HIGHER RANDOMNESS

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Abstract. We present an overview of higher randomness and its recent developments. In particular we present the main higher randomness notions, show how to separate them and study their corresponding lowness classes. We study more specifically Π^1_1-Martin-Löf randomness, the higher analogue of the most well-known and studied class in classical algorithmic randomness, and Π^1_1-randomness, a notion which present many remarkable properties and does not have any analogue in classical randomness.

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1. Introduction

Mathematical objects often have a general definition which has no regard for any method or procedure that can describe it. For instance, a function is defined as an arbitrary correspondence between objects, but nothing in the definition requires that we are given a way to construct the correspondence. Nonetheless, when the modern definition of functions (often credited to Dirichlet) appeared, it was obvious that all the actual functions that were studied in practice were determined by simple analytic expressions, such as explicit formulas or infinite series.

In the early days of logic, some mathematicians tried to delineate the functions which could be defined by such accepted methods and they searched for their characteristic properties, presumably nice properties not shared by all functions. Baire was first to introduce in his Thesis [1] what we now call Baire functions, the smallest set which contains all continuous functions and is closed under the taking of (pointwise) limits. His work was then pursued by Lebesgues [28], who initiated the first systematic study of definable functions. According to Moschovakis [37] Lebesgue's paper truly started the subject of descriptive set theory.

At the time, the modern notions of computability and definability were yet to appear, but we can see, through the work of Borel, Baire and Lebesgues, the necessity of giving a precise meaning to the intuition we have of objects we can "describe" or "understand". A couple of years later, Godel's work around his famous incompleteness theorems constituted certainly a key step leading to the understanding of what is a computable object and to the understanding of definability in general. This work was then pursued in the thirties, by Church with Lambda calculus, and by Turing with his famous eponymous machine. The modern notion of computable function was made clear and all the researchers were soon convinced of the rather philosophical following statement, known as the Turing-Church thesis : "A function is computable (using either of the numerous possible equivalent mathematical definitions) iff its values can be found by some purely mechanical process".

Let us now go back to the early days of descriptive set theory. The study of the hierarchy of functions initiated by Baire and pursued by Lebesgue naturally led to the notion of Borel sets. One goal here was again to refine the very general definition of sets (say of reals) in order to work with objects we can understand and describe. The notion of Borel sets takes care of one aspect of sets complexity, their complexity with respect to their "shape" : The sets of reals with simplest shape complexity are the open sets ($\mathbf{\Sigma^0_1}$ sets) and their complement, the closed sets ($\mathbf{\Pi^0_1}$ sets). The first ones are merely unions of interval and the second ones complements of unions of interval. We then obtain sets of higher and higher complexity by taking countable unions or countable intersections of sets of lower complexity. We obtain a hierarchy of sets, each of them having nice properties, such as for instance being measurable or having the Baire property. However these hierarchy of complexity is still unsatisfactory, because even a set of simple shape, like an open set, can be very complex from the viewpoint of effectiveness: A set may be open, but there may be no way to describe the intervals which compose it. It is Kleene, a student of Church -like Turing- who reintroduced computability in the study of Borel sets. We now want to work only with open sets that can be described in some effective way. Then when we consider a countable intersection or a countable union, we also want to be able to describe in some effective way which sets take part in this union or intersection. This led to the very nice and beautiful theory of effectively Borel sets, and of effectively analytic and co-analytic sets, which constitute one of the core material of higher randomness.

Computability and definability could be used successfully in the study of sets of reals. But it was primarily designed to study sets of integers. Interestingly, the effective sets of reals proved themselves useful to conduct a study of the sets of integers which are far from being describable or understandable as single objects. This is the purpose of Algorithmic randomness. This field tries to resolve an apparent paradox that probability theory is helpless with: If one flip a fair coin twenty times in a row, a result like this 01001011011010101110 will seem rather "normal", whereas a result like this one : 00000000000000000000 will appear as non-random and extraordinary, to the point that one would probably check if the coin is valid. However, these two outcomes have the same probability of occurrence. So why one of them seems more random than the other one? It is simply because one is hard to describe whereas the other one is simple to describe. This is an extreme case, and it is not always the case that strings which seem non-random (with respect to a fair-coin fliping) are simple to describe. Consider for instance a long string with twice more 0's than 1's, but chaotic enough with regards to any other aspect you could think of. This string is not necessarily simple to describe, but it belongs to a small set that is simple to describe : the set of strings with twice more 0's than 1's, which has small measure by the concentration inequalities, like the Chernoff bounds. The mathematical formalization of this idea was a long process through the 20's century, started by Kolmorogov and Solomonov [45, 25]. Martin-Löf was the first in 1966 [32] to use the above paradigm to define randomness of infinite binary sequences: Such a sequence is random if it belongs to no set of measure 0, for a given class of set which should be describable in some way. Whichever notion of "being describable" is used, the only requirement is that at most countably many sets are describable for this notion. This way the set of randoms still has measure one, by the countable additivity of measures.

The field of higher randomness deals with effectively Borel, analytic and co-analytic sets. The work conduct by various researchers in this area follows two different directions. The first direction goes into the study of notions analogous to these of classical algorithmic randomness, which had already led to a very rich theory. Most of the work done in algorithmic randomness carries through higher randomness, but most of the time the proofs needs to be adapted to the new phenomenons that appears in higher computability, in particular the lack of continuity. The second direction goes into the study of notions which are new and specific to higher randomness, in particular the notion of Π^1_1-randomness. We will present here an overview of the work achieved by various authors in this field. The presentation is however not exhaustive, and here in particular is a list of subjects that we will not cover:

- The study of higher Kurtz randomness (see [24]).
- In [2] the authors emphasize that precautions must be taken with continuous relativization of Turing reductions and continuous relativization of randomness. A more detailed study of these issues is not given here, and is available in Chapter 7 of [35].
- The study of Δ^1_2, Σ^1_2 and Σ^1_2-Martin-Löf-randomness (see [7]).
- The study of randomness with infinite time Turing machines (see [3]).

2. Higher computability

2.1. **Background.** We assume the reader is familiar with the notions of Δ^1_1, Π^1_1 and Σ^1_1 sets of integers or of reals, and with admissibility and computability over $L_{\omega_1^{ck}}$. We simply recall here the notations and basic things that we are going to use.

2.1.1. *Computable ordinals and Borel sets.*

Definition 2.1. An ordinal α is computable if there exists a computable binary relation on elements of ω with order-type α. We let ω_1^{ck} denote the first non-computable ordinal.

The notion relativizes to any $X \in 2^{\mathbb{N}}$. We write ω_1^X for the smallest non X-computable ordinal.

The Δ^1_1 subsets of $\mathbb{N}$ are elements of $L_{\omega_1^{ck}} \cap \mathbb{N}$, that is, elements constructed with successive uniform unions and intersections of set of lower complexity.

Definition 2.2. The effective Kleene's hierarchy is defined by induction over the computable ordinals as follows:

- A Σ^0_1-index is given by a pair $\langle 0, e\rangle$. The set A corresponding to $\langle 0, e\rangle$ is given by $A = W_e$, the e-th Σ^0_1 set.
- A Π^0_α-index is given by a pair $\langle 1, e\rangle$ where e is a Σ^0_α-index. The set A corresponding to $\langle 1, e\rangle$ is given by $A = \mathbb{N} - B$ where B is the set corresponding to e.
- A Σ^0_α-index is given by a pair $\langle 2, e\rangle$ where W_e is not empty and enumerates only $\Pi^0_{\beta_n}$-indices for $\beta_n < \alpha$, with $\sup_n(\beta_n + 1) = \alpha$. The set A corresponding to $\langle 2, e\rangle$ is given by $\bigcup_n A_n$, where A_n is the set corresponding to the n-th index enumerated by W_e.

We say that a set A is Σ^0_α (resp. Π^0_α) if for some Σ^0_α-index (resp. Π^0_α-index) e, A is the set corresponding to e. We say that a set A is Δ^0_α if it is both Σ^0_α and Π^0_α. Finally we say that a set is $\Sigma^0_{<\alpha}$ (resp. $\Pi^0_{<\alpha}$) if it is Σ^0_β (resp. Π^0_β) for some $\beta < \alpha$.

For any α, there exists a complete Σ^0_α set, that is, a set which is Σ^0_α and such that any Σ^0_α is many-one reducible to it:

Definition 2.3. For any $\alpha < \omega_1^{ck}$, we denote by $\emptyset^\alpha$ a complete set for the Σ^0_α sets. We denote by $\emptyset^{<\alpha}$ a complete set for the $\Sigma^0_{<\alpha}$ sets.

Note that there is not necessarily a canonical way to define $\emptyset^\alpha$ or $\emptyset^{<\alpha}$. A way to define them is to use codes of computable ordinals.

Definition 2.4. A code for an ordinal α is given by the code of a Turing machine which computes a relation on ω or order-type α. We denote by O the set of codes for computable ordinals. For $\alpha < \omega_1^{ck}$ we denote by $O_{<\alpha}$ the set of codes of ordinal strictly smaller than α.

The notion relativizes to any $X \in 2^{\mathbb{N}}$. We write O^X for the set of codes which computes an ordinal using X as an oracle. Similarly for $O^X_{<\alpha}$.

For $a \in O$ (resp. $a \in O^X$) we may denote by $|a|_o$ (resp. $|a|_o^X$) the ordinal coded by a.

A precise study of the complexity and completeness of the sets $O_{<\alpha}$ is given in [35]. This gives an alternative way to define Δ^1_1 sets of integer is to see them as the sets which are Turing reducible to $O_{<\alpha}$ for some $\alpha < \omega_1^{ck}$.

We now similarly define Δ^1_1 subsets of $2^{\mathbb{N}}$:

Definition 2.5. The effective Borel hierarchy is defined by induction over the computable ordinals as follows:

- A Σ^0_1-index is given by a pair $\langle 0, e\rangle$. The set corresponding to this Σ^0_1-index is given by $\bigcup_{\sigma \in W_e} [\sigma]$.
- A Π^0_α-index is given by a pair $\langle 1, e\rangle$ where e is a Σ^0_α-index. The set corresponding to this Π^0_α-index is given by $2^{\mathbb{N}} - \mathcal{B}$ where $\mathcal{B}$ is the set corresponding to the index e.
- A Σ^0_α-index is given by a pair $\langle 2, e\rangle$ where W_e is not empty and enumerate only $\Pi^0_{\beta_n}$ indices, with $\sup_n(\beta_n + 1) = \alpha$. The set corresponding to this Σ^0_α-index is given by $\bigcup_n \mathcal{B}_n$ where $\mathcal{B}_n$ is the set corresponding to the n-th index enumerated by W_e.

We say that a set $\mathcal{B}$ is Σ^0_α (resp. Π^0_α) if for some Σ^0_α-index (resp. Π^0_α-index) e, $\mathcal{B}$ is the set corresponding to e. We say that a set $\mathcal{B}$ is Δ^0_α if it is both Σ^0_α and Π^0_α. Finally we say that a set is $\Sigma^0_{<\alpha}$ (resp. $\Pi^0_{<\alpha}$) if it is Σ^0_β (resp. Π^0_β) for some $\beta < \alpha$.

The following say that O is complete for the Π^1_1 sets. In particular a Π^1_1 set of integers can be seen as a uniform union of Δ^1_1 sets along ω_1^{ck}, and a Π^1_1 set of reals can be seen as a uniform union of Borel sets along ω_1:

Proposition 2.6. *A set of integer A is Π^1_1 iff there is a computable function f such that "$n \in A$ iff $f(n) \in O$". In particular $A = \bigcup_{\alpha<\omega_1^{ck}} \{n : f(n) \in O_{<\alpha}\}$.*

A set of reals $\mathcal{A}$ is Π^1_1 iff there is an integer e such that $X \in \mathcal{A}$ iff "$e \in O^X$". In particular $\mathcal{A} = \bigcup_{\alpha<\omega_1} \{X : e \in O^X_{<\alpha}\}$.

We will also use a lot what we call a projectum function, that is, a Π^1_1 injection from ω_1^{ck} into $\mathbb{N}$. Formally Π^1_1 functions are defined on integers and not ordinals. There are two ways to consider this: Either we work with actual ordinals and see Π^1_1 functions as being Σ_1-definable over $L_{\omega_1^{ck}}$, or we consider functions which are defined on a Π^1_1 set of unique codes of computable ordinals (that is a Π^1_1 set $O_1 \subseteq O$ such that for any $\alpha < \omega_1^{ck}$ there exists exactly one code of α in O_1).

Proposition 2.7. *There is a Π^1_1 function $p : \omega_1^{ck} \to \mathbb{N}$ which is one-to-one. We call p a projectum function.*

Note that a Π^1_1 set of unique codes of computable ordinals, can actually be considered as a projectum function.

2.1.2. Π^1_1 *as an analogue of c.e.* We will consider Π^1_1 predicates from the computability theorist's viewpoint, that is, we will see them as enumerations of objects along computable ordinal stages of computation. Let us cite a section of Sack's book ([44, V.3.3]) that explains what we gain in doing so:

"Post in a celebrated paper ([42]) liberated classical recursion theory from formal arguments by presenting recursive enumerability as a natural mathematical notion safely handled by informal mathematical procedures. He also stressed what may be called a dynamic view of recursion theory. For example, he proves the existence of a simple set S by giving instructions in ordinary language for the enumeration of S and then verifying that the instructions do in fact produce a simple set. A formal approach to S would refer to formulas or equations from some formal system. A static approach would attempt to define S by some explicit formula. The advantages of Post's informal, dynamic method are considerable. Without it arguments in classical recursion theory would be lengthy and hard to devise. His method, and its advantages, lift to metarecursion theory."

Metarecursion theory attacks the problem of transposing notions of classical recursion theory, that take place in the world of integers, into the world of computable ordinals, where elements of the Cantor space are now replaced by functions from ω_1^{ck} to $\{0,1\}$ (sequences of "length" ω_1^{ck}) and where times of computation are now computable ordinals.

We will not deal here with Metarecursion theory, as we still want to work with sequences of the Cantor space. Measure-theoretic notions and therefore algorithmic randomness are indeed well-defined for sequences of length ω, but it is not clear at all if one can extend these notions to sequences of ordinal length. For this reason, what we keep from Metarecursion theory are just the ordinal times of computation.

In this settings, any Δ^1_1 set of integer should be considered as a finite object. Any Π^1_1 set A should be seen as enumerable along the ordinal times of computation. The construction of a c.e. set A is often done step by step, by describing A_s at computational step s, where A_s possibly depends on the values of A_t for $t < s$, and by then defining $A = \bigcup_{s<\omega} A_s$. A formal description of A can then be given by $n \in A \leftrightarrow \exists s\ n \in A_s$. As each set A_s is Δ^0_1 uniformly in s, the description can then be formally written as a Σ^0_1 predicate.

We can similarly build a Π^1_1 set A by describing A_s for each ordinal computational step $s < \omega_1^{ck}$, where A_s possibly depends on the values of A_t for $t < s$, and then by defining $A = \bigcup_{s<\omega_1^{ck}} A_s$. If one wants A to be formally Π^1_1, one has to use codes for computable ordinal in order to give an actual Π^1_1 description of A. The definition should of course not depend on the code that we use, but only the the ordinal represented by the code (and this will always be the case in what we do). A way to go around this is otherwise to see the predicate $n \in A$ as being Σ^0_1 over $L_{\omega_1^{ck}}$.

2.1.3. *Admissibility.* As explained in the previous section, we will use the informal argument of recursion theory to enumerate sets along the computable ordinal, possibly using what happened at previous steps of enumerations. The reason we can do that, is the admissibility of $L_{\omega_1^{ck}}$. For short, given $\alpha < \omega_1^{ck}$, there is no function $f : \alpha \to \omega_1^{ck}$ which is Σ^0_1-definable in $L_{\omega_1^{ck}}$ (with parameter in L_α). In particular, inside admissible sets, we can safely make recursive definitions along the ordinals. Another way to see admissibility is to consider Spector's Σ^1_1 boundedness principle : Let $A \subseteq O$ be a Σ^1_1 set. Then there exists α such that $A \subseteq O_\alpha$.

Admissibility will be mainly use as follow for us: whenever there is a $\Pi^1_1(X)$ total function f from $\alpha < \omega_1^X$ into ω_1^X, then we must have $\sup_n f(n) < \omega_1^X$.

2.1.4. *Notations.* We denote the Cantor space by $2^{\mathbb{N}}$ and the set of strings of the Cantor space by $2^{<\mathbb{N}}$. We denote the Baire space by $\mathbb{N}^{\mathbb{N}}$ and the set of strings of the Baire space by $\mathbb{N}^{<\mathbb{N}}$. Given $\sigma \in 2^{<\mathbb{N}}$ we write $[\sigma]$ for its corresponding cylinder, that is, the set $\{X \in 2^{\mathbb{N}} : \sigma \prec X\}$. An open set is a union of cylinders. Given $W \subseteq 2^{<\mathbb{N}}$ we write $[W]^{\prec}$ for the set $\bigcup_{\sigma \in W} [\sigma]$. In particular we will consider a lot open sets of the following type:

Definition 2.8. An open set $\mathcal{U}$ is a Π^1_1-open set if there is a Π^1_1 set $W \subseteq 2^{<\mathbb{N}}$ such that $\mathcal{U} = [W]^{\prec}$. A Σ^1_1-closed set is the complement of a Π^1_1-open set.

We will denote the Lebesgue measure on the Cantor space by λ. We then have $\lambda([\sigma]) = 2^{-|\sigma|}$ for any $\sigma \in 2^{<\mathbb{N}}$. Given a measurable set $\mathcal{A}$ we also write $\lambda(\mathcal{A}|\sigma)$ for the measure of $\mathcal{A}$ inside σ, that is, the quantity $\lambda(\mathcal{A} \cap [\sigma])/\lambda([\sigma])$.

Given the enumeration of an object A long the computable ordinals, we can write A_s or $A[s]$ for the current enumeration of A up to stage s. We will especially use the latter with the measure of objects. For instance, given a Π^1_1-open set $\mathcal{U}$, we may write $\lambda(\mathcal{U})[s]$ for the measure of $\mathcal{U}$ at stage s. We also sometimes write $A[< s]$ for the current enumeration of an object up to stage s (but without stage s).

The notation A_s will mainly be used when one want to refer as the enumeration up to stage s as a specific object. In particular we will sometimes use the following terminology:

Definition 2.9. A higher computable sequence $\{f_s\}_{s<\omega_1^{ck}}$ is a sequence of uniformly Δ^1_1 functions f_s, that is, each f_s is Δ uniformly in s.

2.2. **Continuity in higher computability.** In higher computability, reductions and relativization are not continuous notions (unlike with normal computability):

Definition 2.10. We write that $X \geqslant_h Y$ and say that Y is hyperarithmetically reducible to X is Y is $\Delta^1_1(X)$.

For instance if $X \geqslant_h Y$, infinitely many bits of X may be needed to determine one bit of Y. The insight that randomness and traditional relative hyperarithmetic reducibility do not interact well goes back to Hjorth and Nies [17], but it is in [2] that Bienvenu, Greenberg and Monin enlighten the centrality of continuous reductions to the theory of randomness.

2.2.1. *Higher Turing reductions.* In order to study analogues of classical randomness notions in the higher settings, we will need a continuous higher analogue of Turing reducibility. Recall that a functional can be seen as a set of pairs (τ, σ) of finite binary strings. If Φ is a functional then for any $X \in 2^{\leqslant\omega}$ (finite or infinite) we have that Φ is defined on X if:

(1) Φ is consistent on prefixes of X, that is, if $\sigma_1 \prec X$ and $\sigma_2 \prec X$ are comparable and if (σ_1, τ_1) and (σ_2, τ_2) are in Φ then τ_1 must be comparable with τ_2.
(2) Φ is total on X, that is, for any n, there exists $\sigma \prec X$ such that σ is mapped to a string of length at least n.

When (1) and (2) are met there is a unique limit point $Y \in 2^{\mathbb{N}}$ of $\bigcap\{[\sigma] \ : \ \exists n \ (X\upharpoonright_n, \sigma) \in \Phi\}$. We then write $\Phi(X) = Y$. This motivates the following definition:

Definition 2.11 (Bienvenu, Greenberg, Monin [2])**.** A higher Turing reduction Φ is a Π^1_1 partial map from $2^{<\mathbb{N}}$ to $2^{<\mathbb{N}}$. For a string σ, if Φ is consistent on prefixes of σ, we write $\Phi(\sigma) = \tau$ where τ is the longest string that prefixes of σ are mapped to in Φ; otherwise $\Phi(\sigma)$ is said to be undefined. Given a sequence X, suppose the set:

$$\bigcap\{[\sigma] \ : \ \exists n \ \Phi(X\upharpoonright_n) = \sigma\}$$

contains exactly one sequence Y, we write $\Phi(X) = Y$. Otherwise the functional Φ is said to be undefined on X. If $\Phi(X) = Y$ for some higher Turing reduction Φ we write $X \geqslant_{\omega_1^{\mathrm{ck}}\mathrm{T}} Y$ and say that X higher Turing computes Y.

Hjorth and Nies were the first to define in [17] a notion of continuous higher reduction, that they called fin-h reduction. The fin-h reduction was define analogously to higher Turing reduction, with the additional restriction that the mapping must be both consistent and closed under prefixes. It appears that the fin-h reduction is too restrictive for most theorems of higher randomness that requires a higher continuous reduction.

Note that with normal Turing reductions, one can always required a c.e. set of pairs Φ to be consistent everywhere, that is, one can uniformly transform Φ into a c.e. set Ψ such that Ψ is consistent everywhere and such that if $\Phi(X) = Y$ then also $\Psi(X) = Y$. Such a thing is not necessarily possible with higher Turing reductions. In particular there are some X, Y such that X higher Turing compute Y but such that X does not fin-h compute Y. For more details about this, the reader can refer to Chapter 7 of [35].

So inconsistency cannot always be removed, but it can be made of measure as small as we want:

Lemma 2.12 (Bienvenu, Greenberg, Monin [2]). From any higher functional Φ one can obtain effectively in ε a higher functional Ψ so that:

(1) The correct computations are unchanged in Ψ: For all X, Y such that $\Phi(X) = Y$, we also have $\Psi(X) = Y$
(2) The measure of the Π^1_1-open set on which Ψ is inconsistent is smaller than ε:

$$\lambda(\{X \mid \exists n_1, n_2 \ \exists \tau_1 \perp \tau_2 \ (X\upharpoonright_{n_1}, \tau_1) \in \Psi \wedge (X\upharpoonright_{n_2}, \tau_2) \in \Psi\}) \leqslant \varepsilon$$

Proof. Let us build Ψ uniformly in Φ and ε. Recall that $p : \omega_1^{ck} \to \omega$ is the projectum function. We can assume that at most one pair enters Φ at each stage. At stage s, if (σ_1, τ_1) enters $\Phi[s]$, we compute the Δ^1_1 set of strings:

$$\mathcal{U}_s = \{\sigma_2 \ : \ \sigma_2 \text{ is compatible with } \sigma_1 \text{ and } (\sigma_2, \tau_2) \in \Psi[< s] \text{ for some } \tau_2 \perp \tau_1)\}$$

We then find uniformly in $\mathcal{U}_s$ and s a finite set of strings C with $[C]^\prec \subseteq [\sigma_1]$, such that $[C]^\prec \cup \mathcal{U}_s$ covers $[\sigma_1]$ and such that $\lambda([C]^\prec \cap \mathcal{U}_s) \leqslant 2^{-p(s)}\varepsilon$. Then we put in $\Psi[s]$ all the pairs (σ, τ_1) for $\sigma \in C$.

We shall prove that (1) and (2) are satisfied. Suppose $\Phi(X) = Y$ and that $(X\upharpoonright_{n_1}, Y\upharpoonright_{n_2})$ enters $\Phi[s]$ at stage s. By definition of $\Phi(X) = Y$, we have no m and no $\tau \perp Y\upharpoonright_{n_2}$ such that $(X\upharpoonright_m, \tau)$ is in $\Phi[< s]$. Then also we have no m and no $\tau \perp Y\upharpoonright_{n_2}$ such that $(X\upharpoonright_m, \tau)$ is in $\Psi[< s]$, because $(X\upharpoonright_m, \tau) \in \Psi$ implies $(X\upharpoonright_n, \tau) \in \Phi$ for $n \leqslant m$. Therefore $X \notin \mathcal{U}_s$ and as $\mathcal{U}_s \cup C$ covers $X\upharpoonright_{n_1}$, we then have a prefix of X that is mapped to $Y\upharpoonright_{n_2}$ in $\Psi[s]$. Then we have (1). Also by construction, at stage s, we add a measure of at most $2^{-p(s)}\varepsilon$ of inconsistency. Then the total inconsistency is at most of ε, which gives us (2). ∎

2.2.2. *Continuous relativization of* Π^1_1. The higher continuous version of Turing reduction is a way to say that some sequence Y is continuously Δ^1_1 in X. We will also need a way to say that some objects are continuously Π^1_1 in X. This will be used mainly for continuous relativization of randomness notions.

Definition 2.13 (Bienvenu, Greenberg, Monin [2])**.** An oracle-continuous Π^1_1 set of integers is given by a set $W \subseteq 2^{<\mathbb{N}} \times \mathbb{N}$. For a string σ we write W^σ to denote the set $\{n \ :\ \exists \tau \preccurlyeq \sigma\ (\tau, n) \in W\}$. For a sequence X we write W^X to denote the set $\{n \ :\ \exists \tau \prec X\ (\tau, n) \in W\}$. The set W^X is then called an X-continuous Π^1_1 set of integers.

2.3. **Refinement of the notion of higher Δ^0_2.** In this section we discuss the higher analogue of the notion of being Δ^0_2. We will identify in particular various restrictions of this notion, in order to have sufficient conditions for higher Δ^0_2 elements to collapse ω_1^{ck}. This work will be useful to show several theorem. In particular that every non-Δ^1_1 higher K-trivial collapses ω_1^{ck}, and to separate higher weak-2-randomness from Π^1_1-randomness. Let us first give a higher version of Shoenfield's limit lemma:

2.3.1. *The higher limit lemma.*

Proposition 2.14 (Bienvenu, Greenberg, Monin [2])**.** *Let $A \in 2^{\mathbb{N}}$. The following are equivalent for $f \in \mathbb{N}^{\mathbb{N}}$.*

1. *O higher Turing computes f.*
2. *O Turing computes f.*
3. *There is a higher computable sequence $\{f_s\}_{s<\omega_1^{ck}}$ of functions from $\mathbb{N}$ to $\mathbb{N}$ with $\lim_{s\to\omega_1^{ck}} f_s = f$.*

Proof. (1) $\Longrightarrow$ (2). Let Ψ be a higher Turing functional such that $\Psi(O)$ is defined. We define the Turing functional Φ which using O, on each n, searches for the first pair m, k such that $\exists t\ \Psi(O\restriction_m, n)[t] = k$.

(2) $\Longrightarrow$ (3). Let Ψ be a Turing functional such that $\Psi(O) = f$. We simply let f_s such that $f_s(n) = 1$ iff $\Psi(O_s, n) = 1$ and $f_s(n) = 0$ otherwise.

(3) $\Longrightarrow$ (1). We use the projectum function $p : \omega_1^{ck} \to \omega$. Given $n \in \mathbb{N}$, for each $m \in \mathbb{N}$ with $s_m = p^{-1}(m)$, we ask to O if $\exists t > s_m\ f_t(n) \neq f_{s_m}(n)$, until we find some m such that this is not the case. Then we set $f(n) = f_{s_m}(n)$. □

Such a function is said to be a higher Δ^0_2 function. There is a topological difference between a Δ^0_2 approximation $\{f_s\}_{s<\omega}$ and a higher Δ^0_2 approximation $\{g_s\}_{s<\omega_1^{ck}}$. In the first case the set $\{f\} \cup \{f_s \ :\ s < \omega\}$ is a closed set, whereas in the second case, the set $\{g\} \cup \{g_s \ :\ s < \omega_1^{ck}\}$ needs not to be closed. Also we present in this section various restrictions of the notion of higher Δ^0_2 functions, introduced in [2], and that are built around this crucial point.

2.3.2. *Collapsing approximations.* Gandy showed that in any non-empty Σ^1_1 set of reals, there is an element $X \leqslant_T O$ such that $\omega_1^X = \omega_1^{ck}$ (see [44]). As the set of non-Δ^1_1 elements is Σ^1_1, it follows that some non-Δ^1_1 higher Δ^0_2 sequence does not collapse ω_1^{ck}. We present here a natural restriction of being higher Δ^0_2, which is enough already for non-Δ^1_1 such approximable elements, to collapse ω_1^{ck}.

Definition 2.15 (Bienvenu, Greenberg, Monin [2])**.** A collapsing approximation of a function f is a higher Turing computable sequence $\{f_s\}_{s<\omega_1^{ck}}$ converging to f and such that for every stage s, the function f is not in the closure of $\{f_t \ :\ t < s\}$ unless it is already an element of $\{f_t \ :\ t < s\}$.

Theorem 2.16 (Bienvenu, Greenberg, Monin [2])**.** *If $f \in \mathbb{N}^{\mathbb{N}}$ is not Δ^1_1 and has a collapsing approximation then $\omega_1^f > \omega_1^{ck}$.*

Proof. Suppose f has a collapsing approximation $\{f_s\}_{s<\omega_1^{ck}}$. We can define the $\Pi^1_1(f)$ total function $g : \omega \to \omega_1^{ck}$ which to n associates the smallest ordinal s_n so that $f_{s_n}\restriction_n = f\restriction_n$. Then we have that f is in the closure of $\{f_t\}_{t<s}$ for $s = \sup s_n$. Therefore we have $\sup s_n = \omega_1^{ck}$. Also as g is $\Pi^1_1(f)$ and total it is also $\Delta^1_1(f)$. Then we can define a $\Delta^1_1(f)$ sequence of computable ordinals, unbounded in ω_1^{ck} which implies $\omega_1^f > \omega_1^{ck}$, by admissibility. □

Note that this is not the most general way for higher Δ^0_2 elements to collapse ω_1^{ck}. Bienvenu, Greenberg and Monin showed [2] that there is a higher Δ^0_2 sequence X such that $\omega_1^X > \omega_1^{ck}$ and such that X does not have a collapsing approximation.

2.3.3. *Higher finite-change approximations.* In the lower setting, any Δ^0_2 approximation $\{f_s\}_{s\in\mathbb{N}}$ is collapsing simply because at every step t, there are only finitely many versions f_s for $s \leqslant t$. We restrict here the collapsing approximations to these which share this property with the Δ^0_2 approximations indexed by $\mathbb{N}$.

Definition 2.17. A higher finite-change approximation of a function f is a higher computable sequence $\{f_s\}_{s<\omega_1^{ck}}$ such that $\lim_s f_s = f$ and such that for any n, the sequence $\{f_s(n)\}_{s<\omega_1^{ck}}$ changes finitely often.

2.3.4. *Higher left-c.e. approximations.* We now define the strongest restriction of higher Δ^0_2, which can be seen as a higher analogue of left-c.e.

Definition 2.18. A higher left-c.e. approximation of a function f is a higher computable sequence $\{f_s\}_{s<\omega_1^{ck}}$ such that $\lim_s f_s = f$ and such that for any stages $s_1 < s_2$ we have f_{s_1} smaller than f_{s_2} for the lexicographic order.

Note that if $\{f_s\}_{s<\omega_1^{ck}}$ is a higher left-c.e. approximation, then $\{f_s(n)\}_{s<\omega_1^{ck}}$ changes at most 2^n times and then $\{f_s\}_{s<\omega_1^{ck}}$ is higher finite-change.

Just like left-c.e. binary sequences are exactly the leftmost path of Π^0_1 sets, it is not hard to see that higher left-c.e. binary sequences are the leftmost path of Σ^1_1-closed sets.

3. Overview of the different classes in higher randomness

We present in this section the main higher randomness classes. These notions are obtained by extending the definability power one can use to capture non-random sequences in nullsets.

3.1. Δ^1_1 **randomness.** Perhaps the simplest higher randomness notion, and also the first that has been introduced is obtained by defining that a sequence is random if it belongs to no effectively Borel set of measure 0:

Definition 3.1 (Sacks, [44] IV.2.5)**.** We say that $Z \in 2^{\mathbb{N}}$ is Δ^1_1-random if it is in no Δ^1_1 nullset.

Martin-Löf was actually the first to promote this notion (see [33]), suggesting that it was the appropriate mathematical concept of randomness. Even if his first definition undoubtedly became the most successful over the years, this other definition got a second wind recently on the initiative of Hjorth and Nies who started to study the analogy between the usual notions of randomness and theirs higher counterparts. One could also define the randomness notion obtained by considering Σ^1_1 nullsets, but this turns out to be equivalent to Δ^1_1-randomness.

Theorem 3.2 (Sacks [44] IV.2.5)**.** *A Δ^1_1-random sequence is in no Σ^1_1 nullset. Therefore Σ^1_1-randomness coincides with Δ^1_1-randomness.*

Proof. Let $\mathcal{A} = \bigcap_{\alpha<\omega_1} \mathcal{A}_\alpha$ be a Σ^1_1 nullset. Note that we can suppose that the intersection is decreasing. By Theorem 3.11 we have that $\bigcap_{\alpha<\omega_1^{ck}} \mathcal{A}_\alpha$ is already of measure 0. Then we can define the Π^1_1 function $f : \omega \to \omega_1^{ck}$ which associates to n the smallest ordinal α such that $\lambda(\mathcal{A}_\alpha) \leqslant 2^{-n}$. As f is total, it is actually a Δ^1_1 function, and then its range is a Δ^1_1 set of computable ordinals, which is then bounded by some computable ordinal β, by the Σ^1_1-boundedness principle. Therefore we have $\lambda(\bigcap_{\alpha<\beta} \mathcal{A}_\alpha) = 0$ and then $\mathcal{A}$ is contained in a Δ^1_1 set of measure 0. □

3.2. Π^1_1**-Martin-Löf randomness and below.** Hjorth and Nies introduced in [17] a higher analogue of Martin-Löf randomness.

Definition 3.3 (Hjorth, Nies [17])**.** A Π^1_1-Martin-Löf test is given by an intersection of open sets $\bigcap_n \mathcal{U}_n$, such that $\lambda(\mathcal{U}_n) \leqslant 2^{-n}$ for each n and such that each $\mathcal{U}_n$ is Π^1_1 uniformly in n. A sequence X is Π^1_1-Martin-Löf-random if it is in no Π^1_1-Martin-Löf test.

It will be sometimes convenient to use a higher version of Solovay tests:

Definition 3.4. A higher Solovay test is given by a sequence $\{\mathcal{U}_n\}_{n\in\mathbb{N}}$ of uniformly (in n) Π^1_1-open sets such that $\sum_{n\in\mathbb{N}} \lambda(\mathcal{U}_n)$ is finite. A sequence X passes the higher Solovay test if it belongs to only finitely many $\mathcal{U}_n$.

The proof that X is Π^1_1-Martin-Löf random iff it passes all the Solovay tests works as in the lower setting. An interesting possibility with higher Solovay tests, that will be used sometimes, is that we can index each open set with a computable ordinal instead of indexing it with an integer.

Formally, given a sequence of Π^1_1-open sets $\{\mathcal{U}_s\}_{s<\omega_1^{ck}}$, we can build the higher Solovay test $\mathcal{V}_n$ where each $\mathcal{V}_n$ starts with an empty enumeration, until n is witnessed to be a code for the ordinal s, in which case $\mathcal{V}_n$ becomes equal to $\mathcal{U}_s$. It is clear that the notion of being captured in unchanged between $\{\mathcal{U}_s\}_{s<\omega_1^{ck}}$ and $\{\mathcal{V}_n\}_{n\in\mathbb{N}}$.

We now discuss the relationship between Π^1_1-Martin-Löf randomness and Δ^1_1-randomness. Theorem 3.6 implies that the set of Π^1_1-Martin-Löf randoms is included in the set of Δ^1_1-randoms. In other words, the notion of Π^1_1-Martin-Löf randomness is stronger than or equal to the notion of Δ^1_1 randomness. To see that, we simply need to make effective the Lebesgue's theorem stating that any Borel set of arbitrary complexity is approximable from above by $\mathbf{\Pi^0_2}$ sets of the same measure, and from below by $\mathbf{\Sigma^0_2}$ sets of the same measure. Such an effective version of the theorem has been done for the arithmetical hierarchy in Kurtz's thesis [27] and in Kautz [19]. We present here the proof of [35] for the whole effective hyperarithmetical hierarchy. We start with the following lemma, which says that "$\mu(\mathcal{A}) > q$" is a Σ^0_α predicate for $\mathcal{A}$ a Σ^0_α set.

Lemma 3.5. Let μ be a computable Borel probability measure. Let $\mathcal{A} \subseteq 2^{\mathbb{N}}$ be a Σ^0_α set. The set $\{q \in \mathbb{Q} \cap [0,1] \ : \ \mu(\mathcal{A}) > q\}$ is a Σ^0_α set, uniformly in μ, and in an index for $\mathcal{A}$.

Proof. The proof goes by induction on computable ordinals. If $\mathcal{A}$ is a Σ^0_1 set, the predicate $\mu(\mathcal{A}) > q$ is equivalent to $\exists t\ \mu(\mathcal{A}[t]) > q$, which is Σ^0_1 as $\mathcal{A}[t]$ is a clopen set. Everything is clearly uniform.

Suppose that for an ordinal α, any $\Sigma^0_{<\alpha}$ set $\mathcal{A}$ and any rational $q > 0$, the set $\{q \in \mathbb{Q} \cap [0,1] \ :\ \mu(\mathcal{A}) > q\}$ is a $\Sigma^0_{<\alpha}$ set uniformly in an index for $\mathcal{A}$. Consider the Σ^0_α set $\mathcal{A} = \bigcup_n \mathcal{B}_n$ where each $\mathcal{B}_n$ is $\Pi^0_{<\alpha}$ uniformly in n. The predicate $\mu(\mathcal{A}) > q$ is equivalent to $\exists m\ \mu(\bigcup_{n\leqslant m} \mathcal{B}_n) > q$. Also for each m the set $\bigcup_{n\leqslant m} \mathcal{B}_n$ is a $\Pi^0_{<\alpha}$ set uniformly in m. By induction hypothesis, it follows that $\{q \in \mathbb{Q} \cap [0,1] \ :\ \mu(\bigcup_{n\leqslant m} \mathcal{B}_n) > q\}$ is a $\Pi^0_{<\alpha}$ set for every m and uniformly in m. It follows that the set $\{q \in \mathbb{Q} \cap [0,1] \ :\ \exists m\ \mu(\bigcup_{n\leqslant m} \mathcal{B}_n) > q\}$ is a Σ^0_α set. □

Theorem 3.6. *For any Σ^0_α set $\mathcal{A} \subseteq 2^{\mathbb{N}}$, any positive rational q and any computable Borel probability measure μ, there is:*

(1) *A $\Sigma^0_1(\emptyset^{<\alpha})$ set $\mathcal{U}$ with $\mathcal{A} \subseteq \mathcal{U}$ such that $\mu(\mathcal{U} - \mathcal{A}) \leqslant q$*
(2) *A $\Pi^0_1(\emptyset^{<\alpha})$ set $\mathcal{F}$ for some $\beta < \alpha$, with $\mathcal{F} \subseteq \mathcal{A}$ such that $\mu(\mathcal{A} - \mathcal{F}) \leqslant q$*

Moreover an index for $\mathcal{U}$ can be found uniformly in q and in an index for $\mathcal{A}$, and an index for $\mathcal{F}$ can be found uniformly in q, in an index for $\mathcal{A}$ and in $\emptyset^\alpha$.

Proof. The proof goes by induction on computable ordinals. For a Σ^0_1 set $\mathcal{A}$, the Σ^0_1 set $\mathcal{U}$ is trivially $\mathcal{A}$ itself for any q. The Π^0_1 set $\mathcal{F}$ is $\mathcal{U}[t]$ for t the smallest integer such that $\mu(\mathcal{U} - \mathcal{U}[t]) \leqslant q$. As $\mathcal{U} - \mathcal{U}[t]$ is a Σ^0_1 set, from Lemma 3.5 we have that $\mu(\mathcal{U} - \mathcal{U}[t]) \leqslant q$ is a Π^0_1 predicate, making t computable in $\emptyset^1$, uniformly in q and an index for $\mathcal{U}$. This makes $\mathcal{U}[t]$ a Π^0_1 set whose index can be uniformly obtained in an index for $\mathcal{A}$, in q and in $\emptyset^1$.

Suppose that the theorem is true below ordinal α and let us prove that it is true at ordinal α. Let $\mathcal{A} = \bigcup_n \mathcal{B}_n$ be a Σ^0_α set, with each $\mathcal{B}_n$ a $\Pi^0_{<\alpha}$ set. By induction hypothesis (2), for each $\mathcal{B}_n$ and each positive rational q, we can find a $\Sigma^0_1(\emptyset^{<\alpha})$ set $\mathcal{U}_n \supseteq \mathcal{B}_n$ uniformly in q, in n and in $\emptyset^{<\alpha}$ such that $\mu(\mathcal{U}_n - \mathcal{B}_n) \leqslant q$. Now by induction hypothesis (1), for each $\mathcal{B}_n$ and each positive rational q, we can find a $\Pi^0_1(\emptyset^{<\alpha})$ set $\mathcal{F}_n \subseteq \mathcal{B}_n$ uniformly in q, and in n, such that $\mu(\mathcal{B}_n - \mathcal{F}_n) \leqslant q$.

For any q, fix a computable sequence $\{q_n\}_{n<\omega}$ such that $\sum_n q_n \leqslant q$. The desired $\Sigma^0_1(\emptyset^{<\alpha})$ set $\mathcal{U}$ is then the union of the $\Sigma^0_1(\emptyset^{<\alpha})$ sets $\mathcal{U}_n \supseteq \mathcal{B}_n$ such that $\mu(\mathcal{U}_n - \mathcal{B}_n) \leqslant q_n$. As each open set $\mathcal{U}_n$ is obtained uniformly in an index for $\mathcal{B}_n$, in q_n and in $\emptyset^{<\alpha}$, their union is a $\Sigma^0_1(\emptyset^{<\alpha})$ set, uniformly in an index for $\mathcal{A}$ and in q.

Still using the computable sequence $\{q_n\}_{n<\omega}$ such that $\sum_n q_n \leqslant q$, the desired $\Pi^0_1(\emptyset^\beta)$ set $\mathcal{F}$ is equal to $\bigcup_{n<m} \mathcal{F}_n$ where m is the smallest integer such that $\mu(\mathcal{A} - \bigcup_{n\leqslant m} \mathcal{B}_n) \leqslant q_0$ and with $\mathcal{F}_n \subseteq \mathcal{B}_n$ and $\mu(\mathcal{B}_n - \mathcal{F}_n) \leqslant q_{n+1}$. As each closed set $\mathcal{F}_n$ is $\Pi^0_1(\emptyset^{<\alpha})$ and as there are only finitely many of them, then their union is still a $\Pi^0_1(\emptyset^{<\alpha})$ set. Besides $\mathcal{A} - \bigcup_{n\leqslant m} \mathcal{B}_n$ is a $\Sigma^0_1(\emptyset^\alpha)$ set uniformly in m and therefore, using Lemma 3.5, the integer m can be found uniformly in $\emptyset^\alpha$, in q and in an index for $\mathcal{A}$. We also have that $\mathcal{A} - \mathcal{F} \subseteq \bigcup_{n<m}(\mathcal{B}_n - \mathcal{F}_n) \cup (\mathcal{A} - \bigcup_{n\leqslant m} \mathcal{B}_n)$ and therefore $\mu(\mathcal{A} - \mathcal{F}) \leqslant \sum_{n<m} \mu(\mathcal{B}_n - \mathcal{F}_n) + \mu(\mathcal{A} - \bigcup_{n\leqslant m} \mathcal{B}_n) \leqslant q$. □

We now easily deduce the following:

Proposition 3.7 (Hjorth, Nies [17])**.** *If Z is Π^1_1-Martin-Löf random, then Z is Δ^1_1-random.*

Proof. Suppose Z is in a Δ^1_1 nullset $\mathcal{A}$. This nullset is Σ^0_α for some computable α. Now using Theorem 3.6, we can find uniformly in n a $\Sigma^0_1(\emptyset^{<\alpha})$ set of measure less than 2^{-n}, and containing $\mathcal{A}$. Also a $\Sigma^0_1(\emptyset^{<\alpha})$-open set is clearly a Π^1_1-open set and we can then build a Π^1_1-Martin-Löf test capturing Z. □

We shall see in Section 4.1 that Π^1_1-Martin-Löf randomness is strictly stronger than Δ^1_1-randomness.

3.3. **Higher weak-2 and difference randomness.** The higher analogue of weak-2-randomness has also been studied by Chong and Yu in [6]. This notion received quite many different names in the literature. Chong and Yu refereed to it as Strong-Π^1_1-Martin-Löf randomness, Monin [36, 35] refereed to it as weak-Π^1_1-randomness and Bienvenu, Greenberg and Monin [2] as higher weak-2-randomness. We stick here with this last name, which echoes to its well-know analogue in classical randomness.

Definition 3.8 (Nies [38] 9.2.17)**.** We say that Z is higher weakly-2-random if it belongs to no uniform intersection of Π^1_1-open sets $\bigcap_n \mathcal{U}_n$, with $\lambda(\bigcap_n \mathcal{U}_n) = 0$.

It is clear that the notion of higher weakly-2-randomness is stronger than the notion of Π^1_1-Martin-Löf randomness. We shall see later that it is strictly stronger. In fact we will even see another notion of randomness which is strictly between Π^1_1-Martin-Löf randomness and higher weak-2-randomness: Franklin and Ng defined in [11] a notion of test in classical randomness, which exactly captures the sequences which are either not Martin-Löf random, or Turing compute the halting problem. They called difference randomness this notion of randomness, which has been very useful to prove various theorems.

Something analogous can be done in higher randomness, to capture exactly the Π^1_1-Martin-Löf random sequences which higher Turing compute O.

Definition 3.9 (Yu [39])**.** A sequence X is not higher difference random if there is a Σ^1_1-closed set $\mathcal{F}$ and a uniform sequence of Π^1_1-open sets $\{\mathcal{U}_n\}_{n\in\mathbb{N}}$ such that $\lambda(\mathcal{U}_n \cap \mathcal{F}) \leqslant 2^{-n}$ and such that $X \in \bigcap_n(\mathcal{U}_n \cap \mathcal{F})$.

Yu [39] showed that a Π^1_1-Martin-Löf random sequence is not higher difference random iff it higher Turing computes O. We will see this in Section 6.1.

3.4. **Π^1_1-randomness.** So far, the full descriptive power of Π^1_1 or Σ^1_1 predicates has not been used. When Sacks introduced Δ^1_1-randomness, he also introduced a notion stronger than any presented so far : the tests are now the Π^1_1 nullsets. Note that a Π^1_1 set is not necessarily Borel. Lusin showed however that they remain all Lebesgue-measurable, that is, any Π^1_1 set is the union of a Borel set and of a set which is included in a Borel set of measure 0. It is shown using the fact that any Π^1_1 set $\mathcal{A}$ is a uniform union of Borel sets $\mathcal{A}_\alpha$ over $\alpha < \omega_1$ (formally for any Π^1_1 set, there exists $e \in \mathbb{N}$ such that $\mathcal{A} = \bigcup_{\alpha<\omega_1} \mathcal{A}_\alpha$ with $\mathcal{A}_\alpha = \{X \ : \ e \in O^X_{<\alpha}\}$).

Theorem 3.10 (Lusin)**.** *There is an ordinal γ and a Borel set $\mathcal{B}$ of measure 0 such that for any Π^1_1 set $\mathcal{A} = \bigcup_{\alpha<\omega_1} \mathcal{A}_\alpha$, the set $\mathcal{A} - \mathcal{A}_\gamma$ is contained in $\mathcal{B}$. In particular any Π^1_1 set is measurable.*

Sacks proved later that the ordinal γ of the previous theorem actually equals ω_1^{ck}, making the set $\{X \ : \ \omega_1^X > \omega_1^{ck}\}$ a set of measure 0:

Theorem 3.11 (Sacks [44])**.** *The set $\{X \ : \ \omega_1^X > \omega_1^{ck}\}$ has measure 0. This set is in fact a Borel set $\mathcal{B}$ of measure 0 such that for any Π^1_1 set $\mathcal{A} = \bigcup_{\alpha<\omega_1} \mathcal{A}_\alpha$, we have that $\mathcal{A} - \mathcal{A}_{\omega_1^{ck}}$ is contained in $\mathcal{B}$.*

Proof. Suppose $\omega_1^X > \omega_1^{ck}$. Then there must be an integer of O^X coding for ω_1^{ck}. In particular, there must be a functional $\Phi : 2^{\mathbb{N}} \times \omega \to \omega$, such that $\Phi(X)$ is total on ω and whose range is a set of codes for X-computable ordinals, unbounded below ω_1^{ck}. Given any functional Φ, let $\mathcal{P}_{n,\alpha} = \{X \mid \Phi(X, n) \in O^X_\alpha\}$. Note that $\mathcal{P}_{n,\alpha}$ is Δ^1_1 uniformly in n and α. If $\omega_1^X > \omega_1^{ck}$ is witnessed in the way stated above, via the functional Φ, we must have $X \in \bigcap_n \bigcup_{\alpha<\omega_1^{ck}} \mathcal{P}_{n,\alpha} - \bigcup_{\alpha<\omega_1^{ck}} \bigcap_n \mathcal{P}_{n,\alpha}$. Let us show $\lambda(\bigcap_n \bigcup_{\alpha<\omega_1^{ck}} \mathcal{P}_{n,\alpha} - \bigcup_{\alpha<\omega_1^{ck}} \bigcap_n \mathcal{P}_{n,\alpha}) = 0$.

Let $r = \lambda(\bigcap_n \bigcup_{\alpha<\omega_1^{ck}} \mathcal{P}_{n,\alpha})$. For any rational $q < r$, let $f_q : \omega \to O$ be the Π^1_1 function defined by:

$$f_q(n) = \min_{\alpha<\omega_1^{ck}} \text{ s.t. } \lambda\left(\bigcap_{k\leqslant n} \mathcal{P}_{k,\alpha}\right) > q$$

It is clear that f_q is total. Let $\alpha_q = \sup_n f_q(n)$. By admissibility, we have $\alpha_q < \omega_1^{ck}$. We have in particular $\lambda(\bigcap_n \mathcal{P}_{n,\alpha_q}) \geqslant q$. As we can do this for any rational $q < r$, it follows that we have $\lambda(\bigcup_{\alpha<\omega_1^{ck}} \bigcap_n \mathcal{P}_{n,\alpha_q}) = r$.

So for any functional Φ, the set of $X \in 2^{\mathbb{N}}$ for which Φ witnesses $\omega_1^X > \omega_1^{ck}$, is of measure 0. As there are only countably many functionals, the set of X such that $\omega_1^X > \omega_1^{ck}$ is a set of measure 0. □

The proof of the previous theorem details a Borel description of the set $\{X : \omega_1^X > \omega_1^{ck}\}$. Steel actually showed that this set is $\mathbf{\Sigma}^0_{\omega_1^{ck}+2}$ and not $\mathbf{\Pi}^0_{\omega_1^{ck}+2}$. A full proof can be found in [35]. We have an interesting corollary:

Theorem 3.12 (Sacks [44])**.** *If X is not Δ^1_1, then $\lambda(\{Y : Y \geqslant_h X\}) = 0$.*

Proof. We have $Y \geqslant_h X$ iff $O^X_\alpha \geqslant_T X$ for some $\alpha < \omega_1^X$. Suppose $\lambda(\{Y : Y \geqslant_h X\}) > 0$. As the set $\{X : \omega_1^X > \omega_1^{ck}\}$ has measure 0, we can suppose that there is some $\alpha < \omega_1^{ck}$ and a Turing functional Φ such that $\lambda(\{Y : \Phi(O^Y_\alpha) = X\}) > 0$. As the set $\{Y : \Phi(O^Y_\alpha) = X\}$ is Borel, by the Lebesgue density theorem, there is a string σ such that $\lambda(\{Y : \Phi(O^Y_\alpha) = X\} \mid \sigma) > 1/2$. To know the value of $X(n)$, we simply compute the values $\lambda(\{Y : \Phi(O^Y_\alpha, n) = 1\} \mid \sigma)$ and $\lambda(\{Y : \Phi(O^Y_\alpha, n) = 0\} \mid \sigma)$. Whichever measure is bigger than 1/2 gives us the correct value of $X(n)$, and thus X is Δ^1_1. □

Let us quickly argue that the set $\{X : \omega_1^X > \omega_1^{ck}\}$ is also Π^1_1. We have that $\omega_1^X > \omega_1^{ck}$ iff "$\exists e \in O^X \wedge \forall n\, \forall f$ f is not an order-isomorphism between the order coded by e and the one coded by n". This is a Π^1_1 statement.

The fact that every Π^1_1 set is measurable, even though it is not necessarily Borel, gives the possibility of another notion of higher randomness, which will appear to have many remarkable properties, and no counterpart in classical randomness:

Definition 3.13 (Sacks [44] IV.2.5)**.** We say that $Z \in 2^{\mathbb{N}}$ is Π^1_1-random if it is in no Π^1_1 nullset.

This last notion is very interesting for many reasons. One of them is that no X such that $\omega_1^X > \omega_1^{ck}$ is Π^1_1-random, and we shall see now that this is the best we can do, as any randomness notion weaker than Π^1_1-randomness contains elements that make ω_1^{ck} a computable ordinal. This is achieved through the following simple and yet beautiful theorem of Chong, Nies and Yu (see [5]):

Theorem 3.14 (Chong, Nies, Yu [5])**.** *A sequence Z is Π^1_1-random iff it is Δ^1_1-random and $\omega_1^Z = \omega_1^{ck}$.*

Proof. Suppose Z is Δ^1_1-random. If $\omega_1^Z > \omega_1^{ck}$ then by Theorem 3.11, Z is not Π^1_1-random.

Suppose now that Z is not Π^1_1-random and then captured by a Π^1_1 set $\mathcal{A} = \bigcup_{\alpha<\omega_1} \mathcal{A}_\alpha$ of measure 0. If there is a computable α such that $Z \in \mathcal{A}_\alpha$ then Z is not Δ^1_1-random as $\mathcal{A}_\alpha$ is a Δ^1_1 set of measure 0. Otherwise $Z \in \mathcal{A} - \bigcup_{\alpha<\omega_1^{ck}} \mathcal{A}_\alpha$ and then $\omega_1^Z > \omega_1^{ck}$. □

Another important property of Π^1_1-randomness is certainly the existence of a universal Π^1_1 nullset, in the sense that it contains all the others. Kechris was the first to prove this, in [20], and he actually proved a more general result, implying for example also the existence of a largest Π^1_1 thin set (a largest Π^1_1 set which contains no perfect subset). We will discuss the relation with this largest Π^1_1 thin set and higher randomness in Section 7.4. Later, Hjorth and Nies gave in [17] an explicit construction of this Π^1_1 nullset.

Theorem 3.15 (Kechris [20] Hjorth, Nies [17])**.** *There is a largest Π^1_1 nullset.*

Proof. Let $\{P_e\}_{e\in\omega}$ be an enumeration of the Π^1_1 sets, with $P_e = \bigcup_{\alpha<\omega_1} P_{e,\alpha}$. Recall from above that each set $P_e - \bigcup_{\alpha<\omega_1^{ck}} P_{e,\alpha}$ is always null and contained in the nullset $\{X \mid \omega_1^X > \omega_1^{ck}\}$. Let us argue that uniformly in e, one can transform the set $\bigcup_{\alpha<\omega_1^{ck}} \mathcal{P}_{e,\alpha}$ into a set $\bigcup_{\alpha<\omega_1^{ck}} \mathcal{Q}_{e,\alpha}$ (where each $\mathcal{Q}_{e,\alpha}$ is Δ^1_1 uniformly in e and a code of $O_{=\alpha}$) such that $\lambda(\bigcup_{\alpha<\omega_1^{ck}} \mathcal{Q}_{e,\alpha}) = 0$, and such that if $\lambda(\bigcup_{\alpha<\omega_1^{ck}} \mathcal{P}_{e,\alpha}) = 0$ then $\bigcup_{\alpha<\omega_1^{ck}} \mathcal{Q}_{e,\alpha} = \bigcup_{\alpha<\omega_1^{ck}} \mathcal{P}_{e,\alpha}$.

To do so we simply set $\mathcal{Q}_{e,\alpha} = \mathcal{P}_{e,\alpha}$ if $\lambda(\mathcal{P}_{e,\alpha}) = 0$ (recall that the measure of a Δ^1_1 set is uniformly Δ^1_1) and $\mathcal{Q}_{e,\alpha} = \varnothing$ otherwise. Then we define $\mathcal{Q}$ to be $\bigcup_e \bigcup_{\alpha<\omega_1^{ck}} \mathcal{Q}_{e,\alpha}$ together with the set $\{X \mid \omega_1^X > \omega_1^{ck}\}$. The set $\mathcal{Q}$ is clearly Π^1_1, and by construction it is a nullset containing every Π^1_1 nullset. □

Chong and Yu proved in [6] that higher weak-2-randomness is strictly stronger than Π^1_1-Martin-Löf-randomness (see Section 6.1). Bienvenu, Greenberg and Monin later showed that Π^1_1-randomness is strictly stronger than higher weak-2-randomness (see Section 6.2).

4. Δ^1_1-randomness

4.1. **Separation with Π^1_1-Martin-Löf randomness.** We shall see now that Π^1_1-Martin-Löf randomness is strictly stronger than Δ^1_1-randomness. This was proved by Chong, Nies and Yu in [5] using the notion of higher Kolmogorov complexity that we will introduce later. The proof they gave can be seen as a higher analogue of the separation between computable randomness and Martin-Löf randomness. We give here a similar proof, without using higher Kolmogorov complexity, but rather a combination between higher priority method and forcing with closed sets of positive measure. A similar technique will be reused for Theorem 6.7.

Theorem 4.1 (Chong, Nies, Yu [5])**.** *There is a sequence X which is Δ^1_1-random and not Π^1_1-Martin-Löf random.*

Proof. Let $\{\mathcal{A}_s\}_{s<\omega_1^{ck}}$ be an enumeration of the Δ^1_1 sets of measure 1. To get this enumeration, recall that the Δ^1_1 sets are the Σ^0_α sets, and that the measure of a Σ^0_α set is Δ^1_1, uniformly in α. Recall that $p : \omega_1^{ck} \to \omega$ is the projectum function, let $O^1_{\leqslant s} = \{p(t) \ : \ t \leqslant s\}$, and for $m \in O^1_{\leqslant s}$ let $O^1_{\leqslant s}\upharpoonright_m = \{n \in O^1_{\leqslant s} \ : \ n < m\}$.

The construction:

We can suppose without loss of generality that $\mathcal{A}_0 = 2^{\mathbb{N}}$. At stage 0 we define for each n the set $\mathcal{F}^n_0$ to be $2^{\mathbb{N}}$ and the string σ^n_0 to be the string consisting of $2n$ 0's.

Suppose that at every stage $t < s$ we have defined for each $n \in \mathbb{N}$ a Δ^1_1 closed set $\mathcal{F}^n_t$ and a string σ^n_t such that $\sigma^n_t \prec \sigma^{n+1}_t$ and with $|\sigma^n_t| = 2n$. Suppose also that for each m we have $\lambda(\bigcap_{n\leqslant m} \mathcal{F}^n_t \cap [\sigma^m_t]) > 0$ and that if $m \in O^1_{\leqslant t}$ we have $\mathcal{F}^m_t \subseteq \mathcal{A}_{p^{-1}(m)}$.

Suppose first that s is successor and let us define $\mathcal{F}^m_s$ and σ^m_s for each $m \in \mathbb{N}$. For each $m < p(s)$ we define $\sigma^m_s = \sigma^m_{s-1}$ and $\mathcal{F}^m_s = \mathcal{F}^m_{s-1}$. For each $m \geqslant p(s)$ in increasing order, if $m \in O^1_{\leqslant s}$, let $t = p^{-1}(m)$ and let us compute an increasing union of Δ^1_1 closed sets $\bigcup_n \mathcal{F}_n \subseteq \mathcal{A}_t$ with $\lambda(\mathcal{A}_t - \bigcup_n \mathcal{F}_n) = 0$. Let $\mathcal{F}^m_s$ be the first closed set of the union $\bigcup_n \mathcal{F}_n$ such that $\lambda(\bigcap_{n<m} \mathcal{F}^n_s \cap \mathcal{F}^m_s \cap [\sigma^{m-1}_s]) > 0$. If $m \notin O^1_{\leqslant s}$, let $\mathcal{F}^m_s = 2^{\mathbb{N}}$. Then let σ^m_s be the first string of length $2m$ which extends σ^{m-1}_s, such that $\lambda(\bigcap_{n\leqslant m} \mathcal{F}^n_s \cap [\sigma^m_s]) > 0$.

Finally, for a stage s limit we define for each n the string σ^n_s to be the limit of the sequence $\{\sigma^n_s\}_{t<s}$ and the closed set $\mathcal{F}^n_s$ to be the limit of the sequence $\{\mathcal{F}^n_s\}_{t<s}$. We shall argue that later that such a limit always exists.

The verification:

For every m there is a stage s such that $\{O^1_{\leqslant t}\upharpoonright_m\}_{s\leqslant t<\omega_1^{ck}}$ is stable. Furthermore, for each m, the sequence $\{O^1_{\leqslant t}\upharpoonright_m\}_{t<\omega_1^{ck}}$ can change at most m times, because at most m values can be enumerated in $O^1\upharpoonright_m$. It follows that at every limit stage s and for every m, the sequences $\{\sigma^m_s\}_{t<s}$ and $\{\mathcal{F}^m_s\}_{t<s}$ also can change at most m times, and then converge. Let $\mathcal{F}^m$ the convergence value of $\{\mathcal{F}^m_s\}_{t<s}$.

Also by design for every $s \leqslant \omega_1^{ck}$ such that $O^1_{\leqslant s}$ is infinite, the unique limit point X_s of $\{[\sigma^n_s]\}_{n\in O^1_{\leqslant s}}$ belongs to $\bigcap_n \mathcal{F}^n_t \subseteq \bigcap_{t\leqslant s} \mathcal{A}_t$. Let X be the limit of the sequence $\{X_s\}_{s<\omega_1^{ck}}$.

Let us argue that X is Δ^1_1-random. Let s_k be the smallest stage such that $\{\mathcal{F}^m_s\}_{s_k\leqslant t<\omega_1^{ck}}$ is stable for every $m \leqslant k$. It is clear that the sequences $X \cup \{X_{s_k}\}_{k\in\mathbb{N}}$ is a closed set. Also for every k we have that $\bigcap_{n\leqslant k} \mathcal{F}_n \cap (X \cup \{X_{s_k}\}_{k\in\mathbb{N}})$ is not empty because $X_{s_k} \in \bigcap_{n\leqslant k} \mathcal{F}_n$. It follows that $\bigcap_n \mathcal{F}_n \cap (X \cup \{X_{s_k}\}_{k\in\mathbb{N}})$ is not empty and then that $\bigcap_n \mathcal{F}_n$ contains X, the only non Δ^1_1 point of $\{X_s\}_{s<\omega_1^{ck}} \cup X$. Therefore $X \in \bigcap_{t<\omega_1^{ck}} \mathcal{A}_t$ and X is Δ^1_1-random.

Let us argue that X is not Π^1_1-Martin-Löf random. We argued already that $\{\sigma^m_t\}_{t<\omega_1^{ck}}$ can change at most m times. Then we can put each string σ^m_s of length $2m$, into the m-th component of a Π^1_1-Martin-Löf test which has measure smaller than $m \times 2^{-2m} \leqslant 2^{-m}$. ☐

4.2. **Lowness for Δ^1_1-randomness.** Chong, Nies and Yu studied in [5] lowness for Δ^1_1-randomness. They showed that it coincides with the notion of Δ^1_1-traceability, that they also defined:

Definition 4.2 (Chong, Nies and Yu [5])**.** A sequence $X \in 2^{\mathbb{N}}$ is Δ^1_1-traceable if there is a Δ^1_1 function g such that for any function $f \leqslant_h X$, there is a Δ^1_1 trace $\{T_n\}_{n\in\mathbb{N}}$ such that:

(1) $\forall n\ f(n) \in T_n$
(2) $\forall n\ |T_n| \leqslant g(n)$

Traceability notions have also been studied in set theory. In these field, traces are called slalom. The notion of Δ^1_1-traceable can also be seen the higher analogue of the notion of computably traceable. Also Kjos-Hanssen, Nies and Stephan showed [23] that a sequence is computably traceable iff it is low for Schnorr randomness. The proof that X is low for Δ^1_1-randomness iff X is Δ^1_1-traceable, works analogously. We first start with an easy lemma, whose analogue for computable traceability is well known

Lemma 4.3. Let X be Δ^1_1-traceable with bound g. Then for any Δ^1_1 order function g', the sequence X is Δ^1_1 traceable with bound g'.

Proof. Let $f \leqslant_h X$. let $h : \mathbb{N} \to \mathbb{N}$ be the Δ^1_1 function such that $h(0) = 0$ and $h(n)$ is the smallest greater than $h(n-1)$ for which $\forall n\ \forall k \geqslant h(n)\ g'(k) > g(n)$ (which is possible as g' is an order function).

Let f' be such that $f'(n)$ is an encoding the the values of f from $f(h(n))$ to $f(h(n+1)-1)$. Note that $f' \leqslant_h X$. Also there is a Δ^1_1 trace $\{T'_n\}_{n\in\mathbb{N}}$ of f' with bound g. But this trace can be easily transformed into a Δ^1_1 trace $\{T_n\}_{n\in\mathbb{N}}$ of f with bound g': We set T_k for $0 \leqslant k \leqslant h(1)-1$ so that T_k only contains the value of $f(k)$. Then inductively for each n we set T_k for $h(n) \leqslant k \leqslant h(n+1)-1$ so that each T_k contains the decoding of the $k-h(n)$-th value encoded by each element of T'_n. As we have $g'(k) > g(n)$ for each $k \geqslant h(n)$ and as there are at most $g(n)$ elements in T'_n, then there are at most $g'(k)$ elements in each T_k for $h(n) \leqslant k \leqslant h(n+1)-1$. ☐

Theorem 4.4 (Chong, Nies and Yu [5]). *If $X \in 2^{\mathbb{N}}$ is Δ^1_1-traceable then X is low for Δ^1_1-randomness.*

Proof. Let $\mathcal{A}$ be a $\Delta^1_1(X)$ nullset. From Theorem 3.6 one can find a uniform intersection of $\Delta^1_1(X)$ open sets $\bigcap_m \mathcal{U}_m$ such that:

(1) $\mathcal{A} \subseteq \bigcap_m \mathcal{U}_m$
(2) $\lambda(\mathcal{U}_m) = 2^{-m}$

Note that Theorem 3.6 only gives us $\lambda(\mathcal{U}_m) < 2^{-m}$. One easily complete the set $\mathcal{U}_m$ by adding in a Δ^1_1 way countably many string to that the measure equals 2^{-m}.

For each open set $\mathcal{U}_m$ there is a $\Delta^1_1(X)$ function $f_m : \mathbb{N} \to 2^{<\mathbb{N}}$ such that $\mathcal{U}_m = \bigcup_n [f_m(n)]$. Let us define a $\Delta^1_1(X)$ function $h_m : \mathbb{N} \to \mathbb{N}$ such that:

$$\begin{array}{llll} A^m_1 = \{f_m(k)\ :\ 0 \leqslant k < h(1)\} & \text{with} & r_1 & = \lambda([\mathcal{A}_1]^{\prec}) \geqslant 1/2 \times 2^{-m} \\ A^m_{n+1} = \{f_m(k)\ :\ h(n) \leqslant k < h(n+1)\} & \text{with} & r_{n+1} & = \lambda([A_{n+1}]^{\prec}) \geqslant 1/2 \times (2^{-m} - \sum_{i\leqslant n} r_i) \end{array}$$

Note in particular that $\lambda(A^m_n) \leqslant 2^{-n+1}2^{-m}$ for $n \geqslant 1$. Now let f be defined so that $f(\langle n,m\rangle) = A^m_n$. Let g be a computable order function such that for every m we have $\sum_n g(\langle n,m\rangle)2^{-n+1}2^{-m} \leqslant 2^{-m+2}$. Note that this is possible as $\langle n,m\rangle$ is polynomial in n and m. As X is Δ^1_1-traceable there is a trace $\{T_n\}_{n\in\mathbb{N}}$ of f with bound g.

To compute each Δ^1_1 open set V_m we proceed as follow : For each $T_{\langle n,m\rangle}$ for some n, we consider all its elements of measure smaller than $2^{-n+1}2^{-m}$ and we put there union in V_m. As we have $\sum_n g(\langle n,m\rangle)2^{-n+1}2^{-m} \leqslant 2^{-m+2}$, then the measure of V_m is smaller than 2^{-m_k+2}. As $\lambda(A^m_n) \leqslant 2^{-n+1}2^{-m}$ then $[A^m_n]^{\prec} \subseteq V_m$. It follows that $\bigcap_m V_m$ is a Δ^1_1 set of measure 0 which contains $\mathcal{A}$. Then X is low for Δ^1_1-randomness. ☐

Theorem 4.5 (Chong, Nies and Yu [5]). *If $X \in 2^{\mathbb{N}}$ is low for Δ^1_1-randomness, then X is Δ^1_1-traceable.*

Proof. Let $f \leqslant_h X$. For technical reasons, we suppose that for every n we have that n divides $f(n)$. Note that this hypothesis is harmless, as if this is not the case, we can instead deal with the function $n \mapsto n \times (f(n)+1)$. Also note that any trace for such a function can also be transformed into a trace for f.

Let $B_{n,k} = \{\sigma 0^n\ :\ |\sigma| = k\}$. Note that for any n,k we have $\lambda([B_{n,k}]^{\prec}) = 2^{-n}$. We define the $\Delta^1_1(X)$ open set $V_n = \bigcup_{m\geqslant n} B_{m,f(m)}$. Note that we have $\lambda(V_n) \leqslant \sum_{m\geqslant n} \lambda([B_{n,g(n)}]^{\prec}) \leqslant 2^{-n+1}$. It follows that $\bigcap_n [V_n]^{\prec}$ is a Δ^1_1 set of measure 0. By hypothesis there is a Δ^1_1 nullset $\mathcal{A}$ which contains $\bigcap_n [V_n]^{\prec}$. Also by Theorem 3.6 there is a Δ^1_1 open sets $\mathcal{U}$ such that $\bigcap_n [V_n]^{\prec} \subseteq \mathcal{U}$ and with $\lambda(\mathcal{U}) = 1/4$.

Claim : There exists a string σ and an integer n such that $\lambda(\mathcal{U} \mid \sigma) < 1/4$ and such that $\lambda([\mathcal{V}_n]^{\prec} - \mathcal{U} \mid \sigma) = 0$.

We suppose otherwise. Then we build a sequence $\sigma_0 \prec \sigma_1 \prec$ whose limit point Z is in $\bigcap_n [V_n]^{\prec}$ but not in $\mathcal{U}$. Let σ_0 be the empty string. Suppose σ_n has been defined such that $\lambda(\mathcal{U} \mid \sigma_n) < 1/4$. As the claim is suppose false, we then have $\lambda([V_n]^{\prec} - \mathcal{U} \mid \sigma_n) > 0$. So we can choose $\tau \in \mathcal{V}_n$ with $\tau \geqslant \sigma_n$ such that $\lambda([\tau] - \mathcal{U} \mid \sigma_n) > 0$. By the Lebesgue density theorem there exists an extension σ_{n+1} of τ such that $\lambda([\tau] - \mathcal{U} \mid \sigma_{n+1}) > 3/4$ and then such that $\lambda(\mathcal{U} \mid \sigma_{n+1}) < 1/4$. The limit point Z of the sequence $\{\sigma_n\}_{n\in\mathbb{N}}$ has the property that none of its prefix σ_n is such that $[\sigma_n] \subseteq \mathcal{U}$ (because $\lambda(\mathcal{U} \mid \sigma_n) < 1/4$). But then as $\mathcal{U}$ is open, Z is not in $\mathcal{U}$ and yet $Z \in \bigcap_n [V_n]^{\prec}$ which is a contradiction.

So we pick a prefix σ and an integer a such that $\lambda(\mathcal{U} \mid \sigma) < 1/4$ and such that $\lambda([V_a]^{\prec} - \mathcal{U} \mid \sigma) = 0$. The trace T_n is defined as follow:

$$\begin{array}{lcll} T_n & = & \{k \,:\, \lambda([B_{n,k}]^{\prec} - \mathcal{U} \mid \sigma) = 0 \text{ and } n \text{ divides } k\} & \text{if} \quad n > a \\ T_n & = & \{f(n)\} & \text{if} \quad n \leqslant a \end{array}$$

It is clear that $\{T_n\}_{n\in\mathbb{N}}$ traces f. We shall now prove that $|T_n| < 2^n$ for every n. It is here that we use the fact that n divides $g(n)$. Recall that we have $\lambda([B_{n,k}]^{\prec}) = 2^{-n}$. Therefore if $\mathcal{U}$ covers $[B_{n,k}]^{\prec}$ it must measure at least 2^{-n}. Now given a finite set E of multiple of n, the events "covering $B_{n,k}$" are independent events for different k. In particular we have:

$$\lambda\left(\bigcap_{k\in E}(2^{\mathbb{N}} - [B_{n,k}]^{\prec})\right) = \Pi_{k\in E}(1 - \lambda([B_{n,k}]^{\prec}))$$

As $\lambda(B_{n,k}) = 2^{-n}$ we then have that $\lambda(\bigcup_{k\in E}[B_{n,k}]^{\prec}) = 1 - (1-2^{-n})^{|E|}$. For $|E|$ large enough we then have $\lambda(\bigcup_{k\in E}[B_{n,k}]^{\prec}) > 1/4$. In particular we need $|E|$ to be large enough so that $(1-2^{-n})^{|E|} < 3/4$ iff $((2^n-1)/2^n)^{|E|} < 3/4$ iff $(2^n/(2^n-1))^{|E|} \geqslant 4/3$. Now for $|E| = 2^n$ we have $(2^n)^{2^n} \geqslant 2(2^n-1)^{2^n}$ which implies that $(2^n/(2^n-1))^{2^n} \geqslant 2 > 4/3$. It follows that we must have $|T_n| < 2^n$ as otherwise we have $\lambda(\mathcal{U}) > 1/4$. □

5. Π^1_1-Martin-Löf randomness

5.1. The higher Kučera-Gács theorem.

Hjorth and Nies showed that for every $X \in 2^{\mathbb{N}}$, there is a Π^1_1-Martin-Löf random $Z \geqslant_h X$. They actually even show something stronger in that the reduction can be made continuous in the sense of Definition 2.11. The proof is the same as the one from Kučera in the lower settings. We first need the following combinatorial lemma:

Lemma 5.1. let σ be a string and $\mathcal{F}$ a closed set so that $\lambda(\mathcal{F} \mid \sigma) \geqslant 2^{-n}$. Then there are at least two extensions τ_1, τ_2 of σ of length $|\sigma| + n + 1$ so that for $i \in \{1, 2\}$ we have $\lambda(\mathcal{F} \mid \tau_i) \geqslant 2^{-n-1}$.

Proof. Let C be the set of strings of length $|\sigma| + n + 1$ that extend σ. We have that $\lambda(\mathcal{F} \cap [\sigma]) = \sum_{\tau\in C} \lambda(\mathcal{F} \cap [\tau])$. Suppose that for strictly less than two extensions of length $|\sigma| + n + 1$ we have $\lambda(\mathcal{F} \cap [\tau_i]) \geqslant 2^{-|\tau_i|-n-1}$. Then we have:

$$\begin{array}{rcl} \sum_{\tau\in C} \lambda(\mathcal{F} \cap [\tau]) & \leqslant & 2^{-|\sigma|-n-1} + (2^{n+1} - 1)2^{-|\tau_i|-n-1} \\ & \leqslant & 2^{-|\sigma|-n-1} + 2^{n+1}2^{-|\sigma|-2n-2} - 2^{-|\sigma|-2n-2} \\ & \leqslant & 2^{-|\sigma|-n-1} + 2^{-|\sigma|-n-1} - 2^{-|\sigma|-2(n+1)} \\ & < & 2^{-|\sigma|-n} \end{array}$$

which contradicts $\lambda(\mathcal{F} \mid \sigma) \geqslant 2^{-n}$. □

We now prove the higher analogue of Kučera-Gács theorem:

Theorem 5.2 (Hjorth, Nies [17])**.** *For any sequence X and any Σ^1_1 closed set $\mathcal{F} \subseteq 2^{\mathbb{N}}$ of positive measure, there exists $Z \in \mathcal{F}$ such that Z higher Turing computes X.*

Proof. Consider a Σ^1_1 closed set $\mathcal{F} \subseteq 2^{\mathbb{N}}$ with $\lambda(\mathcal{F}) \geqslant 2^{-c}$ and a sequence X. According to what Lemma 5.1 tells us, we define some length $m_0 = 0$ and inductively $m_{n+1} = m_n + c + n + 1$.

We define σ_0 to be the empty word. Assuming σ_n of length m_n is defined with $\lambda(\mathcal{F} \mid \sigma_n) \geqslant 2^{-c-n}$, we will define an extension σ_{n+1} of σ_n with the same property. From Lemma 5.1 there are at least two extensions τ of σ_n of length $m_n + c + n + 1 = m_{n+1}$ such that $\lambda(\mathcal{F} \mid \tau) \geqslant 2^{-c-(n+1)}$. Also if $X(n) = 0$ let σ_{n+1} be the leftmost of those extensions and if $X(n) = 1$ let σ_{n+1} be the rightmost of those extensions. The unique limit point Z of $\{[\sigma_n]\}_{n\in\mathbb{N}}$ is our candidate. We shall now show how we use it to compute X, by describing the reduction $\Phi \subseteq 2^{<\mathbb{N}} \times 2^{<\mathbb{N}}$.

At stage 0, we map the empty word to the empty word in Φ. Then at successor stage s, and substage $n+1$, for each string σ of length m_n which is mapped to τ in Φ_{s-1}, if there are distinct leftmost and a rightmost extensions σ_1, σ_2 of σ of length m_{n+1} such that $\lambda(\mathcal{F} \mid \sigma_i)[s] \geqslant 2^{-c-(n+1)}$ for $i \in \{0,1\}$, we map the leftmost one to $\tau 0$ in Φ at stage s; then we map the rightmost one to $\tau 1$ in Φ at stage s. At limit stage s we let Φ_s to be the union of Φ_t for $t < s$.

By design, the functional Φ is consistent everywhere because for any two strings $\sigma_2 \succ \sigma_1$ which are mapped to something in Φ, the string σ_2 is always mapped to an extension of what the string σ_1 is mapped to. We also clearly have $\Phi(Z) = X$, because for any prefix σ_1 of Z of length m_n which is mapped to $X\upharpoonright_n$, there is always a stage at which the prefix σ_2 of Z of length m_{n+1} will be witnessed to be either the leftmost or the rightmost path of $\mathcal{F}$ that extends σ_1 and such that $\lambda(\mathcal{F} \mid \sigma_2)[s] \geqslant 2^{-c-(n+1)}$, in which case it will be mapped to $X\upharpoonright_{n+1}$. $\square$

5.2. **Higher Kolmogorov complexity.** In this section, we introduce a higher version of the notion of prefix-free Kolmogorov complexity, a fundamental concept of classical randomness. For a very complete survey on the subject of lower Kolmorogov complexity, the reader can refer to [10] [38] or [30].

While defining the notion of Π^1_1-Martin-Löf randomness in [17], Hjorth and Nies also defined the notion of Π^1_1-Kolmorogov complexity, in order to study higher analogies of theorems occurring in classical randomness.

Definition 5.3 (Hjorth, Nies [17])**.** A Π^1_1-machine M is a Π^1_1 partial function $M : 2^{<\mathbb{N}} \to 2^{<\mathbb{N}}$. A Π^1_1-prefix-free machine M is a Π^1_1 partial function $M : 2^{<\mathbb{N}} \to 2^{<\mathbb{N}}$ whose domain of definition is a prefix-free set of strings. We denote by $\mathrm{K}_M(\sigma)$ the Π^1_1-Kolmorogov complexity of a string σ with respect to the Π^1_1-machine M, defined to be the length of the smallest string τ such that $M(\tau) = \sigma$, if such a string exists, and by convention, ∞ otherwise.

The proof that there is a universal computable prefix-free machine works similarly with Π^1_1-prefix-free machine:

Theorem 5.4 (Hjorth, Nies [17])**.** *There is a universal Π^1_1-prefix-free machine U, that is, for each Π^1_1-prefix-free machine M, there exists a constant c_M such that $\mathrm{K}_U(\sigma) \leqslant \mathrm{K}_M(\sigma) + c_m$ for any string σ.*

Proof. We first have to make sure that we can enumerate the Π^1_1-prefix-free machines: we have a total computable function such that for any e, the integer $f(e)$ is always an index for a Π^1_1-prefix-free machine, and if e is already an index for a Π^1_1-prefix-free machine, then $f(e)$ is an index for the same machine.

We see the machine M_e as an enumeration of pairs (σ, τ) (if $M(\sigma) = \tau$) along the computable ordinal times of computation. Given the machine M_e, suppose that (σ, τ) is enumerated in M_e at stage s. If $M_{f(e)}[< s]$ contains (σ', τ') such that σ' is compatible with σ, then we enumerate nothing in $M_{f(e)}$ at stage s. Otherwise we enumerate (σ, τ) in $M_{f(e)}$ at stage s.

Then we simply define U to be the machine which enumerates $(0^e \frown 1 \frown \sigma, \tau)$ for each e, σ and τ such that (σ, τ) is enumerated in $M_{f(e)}$. For each machine M of index $f(e)$, the constant c_M is given by $e+1$. $\square$

Definition 5.5. For a string σ, we define $\mathrm{K}(\sigma)$ to be $\mathrm{K}_U(\sigma)$ for a universal Π^1_1-prefix-free machine U, fixed in advance.

Hjorth and Nies [17] gave a general technique, used to build Π^1_1-prefix-free machines, that is, a higher version of the well-known KC theorem. For this purpose we need the following definitions.

Definition 5.6. Given a set $A \subseteq \mathbb{N} \times 2^{<\mathbb{N}}$, the weight of A, denoted by $\mathrm{wg}(A)$, refers to the quantity $\sum_{(l,\sigma)\in A} 2^{-l}$ if this quantity is finite, and refers to ∞ otherwise. A set $A \subseteq \mathbb{N} \times 2^{<\mathbb{N}}$ such that $\mathrm{wg}(A) \leqslant 1$ is called a bounded request set.

In classical randomness, given a computably enumerable bounded request set A, we can effectively build a prefix-free machine M such that as long as $(l, \sigma) \in A$, then also $M(\tau) = \sigma$ for some string τ of length l. Here is a higher version of the KC theorem:

Theorem 5.7 (Hjorth, Nies [17])**.** *For any Π^1_1-bounded request set A, there is a Π^1_1-prefix-free machine M such that for any string σ, if $(l, \sigma) \in A$, then for a string τ of length l we have $M(\tau) = \sigma$.*

Proof. The prefix-free machine M can be found uniformly in A. However, handling the case where A is a finite set such that $\text{wg}(A) = 1$ makes the proof slightly more complicated. To keep things as simple as possible, we assume $\text{wg}(A) < 1$ (see below how this hypothesis is used). Except for the sake of uniformity (which again can be achieved with a bit more work), such an assumption is harmless, because if $\text{wg}(A) = 1$, by admissibility, there exists a computable stage s at which $\text{wg}(A_s) = 1$ already, and we can then directly define a Π^1_1-prefix-free machine M that matches the conditions of the theorem with respect to the Δ^1_1 bounded request set A_s.

At each stage s, for each length $l \geqslant 1$ we define a string σ^l_s either of length l or equal to the empty word, and a sequence $r_s \in 2^{\mathbb{N}}$. The strings σ^l_s which are different from the empty word, correspond to the strings available for a mapping at stage $s+1$. The role of r_s is double. First, the real number represented by r_s in a binary form, will be equal to the weight of A_s, which is also the measure of the set of strings that is mapped to something in M_s. Then, if the $(n-1)$-th bit of r_s is 0 (starting at position 0), it will also mean that the string σ^n_s is different from the empty word and available for a future mapping. We need to ensure at each stage s that:

(1) The set of strings currently mapped in M_s, together with each σ^l_s different from the empty word, forms a prefix free set of strings.
(2) r_s is a binary representation of the weight of A_s, which is also the measure of the set of strings mapped to something in M_s.
(3) If $r_s(n-1) = 0$, the string σ^n_s is a string of length n. Otherwise it is the empty word.

At stage 0, we define $\sigma^l_0 = 0^{l-1}\hat{\ }1$ and r_0 to be only 0's. We have that (1), (2) and (3) are verified at stage 0.

At successor stage s suppose (l, τ) enters A_s. If $r_{s-1}(l-1) = 0$ we put (σ^l_{s-1}, τ) into M_s, we set σ^l_s to the empty word and $r_s(l-1)$ to 1. For $i \neq l$ and $i \geqslant 1$ we set $r_s(i-1) = r_{s-1}(i-1)$ and $\sigma^i_s = \sigma^i_{s-1}$. We can easily verify by induction that (1), (2) and (3) are true at stage s. Otherwise, if $r_{s-1}(l-1) = 1$, let n be the largest integer bigger than 0 and smaller than l such that $r_{s-1}(n-1) = 0$. We should argue that such an integer always exists. Suppose otherwise, then either $r_{s-1} = 1000\ldots$, $l = 1$ and $\text{wg}(A_{s-1}) + 2^{-l} = 1$, which is not possible by our special assumption, or $\text{wg}(A_{s-1}) + 2^{-l} > 1$, which is not possible because A is a bounded request set. Thus such an integer n exists. We then set σ^n_s to be the empty string and $r_s(n-1) = 1$. Then for every $n < i \leqslant l$, we set σ^i_s to $\sigma^n_{s-1}\hat{\ }0^{i-n-1}\hat{\ }1$ and $r_s(i-1) = 0$. Then we map $\sigma^n_{s-1}\hat{\ }0^{l-n-1}\hat{\ }0$ to τ in M_s. For $1 \leqslant i < n$ and $i > l$ we set $r_s(i-1) = r_{s-1}(i-1)$ and $\sigma^i_s = \sigma^i_{s-1}$. We can easily verify by induction that (1), (2) and (3) are true at stage s.

At limit stage s we set r_s to the pointwise limit of $\{r_t\}_{t<s}$. Then we set each σ^n_s to the convergence value of the sequence $\{\sigma^n_s\}_{t<s}$. We shall argue that those convergence values always exist. When for some n and some stage s we have $r_s\upharpoonright_n \neq r_{s+1}\upharpoonright_n$, then $r_{s+1}\upharpoonright_n$ is bigger than $r_s\upharpoonright_n$ in the lexicographic order, but as there are at most 2^n strings of length n, the sequence $\{r_s\upharpoonright_n\}_{s<\omega_1^{ck}}$ can change at most 2^n time. Then for any s, a convergence value for $\{r_t\}_{t<s}$ always exists. Also when for some n and some s we have $\sigma^n_{s+1} \neq \sigma^n_s$, then also $r_{s+1}\upharpoonright_n \neq r_s\upharpoonright_n$. But as the sequence $\{r_s\upharpoonright_n\}_{s<\omega_1^{ck}}$ can change at most 2^n times, then also the sequence $\{\sigma^n_s\}_{s<\omega_1^{ck}}$ can change at most 2^n times. We can easily verify by induction that (1), (2) and (3) are true at stage s.

Because (1) is true at every stage s, we then have that M is a Π^1_1-prefix-free machine, also by construction we clearly have that if $(l, \sigma) \in A$, then $M(\tau) = \sigma$ for a string τ of length l. □

For a given Π^1_1 prefix-free machine M, we can consider the probability that M outputs a given string σ. One can imagine the following process : We flip a fair coin to get a bit, either 0 or 1, and we repeat the process endlessly. So we get bigger and bigger strings $\sigma_1 \prec \sigma_2 \prec \sigma_3 \prec \ldots$. In the meantime we test each of our strings σ_i available so far, as an input for our machine M. If at some point $M(\sigma_i)$ halts for one i (and it can be at most one i), then we stop the process.

It is clear that following the previous protocol, the probability that we output a given string τ is given by $\sum\{2^{-|\sigma|} \ :\ M(\sigma) = \tau\}$. Note that this all make sense, thanks to the prefix-free requirement we have for our machine.

Definition 5.8. For a Π^1_1 prefix-free machine M, we denote by $P_M(\sigma)$ the probability that M outputs σ, that is, $\sum\{2^{-|\tau|} \ :\ M(\tau) = \sigma\}$.

We now have the following higher analogue of the coding theorem, which is useful for the study of lowness for Π^1_1-Martin-Löf randomness.

Theorem 5.9 (Hjorth, Nies [17])**.** *For any Π^1_1-prefix-free machine M, we have a constant c_M such that $P_M(\sigma) \leqslant 2^{-\mathrm{K}(\sigma)} \times c_M$ for any σ.*

Proof. We build a Π^1_1-bounded request set A from our machine M. At successor stage s, for every string σ such that $P_M(\sigma)[s] \neq 0$, we simply put into A the pair (m, σ) for $m = \lceil -\log(P_M(\sigma)[s]) \rceil + 1$ (as long as (m, σ) is not already in $A[s]$). At limit stage s, we define $A[s]$ to be $\bigcup_{t<s} A[t]$.

For a given σ suppose that $P_M(\sigma) = r$ for r a real number and let n be the smallest integer such that $2^{-n} \leqslant r$. By construction the weight corresponding to σ in A is of at most $\sum_{m \geqslant n} 2^{-m-1} = 2^{-n} \leqslant r$. Also because $\sum_\sigma P_M(\sigma) \leqslant 1$ we have that A is a bounded request set for which we can build a prefix-free machine N. Also for each string σ with $P_M(\sigma) = r$ and 2^{-n} the greatest power of 2 such that $2^{-n} \leqslant r$, we have that $(n+1, \sigma)$ is enumerated in A and then that $P_M(\sigma) \leqslant 2^{-n+1} \leqslant 2^{-n-1} \times 4 = 2^{-\mathrm{K}_N(\sigma)} \times 4 \leqslant 2^{-\mathrm{K}(\sigma)} \times c_M$ for c_M a constant depending on M. □

5.3. **Equivalent characterizations of Π^1_1-Martin-Löf randomness.** We shall now see an important lemma. It is clear that any Σ^0_1 set can be described by a Σ^0_1 prefix-free set of strings. But this does not hold anymore in the higher setting. Nonetheless, from a measure theoretical point of view, a Π^1_1-open set can be described by a set of strings which is as close as we want from being prefix-free.

Definition 5.10. We say that a set of strings W is ε-prefix-free if $\sum_{\sigma \in W} \lambda([\sigma]) \leqslant \lambda([W]^{\prec}) + \varepsilon$.

Lemma 5.11. For any Π^1_1-open set $\mathcal{U}$, one can obtain uniformly in ε and in an index for $\mathcal{U}$, a ε-prefix-free Π^1_1 set of strings W with $[W]^{\prec} = \mathcal{U}$.

Proof. We use here the projectum function $p : \omega_1^{ck} \to \omega$. Let U be a Π^1_1 set of strings describing $\mathcal{U}$. At successor stage s, if σ enters U, we find a finite prefix-free set of strings C_s, each of them extending σ, such that $[\sigma] \subseteq [W_{s-1}]^{\prec} \cup [C_s]^{\prec}$ and such that $\lambda([W_{s-1}]^{\prec} \cap [C_s]^{\prec}) \leqslant 2^{-p(s)} \times \varepsilon$ (and if nothing enters U we define $C_s = \varnothing$). We then add each string of C_s to W_s. At limit stage s we define W_s to be $\bigcup_{t<s} W_t$.

It is clear by construction that we have $\mathcal{U} = [W]^{\prec}$. Moreover, we have $\sum_{\sigma \in W} \lambda([\sigma]) \leqslant \lambda(\mathcal{U}) + \sum_{s<\omega_1^{ck}} [W_{s-1}]^{\prec} \cap [C_s]^{\prec} \leqslant \lambda(\mathcal{U}) + \varepsilon \sum_{s<\omega_1^{ck}} 2^{-p(s)} \leqslant \lambda(\mathcal{U}) + \varepsilon$. □

We can now show the higher equivalent of the well known Levin-Schnorr theorem, in classical randomness.

Theorem 5.12 (Hjorth, Nies [17])**.** *Given a sequence Z, the following statements are equivalent.*

(1) *The sequence Z is Π^1_1-Martin-Löf-random.*
(2) *There is a constant c such that for every n we have $\mathrm{K}(Z\restriction_n) \geqslant n - c$.*

Proof. (1) $\Longrightarrow$ (2): Let us show that $\neg$(2) implies $\neg$(1). Uniformly in $c \in \mathbb{N}$, we define $\mathcal{U}_c = \{X \mid \exists n \ \ \mathrm{K}(X\restriction_n) < n - c\}$. Each $\mathcal{U}_c$ is a Π^1_1-open set and $\bigcap_c \mathcal{U}_c$ contains all the sequences that do not verify (2). It remains to prove $\lambda(\mathcal{U}_c) \leqslant 2^{-c}$ to deduce that none of them is Π^1_1-Martin-Löf random. Suppose for contradiction that $\lambda(\mathcal{U}_c) > 2^{-c}$ and let W be the (non effective) prefix-free set of strings which describes $\mathcal{U}_c$ and which is minimal under the prefix ordering. We have $1 \geqslant \sum_{\sigma \in W} 2^{-\mathrm{K}(\sigma)} \geqslant \sum_{\sigma \in W} 2^{-|\sigma|} 2^c \geqslant \lambda(\mathcal{U}_c) 2^c > 1$, which contradicts that μ is a Π^1_1-continuous semi-measure.

(2) $\Longrightarrow$ (1): Consider now a Π^1_1-Martin-Löf-test $\bigcap_n \mathcal{U}_n$ and let us build a Π^1_1-prefix-free machine M such that for every $X \in \bigcap_n \mathcal{U}_n$ and every c we have some n with $\mathrm{K}_M(X\restriction_n) < n - c$. Using Lemma 5.11, we can get a Π^1_1 set of strings W_n, uniformly in n, such that $\mathcal{U}_n = [W_n]^{\prec}$ and such that $\sum_{\sigma \in W_n} \lambda([\sigma]) \leqslant \lambda(\mathcal{U}_n) + 2^{-n}$.

Then to define M, we first define the Π^1_1-bounded request set A by enumerating $(|\sigma| - n, \sigma)$ for each n and each $\sigma \in W_{2n+2}$. We have that A is a bounded request set because $\mathrm{wg}(A) \leqslant \sum_n \sum_{\sigma \in W_{2n+2}} 2^{-|\sigma|+n} \leqslant \sum_n 2^n \sum_{\sigma \in W_{2n+2}} 2^{-|\sigma|} \leqslant \sum_n 2^n (\lambda(\mathcal{U}_{2n+2}) + 2^{-2n-2}) \leqslant \sum_n 2^n 2^{-2n-1} \leqslant \sum_n 2^{-n-1} \leqslant 1$. Also we have for any $X \in \bigcap_n \mathcal{U}_n$ and any n, a prefix of X in W_{2n+2} which is compressed by at least n, with the Π^1_1 prefix-free machine defined from A. Therefore for every c there is an n such that $\mathrm{K}(X\restriction_n) < n - c$. □

We can also deduce from Lemma 5.11 a characterization of Π^1_1-Martin-Löf-randomness, an analogue of a result of Kučera's [26].

Proposition 5.13. *A sequence Z is Π^1_1-Martin-Löf-random if and only if Z has a tail in every non-null Σ^1_1 closed set.*

Proof. Suppose that Z is not Π^1_1-Martin-Löf-random. Then every tail of Z is not Π^1_1-Martin-Löf-random, so Z and all of its tails miss every Σ^1_1 closed set consisting only of Π^1_1-Martin-Löf-random sequences (e.g. complements of components of the universal Π^1_1-Martin-Löf-test).

Suppose that Z is Π^1_1-Martin-Löf-random. Let $\mathcal{P}$ be Σ^1_1 closed and non-null, and let $\mathcal{V}$ be the complement of $\mathcal{P}$. Let ε be such that $\lambda(\mathcal{V}) + \varepsilon < 1$. By Lemma 5.11, let V be a ε-prefix-free Π^1_1 set of strings which generates $\mathcal{V}$. We let $\mathcal{V}^m = [V^m]^{<}$, where V^m is the set of concatenations of m strings, all from V. We have $\sum_{\sigma \in V^m} \lambda([\sigma]) \leqslant (\sum_{\sigma \in V} \lambda([\sigma]))^m$, and the measure of $\mathcal{V}^m$ is bounded by the weight of V^m. The important point is that $\lambda(\mathcal{V}^m)$ goes to 0 computably, so $\bigcap_m \mathcal{V}^m$ is a Π^1_1-Martin-Löf test. Let m be least such that $X \notin \mathcal{V}^m$; as $\mathcal{V}^0 = 2^{\mathbb{N}}$, $m > 0$. Let $\sigma \in V^{m-1}$ which is a prefix of X. Let Y be such that $X = \sigma Y$. Then $Y \in \mathcal{P}$. □

5.4. **Lowness for Π^1_1-Martin-Löf randomness.** The sequences which are low for Martin-Löf randomness have been extensively studied. We shall transpose in this section the main results of the lower setting to the higher setting, using continuous relativization.

In general, given a randomness notion C whose definition relativizes to any oracle X, we say that X is low for C if $C^X = C$.

Definition 5.14 (Hjorth, Nies [17])**.** We say that A is low for Π^1_1-Martin-Löf randomness iff every Π^1_1-Martin-Löf random Z is also $\Pi^1_1(A)$-Martin-Löf random.

5.4.1. *higher* K*-trivial sequences.*

Definition 5.15 (Hjorth, Nies [17])**.** A sequence A is higher K-trivial if for some constant d, $\mathrm{K}(A\upharpoonright_n) \leqslant \mathrm{K}(n) + d$.

It is obvious that any Δ^1_1 sequence is higher K-trivial, because up to an index for such a sequence A, the information about the length of a prefix of A is enough to retrieve that prefix. We shall see that just like for the lower setting, there are non-Δ^1_1 and higher K-trivial sequences. Solovay was the first in [46] to build an incomputable K-trivial sequence. Later, Hjorth and Nies showed that similarly, there are non-Δ^1_1 higher K-trivial sequences. Both proofs are similar in the lower and in the higher setting.

Theorem 5.16 (Hjorth, Nies [17])**.** *There is a higher* K*-trivial which is not* Δ^1_1.

Proof. **The construction :**

We want to build a Π^1_1 higher K-trivial sequence X which is co-infinite and which intersect any infinite Π^1_1 set. Let $\{P_e\}_{e\in\mathbb{N}}$ be an enumeration of the Π^1_1 sets and let U be a universal Π^1_1-prefix-free machine. We enumerate X and build at the same time a Π^1_1-bounded request set M such that $\inf\{m : (m, X\upharpoonright_n) \in M\} \leqslant \mathrm{K}_U(n) + 1$. We keep track of a set of Boolean values R_e, initialized to false and meaning that X does not intersect P_e yet.

At successor stage s, at substage e for which R_e is false, if there is $n \in P_{e,s}$ with $n \geqslant 2e$ and such that the weight of M at stage s and substage $e - 1$, restricted to strings of length bigger than n, is smaller than 2^{-e-1}, then we enumerate n in X at stage s, we set R_e to true, and for every pair $(l, X_{s-1}\upharpoonright_m)$ in M at stage s and substage $e - 1$, we put $(l, X_s\upharpoonright_m)$ in M at stage s and substage e.

After all substages e, if (σ, n) is enumerated in U at stage s, we enumerate $(|\sigma| + 1, X_s\upharpoonright_n)$ in M at stage s.

The verification :

We should prove that $\mathrm{wg}(M) \leqslant 1$. The weight of all the pairs we enumerate in M because of some (σ, n) in U, is bounded by $1/2$ (because $\sum_{(\sigma,n)\in U} 2^{-|\sigma|} \leqslant 1$ and because for each $(\sigma, n) \in U$ we increase the weight of M by at most $2^{-|\sigma|-1}$). Then for each e, the additional weight we put in is bounded by 2^{-e-1}. Therefore the weight of M is bounded by 1.

We should now prove that X is not Δ^1_1. It is clearly co-infinite, as for each e we add in X at most one integer bigger than $2e$. Suppose that P_e is infinite. Then at some stage s it is already infinite, by admissibility. Also at any stage t we have $\mathrm{wg}(M[t]) \leqslant 1$. Therefore there is a smallest length n such that the weight of M at stage s, restricted to strings of length bigger than n, is smaller than 2^{-e-1}. At this point, the integer n is enumerated in X if R_e is still false. So X intersects every infinite Π^1_1 set.

Also by construction it is clear that $\inf\{m : (m, X\upharpoonright_n) \in M\} \leqslant \mathrm{K}_U(n) + 1$. Therefore X is higher K-trivial. □

Chaitin proved in [4] that there are only countably many K-trivial sequences. With a similar proof, we also have that there are only countably many higher K-trivial sequences.

Theorem 5.17 (Hjorth, Nies [17])**.** *There is a constant c, such that for each constant d and each n, there are at most $c \times 2^d$ many strings σ of length n such that* $\mathrm{K}(\sigma) \leqslant \mathrm{K}(|\sigma|) + d$.

Proof. Let M be the machine which on a string τ outputs $|U(\tau)|$. If τ is a short description for any string of length n via U, then τ is a short description for n, via the machine M. Also by the coding theorem (Theorem 5.9) we have $P_M(n) < 2^{-\mathrm{K}(n)} \times c_M$ for some constant c_M (recall P_M from Definition 5.8). We now claim that for any length n and any d, there are at most $c_M \times 2^d$ strings σ of length n such that $\mathrm{K}(\sigma) \leqslant \mathrm{K}(n) + d$. Suppose otherwise for a given length n. Then $P_M(n) \geqslant c_M \times 2^d \times 2^{-\mathrm{K}(n)-d} = c_M \times 2^{-\mathrm{K}(n)}$, which is a contradiction. □

Corollary 5.18 (Hjorth, Nies [17])**.** *There is a constant c, such that for each constant d there are at most $c \times 2^d$ many sequences X such that $\mathrm{K}(X{\upharpoonright}_n) \leqslant \mathrm{K}(n) + d$ for every n. In particular there are at most countably many higher* K*-trivial sequences.*

Proof. With c the constant of the previous theorem, if there are more than $c \times 2^d$ many sequences X such that $\mathrm{K}(X{\upharpoonright}_n) \leqslant \mathrm{K}(n) + d$ for every n, then also for n large enough, there are more than $c \times 2^d$ many strings σ of length n such that $\mathrm{K}(\sigma) \leqslant \mathrm{K}(|\sigma|) + d$. □

The previous theorem will allow us to determine that higher K-trivial sequences are actually fairly simple to describe: They are all higher Δ^0_2 sequences. Also we can even put them in the sharper class of sequences with a collapsing approximation.

Theorem 5.19 (Hjorth, Nies [17])**.** *Every higher* K*-trivial sequence A has a collapsing approximation.*

Proof. Suppose that A is higher K-trivial with constant d. For each stage $s < \omega_1^{ck}$, let us define the Δ^1_1 function $f_s : 2^{<\mathbb{N}} \to \mathbb{N}$ by $f_s(\sigma) = 1$ if $\forall \tau \preccurlyeq \sigma\ \ \mathrm{K}(\tau)[s] \leqslant \mathrm{K}(|\tau|)[s] + d$ and $f_s(\sigma) = 0$ otherwise. Note first that $T_s = \{\sigma\ :\ f_s(\sigma) = 1\}$ is a tree, that is, if $f_s(\sigma) = 1$ then also we must have $f_s(\tau) = 1$ for $\tau \preccurlyeq \sigma$. Let us show that $\{f_s\}_{s<\omega_1^{ck}}$ is a finite-change approximation converging to some function f. Suppose otherwise and let σ be minimal for the prefix ordering, such that $\{f_s(\sigma)\}_{s<\omega_1^{ck}}$ changes infinitely often. By minimality of σ we have stages $s_1 < s_2 \leqslant \omega_1^{ck}$ such that $\{f_s(\tau)\}_{s_1\leqslant s\leqslant s_2}$ is stable for any $\tau \prec \sigma$ but such that $\{f_s(\sigma)\}_{s_1\leqslant s\leqslant s_2}$ changes infinitely often. Note also that in this case we must have $f_s(\tau) = 1$ for every $\tau \prec \sigma$ and every $s \in [s_1, s_2]$ because any set T_s is a tree and because we must have infinitely many stages $s \in [s_1, s_2]$ with $f_s(\sigma) = 1$. This imply in particular that $\forall s \in [s_1, s_2]\ \forall \tau \prec \sigma\ \ \mathrm{K}_s(\tau) \leqslant \mathrm{K}_s(|\tau|) + d$. Also we must have infinitely stages $t_0 < t_1 < t_2 < \cdots \in [s_1, s_2]$ such that $f_{t_i}(\sigma) = 1$ but $f_{t_i+1}(\sigma) = 0$ for $i \in \mathbb{N}$. For each stage t_i we have $\mathrm{K}(\sigma)[t_i] \leqslant \mathrm{K}(|\sigma|)[t_i] + d$ but $\mathrm{K}(\sigma)[t_i + 1] > \mathrm{K}(|\sigma|)[t_i + 1] + d$. As K is decreasing it means that $\mathrm{K}(|\sigma|)[t_i + 1] < \mathrm{K}_s[t_i]$. But then we have $\mathrm{K}(|\sigma|)[t_0] > \mathrm{K}(|\sigma|)[t_1] > \mathrm{K}(|\sigma|)[t_2] > \dots$ which is a contradiction.

Thus $\{f_s\}_{s<\omega_1^{ck}}$ is a finite-change approximation converging to some function f. This implies that the sequence of trees $\{T_s\}_{s<\omega_1^{ck}}$ converges pointwise to a tree T whose paths are exactly the sequences which are higher K-trivial with constant d. In particular $A \in [T]$. As $[T]$ contains finitely many elements, then there must be a prefix σ of A such that A is the only element of $[T]$. Now let A_1 be the set of stages such that $s \in A_1$ iff for every n, T_s contains at most $c \times 2^d$ strings of length n. Let A_2 be the set of stages such that $s \in A_2$ iff T_s contains at least one infinite path extending σ. By admissibility, we have that both A_1 and A_2 are unbounded below ω_1^{ck}. Also As $\{f_s\}_{s<\omega_1^{ck}}$ is a finite-change approximation, we also have that both A_1 and A_2 are closed. Thus $A_1 \cap A_2$ is a closed unbounded set of stages. Let $\{T_s\}_{s<\omega_1^{ck}}$ be the approximation of T restricted to stages $s \in A_1 \cap A_2$. As stage s let A_s be the leftmost path of T_s extending σ.

It is clear that $\{A_s\}_{s<\omega_1^{ck}}$ converges to A, because there is only one infinite path extending σ in T, and because $\{f_s\}_{s<\omega_1^{ck}}$ is a finite-change approximation. Let us show that $\{A_s\}_{s<\omega_1^{ck}}$ is collapsing. For contradiction, suppose otherwise, that is for some lengths $n_1 < n_2 < \dots$ and some stages $s_1 < s_2 < \dots$ such that $s = \sup_i s_i < \omega_1^{ck}$, we have $A{\upharpoonright}_{n_i} \prec A_{s_i}$ for each $i \in \mathbb{N}$. As As the sequence $\{f_s\}_{s<\omega_1^{ck}}$ is a finite-change approximation, we must have $A \in T_s$. But as $s \in A_1 \cap A_2$ we must have that T_s contains at most $c \times 2^d$ many path and thus that A is Δ^1_1. □

Using Theorem 2.16, the following is immediate:

Corollary 5.20 (Hjorth, Nies [17])**.** *If X is higher* K*-trivial and X is not Δ^1_1, then $\omega_1^X > \omega_1^{ck}$.*

5.4.2. *Lowness and continuity.* Hjorth and Nies showed [17] that A is low for Π^1_1-Martin-Löf randomness iff A is Δ^1_1. In order to see that, we will restrict the notion of relativization in the same way we restricted the notion of hyperarithmetical reducibility : by forcing to keep continuity. In the lower settings, any $\Sigma^0_1(X)$ set of reals $\mathcal{U}$, can also be seen as a c.e. set of pairs $W \subseteq 2^{<\mathbb{N}} \times 2^{<\mathbb{N}}$, such that $\mathcal{U} = \bigcup\{[\tau] \ : \ (\sigma, \tau) \in W \text{ and } \sigma \prec X\}$. Note that such a set W gives $\Sigma^0_1(Y)$ sets of reals for every $Y \in 2^{\mathbb{N}}$.

Definition 5.21 (Bienvenu, Greenberg, Monin [2]). An open set $\mathcal{U}$ is X-continuously Π^1_1 if there is an X-continuous Π^1_1 set of strings W such that $\mathcal{U} = [W^X]^{\prec}$.

Definition 5.22 (Bienvenu, Greenberg, Monin [2]). An X-continuous Π^1_1-Martin-Löf test is given by a uniform sequence of X-continuous Π^1_1 open sets $\{\mathcal{U}_n\}_{n\in\mathbb{N}}$, such that for any n we have $\lambda(\mathcal{U}^X_n) \leqslant 2^{-n}$.

In the lower settings, given c.e. description $W \subseteq 2^{<\mathbb{N}} \times 2^{<\mathbb{N}}$ of a $\Sigma^0_1(X)$ set of reals $\mathcal{U}$ such that $\lambda(\mathcal{U}) \leqslant \varepsilon$, it is possible to uniformly transform W into $V \subseteq 2^{<\mathbb{N}} \times 2^{<\mathbb{N}}$, such that $\forall X\ \lambda(V^X) \leqslant \varepsilon$ and such that $\forall X\ \lambda(W^X) \leqslant \varepsilon \rightarrow [W]^X = [V]^X$. Note that this is not always possible with X-continuous Π^1_1-open sets. In particular, there are some oracle X such that there exists no universal X-continuous Π^1_1-Martin-Löf test (see Chapter 7 of [35]). The fact that continuous relativization lacks such convenient properties, diminishes its interest. It is nonetheless still a well-defined notion, and it will find its use in the study of lowness for Π^1_1-Martin-Löf randomness. In particular we define:

Definition 5.23 (Bienvenu, Greenberg, Monin [2]). A sequence A is continuously low for Π^1_1-Martin-Löf randomness if the A-continuous Π^1_1-Martin-Löf randoms coincide with the Π^1_1-Martin-Löf randoms.

It is clear that if A is low for Π^1_1-Martin-Löf randomness, then also it must be continuously low for Π^1_1-Martin-Löf randomness. Also we will now see that higher K-triviality coincides with continuous lowness for Π^1_1-Martin-Löf randomness. We will then see that no non-Δ^1_1 higher K-trivial is low for Π^1_1-Martin-Löf randomness (using this time full relativization), which will imply that only the Δ^1_1 sets are low for Π^1_1-Martin-Löf randomness.

We have defined continuous lowness for Π^1_1-Martin-Löf randomness. Let us now define the analogue notion for the higher Kolmogorov complexity.

Definition 5.24 (Bienvenu, Greenberg, Monin [2]). A sequence X is continuously low for K if for any X-continuous Π^1_1 prefix-free machine M we have a constant c_M such that $\mathrm{K}(\sigma) \leqslant \mathrm{K}^X_M(\sigma) + c_M$ for every σ.

Lemma 5.25 (Bienvenu, Greenberg, Monin [2]). Given an oracle-continuous Π^1_1-open set $\mathcal{U} \subseteq 2^{<\mathbb{N}} \times 2^{<\mathbb{N}}$ one can define uniformly in $n \in \mathbb{N}$ and in $\varepsilon \in \mathbb{Q}^+$ an oracle-continuous Π^1_1-open set $\mathcal{V} \subseteq 2^{<\mathbb{N}} \times 2^{<\mathbb{N}}$ such that:

- If $\lambda(\mathcal{U}^X) \leqslant 2^{-n}$ then $\mathcal{U}^X = \mathcal{V}^X$.
- $\lambda(\{X \ : \ \lambda(\mathcal{V}^X) > 2^{-n}\}) \leqslant \varepsilon$.

Proof. Let n be fixed. Recall that $p : \omega_1^{ck} \rightarrow \omega$ is the projectum function. At stage 0 we set $\mathcal{V}_0 = \varnothing$. At successor stage s, suppose that (σ, τ) is enumerated in $\mathcal{U}$. Let us consider the Δ^1_1-open set $\mathcal{W} = \{X \ : \ \lambda(\mathcal{V}^X_{s-1} \cup [\tau]) > 2^{-n}\}$. Let us find a finite set of strings B such that $[B]^{\prec} \cup \mathcal{W} = [\sigma]$ and such that $\lambda([B]^{\prec} \cap \mathcal{W}) \leqslant \varepsilon \times 2^{-p(s)}$. For any string ρ in B we then add (ρ, τ) in $\mathcal{V}$ at stage s. At limit stage s we define $\mathcal{V}_s$ to be the union of $\mathcal{V}_t$ for $t < s$.

It is obvious that if $\lambda(\mathcal{U}^X) \leqslant 2^{-n}$, then $\mathcal{U}^X = \mathcal{V}^X$. Also by construction, at successor stage s, we add in $\{X \ : \ \lambda(\mathcal{V}^X_{s-1}) > 2^{-n}\}$ something of measure at most $\varepsilon \times 2^{-p(s)}$. It follows that $\lambda(\{X \ : \ \lambda(\mathcal{V}^X) > 2^{-n}\}) \leqslant \varepsilon$. □

Before we continue, we emphisize that continuous relativization can be used, thanks to the previous lemma, to show the higher analogue of the van Lambalgen theorem:

Theorem 5.26 (Bienvenu, Greenberg, Monin [2]). *The sequence $X \oplus Y$ is Π^1_1-Martin-Löf random iff X is Π^1_1-Martin-Löf random and Y is X-continuously Π^1_1-Martin-Löf random.*

Proof. Suppose first that some sequence $X \oplus Y$ is captured by some Π^1_1-Martin-Löf test $\bigcap_n \mathcal{U}_n$. For $\mathcal{U}_n = \bigcup_{\sigma_1,\sigma_2}[\sigma_1 \oplus \sigma_2]$, note that we clearly have $\lambda(\bigcup_{\sigma_1,\sigma_2}[\sigma_1 \oplus \sigma_2]) = \lambda_{\sigma_1,\sigma_2}(\bigcup[\sigma_1] \times [\sigma_2])$. Also we can consider that the pair (X, Y) is not Π^1_1-Martin-Löf random in the product space $2^{\mathbb{N}} \times 2^{\mathbb{N}}$.

Let $\bigcap_n \mathcal{U}_n$ be a uniform intersection of Π^1_1-open sets of $2^{\mathbb{N}} \times 2^{\mathbb{N}}$ with $\lambda(\mathcal{U}_n) \leqslant 2^{-n}$ and $(X, Y) \in \bigcap_n \mathcal{U}_n$. For a string σ and an integer n, let us denote by $\mathcal{U}^\sigma_n$ the Π^1_1-open set $\{Y \ : \ \forall X \succ \sigma\ (X, Y) \in \mathcal{U}_n\}$. Let $\mathcal{V}_n$ be the X-continuously Π^1_1-open set containing Y and equal to $\bigcup_{\sigma \prec X} \mathcal{U}^\sigma_{2n}$. Suppose that for all but finitely many n we have $\lambda(\mathcal{V}_n) \leqslant 2^{-n}$. Then Y is not X-continuously Π^1_1-Martin-Löf random. Otherwise there are infinitely many n such that $\lambda(\mathcal{V}_n) > 2^{-n}$. Also consider now for each

n the Π^1_1-open set $\mathcal{S}_n = \{Z : \lambda(\mathcal{U}^Z_{2n}) > 2^{-n}\}$. Let us show that $\lambda(\mathcal{S}_n) \leqslant 2^{-n}$. Suppose otherwise and let A be a pairwise disjoint set of strings describing $\mathcal{S}_n$. We have $\lambda(\mathcal{U}_{2n}) \geqslant \sum_{\sigma\in A} 2^{-|\sigma|}\lambda(\mathcal{U}^\sigma_{2n}) > 2^{-n}\sum_{\sigma\in A} 2^{-|\sigma|} > 2^{-2n}$, which is a contradiction. Thus $\lambda(\mathcal{S}_n) \leqslant 2^{-n}$ for every n and we have for infinitely many n such that $X \in \mathcal{S}_n$. Also $\{\mathcal{S}_n\}_{n\in\mathbb{N}}$ is a Π^1_1-Solovay test capturing X, which is then not Π^1_1-Martin-Löf random.

Conversely, suppose that X is not Π^1_1-Martin-Löf random or that Y is not X-continuously Π^1_1-Martin-Löf random. It is enough to deal with the last case, as if X is not Π^1_1-Martin-Löf random it is certainly not Y-continuously Π^1_1-Martin-Löf random either. So suppose that Y is in some X-continuous Π^1_1-Martin-Löf test $\bigcap_n \mathcal{U}^X_n$ where each $\mathcal{U}_n$ can be seen as a Π^1_1 subset of $2^{<\mathbb{N}} \times 2^{<\mathbb{N}}$. From Lemma 5.25 we can consider that each $\mathcal{U}_n$ is such that $\lambda(\{Z : \lambda(\mathcal{U}^Z_n) > 2^{-n}\} \leqslant 2^{-n})$ still with $Y \in \bigcap_n \mathcal{U}^X_n$. It is clear that the set $\bigcup_{\tau\in 2^{<\mathbb{N}}} [\tau] \times \mathcal{U}^\tau_n$ is a Π^1_1-open subset of $2^{\mathbb{N}} \times 2^{\mathbb{N}}$, defined uniformly in n and which contains (X, Y). Let us prove that it has measure smaller than 2^{-n+1}. Since for $\tau \preccurlyeq \tau'$ we have $\mathcal{U}^\tau_n \subseteq \mathcal{U}^{\tau'}_n$, we then have $\lambda(\bigcup_{\tau\in 2^{<\mathbb{N}}} [\tau] \times \mathcal{U}^\tau_n) = \sup_m \sum_{|\tau|=m} \lambda([\tau] \times \mathcal{U}^\tau_n)$. Also for each m, the measure of the set of strings τ of length m such that $\lambda(\mathcal{U}^\tau_n) > 2^{-n}$ is of $\varepsilon_m \leqslant 2^{-n}$, whereas on other strings τ of length m we have $\lambda(\mathcal{U}^\tau_n) \leqslant 2^{-n}$. We then have:

$$\sum_{|\tau|=m} \lambda([\tau] \times \mathcal{U}^\tau_n) \leqslant (1-\varepsilon_m)2^{-n} + \varepsilon_m \leqslant 2^{-n+1}$$

It follows that $\lambda(\bigcup_{\tau\in 2^{<\mathbb{N}}} [\tau] \times \mathcal{U}^\tau_n) \leqslant 2^{-n+1}$ and we then have a Π^1_1-Martin-Löf test capturing (X, Y). □

5.4.3. *Low for* K *and low for* Π^1_1*-Martin-Löf randomness.*

Proposition 5.27 (Bienvenu, Greenberg, Monin [2])**.** *If a sequence X is continuously low for* K*, then it is higher* K*-trivial.*

Proof. Let U be a universal Π^1_1-prefix-free machine and let M be the Π^1_1 set of triples where we enumerate $\{\sigma, \tau, \sigma\}$ in M at stage s if $U(\tau) = |\sigma|$ at stage s. We have for every oracle X that M^X is a prefix-free machine. We also have for any X and any $\sigma \prec X$ that $\mathrm{K}^X_M(\sigma) = \mathrm{K}(n)$. Now because X is low for K we have $\mathrm{K}(X\upharpoonright_n) \leqslant \mathrm{K}^X_M(X\upharpoonright_n) + c_M = \mathrm{K}(n) + c_M$ which makes X higher K-trivial as well. □

It is clear that continuous lowness for K implies continuous lowness for Π^1_1-Martin-Löf randomness. The converse also holds but requires some work. We shall show that just as in the lower settings, continuous lowness for Π^1_1-Martin-Löf randomness implies continuous lowness for K. A direct proof of that would certainly be possible, but we will instead show more, by using the following notion:

Definition 5.28 (Bienvenu, Greenberg, Monin [2])**.** The sequence A is a continuous base for Π^1_1-Martin-Löf randomness if there is some A-continuous Π^1_1-Martin-Löf random sequence Z such that $Z \geqslant_{\omega_1^{\mathrm{ck}}\mathrm{T}} A$.

We can first observe that any sequence which is continuously low for K is also a continuous base for Π^1_1-Martin-Löf randomness.

Proposition 5.29 (Bienvenu, Greenberg, Monin [2])**.** *If A is continuously low for* K*, then A is a continuous base for Π^1_1-Martin-Löf randomness.*

Proof. Being continuously low for K implies being continuously low for Π^1_1-Martin-Löf randomness. Also by Theorem 5.2, for any sequence A, there is a Π^1_1-Martin-Löf random Z such that Z higher Turing computes A. Also as A is continuously low for Π^1_1-Martin-Löf randomness, the sequence Z is A-continuously Π^1_1-Martin-Löf random. □

Hirschfeldt, Nies and Stephan proved in [16] that the two notions actually coincide in the lower setting. The result can be transfered in the higher setting, but the proof needs to be modified due to the usual topological issues of higher computability.

Theorem 5.30 (Bienvenu, Greenberg, Monin [2])**.** *If A is a base for continuous Π^1_1-Martin-Löf randomness, then A is continuously low for* K*.*

Proof. Suppose that Z is A-continuously Π^1_1-Martin-Löf random and suppose that $\Phi(Z) = A$ for some higher Turing functional Φ. We can assume that if (τ, σ) is in Φ then Φ also contains (τ, σ') for each $\sigma' \preccurlyeq \sigma$. Let M be any higher A-continuous prefix-free machine. Note that we see M as a Π^1_1 subset of $2^{<\mathbb{N}} \times 2^{<\mathbb{N}} \times 2^{<\mathbb{N}}$ such that M^X is a prefix-free machine. Note also that M^Y need not to be a prefix-free machine for any oracle Y. We can assume that each triple (τ, σ, ρ) is enumerated ω_1^{ck}-cofinally many times in M. For each integer d we will describe an algorithm having d as a parameter. Each instance of the algorithm will enumerate some Π^1_1 set of strings $C_{\tau,\sigma,\rho}$ for each triple $(\tau, \sigma, \rho) \in 2^{<\mathbb{N}} \times 2^{<\mathbb{N}} \times 2^{<\mathbb{N}}$ (so called 'hungry sets' by Hirschfeldt, Nies and Stephan) and will enumerate a Π^1_1 bounded request set $N \subseteq \mathbb{N} \times 2^{<\mathbb{N}}$.

The algorithm for a parameter d:

Before giving the algorithm, let us first fix for each triple (τ, σ, ρ) a rational $\delta_{\tau,\sigma,\rho}$ such that $\sum_{\tau,\sigma,\rho} \delta_{\tau,\sigma,\rho} \leqslant 1$. Recall also that $p : \omega_1^{ck} \to \omega$ is the projectum function.

At the beginning of the algorithm, for each triple (τ, σ, ρ) we set $C^0_{\tau,\sigma,\rho} = \varnothing$. Then at successor stage $s + 1$ of the algorithm, let (τ, σ, ρ) be the new triple enumerated in M_s. Look at all pairs (η, τ) enumerated in Φ at stage $t < s$ until two conditions are met: First the string η should not be marked as used (as defined below). Then we must have $\lambda([C^s_{\tau,\sigma,\rho}]^{\prec}) + 2^{-|\eta|} \leqslant 2^{-d}2^{-|\sigma|}$. If no such pair (η, τ) is found then we go to the next stage.

Otherwise we want to add η to $C^s_{\tau,\sigma,\rho}$. But we also want to keep all the open sets described by each $C^s_{\tau,\sigma,\rho}$ pairwise disjoint. Since it is not be always possible, we keep them 'mostly disjoint'. Let U^s be the set of all the strings in any of the $C^s_{\tau,\sigma,\rho}$ which are compatible with η. It is possible that $[\eta] - [U^s]^{\prec}$ is not an open set. To remedy this, just like in the proof of Lemma 5.11, let B^s be a finite set of strings such that $[B^s]^{\prec} \cup [U^s]^{\prec} = [\eta]$ and such that $\lambda([B^s]^{\prec} \cap [U^s]^{\prec}) \leqslant 2^{-p(s)}\delta_{\tau,\sigma,\rho}$. Note that it is Δ^1_1 uniformly in s to find such a set B^s. Then we mark η and all strings extending η as 'used' and we set $C^{s+1}_{\tau,\sigma,\rho} = C^s_{\sigma,x,q} \cup B^s$. Then if $\lambda([C^{s+1}_{\tau,\sigma,\rho}]^{\prec}) > 2^{-d-1}2^{-|\sigma|}$ we enumerate the pair $(d + 1 + |\sigma|, \rho)$ into N.

Finally, at limit stage s we set each $C^s_{\tau,\sigma,\rho}$ to be $\bigcup_{t<s} C^t_{\tau,\sigma,\rho}$.

Verification : Bounded request set

We have to prove that for each d, the set N created by the instance of the algorithm with parameter d, is a bounded request set. In other words we have to prove that $\mathrm{wg}(N) = \sum_{(l,\rho)\in N} 2^{-l} \leqslant 1$. It is clear that we have $\mathrm{wg}(N) \leqslant \frac{1}{2}\sum_{\tau,\sigma,\rho} \lambda([C_{\tau,\sigma,\rho}]^{\prec})$ because each $[C_{\tau,\sigma,\rho}]^{\prec}$ has measure at most $2^{-d} \times 2^{-|\sigma|}$, and for each of them we enumerate at most once some $(d+1+|\sigma|, \rho)$ into N. So it is enough to prove that $\sum_{\tau,\sigma,\rho} \lambda([C_{\tau,\sigma,\rho}]^{\prec}) \leqslant 2$. Let

$$E = \bigcup_{(\tau',\sigma',\rho')\neq(\tau,\sigma,\rho)} ([C_{\tau,\sigma,\rho}]^{\prec} \cap [C_{\tau',\sigma',\rho'}]^{\prec})$$

and let $E_{\tau,\sigma,\rho}$ be the open set generated by strings η such that $[\eta]$ is covered by $[C_{\tau,\sigma,\rho}]^{\prec}$ after $[\eta]$ is covered by some $[C_{\tau',\sigma',\rho'}]^{\prec}$ for $(\tau', \sigma', \rho') \neq (\tau, \sigma, \rho)$. Let $E'_{\tau,\sigma,\rho}$ be the open set generated by strings η such that $[\eta] \subseteq E$ and such that $[\eta]$ is covered by $[C_{\tau,\sigma,\rho}]^{\prec}$ before it is covered by other $[C_{\tau',\sigma',\rho'}]^{\prec}$ for $(\tau', \sigma', \rho') \neq (\tau, \sigma, \rho)$. We have:

$$\sum_{\tau,\sigma,\rho} \lambda([C_{\tau,\sigma,\rho}]^{\prec}) \leqslant \sum_{\tau,\sigma,\rho} \lambda([C_{\tau,\sigma,\rho}]^{\prec} - E) + \sum_{(\tau,\sigma,\rho)} \lambda(E'_{\tau,\sigma,\rho}) + \sum_{(\tau,\sigma,\rho)} \lambda(E_{\tau,\sigma,\rho})$$

Clearly $\sum_{\tau,\sigma,\rho} \lambda([C_{\tau,\sigma,\rho}]^{\prec} - E) + \sum_{(\tau,\sigma,\rho)} \lambda(E'_{\tau,\sigma,\rho}) \leqslant 1$ because all the sets involved are pairwise disjoint, by the definition of E and $E'_{\tau,\sigma,\rho}$. Let us prove that $\sum_{(\tau,\sigma,\rho)} \lambda(E_{\tau,\sigma,\rho}) \leqslant 1$. We have:

$$\begin{aligned}
\sum_{(\tau,\sigma,\rho)} \lambda(E_{\tau,\sigma,\rho}) &\leqslant \sum_{(\tau,\sigma,\rho)} \sum_{s<\omega_1^{ck}} \lambda([B^s]^{\prec} \cap [U^s]^{\prec}) \\
&\leqslant \sum_{(\tau,\sigma,\rho)} \sum_{s<\omega_1^{ck}} 2^{-p(s)} \times \delta_{\tau,\sigma,\rho} \\
&\leqslant 1
\end{aligned}$$

Therefore N is a bounded request set.

Verification : Martin-Löf test

Let $C^d_{\tau,\sigma,\rho}$ be the set of strings $C_{\tau,\sigma,\rho}$ created by an instance of the algorithm with d as parameter. Let $C^A_d = \bigcup C^d_{\tau\prec A,\sigma,\rho}$. By construction we have that $\lambda([C^A_d]^{\prec}) \leqslant \sum_{\tau\prec A,\sigma,\rho} \lambda([C_{\tau,\sigma,\rho}]^{\prec}) \leqslant \sum_{\sigma\in\mathrm{dom}(M)} 2^{-d}2^{-|\sigma|}$. As M^A is an A-continuous higher prefix-free machine we have that $\sum_{\sigma\in\mathrm{dom}(M)} 2^{-|\sigma|} \leqslant 1$ and then $\lambda([C^A_d]^{\prec}) \leqslant 2^{-d}$. Then $\bigcap_d [C^A_d]^{\prec}$ is a A-continuous Π^1_1-Martin-Löf test. This implies by hypothesis that there is some d such that $Z \notin [C_d^A]^<$.

Verification : Continuously low for K

First note that if $Z \in C_{\tau,\sigma,\rho}$ for some strings τ, σ, ρ, we necessarily have $\tau \prec A$, because otherwise some prefix of Z would be mapped to something incomparable with A, which is a contradiction. We now only consider the algorithm with d as a parameter where $Z \notin [C_d^A]^<$. We pretend that if (τ, σ, ρ) is enumerated in M for $\sigma \preccurlyeq A$ then $(d + 1 + |\sigma|, \rho)$ will be enumerated in N. Suppose not, then it means that $\lambda([C_{\tau,\sigma,\rho}]^<) \leqslant 2^{-d-1} \times 2^{-|\sigma|}$. Let $\eta \prec Z$ be large enough so that $\lambda([C_{\tau,\sigma,\rho}]^<) + 2^{-|\eta|} < 2^{-d} \times 2^{-|\sigma|}$. There exists s such that (τ, σ, ρ) is enumerated in M at stage s and such that for some $t \leqslant s$ we have (η', τ) which is enumerated in Φ at stage t for $\eta' \succcurlyeq \eta$. At this stage, if η' was marked as used it means that some prefix of η' was already enumerated in another $C^s_{\tau',\sigma',\rho'}$ for $\tau' \prec A$, and so that Z is in $[C_d^A]^<$ which is a contradiction. If η' was not marked as used then some B^s is created such that $\eta' = [B^s]^< \cup [U^s]^<$. If a prefix of Z is in B^s then Z is in $[C^{s+1}_{\tau,\sigma,\rho}]^<$ otherwise Z was already in some $[C^s_{\tau',\sigma',\rho'}]^<$ for $\tau' \prec A$. In either case it is a contradiction. Therefore $(d + 1 + |\sigma|, \rho)$ will be enumerated in N. It follows that from N, we can build a Π^1_1 prefix-free machine that compresses as well as M^A, up to the constant $d + 1$. □

Corollary 5.31 (Bienvenu, Greenberg, Monin [2])**.** *If a sequence A is continuously low for Π^1_1-Martin-Löf randomness, then also it is continuously low for* K*.*

Proof. Suppose A is continuously low for Π^1_1-Martin-Löf randomness. By the higher Kučera-Gács theorem (Theorem 5.2), there is a Π^1_1-Martin-Löf random sequence Z which higher Turing computes A. But Z is also A-continuously Π^1_1-Martin-Löf random, making A a continuous base for Π^1_1-Martin-Löf randomness. Therefore A is continuously low for K. □

Corollary 5.32 (Hjorth, Nies [17])**.** *A sequence A is low for Π^1_1-Martin-Löf randomness, with full relativization, iff it is Δ^1_1.*

Proof. Suppose A is not continuously low for Π^1_1-Martin-Löf randomness, then it is certainly not low for Π^1_1-Martin-Löf randomness using full relativization. Now if it is low for Π^1_1-Martin-Löf randomness it is then continuously low for K and therefore higher K-trivial. If furthermore it is not Δ^1_1, by Corollary 5.20 we then have $\omega_1^A > \omega_1^{ck}$. Also then we have $A \geqslant_h O$ and therefore A hyperarithmetically computes a member in any non-empty Σ^1_1 class. In particular it hyperarithmetically computes a Π^1_1-Martin-Löf random Z which is therefore in a $\Delta^1_1(A)$ nullset. It follows that A is not low for Π^1_1-Martin-Löf randomness. □

So no non-Δ^1_1 sequence is low for Π^1_1-Martin-Löf randomness. It is however possible to show that non-Δ^1_1 sequences are continuously low for Π^1_1-Martin-Löf randomness. Actually it is possible to show that any higher K-trivial is also continuously low for Π^1_1-Martin-Löf randomness. The proof works similarly to the one of Hirschfeldt and Nies in the lower settings, with some additional care due to the continuity problems which comes with higher computability. The proof is rather long and technical, which is why we do not present it here, but the reader who is interested in it can refer to Section of 4.5 of [35].

6. More higher randomness notions

6.1. **Higher difference randomness.** Recall higher difference randomness from Definition 3.9. We shall now show that a Π^1_1-Martin-Löf random is higher difference random iff does not higher Turing computes O.

Lemma 6.1. Let Z be a Π^1_1-Martin-Löf random sequence. Let Φ be a functional. For any ε, let Φ_ε be the transformation of Φ given by Lemma 2.12, so that the open set of sequences on which Φ is not consistent has measure smaller than ε. Then there exists c such that $\lambda(\Phi^{-1}_{2^{-n}}(Z\upharpoonright_n)) \leqslant 2^{-n}2^c$ for every n.

Proof. Let $\mu(\sigma) = \lambda(\Phi^{-1}_{2^{-|\sigma|}}(\sigma))$. Let us show that there must be a constant c such that $\mu(Z\upharpoonright_n) \leqslant 2^{-n}2^c$ for every n. Let $W_c = \{\sigma \mid \mu([\sigma]) > 2^{-|\sigma|}2^{c+1}\}$. Let us show that $\lambda([W_c]^<) \leqslant 2^{-c}$. Suppose otherwise, that is $\lambda([W_c]^<) > 2^{-c}$. Let $\{\sigma_n\}_{n\in\mathbb{N}}$ be a prefix-free set of strings of W_c, minimal for the prefix ordering. Let A_{σ_n} be the open set of strings which are mapped to extensions of σ_n via $\Phi_{2^{-|\sigma_n|}}$. Because $\sigma_n \in W_c$ we have $\lambda(A_{\sigma_n}) > 2^{-|\sigma_n|}2^{c+1}$. Let E_{σ_n} be the open set of string which are in sets $A_{\sigma_n} \cap A_{\sigma_i}$ for $i \neq n$. By hypothesis on $\Phi_{2^{-|\sigma_n|}}$ we have $\lambda(E_{\sigma_n}) \leqslant 2^{-|\sigma_n|}$ and thus that $\lambda(A_{\sigma_n} - E_{\sigma_n}) > 2^{-|\sigma_n|}2^{c+1} - 2^{-|\sigma_n|}$. Also the sets $A_{\sigma_n} - E_{\sigma_n}$ are pairwise disjoint. It follows that:

$$\begin{array}{rcl} \sum_n \lambda(A_{\sigma_n} - E_{\sigma_n}) & \geqslant & \sum_n 2^{-|\sigma_n|}2^{c+1} - 2^{-|\sigma_n|} \\ & \geqslant & 2^{c+1}\sum_n 2^{-|\sigma_n|} - \sum_n 2^{-|\sigma_n|} \\ & \geqslant & (2^{c+1}-1)\sum_n 2^{-|\sigma_n|} \\ & \geqslant & (2^{c+1}-1)2^{-c} \\ & > & 1 \end{array}$$

This is a contradiction. It follows that $\lambda([W_c]^{<}) \leqslant 2^{-c}$. As Z is Π^1_1-martin-Löf random, there exists c such that $Z \notin \mathcal{U}_c$ and thus there exists c such that $\mu(Z\upharpoonright_n) \leqslant 2^{-n}2^c$ for every n. □

Theorem 6.2 (Yu [39]). *Let Z be a Π^1_1-Martin-Löf random sequence. Then Z is not higher difference random iff Z higher Turing computes O.*

Proof. Suppose Z higher Turing compute O. Then also Z higher Turing computes Ω, the leftmost path of a Σ^1_1-closed set containing only Π^1_1-martin-Löf random sequences. Let Φ be such that $\Phi(Z) = \Omega$. From Lemma 6.1 there exists a constant c such that $\lambda(\Phi^{-1}_{2^{-n}}(Z\upharpoonright_n)) \leqslant 2^{-n}2^c$ for every n. In what follows, the notation $\Phi^{-1}([\sigma])$ implicitly means $\Phi^{-1}_{2^{-|\sigma|}}([\sigma])$.

For every n, we define the Π^1_1-open set $\mathcal{U}_n$ to be $\bigcup_{s<\omega_1^{ck}} \Phi^{-1}([\Omega_s\upharpoonright_n])$. Then we define the Π^1_1-open set $\mathcal{V}$ to be $\bigcup_{n\in\mathbb{N}}\bigcup_{s<\omega_1^{ck}}\{\Phi^{-1}([\Omega_s\upharpoonright_n]) \ : \ \Omega_s\upharpoonright_n \neq \Omega_{s+1}\upharpoonright_n\}$. Because Ω is higher left-c.e. we clearly have $Z \in \bigcap_n(\mathcal{U}_n \cap \mathcal{V}^c)$. Also $\mathcal{U}_n \cap \mathcal{V}^c$ is actually equal to $\Phi^{-1}([\Omega\upharpoonright_n])$ and therefore its measure is smaller than $2^{-n}2^c$ for every n. Thus Z is not higher difference random.

For the converse, suppose that a Π^1_1-Martin-Löf random Z belongs to $\bigcap_n(\mathcal{U}_n \cap \mathcal{F})$ with $\lambda(\mathcal{U}_n \cap \mathcal{F}) \leqslant 2^{-n}$. We build a Π^1_1-Solovay test $\{\mathcal{V}_m\}_{m\in\mathbb{N}}$. If m enter O at stage s, we search for the smallest stage $t > s$ such that $\lambda(\mathcal{U}_{m,t} \cap \mathcal{F}_t) \leqslant 2^{-m}$ and we set $\mathcal{V}_m = \mathcal{U}_{m,t} \cap \mathcal{B}_t$ with $\mathcal{B}_t \supseteq \mathcal{F}_t$ a clopen set such that $\lambda(\mathcal{U}_{m,t} \cap \mathcal{B}_t) < 2^{-m+1}$. Note that we can find $\mathcal{B}_t$ uniformly in $\mathcal{U}_{m,t}$, $\mathcal{F}_t$ and m.

As Z is Π^1_1-Martin-Löf random, there is some n such that for all $m \geqslant n$, the sequence Z is not in $\mathcal{V}_m$. Also to know if $m \geqslant n$ is in O, with the help of Z, we search for the smallest stage s such that $Z \in \mathcal{U}_{m,s}$. We claim that $m \in O$ iff $m \in O_s$. Suppose otherwise, that is, $m \in O$ but $m \notin O_s$. Note that for every stage $t \geqslant s$ we have $Z \in \mathcal{U}_{m,t} \cap \mathcal{F}_t$, because otherwise Z could not be in $\mathcal{U}_m \cap \mathcal{F}$. Now for t the smallest stage bigger than s such that $m \in O_t$ and such that $\lambda(\mathcal{U}_{m,t} \cap \mathcal{F}_t) \leqslant 2^{-m}$, we then have that $\mathcal{U}_{m,t} \cap \mathcal{B}_t$ is enumerated in $\mathcal{V}_m$. But then $Z \in \mathcal{V}_m$ which is a contradiction. □

Corollary 6.3 (Yu [39]). *Higher difference randomness is strictly stronger than Π^1_1-Martin-Löf randomness.*

Proof. It is clear that a Π^1_1-Martin-Löf test is also a higher difference test. So the set of higher difference randoms is included in the set of Π^1_1-Martin-Löf randoms.

Also using the higher Kučera-Gács theorem (see Theorem 5.2), there is some Π^1_1-Martin-Löf random sequence which higher Turing computes O and which is then not higher difference random, so the inclusion is strict. □

6.2. **Higher weak-2-randomness.**

6.2.1. *An equivalent test notion.* In order to get a better understanding of higher weak-2-randomness, Bienvenu, Greenberg and Monin [2] developed an equivalent new type of test. We start by generalization of a result from Chong and Yu (see [6]) which says that every higher left-c.e. sequence can be captured by a higher weak-2-test.

Theorem 6.4 (Bienvenu, Greenberg, Monin [2]). *No sequence $X \in 2^{\mathbb{N}}$ with a higher finite-change approximation is higher weakly-2-random.*

Proof. Let $\{X_s\}_{s\leqslant t}$ be a finite-change approximation of X. In particular, note that the set $\mathcal{C} = \{X_s\}_{s\leqslant\omega_1^{ck}}$ is a closed set. Let $\mathcal{U}_n = \bigcup_{s<\omega_1^{ck}}[X_s\upharpoonright_n]$ and let us prove that $\bigcap_n \mathcal{U}_n \subseteq \mathcal{C}$. If an element is in $\mathcal{U}_n$ then its distance to the closed set $\mathcal{C}$ is smaller than 2^{-n} (it shares the same first n bits with an element of $\mathcal{C}$). Thus if it is in all the $\mathcal{U}_n$, its distance to the closed set $\mathcal{C}$ is null and thus it is an element of $\mathcal{C}$. Therefore we have $\bigcap_n \mathcal{U}_n \subseteq \mathcal{C}$ and as $\mathcal{C}$ is countable it has measure 0. Therefore we have that $\bigcap_n \mathcal{U}_n$ is a higher weak-2-test containing X. □

We now bring the technique of Theorem 6.4 to its full generalization, by giving an equivalent notion of higher weak-2-tests, that uses finite-change approximations of elements of the Baire space.

Theorem 6.5 (Bienvenu, Greenberg, Monin [2]). *Let $\{\mathcal{U}_e\}_{e\in\omega}$ be a standard enumeration of the Π^1_1-open sets. For a sequence X we have that the following is equivalent :*

(1) *X is higher weakly-2-random.*
(2) *X is in no uniform intersection of Π^1_1-open sets $\bigcap_n \mathcal{U}_{f(n)}$ where f has a finite change approximation and with $\lambda(\mathcal{U}_{f(n)}) \leqslant 2^{-n}$.*

Proof. (1) $\implies$ (2) : Consider a set $\bigcap_n \mathcal{U}_{f(n)}$ with $\{f_s\}_{s<\omega_1^{ck}}$ a finite-change approximation of f, with $\lambda(\mathcal{U}_{f(n)}) \leqslant 2^{-n}$ and with $X \in \bigcap_n \mathcal{U}_{f(n)}$. Note that that we can consider without loss of generality that $\lambda(\mathcal{U}_{f_s(n)}) \leqslant 2^{-n}$ for any n and any stage s (as we can simply stop enumerating $\mathcal{U}_{f_s(n)}$ if the measure gets too big). Let us prove that X is not higher weakly-2-random. To do so consider the set $\mathcal{A} = \bigcup_{s\leqslant\omega_1^{ck}} \bigcap_{n\in\mathbb{N}} \mathcal{U}_{f_s(n)}$ and the set $\mathcal{B} = \bigcap_{n<\omega} \bigcup_{s<\omega_1^{ck}} \bigcap_{m\leqslant n} \mathcal{U}_{f_s(m)}$.

Let us prove that $\mathcal{B} \subseteq \mathcal{A}$. Suppose that $Y \in \mathcal{B}$. Then for all n there is a smallest stage s_n so that $Y \in \bigcap_{m\leqslant n} \mathcal{U}_{f_{s_n}(m)}$. As f has a finite-change approximation we have that the limit point of $\{f_{s_n}\}_{n\in\mathbb{N}}$ is equal to f_s for some $s = \sup_n s_n$. For any k there is $i \geqslant k$ be such that $f_{s_i}\upharpoonright_k = f_s\upharpoonright_k$ and then such that $\bigcap_{m\leqslant k} \mathcal{U}_{f_{s_i}(m)} = \bigcap_{m\leqslant k} \mathcal{U}_{f_s(m)}$. Now we have by definition of the sequence $\{s_n\}_{n\in\mathbb{N}}$ that $Y \in \bigcap_{m\leqslant i} \mathcal{U}_{f_{s_i}(m)}$ and therefore we have that $Y \in \bigcap_{m\leqslant k} \mathcal{U}_{f_s(m)}$. Since this holds for any k, this shows that Y belongs to $\bigcap_k \mathcal{U}_{f_s(k)}$ and thus we have $Y \in \mathcal{A}$.

Let us prove that $\lambda(\mathcal{B}) = 0$. By measure countable subadditivity we have $\lambda(\mathcal{A}) \leqslant \sum_{s\leqslant\omega_1^{ck}} \lambda\left(\bigcap_n \mathcal{U}_{f_s(n)}\right)$. For each $s \leqslant \omega_1^{ck}$ we have $\lambda(\bigcap_n \mathcal{U}_{f_s(n)}) = 0$ and then that $\lambda(\mathcal{A}) = 0$. But then as $\mathcal{B} \subseteq \mathcal{A}$ we have $\lambda(\mathcal{B}) = 0$.

Let us prove that $X \in \mathcal{B}$. For all n, there is some stage s_n such that $f_{s_n}\upharpoonright_n = f\upharpoonright_n$. Then at stage s_n we have $X \in \bigcap_{m\leqslant n} \mathcal{U}_{f_{s_n}(m)}$. As this is true for all n, we have $X \in \mathcal{B}$. We can then conclude that $\mathcal{B}$ is in a higher weak-2-test containing X.

(2) $\implies$ (1) : Suppose now that X is not higher weakly-2-random in order to prove that it is in some set $\bigcap_n \mathcal{U}_{f(n)}$ where f has a finite change approximation. Suppose that $X \in \bigcap_n \mathcal{V}_n$ with $\lambda(\bigcap_n \mathcal{V}_n) = 0$. We define $f(n)$ to be the smallest m such that $\lambda(\mathcal{V}_m) \leqslant 2^{-n}$. We have for every n that $\lambda(\mathcal{V}_{f(n)}) \leqslant 2^{-n}$ and $X \in \mathcal{V}_{f(n)}$. All we need to prove is that f has a finite change approximation $\{f_s\}_{s<\omega_1^{ck}}$. We simply let $f_s(n)$ be the smallest m such that $\lambda(\mathcal{V}_m[s]) \leqslant 2^{-n}$. Then we clearly have for each n that the set $\{s \;:\; f_s(n) \neq f_{s+1}(n)\}$ is finite. □

Corollary 6.6 (Bienvenu, Greenberg and Monin [2]). *Higher weak-2-randomness is strictly stronger than higher difference randomness.*

Proof. From the previous theorem, we can deduce that higher weak-2-randomness is stronger than higher difference randomness. Consider the leftmost path Ω of a Σ^1_1 closed set containing only Π^1_1-Martin-Löf randoms. In particular Ω is higher left-c.e. and then it is Turing computable by O. Also if Z higher Turing computes O it also higher Turing computes Ω. Let $\{\Omega_s\}_{s<\omega_1^{ck}}$ be a higher left-c.e. sequence converging to Ω. Given Z that is Π^1_1-Martin-Löf random and not higher difference random, let Φ be the higher Turing functional such that $\Phi(Z) = \Omega$. From Lemma 6.1, there exists c such that $\forall n\ \Phi^{-1}_{2^{-n}}(\Omega\upharpoonright_n) \leqslant 2^{-n}2^c$. Using this, we simply define $f_s(n)$ to be the index of the open set $\Phi^{-1}_{2^{-n-c}}(\Omega_s\upharpoonright_{n+c})$. It is clear that $\{f_s\}_{s<\omega_1^{ck}}$ is finite-change, as $\{\Omega_s\}_{s<\omega_1^{ck}}$ is. It is also clear that $Z \in \mathcal{U}_{f(n)}$ for every n and that $\lambda(\mathcal{U}_{f(n)}) \leqslant 2^{-n}$. Thus Z is not higher weakly-2-random.

Now to prove that the inclusion is strict. Let Ω_1, Ω_2 be the two halves of Ω, that is, $\Omega = \Omega_1 \oplus \Omega_2$. By the higher van Lambalgen theorem (see Theorem 5.26) we have that Ω_1 and Ω_2 are higher Turing incomparable. Therefore, neither Ω_1 nor Ω_2 higher Turing compute O. It follows that neither Ω_1 nor Ω_2 is higher difference random. However Ω_1 and Ω_2 still have higher finite-change approximations. Therefore they are not higher weakly 2 random. □

6.2.2. *Separation of higher weak-2-randomness and Π^1_1-randomness.* We now separate the notion of higher weak-2-randomness and the notion of Π^1_1-randomness. This is actually done by building a collapsing approximation of a sequence X which is higher weakly-2-random. To do so we build an approximation $\{X_s\}_{s<\omega_1^{ck}}$ such that for any n, there is no infinite sequence of ordinals $s_0 < s_1 < \dots$ for which $X\upharpoonright_n = X_{s_i}\upharpoonright_n$ and for which $X_{s_i}(n) \neq X_{s_{i+1}}(n)$. It is clear that such an approximation is collapsing when X is not Δ^1_1: Suppose X is in the closure of $\{X_t \;:\; t < s\}$ for some smallest stage s. Then X cannot be the only limit point of $\{X_t \;:\; t < s\}$ as otherwise X would be Δ^1_1. But then there are several limit points and this implies infinitely many changes above some prefix of X.

Theorem 6.7 (Bienvenu, Greenberg and Monin [2]). *There is a higher weak-2-random X with a collapsing approximation. In particular, there is a higher weak-2-random X that is not Π^1_1-random.*

The rest of the section is dedicated to the proof of Theorem 6.7. Let $\{\mathcal{S}_i\}_{i\in\omega}$ be an enumeration of all the higher Σ^0_2 sets. For each $\mathcal{S}_i$ and each j let us define the Σ^1_1 closed set $\mathcal{F}_{i,j}$ so that $\mathcal{S}_i = \bigcup_j \mathcal{F}_{i,j}$.

Sketch of the proof:

We will build X as a limit point of some $\{X_s\}_{s<\omega_1^{ck}}$. Each X_s is built as the unique limit point of a sequence $\{[\sigma^n_s]\}_{n\in\mathbb{N}}$, where $\sigma^1_s \prec \sigma^2_s \prec \dots$. At each stage we will ensure that X_s is in some sense *higher weakly-2-random at stage s*. By this, we mean that for any n, as long as $\lambda(\mathcal{S}_n[s]) = 1$, we believe that X_s should belong to $\mathcal{S}_n[s]$. If at some point we have $\lambda(\mathcal{S}_n[s]) < 1$ (which is by admissibility equivalent to $\lambda(\mathcal{S}_n) < 1$) then n is removed from the set of indices that we use to make X_s higher weakly-2-random.

Concretely we have at each stage s a set of indices $\{e_n\}_{n\in\mathbb{N}}$ which are initialized at stage 0 with $e_n = n$. Suppose that at stage s we have for each n that $\lambda(\mathcal{S}_{e_n}[s]) = 1$. Then it is easy to build a Δ^1_1 sequence X_s in $\bigcap_n \mathcal{S}_{e_n}[s]$: We can suppose that e_0 is such that $\mathcal{F}_{e_0,i} = 2^{\mathbb{N}}$ for all i. So for $d_0 = 0$ and σ_0 equal the empty word, we have $\lambda(\mathcal{F}_{e_0,d_0} \mid \sigma_0) \geqslant 1$. Then, inductively, assuming that for some n we have $\lambda(\bigcap_{k\leqslant n} \mathcal{F}_{e_k,d_k} \mid \sigma_n) \geqslant 2^{-n}$, we then continue the construction as follows:

Step 1: We find one strict extension σ_{n+1} of σ_n so that $\lambda(\bigcap_{k\leqslant n} \mathcal{F}_{e_k,d_k} \mid \sigma_{n+1})[s] \geqslant 2^{-n}$.
Step 2: We find some index d_{n+1} such that $\lambda(\bigcap_{k\leqslant n+1} \mathcal{F}_{e_k,d_k} \mid \sigma_{n+1})[s] \geqslant 2^{-n-1}$.

This way we have an intersection of closed sets containing at most one point X_s. Also by the measure requirement, this intersection is not empty at each step and then we really have $X_s \in \bigcap_n \mathcal{S}_{e_n}[s]$. Note that for the actual construction we will need different lower bounds for the measure requirements. This is due to some technicalities, explained in the next paragraphs.

We only try here to give the general idea. To have that the X_s converge to some X, we have to keep the chosen strings and closed sets at stage $s+1$ equal if possible to those of stage s. When do we have to change them? Three things can happen :

(1) We might have $\lambda(\mathcal{S}_{e_n})[s] = 1$ for all $s < t$ but $\lambda(\mathcal{S}_{e_n})[t] < 1$.
(2) We might have a smallest n such that (3) does not happen up to $n-1$ and such that the measure of $\bigcap_{k\leqslant n} \mathcal{F}_{e_k,d_k}$ inside $[\sigma_{n+1}]$ drops below 2^{-n} at stage t.
(3) We might have a smallest n such that (2) does not happen up to n and such that the measure of $\bigcap_{k\leqslant n+1} \mathcal{F}_{e_k,d_k}$ inside $[\sigma_{n+1}]$ drops below 2^{-n-1} at stage t.

If (1) happens then the index e_n is set to some fixed index a so that $\lambda(\mathcal{S}_a) = 1$, therefore each index e_n can change at most once. If (2) happens, it is the responsibility of the string σ_{n+1} to change, and if (3) happens it is the responsibility of the index d_{n+1} to change.

For (2), we are sure that there exists one extension σ_{n+1} of σ_n of length $|\sigma_n|+1$ such that the measure inside $[\sigma_{n+1}]$ does not drop below 2^{-n}. So as long as the construction is stable 'below the choice of σ_{n+1}, the string σ_{n+1} can change at most once. We will see that in practice we will need extensions of length $|\sigma_n| + 2n$, but for the same reason, the string σ_{n+1} can then change at most finitely often.

For (3), as long as $\lambda(\mathcal{S}_{e_{n+1}}) = 1$, we are sure that we will change only finitely often of index d_{n+1}. However if $\lambda(\mathcal{S}_{e_{n+1}}) < 1$ it can happen that d_{n+1} will change infinitely often at stages $s_1 < s_2 < \dots$, and that $t = \sup_n s_n$ is the first stage for which we witness $\lambda(\mathcal{S}_{e_{n+1}})[t] < 1$ (then at stage t the integer e_{n+1} is set to a the fixed index such that $\lambda(\mathcal{S}_a) = 1$). There is nothing we can do to prevent those infinitely many changes, which could lead as well to infinitely many changes of the string σ_{n+2}. However we can still ensure that if this happens, the string σ_{n+1} will then change, and its previous value will be banished forever, so that the approximation of the sequence X is still collapsing. To do so, we need to take extensions sufficiently long, so that the current closed set still has positive measure inside at least two of them. That way we can afford to banish one of them. So before the formal proof, we recall here Lemma 5.1 that helps us to achieve this:

Lemma 6.8. let σ be a string and $\mathcal{F}$ a closed set so that $\lambda(\mathcal{F} \mid \sigma) \geqslant 2^{-n}$. Then there is at least two extensions τ_1, τ_2 of σ of length $|\sigma| + n + 1$ so that for $i \in \{1, 2\}$ we have $\lambda(\mathcal{F} \mid \tau_i) \geqslant 2^{-n-1}$.

Before the construction:

Let $\{\mathcal{S}_i\}_{i\in\mathbb{N}}$ be an enumeration of all the higher Σ^0_2 sets, with $\mathcal{S}_i = \bigcup_{j\in\mathbb{N}} \mathcal{F}_{i,j}$ where each $\mathcal{F}_{i,j}$ is a Σ^1_1 closed set. We can assume that each union is increasing. We start by deciding in advance the length m_n of each extension. We set $m_0 = 0$ and then recursively we set $m_{n+1} = m_n + (2n+1)$. Finally, let a be an integer so that $\mathcal{F}_{a,i} = 2^{\mathbb{N}}$ for every i.

For each stage s and each n we will define indices e_s^n and d_s^n for the closed set $\mathcal{F}_{e_s^n, d_s^n}$, as well as strings σ_s^n. Also to simplify the reading, we define three predicates:

$$\begin{array}{rll} A(n,s) & \text{means} & \lambda(\bigcap_{k\leqslant n} \mathcal{F}_{e_s^k, d_s^k} \mid \sigma_s^n)[s] \geqslant 2^{-2n} \\ A(n,s,\sigma) & \text{means} & \lambda(\bigcap_{k\leqslant n} \mathcal{F}_{e_s^k, d_s^k} \mid \sigma)[s] \geqslant 2^{-2n-1} \\ A(n,s,\sigma,d) & \text{means} & \lambda(\bigcap_{k\leqslant n} \mathcal{F}_{e_s^k, d_s^k} \cap \mathcal{F}_{e_s^{n+1}, d} \mid \sigma)[s] \geqslant 2^{-2n-2} \end{array}$$

The construction:

At stage 0 we define for each n the set P_0^n to be the set of strings of length m_n, ordered lexicographically. We initialize each string σ_0^n to be the first string of P_0^n (so they are all a range of 0), we initialize e_0^0 to a and e_0^{n+1} to n. Then we initialize to 0 each index d_0^n of the sets $\mathcal{F}_{e_0^n, d_0^n}$.

At successor stage $s+1$ and substage 0, we set $e_{s+1}^0 = e_s^0 = a$, $\sigma_{s+1}^0 = \sigma_s^0$ (always the empty word) and $d_{s+1}^0 = d_s^0 = 0$. Now assume that at substage n we have defined e_{s+1}^k, d_{s+1}^k and σ_{s+1}^k for $k \leqslant n$ and that we have $A(n, s+1)$ is true. Let us now define e_{s+1}^{n+1}, d_{s+1}^{n+1} and σ_{s+1}^{n+1} at substage $n+1$.

Def. of e_{s+1}^{n+1} : If $\lambda(\mathcal{S}_{e_s^{n+1}})[s+1] = 1$, set $e_{s+1}^{n+1} = e_s^{n+1}$ and $P_{s+1}^{n+1} = P_s^{n+1}$, otherwise set $e_{s+1}^{n+1} = a$ and $P_{s+1}^{n+1} = P_s^{n+1} - \{\sigma_s^{n+1}\}$ (the string σ_s^{n+1} is banished).

Def. of σ_{s+1}^{n+1} : If $A(n, s+1, \sigma_s^{n+1})$ and σ_s^{n+1} extends σ_{s+1}^n, set $\sigma_{s+1}^{n+1} = \sigma_s^{n+1}$. Otherwise set σ_{s+1}^{n+1} to be the first string of P_{s+1}^{n+1} extending σ_{s+1}^n such that $A(n, s+1, \sigma_{s+1}^{n+1})$.

Def. of d_{s+1}^{n+1} : If $A(n, s+1, \sigma_{s+1}^{n+1}, d_s^{n+1})$ set $d_{s+1}^{n+1} = d_s^{n+1}$. Otherwise set d_{s+1}^{n+1} to be the smallest integer such that $A(n, s+1, \sigma_{s+1}^{n+1}, d_{s+1}^{n+1})$.

Finally after every substage, define X_{s+1} to be the unique element in $\bigcap_n [\sigma_{s+1}^n]$.

At limit stage s, for each $n \geqslant 0$ set e_s^n to be the convergence value of $\{e_t^n\}_{t<s}$ and set P_s^n to be the convergence value of $\{P_t^n\}_{t<s}$ (among other things we will have to prove that we always have convergence). At substage n, if $\{\sigma_t^n\}_{t<s}$ does not converge, set σ_s^n to be the first string of P_s^n extending σ_s^{n-1}, otherwise set σ_s^n to be the convergence value. If $\{d_t^n\}_{t<s}$ does not converge, set d_s^n to 0, otherwise set it to its convergence value. Finally after every substage, define X_s to be the unique element in $\bigcap_n [\sigma_s^n]$.

The verification:

Claim 1: For every n the sequence $\{e_s^n\}_{s<\omega_1^{ck}}$ can change at most once. In particular, for every s and every n we have that $\{e_t^n\}_{t<s}$ converges.

It is clear because $e_{s+1}^n \neq e_s^n$ only if $\lambda(\mathcal{S}_{e_s^n}[s+1]) < 1$. Also when this happens we have $e_{s+1}^n = a$ and then it can not happen anymore.

Claim 2: For every stage s, any string τ of size m_n and any closed set $\mathcal{F}$ such that $\lambda(\mathcal{F} \mid \tau) \geqslant 2^{-2n}$, there is a string $\sigma \in P_s^{n+1}$ which extends τ so that $\lambda(\mathcal{F} \mid \sigma) \geqslant 2^{-2n-1}$.

Suppose that $\lambda(\mathcal{F} \mid \tau) \geqslant 2^{-2n}$ for $|\tau| = m_n$. Using Lemma 5.1 we have two strings τ_1 and τ_2 of length $m_n + 2n + 1$ so that for $i \in \{1, 2\}$ we have $\lambda(\mathcal{F} \mid \tau_i) \geqslant 2^{-2n-1}$. Also $m_{n+1} = m_n + 2n + 1$ and then $\tau_1, \tau_2 \in P_0^{n+1}$. By construction and by Claim 1, at any stage s we have that P_0^{n+1} contains at most one more string than P_s^{n+1}. Then at any stage s we have at least one string $\sigma \in P_s^{n+1}$ which extends τ and so that $\lambda(\mathcal{F} \mid \sigma) \geqslant 2^{-2n-1}$.

Claim 3: The construction converges, in particular the sequence $\{X_s\}_{s<\omega_1^{ck}}$ converges to X.

There is no difficulty here.

Claim 4: The sequence $\{X_s\}_{s<\omega_1^{ck}}$ is collapsing.

Let $D(s,n)$ be the sentence : "There is an infinite sequence of ordinal $s_0 < s_1 < \dots$ with $\sup_i s_i = s$, such that $X_{s_i}\upharpoonright_n = X_{s_{i+1}}\upharpoonright_n$, and such that $X_{s_i}(n) \neq X_{s_{i+1}}(n)$".

For $\{X_s\}_{s<\omega_1^{ck}}$ to be collapsing, it is enough to prove that for any s and any n, if $D(s,n)$ is true, then $X\upharpoonright_n \neq X_s\upharpoonright_n$. Let s be any stage such that $D(s,n)$ is true for some n. Let n be the smallest integer such that $D(s,n)$ is true, and let $s_0 < s_1 < \dots$ be a sequence of ordinals making $D(s,n)$ true.

Let us prove that there is some i such that $\{X_t\upharpoonright_n\}_{s_i \leqslant t < s}$ is stable. If $n = 1$ it is clear because $X_t\upharpoonright_1 = 0$ for every $t < \omega_1^{ck}$. If $n > 1$, then by minimality of n, we necessarily have that $\{X_t\upharpoonright_2\}_{t<s}$

converges, otherwise $D(s,1)$ would be true. So for some i we have that $\{X_t\upharpoonright_2\}_{s_i\leqslant t<s}$ is stable. We continue inductively to prove that there is some i such that $\{X_t\upharpoonright_n\}_{s_i\leqslant t<s}$ is stable.

Let us now fix the integer m such that $\{\sigma_t^m\}_{s_i\leqslant t<s}$ is stable, and such that $\sigma_{s_j}^{m+1}\neq\sigma_{s_{j+1}}^{m+1}$ for $j\in\mathbb{N}$. We shall now prove that for at least one $k\leqslant m$ (presumably for $k=m$), the sequence $\{d_t^k\}_{s_i\leqslant t<s}$ does not converge. Suppose otherwise, that is, the sequence $\{d_t^k\mid k\leqslant m\}_{s_i\leqslant t<s}$ converges, then there is some $j\geqslant i$ such that $\{d_t^k\mid k\leqslant m\}_{s_j\leqslant t<s}$ is stable. But then for all t with $s_j\leqslant t<s$ we have $A(m,t)$ and then we also have $A(m,s)$. Then using Claim 2 with $\bigcap_{k\leqslant m}\mathcal{F}_{e_s^k,d_s^k}[s]$ as the closed set $\mathcal{F}$, we have at least one string σ in P_s^{m+1} extending σ_s^m such that $A(m,s,\sigma)$ is true and then such that $A(m,t,\sigma)$ is true for every t with $s_j\leqslant t<s$. Also this contradicts that $\{\sigma_t^{m+1}\}_{s_i\leqslant t<s}$ does not converge.

So let $k\leqslant m$ be the smallest integer such that $\{d_t^k\}_{s_i\leqslant t<s}$ does not converge, equivalently $\lim_{t<s}d_t^k=\infty$. In particular we have $A(k-1,s,\sigma_s^k)$, but there is no d large enough such that $A(k-1,s,\sigma_s^k,d)$. This is only possible if $\lambda(\mathcal{S}_{e_s^k})[s]<1$. Then at stage $s+1$ we have that $\sigma_s^k\preccurlyeq\sigma_s^m\prec X_s\upharpoonright_n$ is banished, that is, removed from P_s^k.

It follows that we have $X\upharpoonright_n\neq X_s\upharpoonright_n$. Thus, by minimality of n, for every n' such that $D(s,n')$ is true, we have $X\upharpoonright_{n'}\neq X_s\upharpoonright_{n'}$.

Claim 5: The sequence X is higher weakly-2-random.

It is clear that if $\lambda(\mathcal{S}_n)=1$, then $e^{n+1}=\lim_{s<\omega_1^{ck}}e_s^{n+1}$ is equal to n. Therefore any sequence in $\bigcap_n\mathcal{S}_{e^n}$ is higher weakly-2-random. We shall then simply prove that we have $X\in\bigcap_n\mathcal{S}_{e^n}$.

Let s_n be the smallest ordinal such that $\{(e_t^k,d_t^k)\mid k\leqslant n\}_{s_n\leqslant t<\omega_1^{ck}}$ is stable and equal to $\{(e^k,d^k)\mid k\leqslant n\}$. In particular we have that $\mathcal{A}=\{X_{s_n}\}_{n\in\mathbb{N}}\cup\{X\}$ is a closed set and that $\bigcap_{k\leqslant n}\mathcal{F}_{e^k,d^k}\cap\mathcal{A}$ is not empty because it contains X_{s_n}. Then also $\bigcap_{k\in\mathbb{N}}\mathcal{F}_{e^k,d^k}\cap\mathcal{A}$ is not empty and it then contains X, as it is the only non Δ_1^1 point of $\mathcal{A}$.

7. Π_1^1-RANDOMNESS

7.1. **The Borel complexity of the set of Π_1^1-randoms.** For a while not much was known about Π_1^1-randomness, mainly because the community did not have an angle of attack. This came with the work of Monin [36] who found a decomposition of the largest Π_1^1 nullset into simpler objects, objects that computability theorists are used to work with. Monin defined for this two genericity notions equal respectively to higher weak-2-randomness and Π_1^1-randomness. This then helped to answer several open questions.

Definition 7.1 (Monin [36])**.** We say that X is weakly-Σ_1^1-Solovay-generic if it belongs to all sets of the form $\bigcup_n\mathcal{F}_n$ which intersect with positive measure all the Σ_1^1-closed sets of positive measure, where each $\mathcal{F}_n$ is a Σ_1^1-closed set uniformly in n.

Definition 7.2 (Monin [36])**.** We say that X is Σ_1^1-Solovay-generic if for any set of the form $\bigcup_n\mathcal{F}_n$ where each $\mathcal{F}_n$ is a Σ_1^1-closed set uniformly in n, either X is in $\bigcup_n\mathcal{F}_n$ or X is in some Σ_1^1-closed set of positive measure $\mathcal{F}$, disjoint from $\bigcup_n\mathcal{F}_n$.

Proposition 7.3 (Monin [36])**.** *A sequence X is weakly-Σ_1^1-Solovay-generic iff it is higher weakly-2-random.*

Proof. Note first that X is higher weakly-2-random iff it is in every uniform union of Σ_1^1-closed sets of measure 1. We shall prove that a uniform union of Σ_1^1-closed sets is of measure 1 iff it intersects with positive measure every Σ_1^1-closed set of positive measure.

Let us prove that a uniform union of Σ_1^1 closed sets of measure less than 1 cannot intersect all Σ_1^1-closed sets of positive measure. Let $\bigcup_n\mathcal{F}_n$ be a uniform union of Σ_1^1-closed sets of measure strictly smaller than 1. Let $\bigcap_n\mathcal{U}_n$ be its complement. We shall prove that already for some computable s we have that $\bigcap_n\mathcal{U}_{n,s}$ is of positive measure. We actually have that $\mathcal{A}=\bigcap_n\mathcal{U}_n-\bigcup_{s<\omega_1^{ck}}\bigcap_n\mathcal{U}_{n,s}\subseteq\{X\ :\ \omega_1^X>\omega_1^{ck}\}$. Indeed, if $X\in\mathcal{A}$ then the $\Pi_1^1(X)$ total function which to n associates the smallest s such that $X\in\bigcap_{m\leqslant n}\mathcal{U}_{m,s}$ has its range unbounded in ω_1^{ck}, implying that $\omega_1^X>\omega_1^{ck}$. Also using Theorem 3.11 saying that $\lambda(\{X\ :\ \omega_1^X>\omega_1^{ck}\})=0$ we then have $\lambda(\bigcap_n\mathcal{U}_n)=\lambda(\bigcup_{s<\omega_1^{ck}}\bigcap_n\mathcal{U}_{n,s})$, and as $\lambda(\bigcap_n\mathcal{U}_n)>0$, there exists then some s such that $\lambda(\bigcap_n\mathcal{U}_{n,s})>0$. Also $\bigcap_n\mathcal{U}_{n,s}$ is a Δ_1^1 set of positive measure, and then by Theorem 3.6 there exists a Δ_1^1-closed set of positive measure $\mathcal{F}\subseteq\bigcap_n\mathcal{U}_{n,s}\subseteq\bigcap_n\mathcal{U}_n$. Thus $\bigcup_n\mathcal{F}_n$ does not intersect all Σ_1^1-closed sets of positive measure.

Conversely a uniform union of Σ^1_1-closed sets of measure 1 obviously intersects with positive measure any Σ^1_1-closed set of positive measure. Then the weakly-Σ^1_1-Solovay-generics are exactly the higher weakly-2-randoms. □

We shall now prove that the notion of Σ^1_1-Solovay-genericity coincides with the notion of Π^1_1-randomness. We already know from Theorem 3.14 that if X is higher weakly-2-random but not Π^1_1-random, then $\omega_1^X > \omega_1^{ck}$. We first should prove that if X is Σ^1_1-Solovay-generic then $\omega_1^X = \omega_1^{ck}$ (this is the difficult part of the equivalence).

Note first that $\omega_1^X > \omega_1^{ck}$ iff there is $a \in O^X$ such that $|a|_o^X = \omega_1^{ck}$. In particular, $\omega_1^X > \omega_1^{ck}$ iff there is a Turing functional $\Phi : 2^{<\mathbb{N}} \times \mathbb{N} \to \mathbb{N}$ such that for any n we have $\Phi(X,n) \in O^X_{<\omega_1^{ck}}$ and with $\sup_n |\Phi(X,n)|_o^X = \omega_1^{ck}$. We should show that if X is Σ^1_1-Solovay-generic and if we have some Φ such that $\Phi(X,n) \in O^X_{<\omega_1^{ck}}$ for all n, then $\sup_n |\Phi(X,n)|_o^X < \omega_1^{ck}$. To show this we need an approximation lemma, which can be seen as an extension of Theorem 3.6, saying that any Δ^1_1 set can be approximated from below by a uniform union of Δ^1_1-closed sets of the same measure. We cannot extend this to all Σ^1_1 sets, but we can for a restricted type of Σ^1_1 set:

Lemma 7.4. For a Σ^1_1 set $\mathcal{S} = \bigcap_{\alpha<\omega_1^{ck}} \mathcal{S}_\alpha$ where each $\mathcal{S}_\alpha$ is Δ^1_1 uniformly in α, one can find uniformly in an index for $\mathcal{S}$ and in any n, a Σ^1_1 closed set $\mathcal{F} \subseteq \mathcal{S}$ with $\lambda(\mathcal{S} - \mathcal{F}) \leqslant 2^{-n}$.

Proof. Recall that $p : \omega_1^{ck} \to \omega$ is the projectum function. Using Theorem 3.6, one can find uniformly in $\alpha < \omega_1^{ck}$ a Δ^1_1-closed set $\mathcal{F} \subseteq \mathcal{S}_\alpha$ such that $\lambda(\mathcal{S}_\alpha - \mathcal{F}_\alpha) \leqslant 2^{-p(\alpha)}2^{-n}$. We now define the Σ^1_1-closed set $\mathcal{F}$ to be $\bigcap_\alpha \mathcal{F}_\alpha$. We clearly have $\mathcal{F} \subseteq \mathcal{S}$ and we have:

$$\begin{array}{rcl} \lambda(\mathcal{S} - \mathcal{F}) & = & \lambda(\mathcal{S} - \bigcap_{\alpha<\omega_1^{ck}} \mathcal{F}_\alpha) \\ & = & \lambda(\bigcup_{\alpha<\omega_1^{ck}} (\mathcal{S} - \mathcal{F}_\alpha)) \\ & \leqslant & \lambda(\bigcup_{\alpha<\omega_1^{ck}} (\mathcal{S}_\alpha - \mathcal{F}_\alpha)) \\ & \leqslant & \sum_{\alpha<\omega_1^{ck}} \lambda(\mathcal{S}_\alpha - \mathcal{F}_\alpha) \leqslant 2^{-n}. \end{array}$$

□

We can now prove the desired theorem:

Theorem 7.5 (Monin [36])**.** *If Y is Σ^1_1-Solovay-generic then $\omega_1^Y = \omega_1^{ck}$.*

Proof. Suppose that Y is Σ^1_1-Solovay-generic. For any functional Φ, consider the set

$$\mathcal{P} = \{X \mid \forall n \ \ \exists \alpha < \omega_1^{ck} \ \ \Phi(X,n) \in O^X_\alpha\}$$

Let $\mathcal{P}_n = \{X \mid \exists \alpha < \omega_1^{ck} \ \ \Phi(X,n) \in O^X_\alpha\}$ and $\mathcal{P}_{n,\alpha} = \{X \mid \Phi(X,n) \in O^X_\alpha\}$, so $\mathcal{P} = \bigcap_n \mathcal{P}_n$ and $\mathcal{P}_n = \bigcup_{\alpha<\omega_1^{ck}} \mathcal{P}_{n,\alpha}$.

Note that the complement of each $\mathcal{P}_n$ is a restricted type of Σ^1_1 set, on which we can then apply Lemma 7.4. So we can find uniformly in n a uniform union of Σ^1_1-closed sets included in $\mathcal{P}_n^c$ with the same measure as $\mathcal{P}_n^c$. From this we can find a uniform union of Σ^1_1-closed sets included in $\mathcal{P}^c$ with the same measure as $\mathcal{P}^c$. Suppose that Y is in $\mathcal{P}$. As it is Σ^1_1-Solovay-generic we have a Σ^1_1-closed set $\mathcal{F}$ of positive measure containing Y which is disjoint from $\mathcal{P}^c$ up to a set of measure 0, formally $\lambda(\mathcal{F} \cap \mathcal{P}^c) = 0$. In particular for each n we have $\lambda(\mathcal{F} \cap \mathcal{P}_n^c) = 0$ and then $\lambda(\mathcal{F}^c \cup \mathcal{P}_n) = 1$. Then let f be the Π^1_1 total function which to each pair $\langle n, m\rangle$ associates the smallest computable ordinal $\alpha < \omega_1^{ck}$ such that:

$$\lambda(\mathcal{F}_\alpha^c \cup \mathcal{P}_{n,\alpha}) > 1 - 2^{-m}$$

where $\{\mathcal{F}_\alpha^c\}_{\alpha<\omega_1^{ck}}$ is the co-enumeration of $\mathcal{F}^c$. Let $\alpha^* = \sup_{n,m} |f(n,m)|$. As f is total and Π^1_1, we have by admissibility that $\alpha^* < \omega_1^{ck}$. Also

$$\begin{array}{rrcl} & \forall n \ \ \lambda(\mathcal{F}_{\alpha^*}^c \cup \bigcup_{\alpha<\alpha^*} \mathcal{P}_{\alpha,n}) & = & 1 \\ \to & \forall n \ \ \lambda(\mathcal{F}_{\alpha^*} \cap \bigcap_{\alpha<\alpha^*} \mathcal{P}_{\alpha,n}^c) & = & 0 \\ \to & \forall n \ \ \lambda(\mathcal{F} - \bigcup_{\alpha<\alpha^*} \mathcal{P}_{\alpha,n}) & = & 0 \\ \to & \lambda(\mathcal{F} - \bigcap_n \bigcup_{\alpha<\alpha^*} \mathcal{P}_{\alpha,n}) & = & 0 \end{array}$$

As Y is Σ^1_1-Solovay-generic it is in particular weakly-Σ^1_1-Solovay-generic and then higher weakly-2-random. Thus by Theorem 3.2 it belongs to no Σ^1_1 set of measure 0. Then as $\mathcal{F} - \bigcap_n \bigcup_{\alpha<\alpha^*} \mathcal{P}_{\alpha,n}$ is a Σ^1_1 set of measure 0 we have that Y belongs to $\bigcap_n \bigcup_{\alpha<\alpha^*} \mathcal{P}_{\alpha,n}$ and then $\sup_n |\Phi(Y,n)|_o^Y \leqslant \alpha^* < \omega_1^{ck}$. □

We can now prove the equivalence:

Theorem 7.6 (Monin [36])**.** *The set of Σ^1_1-Solovay-generics coincides with the set of Π^1_1-randoms.*

Proof. Using Theorem 3.14 combined with the previous theorem, we have that the Σ^1_1-Solovay-generics are included in the Π^1_1-randoms. We just have to prove the reverse inclusion.

Suppose Y is not Σ^1_1-Solovay-generic. If $\omega_1^Y > \omega_1^{ck}$ then Y is not Π^1_1-random. Otherwise $\omega_1^Y = \omega_1^{ck}$ and also there is a sequence of Σ^1_1-closed sets $\bigcup_n \mathcal{F}_n$ of positive measure such that Y is not in $\bigcup_n \mathcal{F}_n$ and such that any Σ^1_1-closed set of positive measure which is disjoint from $\bigcup_n \mathcal{F}_n$ does not contain Y. Let $\bigcap_n \mathcal{U}_n$ be the complement of $\bigcup_n \mathcal{F}_n$. As $\omega_1^Y = \omega_1^{ck}$ we have that $Y \in \bigcap_n \mathcal{U}_{n,s}$ for some computable ordinal s (the proof of this is like in the proof of Proposition 7.3). Also as $\bigcap_n \mathcal{U}_{n,s}$ is a Δ^1_1 set, either it is of measure 0 and then Y is not Δ^1_1-random, or it is of positive measure and can then be approximated from below, using Theorem 3.6 by a uniform union of Δ^1_1-closed sets, of the same measure. Also as Y is in none of them it is in their complement in $\bigcap_n \mathcal{U}_{n,s}$, which is a Δ^1_1-set of measure 0. Then Y is not Δ^1_1-random. □

The previous theorem gives a higher bound on the Borel complexity of the Π^1_1-randoms, and then on the Borel complexity of the largest Π^1_1 nullset.

Corollary 7.7 (Monin [36])**.** *The set of Π^1_1-randoms is $\mathbf{\Pi^0_3}$.*

The previous corollary, combined with a result of Liang Yu (see [40]) shows that the complexity of the set of Π^1_1-randoms is exactly $\mathbf{\Pi^0_3}$. Yu's result is an adaptation of one of its earlier result, showing that the set of weakly-2-randoms (in the lower settings) cannot be $\mathbf{\Sigma^0_3}$ [48].

Theorem 7.8 (Yu [40])**.** *Let $\mathbb{P}$ be the set of forcing condition consisting of Σ^1_1-closed sets containing only Π^1_1-Martin-Löf randoms, and ordered by reverse inclusion. Let $\bigcap_n \mathcal{U}_n$ be a $\mathbf{\Pi^0_2}$ set containing only higher weakly-2-randoms. Then the set $\{\mathcal{F} \in \mathbb{P} \mid \bigcap_n \mathcal{U}_n \cap \mathcal{F} = \varnothing\}$ is dense in $\mathbb{P}$.*

Proof. We first show that for any Σ^1_1-closed set $\mathcal{F}$, there is a uniform sequence of Π^1_1-open set $\bigcap_n \mathcal{V}_n$ such that:

(1) For every n we have $\lambda(\mathcal{F} \cap \mathcal{V}_n) \leqslant 2^{-n}$ (so in particular $\mathcal{F} \cap \bigcap_n \mathcal{V}_n$ is a higher difference test).
(2) For any σ, if $\mathcal{F} \cap [\sigma] \neq \varnothing$, then $\mathcal{F} \cap [\sigma] \cap \bigcap_n \mathcal{V}_n \neq \varnothing$

As stage s, for every σ, we put in $\mathcal{V}_n$ the leftmost extension of σ of length $2|\sigma| + n + 1$ which is in $\mathcal{F}[s]$ (if it exists). Note that for every σ, there is at most one string of length $2|\sigma| + n + 1$ which is in $\mathcal{F}$. It follows that $\lambda(\mathcal{F} \cap \mathcal{V}_n) \leqslant \sum_{m \in \mathbb{N}} \sum_{|\sigma|=m} \leqslant 2^{-2m-n-1} \leqslant \sum_{m \in \mathbb{N}} 2^{-m-n-1} \leqslant 2^{-n}$. Note also that for any σ, the set $\bigcap_n \mathcal{V}_n$ contains the leftmost path of $\mathcal{F}$ if this leftmost path exists.

Consider now a Σ^1_1-closed set $\mathcal{F}$ only Π^1_1-Martin-Löf randoms together with the set $\bigcap_n \mathcal{V}_n$ of the previous paragraph. Suppose that for every σ such that $\mathcal{F} \cap [\sigma]$ is not empty, then $\bigcap_n \mathcal{U}_n$ intersects $\mathcal{F} \cap [\sigma]$. Then both $\bigcap_n \mathcal{U}_n$ and $\bigcap_n \mathcal{V}_n$ are dense in $\mathcal{F}$ (for the partial order of strings). In particular $\bigcap_n \mathcal{U}_n \cap \bigcap_n \mathcal{V}_n$ is dense in $\mathcal{F}$ and thus there must be an element $X \in \bigcap_n \mathcal{U}_n \cap \bigcap_n \mathcal{V}_n \cap \mathcal{F}$. As $X \in \mathcal{F} \cap \bigcap_n \mathcal{V}_n$, it follows that X is not higher difference random. But then X is not higher weakly-2-random which contradicts $X \in \bigcap_n \mathcal{U}_n$. It follows that there must exists σ such that $\sigma \cap \mathcal{F}$ is not empty but such that $\bigcap_n \mathcal{U}_n \cap \mathcal{F} \cap [\sigma]$ is empty. □

It follows that the set of higher weakly-2-randoms cannot be $\mathbf{\Sigma^0_3}$ but also that the set of Π^1_1-randoms cannot be $\mathbf{\Sigma^0_3}$, and more generally:

Corollary 7.9 (Yu [40])**.** *No set $\mathcal{A}$ containing the set of Π^1_1-random sequences and contained in the set of higher weakly-2-random sequences is $\mathbf{\Sigma^0_3}$.*

Proof. Suppose that such a set $\mathcal{A}$ is equal to $\bigcup_n \bigcap_m \mathcal{U}_{n,m}$ each $\mathcal{U}_{n,m}$ being open. Let $\mathbb{P}$ be the partial order of Theorem 7.8. For each n let $\mathcal{B}_n = \bigcup\{\mathcal{F} \in \mathbb{P} \mid \bigcap_m \mathcal{U}_{n,m} \cap \mathcal{F} = \varnothing\}$. We have $\bigcap_n \mathcal{B}_n \cap \bigcup_n \bigcap_m \mathcal{U}_{n,m} = \varnothing$. Also each set $\bigcap_m \mathcal{U}_{n,m}$ is a $\mathbf{\Pi^0_2}$ set containing only higher weakly-2-randoms. Therefore by Theorem 7.8 we have that $\bigcap_n \mathcal{B}_n$ contains some Solovay-Σ^1_1-generic element (some Π^1_1-random element), which contradicts that $\mathcal{A} = \bigcup_n \bigcap_m \mathcal{U}_{n,m}$ contains all of them. □

7.2. **Randoms with respect to (plain) Π^1_1-Kolmogorov complexity.** Monin deduced from Corollary 7.9 another interesting theorem. Before stating it, we need to introduce a few notions. In classical randomness, we can define a non prefix-free Kolmogorov complexity $C : 2^{<\mathbb{N}} \to \mathbb{N}$, also called plain complexity. Miller [34] together with Nies, Stephan, and Terwijn [41] proved that a sequence X is 2-random iff infinitely many prefixes of X have maximal plain Kolmogorov complexity. We can make a similar definition in the higher setting:

Definition 7.10. A Π^1_1-machine M is a Π^1_1 partial function $M : 2^{<\mathbb{N}} \to 2^{<\mathbb{N}}$. We denote by $\mathrm{C}_M(\sigma)$ the Π^1_1-Kolmorogov complexity of a string σ with respect to the Π^1_1-machine M, defined to be the length of the smallest string τ such that $M(\tau) = \sigma$, if such a string exists, and by convention, ∞ otherwise.

Just like we proved that there exists a universal Π^1_1-prefix-free machine (see Theorem 5.4) we can prove that there is a universal Π^1_1-machine (we leave the proof to the reader, as it is very similar to the proof of Theorem 5.4):

Theorem 7.11 (Universal Π^1_1-machine theorem)**.** *There is a universal Π^1_1-machine U, that is, for each Π^1_1-machine M, there exists a constant c_M such that $\mathrm{C}_U(\sigma) \leqslant \mathrm{C}_M(\sigma) + c_m$ for any string σ.*

We can then give a meaning to *the* Π^1_1-Kolmorogov complexity of a string:

Definition 7.12. For a string σ, we define $\mathrm{C}(\sigma)$ to be $\mathrm{C}_U(\sigma)$ for a universal Π^1_1-machine U, fixed in advance.

Let us now define the set $\mathcal{A}$ of sequences which have infinitely many prefixes of maximal Π^1_1-Kolmogorov complexity:

$$\mathcal{A} = \{X \mid \exists c\ \forall n\ \exists m \geqslant n\ \mathrm{C}(X{\upharpoonright}_m) \geqslant m - c\}$$

Proposition 7.13. *The set $\mathcal{A}$ contains the Π^1_1-randoms and is contained in the Π^1_1-Martin-Löf randoms.*

Proof. It is clear that $\mathcal{A}$ is a Σ^1_1 set. So to show that it contains the Π^1_1-randoms, it is enough to show that it is of measure 1. For every length n, there are at most $\sum_{i\leqslant n-c-1} 2^i = 2^{n-c}$ strings of length smaller than or equal to $n-c-1$. Thus the number of strings σ of length n such that $\mathrm{C}(\sigma) < n - c$ is at most of 2^{n-c}. Thus the measure of the clopen set generated by these strings is at most of 2^{-c}. It follows that for any c, n we have $\lambda(\{X \mid \forall m \geqslant n\ \mathrm{C}(X{\upharpoonright}_m) < m - c\}) < 2^{-c}$. Also for $n_1 \leqslant n_2$ we have $\{X \mid \forall m \geqslant n_1\ \mathrm{C}(X{\upharpoonright}_m) < m - c\} \subseteq \{X \mid \forall m \geqslant n_2\ \mathrm{C}(X{\upharpoonright}_m) < m - c\}$. Thus we have $\lambda(\{X \mid \exists n\ \forall m \geqslant n\ \mathrm{C}(X{\upharpoonright}_m) < m - c\}) < 2^{-c}$. It follows that the measure of $\mathcal{A}$ must be 1. In particular $\mathcal{A}$ contains the set of Π^1_1-randoms.

Let us argue that $\mathcal{A}$ is contained in the set of Π^1_1-Martin-Löf randoms. Indeed, given a prefix-free machine M such that $\forall c\ \exists n\ \mathrm{K}_M(X{\upharpoonright}_n) < n - c$, one can build the machine N which on any string σ look for strings τ_1, τ_2 with $\sigma = \tau_1\tau_2$ such that $M(\tau_1)\downarrow$ and then output $M(\tau_1)\tau_2$. Now given τ_1 of length smaller $n - c$ such that $M(\tau_1) = X{\upharpoonright}_n$, we clearly have that N compresses every string $X{\upharpoonright}_m$ by at least c for every $m \geqslant n$. ☐

It follows directly from Corollary 7.9 that $\mathcal{A}$ does not coincide with the set of Π^1_1-randoms or with the set of higher weakly-2-randoms:

Proposition 7.14 (Monin [35] Section 6.2)**.** *The set $\mathcal{A}$ strictly contains the set of Π^1_1-randoms. The set $\mathcal{A}$ is not contained in the set of higher weakly-2-randoms.*

Proof. The set $\mathcal{A}$ is easily seen to be $\mathbf{\Sigma^0_3}$. The results follows then from Corollary 7.9. ☐

The following question remains open:

Question 7.15. *Does the set $\mathcal{A}$ contain the higher weakly-2-randoms?*

7.3. Lowness an cupping for Π^1_1-randomness.

7.3.1. *Lowness for Π^1_1-randomness.* Greenberg and Monin could use Theorem 7.6 to solve the question of lowness for Π^1_1-randomness [38, question 9.4.11]: Is there some sequence A which is not Δ^1_1 and such that the largest $\Pi^1_1(A)$ set equals the largest Π^1_1 set? They answered the question by the negative, in a strong sense.

Theorem 7.16 (Greenberg, Monin [15])**.** *If A is not hyperarithmetic, then some Π^1_1-random is not $\Pi^1_1(A)$-Martin-Löf random.*

Greenberg and Monin also improved this result with Theorem 7.21 by showing that a non-hyperarithmetic A can be cupped above O with a Π^1_1-random sequence Z, that is, $Z \oplus A \geqslant_h O$. However the direct proof of Theorem 7.16 is simpler and we believe is interesting in its own right. Indeed the second proof elaborates on the simpler one. The proof can be transfered in a straightforward way to the lower setting, simplifying the proof that a non K-trivial is not low for weak-2-randomness [9].

The proof is also based on Hjorth and Nies's Corollary 5.32 : only the Δ^1_1 sets are low for Π^1_1-Martin-Löf-randomness (with full relativisation). Our first step is a higher version of Kjos-Hanssen's characterization of lowness for Martin-Löf randomness [21].

Lemma 7.17. Suppose that A is not hyperarithmetic. Let $\mathcal{U}$ be a $\Pi^1_1(A)$-open set which contains all reals which are not $\Pi^1_1(A)$-Martin-Löf-random. Then $\mathcal{U}$ intersects with positive measure every Σ^1_1-closed set of positive measure.

Proof. As mentioned, we use the fact that A is not low for Π^1_1-Martin-Löf-randomness. Let X be a Π^1_1-Martin-Löf random which is not $\Pi^1_1(A)$-Martin-Löf-random. Let $\mathcal{P}$ be a non-null Σ^1_1 closed set. By Kučera's Proposition 5.13, there is a tail Y of X in $\mathcal{P}$. Since Y is not $\Pi^1_1(A)$-Martin-Löf-random, $Y \in \mathcal{U}$, so $\mathcal{U} \cap \mathcal{P} \neq \varnothing$. Also this intersection must have positive measure: for $\sigma \prec Y$ and $[\sigma] \subseteq \mathcal{U}$, we have that $[\sigma] \cap \mathcal{P}$ is a non-empty Σ^1_1-closed set containing Y. As Y is Π^1_1-Martin-Löf random then we must have $\lambda([\sigma] \cap \mathcal{P}) > 0$. □

Proof of Theorem 7.16. Let $A \notin \Delta^1_1$; let $\{\mathcal{U}_n\}$ be the universal $\Pi^1_1(A)$-Martin-Löf test. Let $\mathbb{P}$ be the set of forcing condition consisting of Σ^1_1-closed set containing only Π^1_1-Martin-Löf randoms, and ordered by reverse inclusion. By Lemma 7.17, for every n, the set of elements of $\mathbb{P}$ included in $\mathcal{U}_n$, is dense in $\mathbb{P}$. It follows that if a sequence of conditions $p_1 > p_2 > p_3 > \dots$ is sufficiently generic, then every element of $\bigcap_n [p_n]$ is a member of $\bigcap_n \mathcal{U}_n$.

By Theorem 7.6 if a sequence of condition $p_1 > p_2 > p_3 > \dots$ is sufficiently generic, then every element of $\bigcap_n [p_n]$ is Π^1_1-random.

It follows that there are Π^1_1-random elements in $\bigcap_n \mathcal{U}_n$. □

7.3.2. *Cupping with a Π^1_1-random.*

Definition 7.18 (Chong, Nies, Yu [5])**.** A real X is Π^1_1-random cuppable if there is a Π^1_1-random sequence Z such that $X \oplus Z \geqslant_h O$.

Chong, Nies and Yu, together with Harrington and Slaman proved in [5] a theorem making an interesting connection between lowness for Π^1_1-randomness and lowness for Δ^1_1-randomness: A sequence Z is low for Π^1_1-randomness iff it is low for Δ^1_1-randomness and non Π^1_1-random cuppable. Later Greenberg and Monin showed [15] that every non-Δ^1_1 real is Π^1_1-random cuppable. Note that if $A \oplus Z \geqslant_h O$, then $\omega_1^{A \oplus Z} > \omega_1^{ck}$ and that the set $\{Z : \omega_1^{A\oplus Z} > \omega_1^{ck}\}$ is a $\Pi^1_1(A)$ nullset. Thus if A is Π^1_1-random cuppable it is not low for Π^1_1-randomness. It implies that Greenberg and Monin's result strengthen Theorem 7.16. They actually even showed something stronger : If A is not Δ^1_1, then A can join with a Π^1_1-random above any degree. This cupping result is very similar to another cupping result of Greenberg, Miller, Monin and Turetsky [13]; they show that if $A \not\leqslant_{\mathrm{LR}} B$ then A can be cupped (in the Turing degrees) with B-Martin-Löf-randoms arbitrarily high. Before we continue, we need to show two lemmas. The first one is the same as in [13].

Lemma 7.19 (Greenberg, Monin [15]). Let W be a set of strings such that $\lambda([W]^\prec) < 0.1$ and such that $[W]^\prec$ intersects every Σ^1_1-closed set of positive measure. For any string τ and any Σ^1_1-closed set $\mathcal{P}$ such that $\lambda(\mathcal{P} \mid \tau) > 0.1$ there is some $\sigma \in W$ such that $\lambda(\mathcal{P} \mid \tau\sigma) \geqslant 0.8$.

Proof. First we find an extension ρ of τ such that ρ extends no string in τW (where $\tau W = \{\tau\sigma : \sigma \in W\}$), and such that $\lambda(\mathcal{P} \mid \rho) > 0.9$. This is done with the Lebesgue density theorem. Letting $\mathcal{G} = 2^{\mathbb{N}} - [\tau W]^\prec$, as $\lambda(\mathcal{G} \mid \tau) > 0.9$ and $\lambda(\mathcal{P} \mid \tau) > 0.1$, we must have $\lambda(\mathcal{G} \cap \mathcal{P} \mid \tau) > 0$ and by the Lebesgue density theorem there is an extension ρ of τ such that $\lambda(\mathcal{G} \cap \mathcal{P} \mid \rho) > 0.9$. In particular we must have $\lambda(\mathcal{P} \mid \rho) > 0.9$ and $\mathcal{G} \cap [\rho]$ is nonempty. In particular ρ cannot extend a string in τW.

Next we find an extension ν of ρ such that $\nu \in \tau W$ and such that $\lambda(\mathcal{P} \mid \nu) \geqslant 0.8$ as required. We let $\mathcal{Q}$ be the Σ^1_1-closed subset obtained from $\mathcal{P} \cap [\rho]$ by removing all cylinders in which the measure of $\mathcal{P}$ drops below 0.8. Formally

$$\mathcal{Q} = \left\{X \in \mathcal{P} \cap [\rho] : \forall n \geqslant |\rho| \ \left(\lambda(\mathcal{P} \mid X\upharpoonright_n) \geqslant 0.8\right)\right\}.$$

By considering the antichain of minimal strings removed we see that $\lambda(\mathcal{P} - \mathcal{Q} \mid \rho) \leqslant 0.8$. Since $\lambda(\mathcal{P} \mid \rho) > 0.9$ we see that $\lambda(\mathcal{Q} \mid \rho) > 0.1$. In particular, $\mathcal{Q}$ is a positive measure Σ^1_1 subset of $[\tau]$, and so by hypothesis on W, we have that $[\tau W]^\prec$ intersects $\mathcal{Q}$. Choose $\nu \in \tau W$ such that $[\nu] \cap \mathcal{Q} \neq \varnothing$. Note that we must have $\nu \succ \rho$ because ρ extends no string in τW. Thus by the definition of $\mathcal{Q}$ we have $\lambda(\mathcal{P} \mid \nu) \geqslant 0.8$. □

The second one is needed in order to deal with the usual topological issues that one have with higher computability.

Lemma 7.20 (Greenberg, Monin [15]). Let $\mathcal{U}$ be a Π^1_1-open set. Then for every $\varepsilon > 0$ there is a Π^1_1 set of strings W (and a higher effective enumeration $\{W_s\}$ of W) such that:

- $[W]^\prec$ equals $\mathcal{U}$ up to a set of measure 0.
- For every $s < \omega_1^{\mathrm{ck}}$, if $\sigma \in W_{s+1} - W_s$ then $\lambda([W_s]^\prec \mid \sigma) < \varepsilon$.

Proof. Let U be a Π^1_1 set of strings generating $\mathcal{U}$. As above we assume that at most one string enters U at each stage. We enumerate W: say $\sigma \in U_{s+1} - U_s$. Let

$$G_s = \{\tau \succcurlyeq \sigma \ : \ \lambda(\mathcal{U}_s \mid \tau) < \varepsilon\}.$$

This is Δ^1_1. We then eumerate in W_{s+1} a Δ^1_1 prefix-free set of strings which generates $[G_s]^\prec$. Note that $[W_s]^\prec \subseteq \mathcal{U}_s$ (and so $[W]^\prec \subseteq \mathcal{U}$).

By induction on s we show that $\lambda(\mathcal{U}_s - [W_s]^\prec) = 0$. Suppose it is true at stage s and let us show it is true at stage $s+1$. It suffices to show that for $\sigma \in \mathcal{U}_{s+1} - \mathcal{U}_s$ we have that $[\sigma]$ equals $[G_s]^\prec \cup ([W_s]^\prec \cap [\sigma])$ up to a set of measure 0. Suppose not. Then by the Lebegue density theorem there is some $\tau \succcurlyeq \sigma$ such that $\lambda([G_s]^\prec \cup [W_s]^\prec \mid \tau) < \varepsilon$. Since by induction hypothesis we have $\lambda(\mathcal{U}_s - [W_s]^\prec) = 0$ we then have $\lambda(\mathcal{U}_s \mid \tau) < \varepsilon$ which implies that $\tau \in G_s$, which is a contradiction.

It remains to show that $\lambda([W_s]^\prec \mid \tau) < \varepsilon$ for any $\tau \in W_{s+1} - W_s$. But such τ is an element of G_s, so $\lambda(\mathcal{U}_s \mid \tau) < \varepsilon$, and $\mathcal{U}_s$ equals $[W_s]^\prec$ up to a set of measure 0. ☐

We can now show the cupping result:

Theorem 7.21 (Greenberg, Monin [15]). *If A is not Δ^1_1 then for all $Y \in 2^{\mathbb{N}}$ there is some Π^1_1-random Z such that $Y \leqslant_h A \oplus Z$.*

Proof. We are given A which is not hyperarithmetic and some $Y \in 2^{\mathbb{N}}$. Let $\mathcal{U}$ be a $\Pi^1_1(A)$-open set of measure less than 0.1, which contains all reals which are not $\Pi^1_1(A)$-random. Using Lemma 7.20 let W be a Π^1_1 set of strings such that $[W]^\prec$ equals $\mathcal{U}$ up to a set of measure 0 and such that for every $s < \omega_1^{\mathrm{ck}}$, if $\sigma \in W_{s+1} - W_s$ then $\lambda([W_s]^\prec \mid \sigma) < 0.1$. Let also $\mathcal{S}_1, \mathcal{S}_2, \dots$ be a list of $\mathbf{\Sigma^0_2}$ sets which are each the union of Σ^1_1-closed sets, co-null, and such that $\bigcap_k \mathcal{S}_k$ contains only Π^1_1-random sequences (this is given by Theorem 7.6). We construct Z as a sequence $Y(0)\sigma_0 Y(1)\sigma_1 \cdots$ with each $\sigma_n \in W$. To make Z Π^1_1-random we that $Z \in \bigcap_n \mathcal{S}_n$. To make sure that $Z \oplus A$ computes Y, we also makes sure that for each n with $\tau_n = Y(0)\sigma_0 Y(1)\sigma_1 \dots \sigma_{n-1} Y(n)$, we have that σ_{n+1} is the first string in W such that $\tau_n \sigma_{n+1} \prec Z$. The computation then works as follow : Suppose we have retried $Y(0), \dots Y(n)$ and $\sigma_0, \dots \sigma_{n-1}$ with $\tau_n = Y(0)\sigma_0 Y(1)\sigma_1 \dots \sigma_{n-1} Y(n)$. Then using A we enumerate W until we find a string $\sigma \in W$ such that $\tau_n \sigma \prec Z$. Then we must have $\sigma_n = \sigma$ and we must have that $Y(n+1)$ is the bit of Z following $\tau_n \sigma_n$.

We start with $\mathcal{P}_0 = 2^{\mathbb{N}}$ and $\tau_0 = Y(0)$. Suppose that at step n we have defined $\sigma_0, \dots, \sigma_{n-1} \in W$ and $\tau_0, \dots, \tau_n$ with $\tau_i = Y(0)\sigma_0 Y(1)\sigma_1 \cdots Y(n-1)\sigma_{i-1} Y(i)$ for every $i \leqslant n$. Suppose also that we have defined a Σ^1_1-closed set of positive measure $\mathcal{P}_n \subseteq \bigcap_{i \leqslant n} \mathcal{S}_i$ such that:

(1) $\lambda(\mathcal{P}_n \mid \tau_n) > 0.1$.
(2) For any $i \leqslant n$ for $s_i + 1$ the first stage such that $\sigma_i \in W_{s_i+1}$, we have $\mathcal{P}_n \cap ([\sigma_i] - [W_{s_i}]^\prec)$ is empty.

Let us define σ_n, τ_{n+1} and $\mathcal{P}_{n+1}$ such that (1) and (2) are still true at step $n+1$. By Lemma 7.19 there exists a string $\sigma_n \in W$ such that $\lambda(\mathcal{P}_n \mid \tau_n \sigma_n) \geqslant 0.8$. Now let $\tau_{n+1} = \tau_n \sigma_n Y(n+1)$. It is clear that we must have $\lambda(\mathcal{P}_n \mid \tau_n \sigma_n Y(n+1)) \geqslant 0.3$. Then let $\mathcal{P}'_n$ to be the intersection of $\mathcal{P}_n$ together with $[\tau_n \sigma_n] - [\tau_n W_{s_n}]^\prec$ where $s_n + 1$ is the smallest stage such that $\sigma_n \in W$. By the choice of W we have that $\lambda(\mathcal{P}'_n \mid \tau_n \sigma_n) \geqslant 0.2$. Finally we find a Σ^1_1 closed set $\mathcal{F} \subseteq \mathcal{S}_{n+1}$ of measure sufficiently close to 1, so that $\lambda(\mathcal{P}_{n+1} \mid \tau_n \sigma_n) \geqslant 0.1$ for $\mathcal{P}_{n+1} = \mathcal{P}_n \cap \mathcal{F}$. ☐

7.4. **Π^1_1-randomness with respect to different measures.** Algorithmic randomness has been studied with respect to different measures. As long as a measure μ is computable, the definitions of randomness with respect to μ are the same but with replacing λ by μ. When the measure μ is not computable, it makes sense to have access to the measure to define the tests. For instance to show that there is a universal Martin-Löf test, it is important to have access to the measure. The problem is that the measure is a complex object, and in particular, there is not necessarily a smallest representation of a measure in the Turing degree. This has been showed by Day and Miller in [8], building upon some work of Levin [29].

Several authors could overcome this issue in two different ways that turned out to be equivalent (see [12] [18] and [8]) : either one can extend the notions of computability to metric spaces (in particular the metric space of probability measures) and define randomness notions accordingly, or one can define X to be non-random with respect to a measure μ if for any representation $\hat{\mu}$ of μ, the sequence X is captured by a μ-random test that uses $\hat{\mu}$ as an oracle.

Reimann and Slaman showed [43] that X is not computable iff it is Martin-Löf random with respect to a measure μ such that $\mu(\{X\}) = 0$. Reimann and Slaman also showed that there are some non-computable sequences X such that for any measure μ for which $\mu(\{X\}) = 0$, if X is Martin-Löf random with respect to μ, then μ must concentrate positive measure on some single points (we call these points atoms of the measure).

Reimann and Slaman then defined the class NCR of element which are Martin-Löf random with respect to no continuous measure, that is, measures with no atoms. They showed that this class is a subclass of the Δ^1_1 sequences.

Following this work, Chong and Yu [6] studied the class of elements which are not Π^1_1-random with respect to any continuous measure.

Definition 7.22 (Chong, Yu [6])**.** Given a representation $\hat{\mu} \in 2^{\mathbb{N}}$ of a measure μ, we say that X is Π^1_1-random relative to $\hat{\mu}$ if it does not belong to any $\Pi^1_1(\hat{\mu})$ set $\mathcal{A}$ with $\mu(\mathcal{A}) = 0$.

Definition 7.23 (Chong, Yu [6])**.** We say that $Z \in 2^{\mathbb{N}}$ is Π^1_1-random relative to a measure μ if there exists a representation $\hat{\mu}$ of μ such that Z is Π^1_1-random relative to $\hat{\mu}$.

Definition 7.24 (Chong, Yu [6])**.** The class $NCR_{\Pi^1_1}$ is the class of element $X \in 2^{\mathbb{N}}$ such that for any continuous measure μ, X is not Π^1_1-random relative to μ.

A well known set of higher computability is the largest Π^1_1 set which contains no perfect subset. This is the set:

$$\mathcal{C} = \{X \in 2^{\mathbb{N}} \ : \ X \in L_{\omega_1^X}\}$$

Theorem 7.25 (Mansfield [31] Solovay [47])**.** *The set $\mathcal{C}$ is the largest Π^1_1 set which contains no perfect subset.*

Chong and Yu then provided another characterization of $\mathcal{C}$:

Theorem 7.26 (Chong, Yu [6])**.**

$$NCR_{\Pi^1_1} = \mathcal{C}$$

The theorem follows from the two following lemmas:

Lemma 7.27 (Chong, Yu [6]). *$NCR_{\Pi^1_1}$ is a Π^1_1 set which contains no perfect subset. Therefore $NCR_{\Pi^1_1} \subseteq \mathcal{C}$.*

Proof. Let us show that $NCR_{\Pi^1_1}$ is a Π^1_1. Given any representation $\hat{\mu}$ of a measure, one can define uniformly in $\hat{\mu}$ the largest $\Pi^1_1(\hat{\mu})$ set $\mathcal{Q}_{\hat{\mu}}$ such that $\mu(\mathcal{Q}_{\hat{\mu}}) = 0$. To do so we need to adapt the proof of Theorem 3.11 to show that $\{X \ : \ \omega_1^{X \oplus \hat{\mu}} > \omega_1^{\hat{\mu}}\}$ is a $\Pi^1_1(\hat{\mu})$ set of μ measure 0. The proof relativizes with no difficulty. We then need to adapt the construction of the largest Π^1_1 nullset given in the proof of Theorem 3.15. Here again everything relativizes smoothly with no difficulty. Now we have:

$$X \in NCR_{\Pi^1_1} \leftrightarrow \forall \hat{\mu}(\hat{\mu} \text{ is te representation of a continuous measure } \rightarrow X \in \mathcal{Q}_{\hat{\mu}})$$

which is a Π^1_1 predicate.

Let us now show that $NCR_{\Pi^1_1}$ contains no perfect subset. Consider a perfect tree T. We define the measure μ as $\mu(2^{\mathbb{N}}) = 1$ and then inductively:

$$\begin{array}{rcll} \mu([\sigma i]) & = & \mu([\sigma]) & \text{if } \sigma(1-i) \notin T \\ \mu([\sigma i]) & = & 1/2\mu([\sigma]) & \text{otherwise} \end{array}$$

It is clear that μ is a continuous measure. It is also clear that we have $\mu([T]) = 1$. Note that for any representation $\hat{\mu}$ of μ, the set of elements of T which are Π^1_1-random relative to $\hat{\mu}$ is a set of μ-measure 1. It is also clear that μ has a smallest representation $\hat{\mu}$ in the Turing degree, i.e. a computable encoding of the set $\{(\sigma, n) \ : \ \mu(\sigma) = 2^{-n}\}$ (note that not all measure have a smallest representation in the Turing degree). It follows that the set of elements of which are Π^1_1-random relative to μ is the same as the set of element which are Π^1_1-random relative to $\hat{\mu}$. Therefore T contains elements which are not in $NCR_{\Pi^1_1}$. ☐

Lemma 7.28 (Chong, Yu [6]). *$\mathcal{C} \subseteq NCR_{\Pi^1_1}$*

Proof. Suppose $X \in C$. Let μ be any continuous measure with any representation $\hat{\mu}$. Suppose first that $\hat{\mu} \geqslant_h X$. Then $\{X\}$ is in a $\Delta^1_1(\hat{\mu})$ set. Also as μ is continuous, this $\Delta^1_1(\hat{\mu})$ set is of μ-measure 0. Thus X is not Π^1_1-random relative to $\hat{\mu}$. Suppose now $\hat{\mu} \not\geqslant_h X$. Note that $X \in L_{\omega_1^X}$. Also if $\omega_1^{\hat{\mu}} \geqslant \omega_1^X$ we must have $X \in L_{\omega_1^{\hat{\mu}}}$ and thus $\hat{\mu} \geqslant_h X$. It follows that $\omega_1^{\hat{\mu}} < \omega_1^X$. But the set $\{X : \omega_1^X > \omega_1^{\hat{\mu}}\}$ is a $\Pi^1_1(\hat{\mu})$ set of μ-measure 0. Thus X is not Π^1_1-random relative to $\hat{\mu}$. Then for any representation $\hat{\mu}$ of μ, the sequence X is not Π^1_1-random relative to $\hat{\mu}$. As this is true for any measure μ, we then have $X \in NCR_{\Pi^1_1}$. □

7.5. **Π^1_1-randomness and minimal pair with O.** Recall the following theorem of Downey, Nies, Weber and Yu (see [9]) of classical randomness: For a sequence Z Martin-Löf random the following are equivalent:

(1) Z is weakly-2-random.
(2) Z forms a minimal pair with $\emptyset^{(1)}$.
(3) Z does not compute any non-computable c.e. set.

A first higher counterpart of (1) ↔ (2) of the above would be: 'For Z Π^1_1-Martin-Löf random, Z is higher weakly-2-random iff Z forms a higher Turing minimal pair with Kleene's O'. But this cannot be true, as by the Gandy Basis theorem, there is a Π^1_1-random, and therefore a higher weakly-2-random, which is Turing computable by Kleene's O. However, we will be able instead to obtain a higher version of the equivalence (1) ↔ (3), but with Π^1_1-randomness in place of higher weak-2-randomness.

7.5.1. *Π^1_1-randomness and computing Π^1_1 sequences.* We shall prove here that a Π^1_1-Martin-Löf random Z is Π^1_1-random iff it does not higher Turing compute a Π^1_1 sequence which is not Δ^1_1. Note that by the separation of Π^1_1-randomness from higher weak-2-randomness, this implies that some higher weak-2-random sequences compute non-Δ^1_1 Π^1_1 sequences.

Theorem 7.29 (Greenberg, Monin [15])**.** *For a set Z Π^1_1-Martin-Löf random, the following are equivalent:*

(1) *Z is Π^1_1-random.*
(2) *Z does not higher Turing compute a Π^1_1 sequence which is not Δ^1_1.*

Proof. (1) $\Longrightarrow$ (2): This is the easy direction. Suppose that Z higher Turing computes a Π^1_1 sequence A which is not Δ^1_1. As A is Π^1_1, we have an approximation $\{A_s\}_{s<\omega_1^{ck}}$ of A such that for any limit ordinal s we have $\lim_{t<s} A_t = A_s$. As A is not Δ^1_1 it cannot be equal to A_s for some computable s. We can now define the $\Pi^1_1(A)$ total function $f : \omega \to \omega_1^{ck}$ by sending $f(n)$ to the smallest ordinal s such that $A_s\restriction_n = A\restriction_n$. Therefore we have $\sup_n f(n) = \omega_1^{ck}$. Also as A is higher Turing below Z we also have that f is $\Pi^1_1(Z)$, and as f is total it is also $\Delta^1_1(Z)$ and therefore the range of f is a $\Delta^1_1(Z)$ set of ordinals, cofinal in ω_1^{ck}, which implies that $\omega_1^Z > \omega_1^{ck}$.

(2) $\Longrightarrow$ (1): Suppose that Z is Π^1_1-Martin-Löf random but not Π^1_1-random. Then from Theorem 7.6 there is a uniform intersection of Π^1_1-open sets $\bigcap_n \mathcal{U}_n$ so that $Z \in \bigcap_n \mathcal{U}_n$ and so that no Δ^1_1-closed set $\mathcal{F} \subseteq \bigcap_n \mathcal{U}_n$ of positive measure contains Z. Then as Z is Δ^1_1-random we actually have that no Δ^1_1 closed set $\mathcal{F} \subseteq \bigcap_n \mathcal{U}_n$ contains Z. Let $\{W_e\}_{e<\omega}$ be an enumeration of the Π^1_1 subsets of $\mathbb{N}$. We will construct a Π^1_1 sequence A which is not Δ^1_1 and such that Z higher Turing computes A. The usual way to make A not Δ^1_1, is by meeting each requirement:

$$R_e : W_e \text{ infinite } \rightarrow A \cap W_e \neq \varnothing$$

making sure in the meantime that A is co-infinite.

Construction of A:

At stage s, at substage $\langle e, m, k\rangle$, if R_e is actively satisfied, go to the next substage, otherwise if $m \in W_e[s]$ with $m > 2e$, then consider the Δ^1_1 set $\bigcap_n \mathcal{U}_n[s]$ and compute an increasing union of Δ^1_1-closed sets $\bigcup_n \mathcal{F}_n$ with $\bigcup_n \mathcal{F}_n \subseteq \bigcap_n \mathcal{U}_n[s]$ and $\lambda(\bigcup_n \mathcal{F}_n) = \lambda(\bigcap_n \mathcal{U}_n[s])$.

If $\lambda(\mathcal{U}_m[s] - \mathcal{F}_k) \leqslant 2^{-e}$ then enumerate m into A at stage s, mark R_e as 'actively satisfied' and let $\mathcal{V}_{\langle m,e\rangle} = \mathcal{U}_m[s] - \mathcal{F}_k$.

This ends the algorithm. The sets $\mathcal{V}_{\langle m,e\rangle}$ are intended to form a higher Solovay test.

Verification that A is not Δ^1_1:

A is co-infinite because for each e at most one m is enumerated into A and this m is bigger than $2e$. Now suppose that W_e is infinite. By the admissibility there exists $s < \omega_1^{ck}$ so that $W_e[s]$ is infinite. Then there exists $t \geqslant s$ so that $\lambda(\bigcap_n \mathcal{U}_n - \bigcap_n \mathcal{U}_n[t]) < 2^{-e}$. Then there is a Δ_1^1-closed set $\mathcal{F}_k \subseteq \bigcap_n \mathcal{U}_n[t]$ so that $\lambda(\bigcap_n \mathcal{U}_n - \mathcal{F}_k) < 2^{-e}$. Then there exists an integer a such that for all $b \geqslant a$ we have $\lambda(\mathcal{U}_b - \mathcal{F}_k) < 2^{-e}$ and in particular $\lambda(\mathcal{U}_b[r] - \mathcal{F}_k) < 2^{-e}$ for any stage r. But as $W_e[t]$ is infinite we have some $m \in W_e[t]$ with $m > 2e$ such that $\lambda(\mathcal{U}_m[t] - \mathcal{F}_k) < 2^{-e}$. Then at stage t and substage $\langle e, m, k\rangle$, the integer m is enumerated into A, if R_e is not met yet.

Verification that $\{\mathcal{V}_{\langle m,e\rangle}\}_{m,e\in\mathbb{N}}$ is a higher Solovay test:
Note that each $\mathcal{V}_{\langle m,e\rangle}$ is well-defined uniformly in m and e. We implicitly have that $\mathcal{V}_{\langle m,e\rangle}$ enumerates nothing until the algorithm decides otherwise, which can happen at most once for a given pair (m, e), and even at most once for a given e, as when it happens, R_e is actively satisfied. Also as each $\mathcal{V}_{m,e}$ has measure smaller than 2^{-e}, we have a higher Solovay test.

Computation of A from Z:
We now just describe the algorithm to compute A from Z. The verification that the algorithm works as expected is given in the next paragraph. Let p be the smallest integer so that for any $m \geqslant p$, the set Z is in no $\mathcal{V}_{\langle m,e\rangle}$ for any e, which exists because Z passes the Solovay test $\mathcal{V}_{\langle m,e\rangle}$. To decide whether $m \geqslant p$ is in A, we look for the smallest s such that $Z \in \mathcal{U}_m[s]$. Then decide that m is in A iff m is in $A[s]$.

Verification that Z computes A:
Let p be the smallest integer so that for any $m \geqslant p$ the set Z is in no $\mathcal{V}_{\langle m,e\rangle}$ for any e. Suppose for contradiction that we have $m \geqslant p$ and $s < \omega_1^{ck}$ such that $Z \in \mathcal{U}_m[s]$ and $m \notin A[s]$, but $m \in A[t]$ for $t > s$. By construction, it means that we have some e and some Δ_1^1-closed set $\mathcal{F}_k \subseteq \bigcap_n \mathcal{U}_n$ with $\lambda(\mathcal{U}_m[t] - \mathcal{F}_k) < 2^{-e}$ and $\mathcal{V}_{\langle m,e\rangle} = \mathcal{U}_m[t] - \mathcal{F}_k$.

As Z does not belong to $\mathcal{V}_{\langle m,e\rangle}$ and does not belong to $\mathcal{F}_k$, it does not belong to $\mathcal{U}_m[t]$ which contradicts the fact that it belongs to $\mathcal{U}_m[s] \subseteq \mathcal{U}_m[t]$. □

Corollary 7.30 (Greenberg, Monin [15])**.** *Some higher weakly-2-random computes a Π_1^1 set which is not Δ_1^1.*

Proof. This follows from the previous theorem and from Theorem 6.7 saying that the set of Π_1^1-randoms is strictly included in the set of higher weakly-2-randoms. □

Theorem 7.29 can now be used to give another equivalent notion of test for Π_1^1-randomness, in the same spirit as the definition of higher difference randomness.

Theorem 7.31 (Greenberg, Monin [15])**.** *For a sequence X, the following are equivalent:*

(1) *X is captured by a set $\mathcal{F} \cap \bigcap_n \mathcal{U}_n$ with $\lambda(\mathcal{F} \cap \bigcap_n \mathcal{U}_n) = 0$ where $\mathcal{F}$ is a Σ_1^1 set and each $\mathcal{U}_n$ is a Π_1^1-open set uniformly in n.*
(2) *X is not Π_1^1-random.*
(3) *X is captured by a set $\mathcal{F} \cap \bigcap_n \mathcal{U}_n$ with $\lambda(\mathcal{F} \cap \bigcap_n \mathcal{U}_n) = 0$ where $\mathcal{F}$ is a Σ_1^1-closed set and each $\mathcal{U}_n$ is a Π_1^1-open set uniformly in n.*

Proof. (1) $\Longrightarrow$ (2): Suppose first that X is captured by a set $\mathcal{F} \cap \bigcap_n \mathcal{U}_n$ of measure 0. Then either $\omega_1^X > \omega_1^{ck}$, in which case X is not Π_1^1-random, or there exists some stage s for which $X \in \bigcap_n \mathcal{U}_n[s]$. As also $X \in \mathcal{F}$ we then have $X \in \mathcal{U}_n[s] \cap \mathcal{F}$, which is a Σ_1^1 set of measure 0. Therefore X is not Δ_1^1-random and thus not Π_1^1-random.

(2) $\Longrightarrow$ (3): Suppose now that X is not Π_1^1-random. Then by Theorem 7.29, either it is not Π_1^1-Martin-Löf random, in which case we have (3) with $\mathcal{F} = 2^{\mathbb{N}}$ and $\{\mathcal{U}_n\}_{n<\omega}$ a Π_1^1-Martin-Löf test, or it higher Turing computes a Π_1^1 set Y which is not Δ_1^1, via a higher functional Φ. We define $\mathcal{U}_n = \bigcup_s \Phi^{-1}(Y_s{\upharpoonright}_n)$. We now define a Σ_1^1-closed set by defining its complement $\mathcal{F}^c$: We put in $\mathcal{F}^c$ at successor stage $s+1$, the open set $\Phi^{-1}(Y_s{\upharpoonright}_n)$ for every n as soon as we witness $Y_s{\upharpoonright}_n \neq Y_{s+1}{\upharpoonright}_n$. It follows that $\bigcap_n \mathcal{U}_n \cap \mathcal{F}$ contains only the sequences which higher Turing computes Y with the functional Φ, or some sequences on which Φ is not consistent. In particular, by Theorem 3.12, the set of sequences which higher Turing compute Y has measure 0. Therefore the measure of $\bigcap_n \mathcal{U}_n \cap \mathcal{F}$ is bounded by the measure of the inconsistency set of Φ.

Also recall Lemma 2.12 saying that uniformly in ε, we can obtain a version of Φ for which the inconsistency set of Φ has measure smaller than ε. We can then uniformly in ε define a uniform intersection of Π^1_1-open sets $\bigcap_n \mathcal{U}^\varepsilon_n$ such that $\lambda(\bigcap_n \mathcal{U}^\varepsilon_n \cap \mathcal{F}) \leqslant \varepsilon$. Note that we can keep the same set $\mathcal{F}$ for any ε. Then we have $\lambda(\bigcap_{\varepsilon,n} \mathcal{U}^\varepsilon_n \cap \mathcal{F}) = 0$ and $X \in \bigcap_{\varepsilon,n} \mathcal{U}^\varepsilon_n \cap \mathcal{F}$.

(3) $\Longrightarrow$ (1) is immediate. □

7.5.2. Π^1_1-*Martin-Löf[O]-randomness.* Greenberg and Monin [15] also studied a randomness notion which is strong enough to make non-random any higher Δ^0_2 sequence. The motivation for this notion goes back to the notion of test, equivalent in the lower setting to weak-2-tests. The two following are equivalent:

(1) X is weakly-2-random.
(2) X is in no set $\bigcap_n \mathcal{U}_{f(n)}$ with $f : \mathbb{N} \to \mathbb{N}$ a $\emptyset^{(1)}$-computable function such that $\lambda(\mathcal{U}_{f(n)}) \leqslant 2^{-n}$.

This resemble the test notion of Theorem 6.5, except that in Theorem 6.5 we had to restrict ourselves to functions with a finite-change approximation. We study now what we obtain if one can use any higher Δ^0_2 function.

Definition 7.32 (Greenberg, Monin [15])**.** A sequence X is Π^1_1-Martin-Löf[O]-random (to be pronounced, for a mysterious reason: Π^1_1-Martin-Löf 'plop O' randomness) if X is in no set $\bigcap_n \mathcal{U}_{f(n)}$ with f higher Turing computable by O and with $\lambda(\mathcal{U}_{f(n)}) \leqslant 2^{-n}$ for each n.

So as we will see, we don't have the equivalence between Π^1_1-Martin-Löf[O]-randomness and higher weak-2-randomness. Nevertheless there is a way to remove O from this definition, in order to get a better understanding of it:

Proposition 7.33 (Greenberg, Monin [15])**.** *The following are equivalent for a sequence* $X \in 2^{\mathbb{N}}$*:*

(1) *X is Π^1_1-Martin-Löf[O]-random.*
(2) *X does not belong to any test $(\mathcal{U}_s)_{s<\omega_1^{ck}}$ not necessarily nested where each $\mathcal{U}_s$ is a Π^1_1-open set uniformly in s, and such that $\lambda(\bigcap_s \mathcal{U}_s) = 0$.*

Proof. Let us show that (2) implies (1). Let $\bigcap_n \mathcal{U}_{f(n)}$ be an Π^1_1-Martin-Löf[O] test. Recall that $p : \omega_1^{ck} \to \omega$ is the projectum function and let us define $\mathcal{V}_s = \bigcap_{n<p(s)} \bigcup_{t>s} \mathcal{U}_{f_t(n)}$. It is clear that $\bigcap_n \mathcal{U}_{f(n)} \subseteq \bigcap_s \mathcal{V}_s$. To prove that $\lambda(\bigcap_s \mathcal{V}_s) = 0$, let us prove that $\bigcap_s \mathcal{V}_s \subseteq \bigcap_n \mathcal{U}_{f(n)}$. For each n there exists s large enough such that $n \leqslant p(s)$ and $\forall m \leqslant n \quad \bigcup_{t>s} \mathcal{U}_{f_t(m)} = \mathcal{U}_{f(m)}$. Then we have for that n and s that $\mathcal{V}_s \subseteq \bigcap_{m\leqslant n} \mathcal{U}_{f(m)}$ and then $\bigcap_s \mathcal{V}_s \subseteq \bigcap_n \mathcal{U}_{f(n)}$.

Let us show that (1) implies (2). Suppose now that we have a test $(\mathcal{U}_s)_{s<\omega_1^{ck}}$ with $\lambda(\bigcap_s \mathcal{U}_s) = 0$. Then using O we can higher Turing compute the measure of each $\mathcal{U}_s$ uniformly in s. Then for each n, O can higher Turing compute s_n such that $\lambda(\mathcal{U}_{s_n}) \leqslant 2^{-n}$ and then we can find an equivalent Π^1_1-Martin-Löf[O] test, by setting $\mathcal{V}_n = \mathcal{U}_{s_n}$. □

We shall now see that Π^1_1-Martin-Löf[O]-randomness is strictly stronger than Π^1_1-randomness. For this we first prove:

Proposition 7.34 (Greenberg, Monin [15])**.** *If $X \in 2^{\mathbb{N}}$ higher Turing computes a non Δ^1_1 higher Δ^0_2 sequence Y, then X is not Π^1_1-Martin-Löf[O]-random.*

Proof. The set $\mathcal{A} = \bigcap_{n,s} \bigcup_{t\geqslant s} \Phi^{-1}(Y_t\upharpoonright_n)$ is also equal to the set $\bigcap_n \Phi^{-1}(Y_t\upharpoonright_n)$. Also by Sack's theorem (Theorem 3.12), as Y is not Δ^1_1, the set of sequences which higher Turing compute Y is a nullset. However the function Φ can also be inconsistent. Therefore the measure of the set $\mathcal{A}$ is bounded by the measure of the Π^1_1-open set on which Φ is inconsistent. Also by Lemma 2.12 we can transform Φ uniformly in any ε so that the measure of this open set is smaller than ε, without damaging the right computations of Φ. But then uniformly in n we can define the set $\mathcal{A}_n$ like above, but with the measure of $\mathcal{A}_n$ bounded by 2^{-n}. Also by Proposition 7.33, we then have that $\bigcap_n \mathcal{A}_n$ is a Π^1_1-Martin-Löf[O] test, and by design, it contains X. □

Theorem 7.35 (Greenberg, Monin [15])**.** Π^1_1-*Martin-Löf[O]-randomness is strictly stronger than* Π^1_1-*randomness.*

Proof. By the proposition above we have that Π^1_1-Martin-Löf[O]-randomness is either incomparable with Π^1_1-randomness, or strictly stronger than Π^1_1-randomness: Indeed, by the Gandy basis theorem, there is a higher Δ^0_2 sequence which is Π^1_1-random. All that remains to be proved is that Π^1_1-Martin-Löf[O]-randomness is stronger than Π^1_1-randomness.

By Theorem 7.6, if X is Δ^1_1-random but not Π^1_1-random, then there exists a uniform intersection of Π^1_1-open sets $\bigcap_n \mathcal{U}_n$ such that $X \in \bigcap_n \mathcal{U}_n$ but X is in no Σ^1_1-closed set $\mathcal{F}$ with $\mathcal{F} \subseteq \bigcap_n \mathcal{U}_n$. Let us argue that there is an effective enumeration $\{\mathcal{F}_s\}_{s<\omega_1^{ck}}$ of the Σ^1_1-closed sets included in $\bigcap_n \mathcal{U}_n$. For a given Σ^1_1-closed set $\mathcal{F}$, we can build the Π^1_1 function $f : \omega \to \omega_1^{ck}$ which to n associates the least t such that $\mathcal{F}_t \subseteq \bigcap_{m \leqslant n} \mathcal{U}_{m,t}$. If we really have $\mathcal{F} \subseteq \bigcap_n \mathcal{U}_n$ then f is total and then its range is bounded by some computable ordinal t, for which we already have $\mathcal{F}_t \subseteq \bigcap_n \mathcal{U}_{n,t} \subseteq \bigcap_n \mathcal{U}_n$.

So if a Σ^1_1-closed set is included in $\bigcap_n \mathcal{U}_n$ we will know it at some computable ordinal stage. Then we can easily get an effective enumeration $\{\mathcal{F}_s\}_{s<\omega_1^{ck}}$ of the Σ^1_1-closed sets included in $\bigcap_n \mathcal{U}_n$ by checking at each stage t and for each index of a Σ^1_1-closed set $\mathcal{F}$ if we have $\mathcal{F}_t \subseteq \bigcap_n \mathcal{U}_{n,t}$. Also we have that X is in $\bigcap_n \mathcal{U}_n \cap \bigcap_{s<\omega_1^{ck}} \mathcal{F}_s^c$ which is a set of measure 0 and therefore, by Proposition 7.33 a Π^1_1-Martin-Löf$[O]$ test. □

This theorem yields a natural question, which is still open at the moment. We have that no sequence computing a higher Δ^0_2 sequence is Π^1_1-Martin-Löf$[O]$-random. Does the converse hold on Π^1_1-Martin-Löf random sequences? Using Theorem 7.29, we already know that the Π^1_1-Martin-Löf randoms which are not Π^1_1-random can higher Turing computes higher Δ^0_2 sequences (even Π^1_1 sequences). But what about the sequences which are Π^1_1-random but not Π^1_1-Martin-Löf$[O]$-random?

Question 7.36. *Is there some X which is Π^1_1-random, not Π^1_1-Martin-Löf[O]-random, and which does not higher Turing compute any higher Δ^0_2 sequence?*

8. Randomness along a higher hierarchy of complexity of sets

The notion of higher weak-2-randomness deals with uniform intersection of Π^1_1-open sets, the uniformity being along the natural numbers. Also one could think of iterating this notion. We could consider for example uniform union of uniform intersections of Π^1_1 open sets. Recall that we proved in Section 6.2.2 that higher weak-2-randomness is strictly weaker than Π^1_1-randomness, that is, uniform intersections of Π^1_1-open sets, of measure 0, are not enough to cover the largest Π^1_1 nullset.

Greenberg and Monin [15] showed that if we just allow a little bit more descriptional power to define our nullsets, that is allowing more successive intersection and union operations over Π^1_1-open sets, we can then define nullsets that capture every non Π^1_1-random sequence. We start by defining formally the new hierarchy on the complexity of sets, that we will use.

Definition 8.1 (Greenberg, Monin [15])**.** A set is Σ^{ck}_1 if it is a Π^1_1-open set. It is Π^{ck}_1 if it is a Σ^1_1-closed set. It is $\Sigma^{\mathrm{ck}}_{n+1}$ if it is an effective union over $\mathbb{N}$ of a sequence of Π^{ck}_n sets and it is Π^{ck}_{n+1} if it is an effective intersection over $\mathbb{N}$ of a sequence of Σ^{ck}_n sets.

We do not iterate the definition through the computable ordinal, first because we will not use it, and then because it is not clear what should be the meaning of $\Sigma^{\mathrm{ck}}_\omega$. Indeed, this new hierarchy has the unusual property that a Π^{ck}_1 set is not necessarily a Π^{ck}_2 set; more generally, a Π^{ck}_n set is not necessarily Π^{ck}_{n+p} for p odd, and a Σ^{ck}_n set is not necessarily $\Sigma^{\mathrm{ck}}_{n+p}$ for p odd. Indeed, Π^{ck}_n sets for n odd and Σ^{ck}_n for n even are all Σ^1_1 sets, but Π^{ck}_n sets for n even and Σ^{ck}_n for n odd are all Π^1_1 sets. We give here an illustration of this new hierarchy:

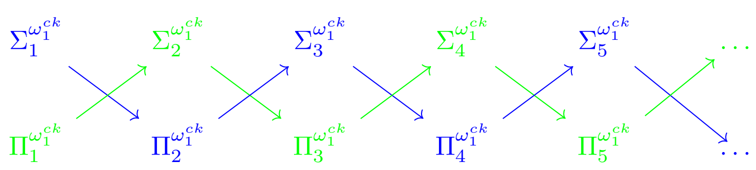

Figure 1. The higher hierarchy of complexity of sets.
The blue complexities correspond to Π^1_1 sets.
The green complexities correspond to Σ^1_1 sets.

With this higher complexity notion, we have by definition that any sequence is higher weakly-2-random iff it is in no null Π^{ck}_2 set. The question we study here is :

What randomness notions do we obtain by considering null Π^{ck}_n sets or null Σ^{ck}_n sets?

Definition 8.2 (Greenberg, Monin [15])**.** We say that X is Σ^{ck}_n-random, respectively Π^{ck}_n-random, if X is in no Σ^{ck}_n nullset, respectively in no Π^{ck}_n nullset.

8.1. **On the Σ^1_1 randomness notions in the higher hierarchy.** It is clear that complexities corresponding to Σ^1_1 sets will give us a notion at least weaker than Σ^1_1-randomness and then than Δ^1_1-randomness. Concretely, the notion of being in no null Σ^{ck}_2 sets, or no null Π^{ck}_3 sets, etc... gives us a notion of randomness at least weaker than Σ^1_1-randomness. The notion of Π^{ck}_1-randomness has been studied by Kjos-hanssen, Nies, Stephan, and Yu in [24], under the name of Δ^1_1-Kurtz randomness. In particular they studied lowness for the notion of Δ^1_1-Kurtz randomness.

The notion of Δ^1_1-randomness where the Borel complexity of the null sets is restrained has also been studied by Chong, Nies and Yu in [5]. In particular, they observed that uniform intersection of Δ^1_1 open sets, effectively of measure 0, are enough to capture any non Δ^1_1-random. What we consider here is different, as we start our successive unions and intersections with Σ^1_1 closed sets.

Theorem 8.3 (Greenberg, Monin [15]). *We have:*

$$\Pi^{\mathrm{ck}}_1\text{-}randomness \leftrightarrow \Sigma^{\mathrm{ck}}_2\text{-}randomness \leftarrow \Pi^{\mathrm{ck}}_3\text{-}randomness = \Delta^1_1\text{-}randomness.$$

The reverse implication is strict. Also it follows from Π^{ck}_3-randomness $=$ Δ^1_1-randomness that Π^{ck}_{3+p}-randomness and $\Sigma^{\mathrm{ck}}_{2+p}$-randomness for p even are all equivalent to Δ^1_1-randomness.

Proof. It is clear that Π^{ck}_1-randomness is the same as Σ^{ck}_2-randomness, because in both cases the non random sequences are those which are in the union of all Σ^1_1-closed null sets.

Let us prove that Π^{ck}_3 nullsets are enough to cover any Δ^1_1 nullsets. Using Theorem 3.6 we can approximate from above any Δ^1_1 set by a uniform intersection of Δ^1_1-open sets $\bigcap_n \mathcal{U}_n$. Also as each $\mathcal{U}_n$ is Δ^1_1 uniformly in n, the predicate $\sigma \subseteq \mathcal{U}_n$ and the predicate $\sigma \nsubseteq \mathcal{U}_n$ are both Δ^1_1 which implies that we can easily define uniformly in n a Δ^1_1 total function $h_n : \omega \to 2^{<\omega}$ such that $\bigcup_m [h_n(m)] = \mathcal{U}_n$. We then define uniformly in (n, m) the Δ^1_1-closed set $\mathcal{F}^n_m$ to be $[h_n(m)]$. We then have $\bigcap_n \bigcup_m \mathcal{F}^n_m = \bigcap_n \mathcal{U}_n$.

Let us prove that Π^{ck}_1-randomness is strictly weaker than Δ^1_1-randomness. The proof is similar to the one that Kurtz-randomness (being in no Π^0_1 sets of measure 0) is strictly weaker than Martin-Löf randomness. We use here some Baire category notions: The set of Π^{ck}_1-randoms is a countable intersection of open sets of measure 1. Also it is clear that an open set of measure 1 is necessarily dense. But then this intersection contains some Cohen generic sequences. Also any X which is generic for even the weakest notion of genericity generally studied, namely weakly-1-generic, is not Martin-Löf random (because each open set of a universal Martin-Löf test is dense), and therefore certainly not Δ^1_1-random.

Now, as Π^{ck}_{3+p} nullsets and $\Sigma^{\mathrm{ck}}_{2+p}$ nullsets are all Σ^1_1 nullsets for p even, the corresponding randomness notions are all equivalent to Σ^1_1-randomness $=$ Δ^1_1-randomness. □

8.2. **On the Π^1_1 randomness notions in the higher hierarchy.** We know that the higher weakly-2-randoms are exactly the elements which are Π^{ck}_2-random. Also it is clear that this notion coincides with Σ^{ck}_3-randomness, as in both case the non-random elements are the unions of all the Π^{ck}_2 null sets. We shall now prove that Π^{ck}_4-randomness coincide with Π^1_1-randomness.

To do so, we will use Π^1_1 functionals Φ from $2^{\mathbb{N}}$ into sequences of computable ordinals, that is, $(\omega_1^{ck})^{\mathbb{N}}$. Concretely such a functional Φ is given by a Π^1_1 subset of $2^{<\mathbb{N}} \times \mathbb{N} \times \omega_1^{ck}$. We then say that Φ is defined on X, if for every n, there exists a unique α such that for some m we have $(X{\upharpoonright}_m, n, \alpha) \in \Phi$.

Note that just like for usual higher Turing reductions, we cannot guarantee that such a functional is consistent everywhere. Also if along some oracle X, some n is mapped to at least two distinct ordinals, then the functional is said to be inconsistent on X. The inconsistency set cannot be completely removed, however, as in Lemma 2.12, it can be made of measure as small as we want. We will prove this formally in Lemma 8.4, but first we give a few notations.

The set of elements on which Φ is defined (and consistent) will be denoted by $\mathrm{Cdom}(\Phi)$. If for some X and n there is some α (not necessarily unique) such that $(X{\upharpoonright}_m, n, \alpha) \in \Phi$ for some m, we write $\Phi(X, n) = \alpha$. One can consider Φ^X as a multivalued function. Note that the equality symbol '=' used in the expression $\Phi(X, n) = \alpha$ does not mean that $\Phi(X, n)$ is equal to α in the strict sense of equality, but more than $\Phi(X, n)$ is mapped to α (among possibly other values). Then the set of elements X such that for any n we have $\Phi(X, n) = \alpha$ for at least one α will be denoted by $\mathrm{dom}(\Phi)$. Formally:

$$\mathrm{dom}(\Phi) = \bigcap_n \{X \ : \ \exists m, \alpha_n \ (X{\upharpoonright}_m, n, \alpha_n) \in \Phi\}$$

One nice thing about $\text{dom}(\Phi)$ is that it is a Π^{ck}_2 set, whereas $\text{Cdom}(\Phi)$ is more complicated. We now prove, as a consequence of Theorem 7.29 (a sequence Z is Π^1_1-Martin-Löf random but not Π^1_1-random iff it higher Turing computes a strictly Π^1_1 sequence) that the measure of the inconsistency set of a functional Φ can be made as small as we want:

Lemma 8.4. If Z is Π^1_1-Martin-Löf random and not Π^1_1-random, one can define uniformly in $\varepsilon \in \mathbb{Q}$ a Π^1_1 functional $\Phi \subseteq 2^{<\mathbb{N}} \times \mathbb{N} \times \omega_1^{ck}$ such that:

- Φ is defined (and consistent) on Z, and $\sup_n \Phi(Z, n) = \omega_1^{ck}$.
- The measure of the Π^1_1 open set on which Φ is not consistent is smaller than ε. Formally:
$$\lambda(\{X \ : \ \exists n, m_1, m_2\ \exists \alpha_1 \neq \alpha_2\ \Phi(X{\upharpoonright}_{m_1}, n) = \alpha_1 \text{ and } \Phi(X{\upharpoonright}_{m_2}, n) = \alpha_2\}) \leqslant \varepsilon$$

Proof. From 7.29 we have a higher Turing functional Ψ so that $\Psi(Z) = A$ for A a Π^1_1 set which is not Δ^1_1. From Lemma 2.12, the measure of the inconsistency set of Φ can be made smaller than ε, uniformly in ε.

To define Φ, we enumerate (σ, n, α) in Φ if there exists τ of length bigger than n and α such that $(\sigma, \tau) \in \Psi$ and α is the first ordinal for which we have $\tau{\upharpoonright}_n = A_\alpha{\upharpoonright}_n$. We verify easily that such a functional Φ has the desired properties. ☐

Using those Π^1_1 functionals, we now state the following theorem, which is the heart of the proof that Π^{ck}_4-randomness coincide with Π^1_1-randomness.

Theorem 8.5 (Greenberg, Monin [15]). *For any Π^1_1 functional $\Phi \subseteq 2^{<\mathbb{N}} \times \mathbb{N} \times \omega_1^{ck}$, One can define, uniformly in an index for Φ, a Π^{ck}_4 nullset $\mathcal{A}$ such that $\{X \in \text{Cdom}(\Phi) \ : \ \sup_n \Phi(X, n) = \omega_1^{ck}\} \subseteq \mathcal{A}$.*

Before proving Theorem 8.5 we see some of its consequences, in particular using Lemma 8.4, it implies that Π^{ck}_4-randomness coincides with Π^1_1-randomness:

Theorem 8.6 (Greenberg, Monin [15]). *We have:*
$$\Pi^{\text{ck}}_2\text{-randomness} \leftrightarrow \Sigma^{\text{ck}}_3\text{-randomness} \leftarrow \Pi^{\text{ck}}_4\text{-randomness} = \Pi^1_1\text{-randomness}.$$
The reverse implication is strict. Also it follows from Π^{ck}_4-randomness $= \Pi^1_1$-randomness that Π^{ck}_{4+p}-randomness and Σ^{ck}_{3+p}-randomness are all equivalent to Π^1_1-randomness for p even and all weaker than Π^1_1-randomness for p odd.

Proof. Let us first prove that Theorem 8.5 implies that Π^{ck}_4-randomness $= \Pi^1_1$-randomness. One direction is obvious as the largest Π^1_1 nullset covers any Π^{ck}_4 nullset. For the other direction, suppose that Z is not Π^1_1-random. If Z is not Π^1_1-Martin-Löf random it is by definition covered by a Π^{ck}_2 nullset. Otherwise we can define using Lemma 8.4 a Π^1_1 functional $\Phi \subseteq 2^{<\mathbb{N}} \times \mathbb{N} \times \omega_1^{ck}$ defined on Z, with $\sup_n \Phi(Z, n) = \omega_1^{ck}$. It follows using Theorem 8.5 that Z can be captured by a Π^{ck}_4 nullset.

We also deduce that Π^{ck}_2-randomness, corresponding to higher weak-2-randomness, is strictly weaker than Π^{ck}_4-randomness, using Theorem 6.7 that separates the two notions. The fact that Σ^{ck}_3-randomness coincide with Π^{ck}_2-randomness is clear. The rest of the proposition follows: For any n the null Σ^{ck}_n or Π^{ck}_n sets are either also null Π^1_1 sets, or covered by some null Π^1_1 sets. ☐

Corollary 8.7 (Greenberg, Monin [15]). *The set of Π^1_1-randoms is Π^{ck}_5.*

Proof. We actually have an effective listing $\{\Phi_e\}_{e \in \mathbb{N}}$ of the Π^1_1 functionals $\Phi_e \subseteq 2^{<\mathbb{N}} \times \mathbb{N} \times \omega_1^{ck}$, as it is simply the listing of all the Π^1_1 subsets of $2^{<\mathbb{N}} \times \mathbb{N} \times \omega_1^{ck}$ (recall that inconsistency is allowed). Then using Theorem 8.5, we can define uniformly in e a Π^{ck}_4 null set $\mathcal{A}_e$ which captures:
$$\{X \in \text{Cdom}(\Phi) \ : \ \sup_n \Phi_e(X, n) = \omega_1^{ck}\}$$
Also using Lemma 8.4 we know that as long as Z is not Π^1_1-random and Π^1_1-Martin-Löf random, it will be captured by some of those set $\mathcal{A}_e$. Therefore, the uniform union of all the sets $\mathcal{A}_e$, itself joined with the universal Π^1_1-Martin-Löf test, is a Σ^{ck}_5 nullset containing the biggest Π^1_1 nullset. And as a Σ^{ck}_5 set is itself Π^1_1, it actually coincides with the biggest Π^1_1 nullset. ☐

It is unkown whether the above Corollary is optimal or not. Bienvenu, Greenberg and Monin [2, Proposition 5.3] showed that the set of Π^1_1 randoms is not Π^{ck}_3, but the following remains open:

Question 8.8. *Is the set of Π^1_1 randoms Σ^{ck}_4 ?*

The rest of this section is dedicated to the proof of Theorem 8.5. So consider a Π^1_1 functional $\Phi \subseteq 2^{<\mathbb{N}} \times \mathbb{N} \times \omega_1^{ck}$. Let us fix some ε and let us assume that the inconsistency set of Φ has measure smaller than ε. From now on, the construction will remain uniform in Φ and then in ε.

The strategy:

The strategy is to define uniformly in each version of Φ that have an inconsistency set of measure smaller ε, a Π^{ck}_4 set $\mathcal{C}$ such that:

- $\{X \in \mathrm{Cdom}(\Phi) \ : \ \sup_n \Phi(X,n) = \omega_1^{ck}\} \subseteq \mathcal{C} \subseteq \mathrm{dom}(\Phi)$.
- $\{X \in \mathrm{Cdom}(\Phi) \ : \ \sup_n \Phi(X,n) < \omega_1^{ck}\} \subseteq 2^{\mathbb{N}} - \mathcal{C}$.

In particular, it will follow that $\mathcal{C}$ contains either some element X such that $\omega_1^X > \omega_1^{ck}$, or some element $X \in \mathrm{dom}(\Phi)$ such that Φ is not consistent on X. As by Theorem 3.11 the measure of the set of X such that $\omega_1^X > \omega_1^{ck}$ is null, it follows that the measure of $\mathcal{C}$ is bounded by ε, the measure of the inconsistency set of Φ. Also uniformly in ε we can define the Π^{ck}_4 set $\mathcal{C}_\varepsilon$ containing $\{X \in \mathrm{Cdom}(\Phi) \ : \ \sup_n \Phi(X,n) = \omega_1^{ck}\}$ and of measure smaller than ε. It follows that the intersection over ε of the sets C_ε is a Π^{ck}_4 nullset containing $\{X \in \mathrm{Cdom}(\Phi) \ : \ \sup_n \Phi(X,n) = \omega_1^{ck}\}$.

Some notations:

In what follows, we denote by R_e the e-th c.e. subset of $\mathbb{N}\times\mathbb{N}$, that is, $(n,m) \in R_e \leftrightarrow \langle n,m\rangle \in W_e$, where W_e is the usual e-th c.e. subset of $\mathbb{N}$. We will consider such a set as a c.e. binary relation. Also for a computable ordinal α we denote by R_α the c.e. binary relation coded by the smallest integer $a \in O$ such that $|a|_o = \alpha$.

We also denote by $R_e\restriction_k$, the binary relation R_e restricted to elements 'smaller' than k in the sense of R, that is, the pair (n,m) is in $R_e\restriction_k$ iff the pair (m,k) and (n,m) are both in R_e ($(n,m) \in R_e$ is intended to be understood as $n < m$ in the sense of R_e). Note that $R_e\restriction_k$ is well defined for any e, but the underlying idea really makes sense when R_e represents an order, and we actually intend to use it only when R_e represents a linear order.

Finally, we say that a function $f : \mathbb{N} \to \mathbb{N}$ is a morphism from a linear order coded by a binary relation R_{e_1} to another linear order coded by a binary relation R_{e_2}, if f is total on $\mathrm{dom}\, R_{e_1}$, with $f(\mathrm{dom}\, R_{e_1}) \subseteq \mathrm{dom}\, R_{e_2}$ and if $(x,y) \in R_{e_1} \to (f(x), f(y)) \in R_{e_2}$. Here $\mathrm{dom}\, R_e$ denotes the support of R_e, that is, the set of integer a such that $(a,b) \in R_e$ or $(b,a) \in R_e$ for some b.

Definition of the Π^{ck}_4 set $\mathcal{C}$

We now do the proof of Theorem 8.5. Let us define uniformly in each integer e the sets $\mathcal{A}_e$ and $\mathcal{B}_e$:

$$\mathcal{A}_e = \left\{ X \in 2^{\mathbb{N}} \ : \ \begin{array}{l} \exists n\ \exists \alpha_n\ \Phi(X,n) = \alpha_n \text{ and} \\ \forall f\ f \text{ is not a morphism from } R_{\alpha_n} \text{ to } R_e \end{array} \right\}$$

and

$$\mathcal{B}_e = \left\{ X \in 2^{\mathbb{N}} \ : \ \begin{array}{l} \exists m\ \forall n\ \exists \alpha_n\ \Phi(X,n) = \alpha_n \text{ and} \\ \forall f\ f \text{ is not a morphism from } R_e\restriction_m \text{ to } R_{\alpha_n} \end{array} \right\}$$

Let us now define the Π^0_2 set G of integers e such that R_e is a linear order of $\mathbb{N}$. We finally define:

$$\mathcal{C} = \bigcap_{e \in G} (\mathrm{dom}(\Phi) \cap (\mathcal{A}_e \cup \mathcal{B}_e))$$

Proof that $\mathcal{C}$ is Π^{ck}_4:

We have that $\mathrm{dom}(\Phi)$ is Π^{ck}_2, that $\mathcal{A}_e$ is Σ^{ck}_1 uniformly in e and that $\mathcal{B}_e$ is Σ^{ck}_3 uniformly in e. Then the set $\mathrm{dom}(\Phi) \cap (\mathcal{A}_e \cup \mathcal{B}_e)$ is Σ^{ck}_3 uniformly in e. As G has a Π^0_2 description, we then have that $\mathcal{C}$ is a Π^{ck}_4 set.

Proof that $\mathcal{C}$ captures enough:

We should prove that $\{X \in \mathrm{Cdom}(\Phi) \ : \ \sup_n \Phi(X,n) = \omega_1^{ck}\} \subseteq \mathcal{C}$. Fix some $Z \in \mathrm{Cdom}(\Phi)$ and suppose that $\sup_n \Phi(Z,n) = \omega_1^{ck}$. Let us prove for any $e \in G$ that $Z \in \mathcal{A}_e \cup \mathcal{B}_e$. It will follow that $Z \in \mathcal{C}$.

Suppose first that R_e is a well-founded relation. As e is already in G we have that R_e is a c.e. well-ordered relation with $|R_e| < \omega_1^{ck}$. But then there is some n so that $\Phi(Z,n) = \alpha_n$ with $|\alpha_n| > |R_e|$ and we cannot have a morphism from R_{α_n} to R_e. Then $Z \in A_e$.

Suppose now that R_e is an ill-founded relation. There is then some m so that $R_e\restriction_m$ is already ill-founded. But as R_{α_n} is well-founded for every $\alpha_n = \Phi(Z,n)$, then for every n we cannot have a morphism from $R_e\restriction_m$ to R_{α_n}, and then $Z \in \mathcal{B}_e$.

Proof that $\mathcal{C}$ does not capture too much:

Let us now prove that for any $X \in \text{Cdom}(\Phi)$, if $\sup_n \Phi(X,n) < \omega_1^{ck}$ then $X \notin \mathcal{C}$. Consider such a sequence X with $\sup_n \Phi(X,n) = \alpha < \omega_1^{ck}$. In particular there exists some integer $e \in G$ so that R_e is a well-order of order-type α. For this e we certainly have for all $\alpha_n = \Phi(X,n)$ a morphism from R_{α_n} into R_e and then $X \notin \mathcal{A}_e$.

Let us now prove that $X \notin \mathcal{B}_e$. For any m we have $|R_e\restriction_m| < \alpha$. But because $\alpha = \sup_n \Phi(X,n)$ there is necessarily some n so that $\Phi(X,n) = \alpha_n > |R_e\restriction_m|$. Thus there is a morphism from $R_e\restriction_m$ into R_{α_n}. Then $X \notin \mathcal{B}_e$, and therefore $X \notin \mathcal{C}$. This ends the proof.

9. Open questions

We sum up in this section the open questions which appear in this paper. We also add three new open questions, one of them not directly related to higher randomness, but more with higher genericity and with higher computability.

9.1. Higher Plain complexity.

In Section 7.2 we defined the set $\mathcal{A}$ of sequences which have infinitely many prefixes of maximal Π^1_1-Kolmogorov complexity:

$$\mathcal{A} = \{X \mid \exists c\ \forall n\ \exists m \geqslant n\ \mathrm{C}(X\restriction_m) \geqslant m - c\}$$

We saw that the set $\mathcal{A}$ contains the Π^1_1-randoms and is contained in the Π^1_1-Martin-Löf randoms. We deduced from that using Corollary 7.9 that we cannot have Π^1_1-randoms $\subseteq \mathcal{A} \subseteq$ weakly-2-randoms. The following question remains open:

Question 9.1. *Does the set $\mathcal{A}$ contain the higher weakly-2-randoms?*

We add this question which is also still unanswered:

Question 9.2. *Does the set $\mathcal{A}$ coincides with the Π^1_1-Martin-Löf randoms ?*

9.2. Higher randomness and minimal pair with O.

The notion of Π^1_1-Martin-Löf$[O]$ randomness defined in Section 7.5.2 removes all the higher Δ^0_2 randoms and even all the randoms which higher Turing compute higher Δ^0_2 sequences. We do not know if this is optimal, that is, we do not know if a Π^1_1 random Z which is not Π^1_1-Martin-Löf$[O]$ has to compute a higher Δ^0_2:

Question 9.3. *Is there some X which is Π^1_1-random, not Π^1_1-Martin-Löf[O]-random, and which does not higher Turing compute any higher Δ^0_2 sequence?*

9.3. Complexity of the set of Π^1_1-randoms.

We showed with Corollary 8.7 that the set of Π^1_1-random is Π^{ck}_5. Bienvenu, Greenberg and Monin showed [2, Proposition 5.3] that every Π^{ck}_3 set of measure 1 contains a sequence X with a finite-change approximation. In particular this sequence cannot be Π^1_1-random and then the set of Π^1_1-randoms cannot be Π^{ck}_3. The proof of Bienvenu, Greenberg and Monin strongly uses the measure 1 assumption, and not just a positive measure assuption. Also it is unkown if there exists a Π^{ck}_3 set of positive measure which contains only Π^1_1-randoms, or more specifically:

Question 9.4. *Is the set of Π^1_1-random Σ^{ck}_4 ?*

9.4. Higher randomness and DNR functions.

Just like for partial computable functions, there is a uniform enumeration $\{\Phi_e\}_{e\in\mathbb{N}}$ of the Π^1_1 partial functions. We can then define a higher version of being DNC : a function $f : \mathbb{N} \to \mathbb{N}$ is a Π^1_1-DNC function if for every e we have $f(e) \neq \Phi_e(e)$. Liang Yu asked the following question:

Question 9.5 (Yu). *Does every Π^1_1-DNR function hyperarithmetically compute a Π^1_1-Martin-Löf random real ?*

In the lower setting, X computes a DNC function iff X computes an infinite subset of a random, and this is provably different from computing a random (see [22] and [14]). It is unknown if things are different in the higher settings.

9.5. **Higher randomness and LR reductions.** Yu asked the following question:

Question 9.6 (Yu)**.** *Suppose $\omega_1^X = \omega_1^{ck}$. Does there exists a $\Delta_1^1(X)$ random sequence Y so that $X \leqslant_h Y$?*

When X is not Δ_1^1, note that if Y is $\Delta_1^1(X)$-random, then for any $\alpha < \omega_1^{ck}$, we cannot have $Y^\alpha \geqslant_T X$. Also a $\Delta_1^1(X)$ sequence Y such that $Y \geqslant_h X$ must be such that $\omega_1^Y > \omega_1^{ck}$.

It follows that a positive answer to the following question would provide a negative answer to Question 9.6.

Question 9.7 (Yu)**.** *Does there exists X such that $\omega_1^X = \omega_1^{ck}$ and such that every $\Delta_1^1(X)$-random is Π_1^1-random ?*

Note that the previous question is connected with a higher version of the LR reduction : X is LR above Y if every X-random is also Y-random. Higher versions of the LR reduction could be, for instance, defined for Δ_1^1 and Π_1^1-randomness, and these reductions have not been studied yet.

9.6. **Genericity and higher computability.** We end with a small digression. In [15] Greenberg and Monin define the notion of Σ_1^1-genericity and show that it is the categorical analogue of Π_1^1-randomness. A characterization of lowness for Σ_1^1-genericity is still unkown:

Question 9.8. *Is there a non-Δ_1^1 sequence which is low for Σ_1^1-genericity ?*

The question is connected to a higher computability question of Liang Yu:

Question 9.9 (Yu)**.** *Let $\mathcal{C}$ be a perfect Σ_1^1 set. Let A be a non Δ_1^1 sequence. Does there necessarily exists $X \in \mathcal{C}$ such that $\omega_1^{A\oplus X} > \omega_1^{ck}$?*

If some non-Δ_1^1 sequence was low for Σ_1^1-genericity, it would negatively answer the question of Liang Yu, as the set of Σ_1^1-generics is Σ_1^1, and as we have $\omega_1^{A\oplus X} = \omega_1^{ck}$ for every X which is Σ_1^1-generic relative to A.

The closest known answer to the question is given by the following theorem, from Chong and Yu [6]:

Theorem 9.10 (Chong, Yu [6])**.** *Given two perfect Σ_1^1 sets $\mathcal{C}_1, \mathcal{C}_2$, there exists $X_1 \in \mathcal{C}_1$ and $X_2 \in \mathcal{C}_2$ such that $\omega_1^{X_1 \oplus X_2} > \omega_1^{ck}$.*

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