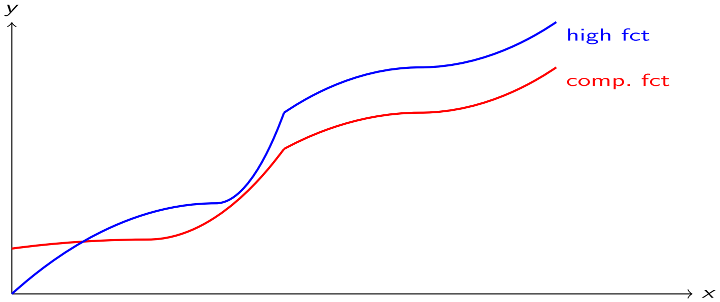
Theorem (Liu)

For every covering $A^1 \cup A^2 \supseteq \omega$*, for some* $i \leqslant 2$ *we have some* $G \in [A^i]^\omega$ *such that* G *is not of PA degree.*

Non high

Definition

A set X is high if it computes a function which eventually grows faster than any computable function.



Theorem (M., Patey)

For every covering $A^0 \cup A^1 \supseteq \omega$, we have some $G \in [A^0]^\omega \cup [A^1]^\omega$ such that G is not high.

Non high

Theorem (Martin)

The following are equivalent for a set X :

1. X *is high*
2. $X' \geqslant_T \emptyset''$

Theorem (M., Patey)

Let $X \subseteq \omega$ be non-$\emptyset'$-computable. For every covering $A^1 \cup A^2 \supseteq \omega$, there exists $G \in [A^0]^\omega \cup [A^1]^\omega$ such that $G' \not\geqslant_T X$.

The proof uses of new forcing technique that builds upon Mathias forcing to control the second jump.

Partition regularity is in particular a key concept of the used forcing.

More cone avoiding forcing

The non-high forcing cannot be extended in a straightforward way to control the truth of Σ^0_n statement for $n > 2$.

We can however bring non-trivial modification in order to show the following :

Theorem (M., Patey)

If B is not $\Delta^0_1(\emptyset^{(\alpha)})$ for $\alpha < \omega_1^{ck}$, for every covering $A^0 \cup A^1 \supseteq \omega$, we have some $G \in [A^0]^\omega \cup [A^1]^\omega$ such that B is not $\Delta^0_1(G^{(\alpha)})$.

Theorem (M., Patey)

If B is not Δ^1_1, for every covering $A^0 \cup A^1 \supseteq \omega$, we have some $G \in [A^0]^\omega \cup [A^1]^\omega$ such that B is not $\Delta^1_1(G)$ (with in particular $\omega_1^G = \omega_1^{ck}$).

Computing random sets

Theorem (Liu)

For every covering $A^0 \cup A^1 \supseteq \omega$, we have some $G \in [A^0]^\omega \cup [A^1]^\omega$ such that G computes no Martin-Löf random sets.

In fact every random set is DNC via a slow-growing DNC function (with $f(n) \leqslant 2^n$). Liu showed that for any computable bound g and every covering $A^0 \cup A^1 \supseteq \omega$, we have some $G \in [A^0]^\omega \cup [A^1]^\omega$ such that G computes no DNC function with bound g.

Computing generic sets

Definition

A set is *weakly-n-generic* if it is in every $\Sigma^0_1(\emptyset^{(n-1)})$ dense open set. It is 1-*generic* if for every $\Sigma^0_1(\emptyset^{(n-1)})$ open set U, it is in U or in the interior of the complement of U.

Theorem

There exists a covering $A^0 \cup A^1 \supseteq \omega$, such that for every $G \in [A^0]^\omega \cup [A^1]^\omega$ we have that G computes a 2-generic.

This is because any function which is not bounded by any Δ^0_3 function can compute a 2-generic. This does not work anymore with weakly-3-genericity and above.

Conjecture

For every covering $A^1 \cup A^2 \supseteq \omega$, we have some $G \in [A^0]^\omega \cup [A^1]^\omega$ such that G computes no weakly 3-generic sets.

The original question

Original Question

Is SRT^2_2 strictly stronger than COH ?

New Question

Let A be any set. Is there an element $G \in [A]^\omega \cup [\omega - A]^\omega$ which computes no p-cohesive set ?

Theorem

A set X computes a p-cohesive set iff X' is $\mathrm{PA}(\emptyset')$, that is, iff X' computes a function $f : \omega \to \{0, 1\}$ such that $f(n) \neq \Phi_e^{\emptyset'}(e)$.

New Question

Let A be any set. Is there an element $G \in [A]^\omega \cup [\omega - A]^\omega$ such that G' is not $\mathrm{PA}(\emptyset')$?

The answer

Theorem

For every Δ^0_2 set A, there is an element $G \in [A]^\omega \cup [\omega - A]^\omega$ such that G' is not $\mathrm{PA}(\emptyset')$.

Note that it is only proved for A a Δ^0_2 set. Fortunately this is sufficient for the following :

Corollary

There is an ω-model of $\mathrm{SRT}^2_2 + \mathrm{RCA}_0$ which is not a model of COH.

Question

Let A be any set (non necessarily Δ^0_2). Is there an element $G \in [A]^\omega \cup [\omega - A]^\omega$ such that G' is not $\mathrm{PA}(\emptyset')$?

The difficulty

The difficulty

The only known technic is Mathias Forcing. The difficulty is that sufficiently generic sets for Mathias forcing are themselves cohesive.

The conditions are (in the basic case) elements (σ, X) such that :

- σ is a string
- X is an infinite set with $X \cap \{0, \ldots, |\sigma|\} = \varnothing$

Forcing extension is $(\sigma, X) \preceq (\tau, Y)$ if :

- σ extends τ with $\sigma - \tau \subseteq Y$
- $X \subseteq Y$

Aspects of the solution

One key step is to be able to control the truth of Σ^0_2 statements.

Another key step is to perform "iterated Mathias forcing".

Let A be any set. Let R be computable with $A \cap R$ and $A \cap \overline{R}$ both infinite.

We build a cohesive generic set $G_0 \subseteq A \cap R$. We then build a cohesive set $G_1 \subseteq A \cap \overline{R}$ which is generic relative to G_0. Then $G_0 \cup G_1$ is not cohesive...