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Higher computability and randomness

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[1] Mixer des haricots rouges et des gousses d'ail puis mettre de côté la purée obtenue après l'avoir assaisonnée. Dans un bol, ajouter de la chair d'avocat, du jus de citron et de la crème fraîche et mélanger jusqu'à l'obtention d'une mixture homogène. Saler et poivrer. Dans un plat allant au four, déposer la purée de haricots, puis la purée d'avocat. Recouvrir d'emmental râpé. Couper de la tomate en petits dés et recouvrir le plat. Enfourner pendant 20 minutes à 150°C et servir accompagné de nachos. Bon appétit !

"Sept personnes portent chacunes un chapeau avec un chiffre entre un et sept inscrit dessus (répétitions possibles). Elles doivent écrire en même temps un chiffre sur un morceau de papier, de telle sorte qu'au moins une d'entre elle écrive le numéro de son chapeau. Notez que chacun voit les numéros des autres chapeaux mais pas celui de son propre chapeau, et que toute forme de communication est interdite ; on suppose bien sûr que les protagonistes ont pu se parler autant qu'ils le voulaient avant de recevoir leurs chapeaux."

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Introduction

Le présent document est un rapport de thèse, résultant de trois années de recherche, menées sous la direction de Laurent Bienvenu, et rédigé en vue d'obtenir le titre de docteur en informatique. Nous commençons par une introduction gentille au domaine des mathématiques dont il est question. Nous donnerons ensuite le résumé détaillé de la thèse et les principales contributions de celle-ci.

Aléatoirité et calculabilité d'ordre supérieur

Cette thèse est une contribution à trois domaines présentant de nombreuses connexions entre eux : la théorie effective de l'aléatoire, la calculabilité d'ordre supérieur et la théorie effective descriptive des ensembles. Nous donnons d'abord une introduction vulgarisée à chacun de ces trois domaines mathématiques, avant d'expliquer l'intérêt de leur étude conjointe. Ces explications déboucheront naturellement sur les problématiques motivées et circonstanciées auxquelles nous nous attaquons.

Théorie effective descriptive des ensembles

Nous ne donnons qu'une idée sur les prémices de la théorie effective descriptive des ensembles, qui va beaucoup plus loin que ce qui est présenté ici. Comme son nom l'indique, la théorie descriptive des ensembles a pour objet d'étude... les ensembles, et comme axe d'étude... leur description. Les ensembles que nous allons considérer seront tous des sous-ensembles issus d'un espace des plus simples : l'espace $2^{\mathbb{N}}$ des suites infinies de 0 et de 1, et nous allons nous intéresser à certains sous-ensembles de $2^{\mathbb{N}}$ qui sont, informellement, 'simples à décrire' :

Considérons une chaîne finie de 0 et de 1, par exemple : 01001010. L'ensemble des éléments de $2^{\mathbb{N}}$ qui commencent par 01001010 admet pour description la chaîne 01001010 elle-même. Ces ensembles seront appelés **intervalles**, et pour une chaîne σ, ils seront dénotés par $[\sigma]$. Nous nous intéressons à présent à une classe d'ensembles un cran plus compliquée : les **ouverts**, c'est à dire les unions dénombrables d'intervalles.

Alors que chaque intervalle est aisé à décrire, la tâche peut être beaucoup plus compliquée pour un ouvert, car un ouvert est constitué d'une infinité d'intervalles. Certains ouverts restent toutefois simples à décrire. Par exemple, de la même manière qu'il est aisé de décrire l'ensemble des nombres pairs (malgré son caractère infini), il est aisé de décrire l'ouvert composé de l'union des intervalles $[0^p\hat{\ }1]$ pour tout nombre pair p, où $0^p\hat{\ }1$ dénote la chaîne composée de p chiffres 0 suivit du chiffre 1. En revanche, de par leur caractère infini, certains ouverts sont 'inaccessibles' en ce sens qu'il est impossible de les définir en un nombre fini de mots, et il en va de même pour certains ensembles de nombres entiers. Nous reviendrons sur ce point dans quelques paragraphes. Pour le moment notre préoccupation restera d'ordre 'géométrique', c'est-à-dire que nous ne faisons pas de distinction entre les ouverts simples ou compliqués à décrire. En revanche un ouvert a par définition une 'forme géométrique' des plus simples : il s'agit d'une union d'intervalles. Nous abordons à présent la question des ensembles ayant une 'forme' plus compliquée.

On s'aperçoit que relativement à un ouvert $\mathcal{U}$, il est aisément possible de définir le complémentaire de $\mathcal{U}$: 'l'ensemble des éléments qui ne sont pas dans $\mathcal{U}$'. Aussi les complémentaires des ouverts seront appelés les **fermés**. La distinction entre ces deux types d'ensemble est légitime. En effet il est aisé de construire des ensembles fermés ne pouvant pas s'écrire comme union d'intervalles. Nous en donnons un exemple, en utilisant la notion de mesure, qui sera détaillée dans la prochaîne section : on peut considérer une liste de toutes les chaînes finies $\{\sigma_i\}_{i\in\mathbb{N}}$, puis un intervalle de mesure $1/4$ contenant la suite $\sigma_0\hat{\ }0^\infty$ (où 0^∞ est la suite constituée d'une infinité de 0), un intervalle de mesure $1/8$ contenant $\sigma_1\hat{\ }0^\infty$, etc... L'union de tous ces intervalles est un ensemble de mesure au plus $1/4 + 1/8 + \cdots = 1/2$. Aussi le complémentaire de cette union est un fermé non vide (puisque de mesure au moins $1/2$) et il ne peut contenir aucun intervalle $[\tau]$, car la suite $\tau\hat{\ }0^\infty$ fait partie de son complémentaire. Qu'en est-il à présent des unions dénombrables de fermés ?

On peut continuer à définir des classes d'ensembles de plus en plus compliquées en considérant les unions dénombrables d'ensembles de la classe précédemment définie, et leur complémentaire, menant à une nouvelle classe d'ensembles, un cran plus compliquée que la précédente. En suivant le schéma que nous venons de décrire, les unions dénombrables de fermés sont dans une nouvelle classe d'ensembles, intuitivement plus complexe que la classe précédente (les ouverts et leur complémentaire).

Afin d'étudier ces classes d'ensembles dans de bonnes conditions, nous allons donner un nom à chacune d'entre elles. La classe des ensembles ouverts sera la classe des ensembles $\boldsymbol{\Sigma^0_1}$. La classe des ensembles fermés sera la classe des ensembles $\boldsymbol{\Pi^0_1}$. Ensuite, pour chaque entier n, la classe des unions dénombrables d'ensembles $\boldsymbol{\Pi^0_n}$ sera la classe des ensembles $\boldsymbol{\Sigma^0_{n+1}}$, et les complémentaires des ensembles $\boldsymbol{\Sigma^0_{n+1}}$ seront appelés ensembles $\boldsymbol{\Pi^0_{n+1}}$.

La hiérarchie précédente s'intéresse à la 'complexité de forme' des ensembles. Cette étude de 'complexité de forme' fait partie de la théorie descriptive des ensembles 'pure'. Aussi nous avons bien précisé que nous parlerions de théorie *effective* descriptive des ensembles. Il s'agit d'un raffinement des classes de complexité que nous venons de définir, à l'aide de la calculabilité. Nous avons donné l'exemple de l'ouvert composé de l'union des intervalles $[0^p\hat{\ }1]$ pour tout nombre pair p, qui est simple à décrire (nous venons précisément de le faire). Qu'en serait-il par exemple de l'ouvert égal à l'union des intervalles $[0^n\hat{\ }1]$ pour tous les n tel que 'pile' est obtenu lors d'un n-ième tirage à pile ou face ? On sent bien intuitivement qu'il y a une 'arnaque' derrière cette définition, même si l'on passe outre son caractère informel : on n'a en fait pas décrit grand chose, puisque les éléments de notre ouvert dépendent du résultat d'une infinité de tirages à pile ou face ; et nous nous trouvons bien incapable de définir un exemple raisonnable d'un tel tirage, avec un nombre fini de mots. Mais puisque l'on n'a rien dit de précis sur cet ouvert, et que l'on ne peut vraisemblablement rien en dire de précis, cela a-t-il même un sens d'en parler ? Nous affirmons que oui, du moins dans une certaine mesure, et nous y reviendrons dans la section suivante. Pour le moment, nous nous cantonnons à souligner la différence entre ces deux ouverts, le premier parfaitement cernable par la pensée, et le deuxième plus insaisissable, et dont l'existence même est sujette à caution. L'idée générale est de faire une distinction entre les ouverts que l'on peut définir avec précision en un nombre fini de mots, et les autres. La calculabilité fournit un cadre à la fois naturel et satisfaisant (en particulier exempt de paradoxe) pour mener à bien cet objectif. Un ouvert sera considéré 'simple à décrire' (on dira effectif) si il existe un programme informatique (en particulier une liste finie d'instructions) qui énumère au fur et à mesure qu'il s'exécute, des intervalles venant composer petit à petit l'ouvert.

Nous nous intéressons maintenant uniquement aux ouverts que l'on peut décrire avec un programme informatique, laissant les autres de coté. On dira donc qu'un ensemble $\mathcal{U}$ est Σ^0_1 si il est ouvert et si il existe un programme informatique énumérant une suite de chaîne $\sigma_1, \sigma_2, \dots$ tel que $\mathcal{U} = \bigcup_n [\sigma_n]$. Un tel programme peut être considéré comme étant une description de l'ensemble $\mathcal{U}$. On continue ensuite inductivement : les ensembles Π^0_1 sont les complémentaires des ensembles Σ^0_1 et admettent comme description la même que celle de leurs complémentaires. On continue en définissant pour tout n les ensembles Σ^0_{n+1} comme étant les unions effectives d'ensembles Π^0_n, c'est à dire que pour un ensemble Σ^0_{n+1}, il doit exister un programme permettant d'énumérer les descriptions des ensembles Π^0_n le composant. Un tel programme fera donc office de description pour chaque nouvel ensemble Σ^0_{n+1} ainsi défini.

On peut montrer que ces deux hiérarchies de complexité sont strictes : un cran de complexité supplémentaire nous permet toujours de décrire strictement plus d'ensembles qu'auparavant. On est pourtant loin d'avoir fait le tour : des ensembles, il y en a *beaucoup*...

Théorie algorithmique de l'aléatoire

Au début du $19^{\text{ème}}$ siècle, Laplace donne dans son ouvrage *Théorie analytique des probabilités* un résumé fort intéressant de cette discipline : "La théorie des hasards consiste à réduire tous les événements du même genre à un certain nombre de cas également possibles, c'est-à-dire tels que nous soyons également indécis sur leur existence, et à déterminer le nombre de cas favorables à l'événement dont on cherche la probabilité. Le rapport de ce nombre à tous les cas possibles est la mesure de cette probabilité qui n'est ainsi qu'une fraction, dont le numérateur est le nombre des cas favorables et dont le dénominateur est le nombre de tous les cas possibles".

La théorie des probabilités nous enseigne que si on répète un très grand nombre de fois une succession de dix tirages à pile ou face, on obtiendra à peu près autant de fois la suite pppppppppp que la suite ppfpffppfp (où 'p'=pile et 'f'=face). Pourtant un joueur obtenant la première de ces suites considèrera certainement cet évènement comme extraordinaire, alors qu'il ne verra rien d'anormal à obtenir la deuxième.

Un examen attentif de la définition que Laplace donne de la 'théorie du hasard' permettrait presque de proposer une explication à ce phénomène : nous classons machinalement, les 'cas également possibles' de suites de piles et de faces dans des groupes, en fonction de certaines règles simples. Ainsi "ne contenir que des piles" est une règle simple à formuler, permettant la création du groupe "des suites ne contenant que des piles". Si une suite obtenue après un tirage de pile ou face tombe dans un groupe à la fois simple à décrire, et contenant peu d'éléments, elle nous semble alors non aléatoire.

La théorie des probabilités est impuissante à décrire ce phénomène. Aussi la théorie effective descriptive des ensembles va-t-elle nous fournir un cadre naturel pour déterminer les "règles" permettant de classer les suites dans les fameux groupes mentionnés ci-dessus.

Martin-Löf propose en 1966 dans [58] une définition qui reste aujourd'hui la plus connue et la plus étudiée. Les "groupes" d'éléments que nous allons considérer pour la définition de Martin-Löf seront simplement les ensembles Π_2^0, c'est à dire d'après la définition donnée dans la section précédente, les complémentaires d'unions effectifs de fermés effectifs. On peut toutefois exprimer les ensembles Π_2^0 plus simplement, comme intersections effectives d'ouverts effectifs. Il nous reste à formaliser la notion de "contenir peu d'éléments".

Pour ce faire, considérons $\mathcal{A}$ un ensemble Π_2^0, c'est à dire un ensemble pouvant être décrit comme une intersection d'ouverts effectifs, dont les descriptions sont énumérables par un programme informatique P (chaque description étant elle-même un programme informatique permettant d'énumérer l'ouvert correspondant). Notons $\mathcal{U}_n$ l'ouvert dont la description est la n-ième à être énumérée par P. On a donc $\mathcal{A} = \bigcap_n \mathcal{U}_n$. On dira que $\mathcal{A}$ est **effectivement de mesure nulle** si pour chaque entier n, la probabilité pour qu'une suite de 0 et de 1 - tirée aléatoirement - appartienne à l'ensemble $\mathcal{U}_n$, est plus petite que 2^{-n}. On suppose bien sûr que les tirages successifs de chaque bit de la suite sont uniformes (autant de chance de tirer 0 que 1). Intuitivement la probabilité pour qu'une suite appartienne à l'ensemble $\bigcap_n \mathcal{U}_n$ est donc de 0, puisque pour tout n, la probabilité pour qu'elle appartienne à l'ensemble $\mathcal{U}_n$ est plus petite que 2^{-n}, et qu'elle appartient par hypothèse à *tous* les $\mathcal{U}_n$. Il est donc justifié de considérer ces suites comme étant non aléatoires, puisque la probabilité de les obtenir est nulle. Les suites aléatoire au sens de Martin-Löf sont précisément celles qui ne sont dans aucun ensemble Π_2^0 qui soit effectivement de mesure nulle.

On formalise cette idée intuitive de probabilité en termes mathématiques à l'aide de la notion de **mesure uniforme** sur $2^{\mathbb{N}}$, que l'on dénote par λ. Par convention la mesure de $2^{\mathbb{N}}$ est de 1, ce qui correspond à la probabilité pour qu'une suite tirée aléatoirement soit dans $2^{\mathbb{N}}$ (l'ensemble de toutes les suites). On écrira alors $\lambda(2^{\mathbb{N}}) = 1$. On définit ensuite naturellement $\lambda([0]) = 1/2$ et $\lambda([1]) = 1/2$, chacune de ces valeurs correspondant à la probabilité pour qu'une suite commence par 0 et respectivement par 1. Plus généralement on définit $\lambda([\sigma]) = 2^{-|\sigma|}$ pour toute chaîne σ, où $|\sigma|$ dénote la taille de σ. On constate aisément que la fonction λ peut être étendue à toute union finie d'intervalles : pour tout n et toute suite finie de chaînes $\sigma_1, \dots, \sigma_n$ que l'on peut supposer deux à deux incomparables, on a $\lambda([\sigma_1] \cup \dots \cup [\sigma_n]) = \lambda([\sigma_0]) + \dots + \lambda([\sigma_n])$, et effectivement, la probabilité pour qu'une suite commence par une des chaînes σ_i pour $1 \leq i \leq n$ est bien la somme des probabilités pour chaque i, que la suite soit dans $[\sigma_i]$.

On peut ensuite étendre la fonction λ à tout ensemble ouvert : Pour un ouvert $\mathcal{U}$ et une description de $\mathcal{U}$ donnée par une suite infinie de chaînes $\sigma_1, \sigma_2, \dots$, que l'on peut supposer deux à deux incomparables, on a $\lambda(\mathcal{U}) = \sup_{i \in \mathbb{N}} \lambda([\sigma_1] \cup \dots \cup [\sigma_i])$, le supremum pour tout i de la mesure de l'union des i premiers intervalles composant l'ouvert. Il est possible de montrer que la mesure d'un ouvert, ainsi définie, est indépendante de la description choisie, et ici encore, la mesure d'un ouvert correspond à la probabilité pour qu'une suite soit un élément de cet ouvert.

Une suite est donc Martin-Löf aléatoire si elle n'appartient à aucun ensemble de type $\bigcap_n \mathcal{U}_n$ (ensemble Π_2^0) qui soit effectivement de mesure nulle, c'est à dire avec $\lambda(\mathcal{U}_n) \leq 2^{-n}$, pour tout n. On peut vérifier que la notion de mesure s'étend bien de manière naturelle aux ensembles Π_2^0 comme étant cette fois-ci l'infimum de la mesure sur les intersections des i premiers ouverts du Π_2^0, pour tout i (une intersection finie d'ouverts étant toujours un ouvert). Formellement on a $\lambda(\bigcap_n \mathcal{U}_n) = \inf_{i \in \mathbb{N}} \lambda(\mathcal{U}_1 \cap \dots \cap \mathcal{U}_i)$. On peut également vérifier que la mesure ainsi définie est indépendante de la manière dont est présenté le

Π_2^0. On peut étendre de la même manière cette notion de mesure à tout ensemble de la hiérarchie définie dans la section précédente, et vérifier que cette mesure respecte toujours l'idée intuitive de ce que doit être une mesure : si $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \dots$ est une suite dénombrable d'ensembles deux à deux disjoints, sur lesquels la mesure est définie, alors la mesure est définie sur leur union et on a $\lambda(\mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{A}_3 \cup \dots) = \lambda(\mathcal{A}_1) + \lambda(\mathcal{A}_2) + \lambda(\mathcal{A}_2) \dots$. En particulier l'union de tous les ensembles Π_2^0 et effectivement de mesure nulle, est lui aussi un ensemble de mesure nulle, et le complémentaire de cette union, c'est à dire l'ensemble des suites Martin-Löf aléatoires, un ensemble de mesure 1. Il y a en un certain sens 'beaucoup plus' de suites aléatoires que de suites 'non aléatoires'. Pourtant toute tentative de donner une description précise (à l'aide d'un programme informatique) d'une suite aléatoire est impossible, car une descripion trop précise permettrait de 'l'enfermer' dans un ensemble Π_2^0 'petit', c'est à dire effectivement de mesure nulle. Les suites aléatoires sont donc des suites sur lesquelles on ne peut rien dire de précis, et elles se trouvent être la majorité des suites.

De nombreuses autres variantes d'aléatoirité ont été étudiées depuis, et Martin-Löf lui-même proposa une autre définition dans [59], beaucoup plus forte, où cette fois-ci les ensembles de mesure nulle, capturant les suites non aléatoires, peuvent être décrits avec la puissance des calculabilités d'ordre supérieur que nous abordons à présent.

Calculabilités d'ordre supérieur

Comme nous l'avons vu, la calculabilité peut être considérée comme une manière de décrire certains objets infinis avec un nombre fini de mots, dans notre cas des programmes informatiques. C'est ensuite le temps de calcul qui permet de dérouler petit à petit cette description finie en un objet de plus en plus grand, qui est l'objet ainsi décrit. Comme l'avaient remarqué les instigateurs de cette nouvelle science, en particulier Gödel et Turing, un tel système de description conduit à des définitions naturelles d'objets incalculables. L'exemple le plus connu est celui de l'arrêt des programmes informatiques : la suite infinie de 0 et de 1 telle que son $n^{\text{ème}}$ bit est égal à 1 si le $n^{\text{ème}}$ programme informatique s'arrête et son $n^{\text{ème}}$ bit est égal à 0 sinon, n'admet pas de description calculable. Si un programme s'arrête on le saura toujours au bout d'un temps fini. Mais s'il ne s'arrête pas on ne le saura peut-être jamais; intuitivement il nous faudrait en effet la possibilité de vérifier que le programme ne s'arrête pas pour n'importe quel temps de calcul, ce qui implique une infinité de choses à vérifier, et qui prendrait donc un temps de calcul infini.

Justement, les calculabilités d'ordre supérieur peuvent être considérées comme utilisant les algorithmes usuels, mais permettant une infinité d'étapes de calcul durant leur exécution. Que peut-on bien vouloir dire par là ? Si l'on s'autorise un temps de calcul infini, ne peut-on pas tout calculer ? Quel sens donner à la notion de temps de calcul infini ? Il y a deux approches possibles pour répondre à ces questions. La première est probablement la meilleure en ce sens qu'elle est celle qui en dit le plus sur le sujet. Elle serait malheureusement trop longue à exposer ici, et nous nous rabattrons donc sur la seconde, plus courte à expliquer.

On distingue les étapes de calcul **limites** des étapes de calcul **successeurs**. La manière dont un programme informatique se comporte à une étape de calcul successeur est identique à la manière dont il se comporte pour la calculabilité usuelle. Mais après toutes les étapes finies de calcul $0, 1, 2, \dots$, on s'autorise une étape de calcul limite, la première d'entre elles. A cette étape en fait aucun calcul ne se passe, et c'est un peu comme si tout recommençait depuis 0, à ceci près qu'on garde la *mémoire* de ce qui a été effectué précédemment. Concrètement on peut imaginer qu'à cette étape, la valeur de chaque bit de mémoire (mémoire que l'on suppose elle aussi infinie), est égale à la valeur de convergence de la valeur de ce bit à chaque étape de calcul précédant cette étape limite. C'est à dire que pour un bit donné, si il existe un temps t tel que pour tout temps de calcul s supérieur à t, la valeur de ce bit est de 1, alors au temps de calcul limite, la valeur de ce bit sera également de 1. Il est évidement possible que la valeur d'un bit ne converge pas, et qu'elle change en permanence avant l'étape limite. Dans ce cas la valeur du bit à l'étape limite revient à 0.

Ainsi après la première étape limite que nous notons ω, on poursuit le calcul normalement aux étapes $\omega+1$, $\omega+2$, avec la mémoire de ce que l'on a fait précédemment. Arrive ensuite la deuxième étape limite $\omega+\omega$, et ainsi de suite. Mais où cela s'arrête-t-il, et au bout de 'combien de temps' (le 'temps' ici est à prendre dans un sens large...) décide-t-on qu'un calcul doit finalement s'arrêter et donner sa réponse ? Si un calcul ne s'arrête pas, il viendra bien un moment où on atteindra la première étape limite d'étapes limites (appelons ces étapes les étapes 2-limites), puis la première étape limite d'étapes limites d'étapes limites (les étapes 3-limites), et ainsi de suite jusqu'à la première étape ω-limite, puis jusqu'à la première étape ω-limite d'étapes ω-limites, etc... On pourrait continuer ainsi longtemps, d'ailleurs sans jamais s'arrêter. Il existe toutefois une borne 'naturelle' à toutes ces étapes limites, que l'on appelle étape limite ω_1^{ck}, et dont nous essayons à présent de donner une idée informelle.

Les notions intuitives que nous venons de donner des étapes limites et successeurs ont été imaginées bien avant l'apparition de la calculabilité, et ces notions sont plus connues sous les noms d'ordinaux limites et ordinaux successeurs. Il est aisé d'avoir une idée intuitive de ce que sont les ordinaux en les construisant tout simplement, comme nous venons de le faire avec les premières étapes limites et successeurs d'un calcul infini. Aussi un ordinal peut être vu de manière plus abstraite comme un ordre entre divers éléments et ayant certaines propriétés. Par exemple l'ordre usuel des entiers $(0 < 1 < 2 < \dots)$ représente ω, le plus petit ordinal limite, alors que l'ordre usuel des entiers pairs, suivi de l'ordre usuel des entiers impairs (c'est à dire $0 < 2 < 4 < \dots < 1 < 3 < 5 < \dots$) représente $\omega+\omega$, le deuxième ordinal limite; et à l'intérieur de cette représentation, l'ensemble des entiers plus petits que 1 représente ω. Il est aisé d'imaginer de telles représentations pour des ordinaux de plus en plus grands, mais le lecteur s'adonnant à cet exercice s'apercevra qu'on finit forcément par se perdre dans ces itérations d'infinis successifs. Aussi parlera-t-on d'**ordinaux calculables** pour les ordinaux que l'on peut représenter par un ordre sur les entiers, avec un programme informatique. C'est-à-dire que pour un ordinal α, le programme en question doit pouvoir énumérer petit à petit une liste de plus en plus grande de conditions de la forme $n < m$ pour n et m deux nombres entiers, de telle manière que l'ordre ainsi défini représente l'ordinal α. On définit alors ω_1^{ck} comme étant le plus petit ordinal non représentable de cette manière. Mais quel type de programme informatique s'autorise-t-on pour représenter ces ordinaux ? Les programmes informatiques classiques ? où ceux qui peuvent justement s'exécuter en temps infini ? Un fait remarquable est que cela n'a en fait pas d'importance. En effet ω_1^{ck} est le plus petit ordinal non calculable par un programme informatique au sens classique, mais c'est aussi le plus petit ordinal non calculable par un programme informatique pouvant effectuer un nombre de calcul infini, mais borné par ω_1^{ck} lui même (c'est à dire que le calcul doit s'arrêter à un temps ordinal $\alpha < \omega_1^{ck}$, l'ordinal α pouvant être choisi arbitrairement grand sous ω_1^{ck}). Cette propriété remarquable est une des raisons pour lesquelles ω_1^{ck} est un candidat naturel pour borner le temps de calcul de nos machines infinies.

Aléatoirité d'ordre supérieur

Nous abordons à présent le sujet de cette thèse, qui se trouve au carrefour des trois notions explicitées dans chacune des trois sections précédentes. Dans cette thèse, nous proposons une étude des diverses notions d'aléatoirité qui ont fait suite à la deuxième définition de Martin-Löf, celle qui utilise les calculabilités d'ordre supérieur, et que nous esquissons à présent.

La notion d'ordinal calculable peut être utilisée pour continuer la hierachie descriptive des ensembles décrite plus haut. Ainsi les ensembles Σ^0_ω sont les unions effectives d'ensembles Π^0_n pour des entiers n arbitrairement grands, les ensembles Π^0_ω leurs complémentaires, les ensembles $\Sigma^0_{\omega+1}$ les unions effectives d'ensembles Π^0_ω, et ainsi de suite. La notion de mesure peut être étendue à tous ces ensembles. Pour sa deuxième définition d'aléatoirité, Martin-Löf s'autorise cette fois-ci tous les ensembles Π^0_α de mesure nulle, pour tout ordinal $\alpha < \omega_1^{ck}$. Cette définition se trouve être équivalente à sa première définition, mais où l'on utiliserait cette fois-ci des programmes informatiques à temps de calcul infini, et borné par un ordinal $\alpha < \omega_1^{ck}$, pour énumérer les descriptions d'un ensemble ouvert (l'ordinal en question pouvant être choisit arbitrairement, mais strictement plus petit que ω_1^{ck}).

Plusieurs autres notions d'aléatoirité, encore plus fortes ont été définies. Par exemple la notion de Π^1_1-Martin-Löf aléatoirité, similaire à la première notion d'aléatoire de Martin-Löf, mais où cette fois-ci on s'autorise un temps de calcul infini, et borné par ω_1^{ck} et non $\alpha < \omega_1^{ck}$. Une autre notion importante, et qui sera plus particulièrement étudiée dans cette thèse, est la notion de Π^1_1-aléatoirité, que nous ne développons pas dans cette introduction.

Résumé de la thèse

Nous commençons cette section en expliquant avec précision sur quoi porte la thèse. Nous expliquons ensuite son contenu chapitre par chapitre, et nous terminons par un résumé des contributions notables des travaux exposés ici.

Le sujet de thèse

Danc cette thèse, nous traitons principalement des notions d'aléatoirité d'ordre supérieur, notamment les notions de Δ^1_1-aléatoirité, de Π^1_1-Martin-Löf aléatoirité, de Π^1_1-aléatoirité faible, et de Π^1_1-aléatoirité, en mettant plus particulièrement l'accent sur cette dernière notion : la Π^1_1-aléatoirité. L'étude de ces notions d'aléatoirité soulève plusieurs problématiques. Nous essayons notamment de comprendre les similarités et les différences entre toutes ces notions, mais aussi entre ces notions et les notions d'aléatoirité classiques, largement étudiées ces quinze dernières années.

Une différence importante entre les notions de calculabilité/aléatoirité d'ordre supérieur et les notions de calculabilité/aléatoirité classique est de nature topologique. Aussi avons-nous concentré nos efforts sur trois des phénomènes à travers lesquels cette différence s'exprime : dans la notion de calcul, dans la notion d'aléatoire relatif, et dans la notion d'approchabilité.

Nous soulignons également les liens étroits entre la notion d'aléatoirité et celle de généricité, que l'on peut considérer comme une version catégorique (au sens de Baire) de l'aléatoirité. Pour cette raison, nous étudions aussi la catégoricité effective d'ordre supérieur et nous mettons en avant les différences et similarités qu'elle présente avec la notion d'aléatoirité.

Structure de la thèse

Nous détaillons ici la structure de la thèse, expliquant brièvement le contenu de chaque chapitre.

Dans le premier chapitre nous définissons les notions de base sur l'espace de Cantor, l'espace de Baire, la calculabilité et les ordinaux. Nous faisons ensuite une étude détaillée des ordinaux calculables, que nous utilisons ensuite pour étudier les hiérarchies boréliennes effective et non effective dans l'espace de Baire. Nous menons ensuite une étude similaire de la hiérarchie de Kleene des ensembles d'entiers, établissant par la suite les connexions entre la hiérarchie borélienne effective et celle de Kleene. Nous terminons ce chapitre par quelques notions de base sur les mesures (afin d'étudier l'aléatoirité) et sur les catégories de Baire (afin détudier la généricité).

Dans le deuxième chapitre, nous présentons l'aléatoire algorithmique, à travers la notion principale du domaine : l'aléatoirité au sens de Martin-Löf ; puis nous itérons cette notion à travers la hiérarchie borélienne effective, pour obtenir une hiérarchie de notions d'aléatoirité. Enfin nous procédons de la même manière pour la notion de généricité, avant d'expliquer en quoi les notions d'aléatoirité et de généricité sont similaires.

Dans le troisième chapitre nous posons les bases de la calculabilité d'ordre supérieur. Nous commençons par définir et étudier les notions d'ensembles Σ^1_1, Π^1_1 et Δ^1_1. Nous définissons et étudions ensuite le nombre $\mathcal{O}$ de Klenne, l'ensemble des codes d'ordinaux constructifs, qui sera central tout au long de la thèse. Nous expliquons ensuite en quoi la notion d'être Π^1_1 est l'analogue d'ordre supérieur de la notion - centrale en calculabilité - d'être Σ^0_1. Nous terminons enfin par une introduction des différentes notions de base d'aléatoirité d'ordre supérieur, en insistant plus particulièrement sur celle de Π^1_1-Martin-Löf aléatoirité, l'analogue d'ordre supérieur de la principale notion d'aléatoirité de la théorie classique.

Dans le quatrième chapitre nous étudions les problèmes issus des différences topologiques entre calculabilité/aléatoirité d'orde supérieur, et calculabilité/aléatoirité classique. Nous définissons la nouvelle notion de réduction Turing d'ordre supérieur, dans le but de conserver la puissance descriptive que nous confère la réduction hyperarithmétique, tout en préservant la continuité inhérente aux réductions Turing classiques. Nous étudions ensuite le comportement de cette nouvelle réduction sur divers types d'éléments, notamment les éléments 'suffisamment' aléatoires ou 'suffisamment' génériques. Puis nous définissons une nouvelle façon de relativiser à un oracle, diverses notions relatives à l'aléatoirité d'ordre supérieur, afin encore une fois de préserver la continuité des relativisations classiques tout en gardant la puissance descriptive des relativisations d'ordre supérieur. Nous utilisons cette nouvelle notion pour prouver un analogue d'ordre supérieur de deux théorèmes importants de la théorie classique : Le théorème XYZ et le théorème de van Lambalgen. Enfin nous définissons plusieurs restrictions à la notion d'approchabilité d'ordre supérieur, toujours dans le but de préserver la continuité des approximations classiques. Pour finir nous utilisons toutes les notions précédemment introduites pour définir les notions de 'low-for-hK' et base pour Π^1_1-Martin-Löf aléatoirité, puis nous démontrons leur équivalence avec les notions d'hK-trivialité.

Dans le cinquième chapitre nous étudions de nouvelles notions d'aléatoirité d'ordre supérieur, par analogie avec les notions d'aléatoirité classique. Nous définissons ainsi la différence-aléatoirité d'ordre supérieur, la Π^1_1-Martin-Löf[$\mathcal{O}$]-aléatoirité et la Π^1_1-aléatoirité faible. Nous démontrons toutes les implications et non implications entre ces différentes notions, et nous séparons notamment la notion de Π^1_1-aléatoirité faible de celle de Π^1_1-aléatoirité. Afin de séparer les deux notions, nous utilisons une des restriction de la notion d'approchabilité d'ordre supérieur : la notion d'approchabilité 'self-unclosed'. Nous terminons ce chapitre par une étude d'étaillée des différentes notions d'approchabilité.

Dans le sixième chapitre nous étudions les notions de Π^1_1-aléatoirité et la notion similaire de Σ^1_1-généricité. Nous commençons par démontrer que la complexité borélienne de l'ensemble des suites Π^1_1-aléatoires est exactement $\boldsymbol{\Pi^0_3}$. Nous utilisons ce résultat pour donner trois autres définitions de la notion de Π^1_1-aléatoirité. Nous introduisons ensuite une nouvelle hiérarchie de complexité effective des ensembles, et nous démontrons que la Π^1_1-aléatoirité coïncide avec la $\Pi_4^{\omega_1^{ck}}$-aléatoirité de cette hiérarchie. Nous montrons aussi que l'ensemble des Π^1_1-aléatoires est au niveau $\Pi_5^{\omega_1^{ck}}$ de cette hiérachie (donnant donc une autre manière de caractériser l'ensemble des suites Π^1_1-aléatoires). Nous démontrons ensuite qu'un élément non trivial (non Δ^1_1) "dé-aléatoirise" toujours un élément Π^1_1-aléatoire, répondant ainsi à une question ouverte de Hjorth et Nies (voir [30]). Nous définissons et étudions également plusieurs notions de genericité d'ordre supérieur, en montrant les similarités et les différences avec les notions d'aléatoirité d'ordre supérieur. Nous terminons ce chapitre par une caractérisation de la complexité borélienne de l'ensemble $\{X : \omega_1^X > \omega_1^{ck}\}$, en utilisant le forcing de Steel.

Dans le septième chapitre, nous étudions les problèmes issus du fait de forcer la continuité dans les calculs d'ordre supérieur ou dans la relativisation de notions d'ordre supérieur. Nous séparons notamment la notion de réduction Turing d'ordre supérieur de celle de réduction fin-h. Nous démontrons ensuite que pour certains oracles, il n'existe pas de de test Π^1_1-Martin-Löf universel relatif à ces oracles. Nous démontrons ensuite malgré tout que pour tout oracle, il existe une suite Π^1_1-Martin-Löf aléatoire relative à cet oracle, qui est 'approchable par la gauche' relativement à l'oracle. Nous finissons par une étude des oracles pour lesquels la relativisation continue se passe bien, notamment les oracles 'self-unclosed', et dans une certaine mesure, les oracles Π^1_1-Martin-Löf aléatoires.

Principales contributions

Nous détaillons ici les principaux résultats originaux de cette thèse :

- La contribution la plus importante de cette thèse est sans doute une meilleure compréhension de l'ensemble des suites Π^1_1-aléatoires, définies implicitement par Kechris dans [33] qui prouva l'existence d'un plus grand ensemble Π^1_1 de mesure nulle, étudiées ensuite par Sacks, puis plus tard par Hjorth et Nies dans [30] qui entamèrent l'étude proprement dite des éléments n'étant pas dans cet ensemble : les Π^1_1-aléatoires. Cette étude fut poursuivie par Chong, Nies et Yu dans [7], puis par Chong et Yu dans [8]. Due à sa nature universelle, l'ensemble des Π^1_1-aléatoires était conjecturé par beaucoup comme ayant une grande complexité borélienne. Nous démontrons dans cette thèse que celle-ci est au contraire relativement basse : $\boldsymbol{\Pi^0_3}$ (voir Corollaire 6.1.1). Nous utilisons ensuite ce résultat pour mener une étude détaillée de cet ensemble, notamment :
  - Nous résolvons par l'affirmative, la question ouverte depuis plusieurs années

(voir [30] et question 9.4.11 de l'ouvrage de référence [70]), de savoir si seules les suites Δ_1^1 sont 'low' pour la Π_1^1-aléatoirité.

- Nous démontrons qu'une suite Π_1^1-Martin-Löf aléatoire n'est pas Π_1^1-aléatoire si et seulement si elle calcule une suite Π_1^1 non triviale (non Δ_1^1).
- Nous séparons la notion de Π_1^1-aléatoirité d'une notion d'aléatoire encore mal comprise dans l'ordre supérieur (contrairement à son analogue classique) : avoir infiniment souvent un préfixe de complexité maximale (voir Corollaire 6.2.1).
- Nous donnons deux autres notions de tests pour la Π_1^1-aléatoirité (voir Théorème 6.3.2 et Théorème 6.3.3).

- Un autre résultat important et contribuant toujours à une meilleure compréhension de l'ensemble des Π_1^1-aléatoires, est la séparation de cette notion avec celle de Π_1^1-aléatoirité faible (voir Théorème 5.3.3). Nous jugeons ce résultat important pour lui-même, ainsi que pour sa démonstration, qui a nécessité l'introduction d'une idée nouvelle et prometteuse : les approximations 'self-unclosed'.

- Toujours afin de mieux comprendre l'ensemble des Π_1^1-aléatoires, nous avons introduit une nouvelle hiérarchie de complexité, motivée par la définition de la Π_1^1-aléatoirité faible, qui selon cette hiérarchie correspond à la notion de $\Pi_2^{\omega_1^{ck}}$-aléatoirité. Nous démontrons que l'ensemble des Π_1^1-aléatoires correspond à la notion de $\Pi_4^{\omega_1^{ck}}$-aléatoirité pour cette hiérarchie, et que les notions d'aléatoirité s'effondrent au delà de $\Pi_4^{\omega_1^{ck}}$ (voir Théorème 6.4.3).

- Un autre pan important de la thèse est l'étude de la continuité pour les réductions d'ordre supérieur et la relativisation des notions d'aléatoirité d'ordre supérieur. Cette étude est principalement menée dans le chapitre 4, puis dans le chapitre 7. Nous donnons ici les principaux résultats relatifs à chacun de ces chapitres :

  - Dans le chapitre 4, nous démontrons que l'on peut utiliser ces nouvelles notions pour énoncer et démontrer de nombreux théorèmes analogues à ceux que l'on trouve en aléatoirité classique. Notamment le théorème XYZ, le théorème de van Lambalgen (voir Théorème 4.3.3 et Théorème 4.3.5). Nous montrons aussi que les équivalences entre les notions de K-trivialité, low-for-K et base for randomness peuvent être définies pour l'aléatoirité d'ordre supérieur et coincident (voir Théorème 4.5.3 et Théorème 4.5.4).
  - Dans le chapitre 7, nous étudions les oracles pour lesquels la relativisation d'ordre supérieur continue et/ou la réduction Turing d'ordre supérieur, posent des problèmes. Nous séparons notamment les notions de réduction Turing d'ordre supérieur et de réduction fin-h. Nous démontrons également que pour certains oracles, il n'existe pas de notion de test Π_1^1-Martin-Löf universel, continument relatif à cet oracle. Nous démontrons ensuite malgré tout que pour tout oracle A, il existe une suite Π_1^1-Martin-Löf aléatoire continument relativement à A, qui est 'approchable par la gauche' continument relativement à A. Nous finissons par une étude des oracles pour lesquels la relativisation continue se passe bien, notamment les oracles 'self-unclosed', et dans une certaine mesure, les oracles Π_1^1-Martin-Löf aléatoires.

- Nous menons une étude circonstanciée des différentes restrictions de la notion d'approchabilité (higher Δ_2^0) dont l'approchabilité self-unclosed, utilisée pour séparer la Π_1^1-aléatoirité de la Π_1^1-aléatoirité faible. Cette étude est d'abord menée dans la section 4.4 où l'on dégage les principales notions, du moins celles qui sont utilisées au sein d'autres théorèmes. Une étude plus approfondie est ensuite menée dans la section 5.4 où l'on étudie ces notions pour elles-mêmes, notamment en prouvant qu'elles sont toutes différentes (voir Section 5.4.6 pour un résumé).

- Nous contribuons aussi à une meilleure compréhension de la Π^1_1-aléatoirité faible, et notamment des différences entre cette notion et son analogue classique : la 2-aléatoirité faible. Nous mettons en particulier en évidence qu'une notion de test correspondant à la 2-aléatoirité faible dans le cas classique, a un analogue d'ordre supérieur distinct de la Π^1_1-aléatoirité faible, donnant lieu à une autre notion d'aléatoirité : la Π^1_1-Martin-Löf[$\mathcal{O}$] aléatoirité (voir Section 5.2). Nous identifions la restriction nécessaire sur la notion de Π^1_1-Martin-Löf[$\mathcal{O}$] aléatoirité, afin de garder l'équivalence avec la Π^1_1-aléatoirité faible (voir Théorème 5.3.2).

- Nous définissons et étudions plusieurs notions de généricité d'ordre supérieur. Nous identifions notamment les similarités et les différences avec les notions d'aléatoirité d'ordre supérieur (voir Section 6.6). Nous montrons en particulier que de manière inattendue, la notion de Σ^1_1-généricité faible coincide avec la notion de Σ^1_1-généricité forte (contrairement aux notions analogues d'aléatoirité, voir Théorème 6.6.4).

- Nous menons dans la section 1.6 une étude originale des ensembles Σ^0_α-complets, par le biais des ensembles de codes d'ordinaux calculables.

- Nous donnons une preuve que la complexité borélienne de l'ensemble $\{X : \omega_1^X > \omega_1^{ck}\}$ est exactement $\boldsymbol{\Sigma}^0_{\omega_1^{ck}+2}$. La preuve en question, dont les grandes lignes ont été esquissées dans [88] est due à Steel, toutefois comme elle n'a jamais été écrite complètement, et qu'elle est loin d'être évidente (et qu'il s'agit d'un très beau résultat), nous avons jugé important d'en écrire les détails ici (voir Section 6.7).

Introduction

This document is a thesis report resulting from three years of research under Laurent Bienvenu's supervision. We start with a gentle introduction to the mathematical fields that we deal with here. We then give a detailed summary of the thesis and its main contributions.

Higher randomness and computability

This thesis contributes to three fields of research which are connected to one another in many ways: effective randomness, higher computability and effective descriptive set theory. We first give an introduction to each of these fields, trying to explain the interest of their joined study. These explanations will naturally lead to the questions we will be dealing with all along the report.

Effective descriptive set theory

We only sketch here a few ideas on the very beginning of descriptive set theory, this field extending way beyond than what is presented here. As its name suggests, descriptive set theory studies... sets, with respect to their... description. The sets we consider here are all subsets of one of the simplest spaces in mathematics: the space of infinite sequences of 0's and 1's, denoted by 2^ω, and we will focus our interest on the subsets of 2^ω which are, informally, simple to describe:

Let us consider a **string**, that is, a finite sequence of 0's and 1's, for instance: 01001010. The set of sequences of $2^\mathbb{N}$ starting with 01001010 can be described by the string 01001010 itself. Those sets will be called **intervals**, and given a string σ, the corresponding interval will be denoted by $[\sigma]$. Let us now increase the difficulty and consider the so called open sets, that is, countable unions of intervals.

Each interval is clearly easy to describe, but the task might be much more complicated for an open set: Indeed, such a set is built with *infinitely* many intervals. Some open sets remain however simple to describe. For example it is easy to describe the set of even numbers (despite the fact that there are infinitely many of them), and it then follows that it is just as easy to describe the open set being the union of the intervals $[0^p\hat{}1]$ for all even numbers p; where $0^p\hat{}1$ denotes the string starting with p times the bit 0 followed by the bit 1. On the contrary, other open sets are 'inaccessible', due to their infiniteness, and it is impossible to describe them with finitely many words; the same being true for some sets of integers. We shall come back to this in a few paragraphs. For now we care only about the 'shape' of a set, without making any further distinctions. Also we emphasize that any open set has a very simple 'shape': it is merely a union of intervals. We continue our study by increasing the 'shape complexity' once more.

It is clear that given an open set $\mathcal{U}$, one can easily define its complement: 'the set of sequences which are not in $\mathcal{U}$'. The complement of an open set will be called a **closed set**. The distinction between open sets and closed sets is legitimate, it is indeed easy to build closed sets which cannot be described as a union of intervals, or which do not even contain a single interval. We give here an example using the notion of measure, that we will detail in the next section: Let us consider a list $\{\sigma_i\}_{i\in\mathbb{N}}$ of all the strings and then, an interval of measure $1/4$ containing the sequence $\sigma_0\,\hat{}\,0^\infty$ (where 0^∞ is the sequence of infinitely many 0's), an interval of measure $1/8$ containing the sequence $\sigma_1\,\hat{}\,0^\infty$, etc... The union of all those intervals is a set of measure at most $1/4 + 1/8 + \cdots = 1/2$. Also the complement of this union is a closed set which is non-empty (as it has measure at least $1/2$) and which contains no interval $[\tau]$ because the sequence $\tau\,\hat{}\,0^\infty$ belongs to its complement. Now, what about countable unions of closed sets?

We can continue to define more and more complicated classes of sets by considering countable unions of sets of the previously defined class, and their complement, leading to a new class of sets which is one step more complex than the previous one. For instance the countable unions of closed sets are in a new class of sets, intuitively more complex than the open sets and their complements.

In order to study those classes of sets in good conditions, we now give a name to each of them. The class of open sets will be the class of $\boldsymbol{\Sigma^0_1}$ sets. The class of closed sets will be the class of $\boldsymbol{\Pi^0_1}$ sets. Then for any integer n, the class of countable unions of $\boldsymbol{\Pi^0_n}$ sets will be the class of $\boldsymbol{\Sigma^0_{n+1}}$ sets, and their complement will be the class of $\boldsymbol{\Pi^0_{n+1}}$ sets.

This hierarchy deals with the 'shape complexity' of sets, which is part of pure descriptive set theory. We will now introduce *effective* descriptive set theory, which is a refinement of the classes of complexity we just defined. We gave above the example of the open set built as the union of the intervals $[0^p\,\hat{}\,1]$ for p even. This open set is very simple to describe accurately (we just gave an accurate description of it). What about the open set which equals the union of intervals $[0^n\,\hat{}\,1]$ for every n such that 'head' is the n-th outcome of infinitely many coin tossing that the reader should perform, starting from now? We feel of course swindled by this definition: it seems that we did not describe much of this open set, because its elements depend on the result of infinitely many upcoming coin tossing; and how could we know those results in advance? We will come back on this in the next section. For now we simply emphasize the difference between those two open sets, the first one accurately definable and the second one, elusive and whose 'existence' is not even clear. The general idea is to make a distinction between the open sets we can accurately define with a finite number of words and the others. Computability is a suitable (in particular paradox-free) and natural framework to pursue this goal. Also an open set will be considered 'simple to describe' - we will say **effective** - if there is a computer program, that is, a finite list of instructions, which enumerates intervals whose growing union converges to the open set.

We are now only interested in the effective open sets, leaving the others aside. We say that a set $\mathcal{U}$ is Σ^0_1 if it is open and if there is a computer program enumerating a list of strings $\sigma_1, \sigma_2, \ldots$ such that $\mathcal{U} = \bigcup_n [\sigma_n]$. Such a program can be seen as a description of the set $\mathcal{U}$. We continue inductively: The Π^0_1 sets are the complements of the Σ^0_1 sets and have the same description as that of their complement. We continue inductively by defining for every n the Σ^0_{n+1} sets as the effective unions of Π^0_n sets, that is, for a Σ^0_{n+1} set

$\mathcal{A}$, there must exists a program enumerating the descriptions of all the Π^0_n sets $\mathcal{A}$ consists of. Such a program is then a description of $\mathcal{A}$.

We can show that both hierarchies of complexity are strict: by increasing the complexity, we can always describe strictly more sets than before. We are nonetheless very far from defining all possible sets this way: there are many sets...

Effective randomness

Probability theory teaches us that if we repeat a large number of times a list of ten coin tossing, we will more or less equally obtain the same number of time the string hhhhhhhhhh and the string hhthtthhth (where 'h'=head and 't'=tail). However a player obtaining the first of those strings will consider this event as extraordinary, whereas he won't see anything special about the second one.

The theory of effective randomness proposes a 'solution' to this apparent paradox. We instinctively classify the possible outcomes of sequences of ten coin tosses into groups, depending on some simple rules. Also the rule "containing only heads" is very simple to formulate and leads to the creation of the group of strings containing only heads. When a string obtained after a sequence of ten coin tosses happens to be into a group being both simple to describe and containing few elements, it then seems not random to a human mind.

Classical probability theory is not designed to study this phenomenon. But we will see that effective descriptive set theory actually is a natural framework to decide what are the 'groups of strings depending on some simple rules' that we mentioned above. Martin-Löf proposed in 1966 [58] a definition of randomness which remains today the most famous and the most studied. The groups of elements which are simple to describe will merely be the Π^0_2 sets, that is, according to the definition we gave, complements of effective unions of Π^0_1 sets. We can however simplify this definition a bit and consider equivalently that Π^0_2 sets are effective intersections of Σ^0_1 sets. What remains to do is to define the notion of containing 'few elements'.

To do so, let us consider a Π^0_2 set $\mathcal{A}$, that is, the set $\mathcal{A}$ can be described as an effective intersection of Σ^0_1 sets (effectively open sets), whose descriptions are enumerable by a computer program P (each description being itself a program that enumerates the corresponding open set). Let $\mathcal{U}_n$ be the open set whose description is the n-th to be enumerated by P. We then have $\mathcal{A} = \bigcap_n \mathcal{U}_n$. We say that $\mathcal{A}$ is **effectively of measure** 0 if for each n, the probability to be in $\mathcal{U}_n$ for a sequence whose bits are successively picked randomly, is smaller than 2^{-n}. We suppose of course that each bit of the sequence is picked randomly and uniformly, that is, we have each time as many chances to get 0 as we have to get 1. Intuitively the probability for a sequence to belong to $\bigcap_n \mathcal{U}_n$ should then be of 0, since for every n, the probability that it belongs to $\mathcal{U}_n$ is smaller than 2^{-n}, and since it belongs by hypothesis to all of them. It then makes sense to consider those sequences as non-random, because the probability to obtain them is null. Also the Martin-Löf random sequences are precisely those which belongs to no Π^0_2 set which is effectively of measure 0.

We can mathematically formalize the intuition we have behind probabilities, with the notion of *uniform measure* on $2^{\mathbb{N}}$, that we denote by λ. By convention the measure of $2^{\mathbb{N}}$ is 1, which corresponds to the probability for a sequence randomly produced to be in $2^{\mathbb{N}}$ (the space of all sequences). We will then write $\lambda(2^{\mathbb{N}}) = 1$. We then naturally define

$\lambda([0]) = 1/2$ and $\lambda([1]) = 1/2$, each of those values corresponding to the probability for a sequence to start by 0 and respectively by 1. More generally we define $\lambda([\sigma]) = 2^{-|\sigma|}$ for every string σ, where $|\sigma|$ denotes the length of σ. We then easily see that the function λ can be extended to every finite union of intervals: for any n and any finite list of pairwise incomparable strings $\sigma_1, \dots, \sigma_n$, we define $\lambda([\sigma_1] \cup \dots \cup [\sigma_n]) = \lambda([\sigma_0]) + \dots + \lambda([\sigma_n])$, and indeed, the probability for a sequence to start by one of the strings σ_i for $1 \leq i \leq n$ is the sum over each i of the probability for a sequence to be in $[\sigma_i]$.

We can even extend the function λ to any open set: For an open set $\mathcal{U}$ and a description of $\mathcal{U}$ given by the countable list of strings $\sigma_1, \sigma_2, \dots$, that we can suppose pairwise incomparable, we have $\lambda(\mathcal{U}) = \sup_{i \in \mathbb{N}} \lambda([\sigma_1] \cup \dots \cup [\sigma_i])$, the supremum over i of the measure of the unions of the i first intervals of the description. Note that an open set can be represented by countable union of pairwise incomparable strings in many different ways, and the we defined the measure on representation of open sets rather than on open sets themselves. Fortunately it is possible to show that the measure of an open set does not depends on the representation we consider. Here again the measure of an open set corresponds to the probability for a sequence to belong to this open set.

A string is then Martin-Löf random if it belongs to no Π^0_2 set $\bigcap_n \mathcal{U}_n$ effectively of measure 0, that is, with $\lambda(\mathcal{U}_n) \leq 2^{-n}$ for every n. We can check that the notion of measure naturally extends to Π^0_2 sets, as the infimum over i of the measure of the i first intersections of the open sets forming the Π^0_2 set. Formally we have $\lambda(\bigcap_n \mathcal{U}_n) = \inf_{i \in \mathbb{N}} \lambda(\mathcal{U}_1 \cap \dots \cap \mathcal{U}_i)$. We can also check that the measure defined this way is independent of the presentation of a Π^0_2 set. We can extend similarly the notion of measure to any set of the hierarchy defined in the previous section and verify that this measure always respects the intuitive idea one should have about measures: Given a countable sequence of pairwise disjoint sets $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \dots$ on which the measure is defined, then the measure is also defined on their union by $\lambda(\mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{A}_3 \cup \dots) = \lambda(\mathcal{A}_1) + \lambda(\mathcal{A}_2) + \lambda(\mathcal{A}_2) \dots$. In particular the union of all the Π^0_2 sets which are effectively of measure 0 is itself a Π^0_2 set effectively of measure 0, and the complement of this union, that is, the set of Martin-Löf randoms, is a set of measure 1. Therefore there are in some sense, much more sequences which are random than sequences which are not. However any attempt to give a specific description (with the help of a computer program) of a random sequence is not possible, as such a description would allow us to capture it in a 'small' Π^0_2 set, that is, a Π^0_2 set effectively of measure 0. The random strings are then those about which nothing specific can be said and yet, they happen to form the majority of sequences...

Many other definitions of effective randomness, all variations of the same idea that some groups of sequences are both 'small' and 'simple to define', have been made over the years. Martin-Löf himself proposed in [59] a much stronger definition, according to which the sets of measure 0 (that capture the non random sequences) can this time be described using the power of higher computability, that we shall now introduce.

Higher computability

As we said, computability can be seen as a way to describe some infinite objects with a finite number of words, in our case, computer programs. All the information contained in such a finite description can then be unfolded along a time of computation, to describe an infinite object, or more precisely a finite object growing endlessly towards an infinite one. The investigators of this new science already noticed in the early days (especially Gödel and Turing), that such a system lead to natural definitions of uncomputable objects. The most famous example is certainly the undecidability of the halting problem of a computer program: the sequence of 0's and 1's having its n-th bit equal to 1 if the n-th computer program stops and equal to 0 otherwise. This sequence does not have a computable description. If a computer program stops, we will know after some time (after it stops), but if it never stops, we might never know for sure that it won't. Intuitively, we would need to check if the program doesn't stop for any time of computation, which would imply infinitely many things to check, and would then take infinitely many computation steps.

Precisely, higher computability can be considered as using usual algorithms, but allowing infinitely many computation steps during their execution. But what meaning can we give to the notion of infinite time of computation? There are two possible approaches to answer this question. The first one is more mathematical and will be developed in this thesis. We sketch here the second one, on which it is probably easier to give some intuition.

We distinguish **limit** steps of computation and **successor** steps of computation. With this new computability, a computer program works analogously at successor steps than it does with usual computability. But after all the finite steps of computation $0, 1, 2, 3, \ldots$, which are all successor steps, we have a limit step of computation. At this first limit step, no computation is actually done, and it is more or less as if we were starting everything again from time 0, except that we keep a trace of what has been done previously. Concretely, we can consider that each bit of the memory (that we suppose infinite), is equal to the convergence value of the sequence of all values taken by this bit during the previous times of computation. For example at a limit step s and for given a bit, if there exists a time $t < s$ such that for every computation step r with $t \leq r < s$, the value of this bit is 1, then the value of this bit at the limit step s will also be 1. It is of course possible for the value of a bit not to converge, that is, to oscillate endlessly between 0 and 1, before the limit step s. In this case, its value at step s is set to 0.

After the first limit step, which we denote by ω, we continue the computation normally at each successor step $\omega + 1$, $\omega + 2$, with the trace of what has been done previously. Then comes the second limit step $\omega + \omega$, and so on. But 'when' does this stop? In classical computability theory, a computer program has to stop at after a finite time of computation $t < \omega$, or otherwise it is (rightfully) considered to be a 'non halting program'. Now ω is a mere step of computation. So what should be the 'time bound' for our halting infinite computation? Suppose a computation 'never' stops. After step ω, the computation continues on steps $\omega + 1$, $\omega + 2$, etc., until the second limit step, $\omega + \omega$, and then $\omega + \omega + \omega$, etc... After some time it will reach the first limit step of limit steps (let us call them 2-limit steps), and then the first limit step of limit steps of limit steps (the 3-limit steps), and so on until the first ω-limit step, then the first ω-limit step of ω-limit steps, etc... We could continue for a while, actually even forever. There exists however a 'natural bound' to all those limit steps which can play the role of a 'new infinite', namely, the limit step ω_1^{ck} about which we now try to give an informal idea.

The intuitive notions we just gave of limit steps and successor steps were imagined long before the emergence of computability theory, and they are better known as limit ordinals and successor ordinals. It is easy to have an intuitive idea of the nature of ordinals, by just building them, as we did with the first limit and successor steps of an infinite computation. But an ordinal can be seen in a more abstract way, as a special order on some elements. For example the usual order on integers $(0 < 1 < 2 < \ldots)$ represents ω, the smallest limit ordinal, whereas the usual order on even integers followed by the usual order on odd integers (that is $0 < 2 < 4 < \cdots < 1 < 3 < 5 < \ldots)$ represents $\omega + \omega$, the second limit ordinal;

and inside this representation, the set of integers smaller than 1 represents ω. It is easy to imagine such representations for bigger and bigger ordinals, but the reader willing to try such an exercise should sooner or later get lost into those endless iterations of infinites. Also we should talk of **computable ordinals** for those that we can represent by an order over the integers, with a computer program. Concretely for a computable ordinal α, the program should enumerate a longer and longer list of conditions of the form $n < m$ for n, m two integers, in a way that the order defined this way represents the ordinal α. We then define ω_1^{ck} as the smallest ordinal that we cannot represent this way. But what type of computability do we allow to represent ordinals? The classical one for which programs run in finite time, or the higher computability for which programs executes in infinite time? A remarkable fact is that this does not matter for the definition of ω_1^{ck}. It is a non trivial fact that ω_1^{ck} is both the smallest ordinal not computable using classical computability, or not computable using a version of higher computability for which ω_1^{ck} is itself the strict upper bound on the time of computation we allow to execute a program (which should then terminates at some ordinal step $\alpha < \omega_1^{ck}$). This remarkable closure property is one of the reason ω_1^{ck} is a natural candidate to be the time bound of infinitary computations.

Higher randomness

The subject of this thesis is at the crossroad of the three notions explained in each of the previous sections. In this thesis we study various randomness notions which followed the second definition of Martin-Löf, the one using higher computability and that we now briefly explain.

The notion of ordinal can be used to pursue the effective descriptive hierarchy described in the first section. The Σ^0_ω sets are the effective unions of the effective intersections of Π^0_n sets for n unbounded in the natural numbers, the Π^0_ω sets are their complement, and the $\Sigma^0_{\omega+1}$ sets are the effective unions of Π^0_ω sets, and so on. The notion of measure can be extended to all those sets. Also for his second definition of randomness, Martin-Löf considers that anything which is captured by a Σ^0_α set of measure 0, for any $\alpha < \omega_1^{ck}$, is not random. This definition happens to be equivalent to his first definition, but where a program is now allowed to describe an open set (by enumerating strings) with infinitely many computation steps up to some step $\alpha < \omega_1^{ck}$ (the step α at which the description should be complete can be taken as big as we want below ω_1^{ck}).

Several other notions of randomness, even stronger, have been made over the years. For instance the notion of Π^1_1-Martin-Löf randomness, similar to Martin-Löf's first randomness notion, but where the time of computation allowed to describe an open set can go all the way up to ω_1^{ck}. Another important notion, which will be intensively studied here, is Π^1_1-randomness, that we do not develop for now.

Thesis summary

We start this section explaining more accurately the content of this thesis. We then develop those explanations chapter by chapter and we end with a summary of the main original contribution of the thesis.

The subject

We mainly deal with higher randomness notions, that is, Δ^1_1-randomnes, Π^1_1-Martin-Löf randomness, weak-Π^1_1-randomness and especially Π^1_1-randomness. We also try to understand the similarities and differences between all those higher randomness notions, but also between the classical randomness notions and the higher ones.

One important difference between the notions of higher computability/randomness and their classical counterparts, is of topological nature. Also we have concentrated our efforts on three different concepts for which this topological difference arises: The notion of computation, the notion of relativization of randomness and the notion of approximation.

We also emphasize the tight connection between randomness notions and genercity notions, as we can consider the latter as a categorical version (in the sense of Baire) of randomness. For this reason we also study higher effective categoricity and we point out the differences and similarities higher categoricity shares with higher randomness.

Structure of the thesis

We detail here the thesis' structure, briefly explaining the content of each chapter.

In the first chapter we define basic notions on the Cantor space, on the Baire space, we define computability theory and ordinals. We then pursue with a detailed overview of computable ordinals that we then use to study the Borel and effective Borel hierarchies in the Baire space. We then study similarly Kleene's hierarchy on sets of integers, establishing its connections with the effective Borel hierarchy. We end this chapter with a few basic notions on measure theory (in order to study randomness) and Baire categoricity (in order to study genericity).

In the second chapter we introduce algorithmic randomness, and in particular the main notion of this field: Martin-Löf randomness; then we iterate this notion through the effective Borel hierarchy and we obtain a corresponding hierarchy of randomness notions. Finally we proceed similarly with the notion of genericity, before explaining some similarities between randomness and genericty notions.

In the third chapter we give the foundations of higher computability. We start by defining and studying the Σ^1_1, Π^1_1 and Δ^1_1 sets. We then define and study Klenne's $\mathcal{O}$ (the set of constructible ordinals), which will be a central notion all along this thesis. We explain why Π^1_1 sets can be considered as a higher analogue of Σ^0_1 sets. We end this chapter by introducing basic higher randomness notions, insisting more on Π^1_1-Martin-Löf randomness, the higher counterpart of Martin-Löf randomness.

In the fourth chapter we study issues arising from the topological differences between higher computability/randomness and classical computability/randomness. We define the notion of higher Turing reduction, in order to keep the descriptive power of the hyperarithmetic reductions and meanwhile to preserve the continuity of classical Turing reductions. We then study the behaviour of this new reduction on various sequences, like the 'sufficiently random' ones or the 'sufficiently generic' ones. We then define a way of relativizing to an oracle, various notions related to higher randomness, in order to preserve the continuity of classical relativization, and meanwhile to get the power of higher relativizations. We then use this notion to prove an analogue of two important theorems of classical randomness: The XYZ theorem and the van Lambalgen theorem. We finally define several restrictions to the notion of higher approximation, still in order to preserve the continuity we have with classical approximations. We end this chapter by using all the previously introduced notions to define 'low-for-hK' and 'continuous base for Π^1_1-Martin-Löf randomness', and we show that there are all equivalent to the notion of hK-triviality.

In the fifth chapter we study new higher randomness notions that are inspired by some classical ones. We define higher difference randomness, Π^1_1-Martin-Löf[$\mathcal{O}$]-randomness and weak-Π^1_1-randomness. We then prove all implications and non-implications between those randomness notions. In particular, we separate the notion of Π^1_1-randomness from weak-Π^1_1-randomness. In order to separate those two classes, we use a restriction of the notion of higher approximations: the self-unclosed approximations. We end this chapter by a detailed study of different approximation notions.

In the sixth chapter we study Π^1_1-randomness and the similar categorical notion of Σ^1_1-genericty. We start by showing that the Borel complexity of the set of Π^1_1-randoms is exactly $\mathbf{\Pi^0_3}$. We use this result to give three other characterizations of Π^1_1-randomness. We introduce a new hierarchy of complexity of sets. We show that Π^1_1-randomness coincides with $\Pi^{\omega_1^{ck}}_4$-randomness of this hierarchy. We also show that the set of Π^1_1-randoms is at level $\Pi^{\omega_1^{ck}}_5$ of this hierarchy. We then show that a non trivial (non Δ^1_1) sequence always derandomizes a Π^1_1-random sequence, answering an open question of Hjorth et Nies (see [30]). We then define and study several higher genericity notions, showing theirs similarities and differences with higher randomness notions. We end this chapter by a caracterization of the Borel complexity of the set $\{X : \omega_1^X > \omega_1^{ck}\}$, using Steel forcing.

In the seventh chapter we study the issues arising from the operation of forcing continuity in higher computations and higher relativizations. In particular we separate the notion of Turing reduction from the one of fin-h reduction. We then show that for some oracles, there is no universal Π^1_1-Martin-Löf test continuously relativized to this oracle. However, we also show that for any oracle there exists a Π^1_1-Martin-Löf random sequence relatively to this oracle and 'left-approximable' relatively to this oracle. We end by a study of oracles for which continuous relativization is not an issue, that is, the self-unclosed approximable oracles and in some sense, Π^1_1-Martin-Löf random oracles.

Main contributions

We detail here the main original contributions of this thesis:

- The most important contribution is probably a better understanding of the set of Π^1_1-random sequences, implicitly defined by Kechris in [33] who identified the existence of a largest Π^1_1 nullset, studied then by Sacks and then by Hjorth and Nies [30] who started to actually study the Π^1_1-randoms for themselves. This study has then been pursued by Chong, Nies and Yu in [7] and by Chong and Yu in [8]. Due to its universal nature, the set of Π^1_1-randoms was conjectured by many to have a hight Borel complexity. We show in this thesis that its Borel complexity is at the contrary relatively low: $\mathbf{\Pi^0_3}$ (see Corollary 6.1.1). We then use this result to conduct a detailed study of this set, in particular:
  - We solve by the affirmative a question which has been open for several years (see [30] and question 9.4.11 of [70]): "Are the Δ^1_1 sequences the only low for Π^1_1-randomness sequences?".
  - We show that a Π^1_1-Martin-Löf random sequence is not Π^1_1-random iff it computes a non trivial (non Δ^1_1) Π^1_1 sequence.
  - We separate the notion of Π^1_1-randomness with a notion still not well-understood (unlike its classical analogue): having infinitely often a prefix of maximal plain higher Kolmogovov complexity (see Corollary 6.2.1).

  - We give two other notions of test for Π^1_1-randomness (see Theorem 6.3.2 and Theorem 6.3.3).

- Another important result, still contributing to a better understanding of Π^1_1-randomness, is the separation of this notion from the one of weak-Π^1_1-randomness (see Theorem 5.3.3). We think this result is important for itself, as well as for its proof, which required the introduction of the new and promising notions of 'self-unclosed' approximation.

- Still to get a better understanding of Π^1_1-randomness, we introduced a new hierarchy of complexity of sets, directly inspired by the definition of weak-Π^1_1-randomness, which on this hierarchy corresponds to the notion of $\Pi_2^{\omega_1^{ck}}$-randomness. We show that the set of Π^1_1-randoms corresponds to the notion of $\Pi_4^{\omega_1^{ck}}$-randomness according to this hierarchy and that the other randomness notions collapse above $\Pi_4^{\omega_1^{ck}}$ (see Theorem 6.4.3).

- Another important part of this thesis is the study of continuity for higher reductions and randomness relativization. This study is mainly conducted in Chapter 4 and Chapter 7. We give here the main results of each of those chapters.

  - In Chapter 4 we show that we can use the new reduction and approximation notions to give a higher counterpart of many important theorems of classical randomness, in particular the XYZ theorem and the van Lambalgen theorem (see Theorem 4.3.3 and Theorem 4.3.5). We also show that the equivalent notions of K-triviality, low-for-K et base for randomness can be defined for higher randomness, and are also all equivalent (see Theorem 4.5.3 and Theorem 4.5.4).
  - In Chapter 7 we study oracles for which the continuous higher relativization raises some issues. In particular we separate the notions of higher Turing reduction and the notion of fin-h reduction. We also show that for some oracles there is no universal Π^1_1-Martin-Löf test continuously relativized to this oracle. We then show that however, for any oracle A, there is always a sequence which is Π^1_1-Martin-Löf random continuously relativized to A and approximable from the left, continuously relatively to A. We end this chapter by a study of the oracles for which the continuous relativization raises no issues, that is, the self-unclosed oracles, and in some sense the Π^1_1-Martin-Löf random oracles

- We study the different restrictions of the notion of higher approximations (higher Δ^0_2). In particular the self-unclosed approximations, used to separate Π^1_1-randomness from weak-Π^1_1-randomness. This study is first done in Section 4.4 in which we identify the main notions, or at least those which are used in other theorems. A further study of those notions for themselves is done in Section 5.4 in which we separate each notion with others.

- We also contribute to a better understanding of the notion of weak-Π^1_1-randomness, in particular we study the differences between this notion and its classical analogue: weak-2-randomness. We identify a test notion corresponding to weak-2-randomness in the lower setting, but different from weak-Π^1_1-randomness in the higher setting. This naturally leads to a new notion of randomness: Π^1_1-Martin-Löf[$\mathcal{O}$] randomness (see Section 5.2). We also identify the restriction we need on the notion of Π^1_1-Martin-Löf[$\mathcal{O}$] randomness in order to keep the equivalence with weak-Π^1_1-randomness (see Theorem 5.3.2).

- We define and study several higher genericity notions. We emphasize the similarities and differences with higher randomness notions (see Section 6.6). We also show that unexpectedly, the notion of weak-Σ^1_1-genericity coincides with the one of Σ^1_1-genericity (unlike the corresponding randomness notions see Theorem 6.6.4).

- In Section 1.6 we pursue an original study of the Σ^0_α-complete sets, with respect to sets of codes of computable ordinals.

- We give a proof that the Borel complexity of the set $\{X : \omega_1^X > \omega_1^{ck}\}$ is exactly $\boldsymbol{\Sigma^0_{\omega_1^{ck}+2}}$. The proof we give, sketched in [88], is due to Steel but it has never been fully written down. Since it is not an obvious result (and is a beautiful one!) we judged that it was worth providing a detailed proof here (see Section 6.7).

Chapter 1

Background

> Le savant n'étudie pas la nature parce que cela est utile ; il l'étudie parce qu'il y prend plaisir et il y prend plaisir parce qu'elle est belle. Si la nature n'était pas belle, elle ne vaudrait pas la peine d'être connue, la vie ne vaudrait pas la peine d'être vécue. Je ne parle pas ici, bien entendu, de cette beauté qui frappe les sens, de la beauté des qualités et des apparences ; non que j'en fasse fi, loin de là, mais elle n'a rien à faire avec la science ; je veux parler de cette beauté plus intime qui vient de l'ordre harmonieux des parties, et qu'une intelligence pure peut saisir.
>
> Science et méthode, Henri Poincaré

1.1 Basic spaces and structures

In this thesis we will mainly work with either the Cantor space or the Baire space. Both of them deserve a small subsection, in which we try to sum up all the basic things the reader needs to know about them. We then have a last subsection describing basic vocabulary and definitions about trees.

1.1.1 The Cantor space

Basic vocabulary

In this thesis, we call **strings** finite sequences of zeros and ones. The empty word, denoted by ϵ is also considered to be a string. The space of strings is denoted by $2^{<\mathbb{N}}$, and a string itself will be denoted by σ, τ or ρ. For a string σ, we denote the length of σ by $|\sigma|$. An infinite sequence of zeros and ones will be simply called a **sequence** and we typically use letters X, Y or Z, to name sequences. The **Cantor space**, denoted by $2^{\mathbb{N}}$ is the set of all sequences.

For a string σ and a sequence X we write $\sigma < X$ and we say 'X **extends** σ' or that 'σ is a **prefix** of X', if the $|\sigma|$ first bits of X are equal to σ. Similarly, for two strings σ and τ, we say that $\sigma \leq \tau$ if $|\sigma| \leq |\tau|$ and if the $|\sigma|$ first bits of τ are equal to σ. If we want the extension to be strict we write $\sigma < \tau$. If two strings σ and τ are such that $\sigma \nleq \tau$ and $\tau \nleq \sigma$, we say that σ and τ are **incomparable**, and we write $\sigma \perp \tau$. Conversely, if σ and τ are comparable we will write $\sigma \parallel \tau$. For a string σ, a sequence X, any n with $0 \leq n < |\sigma|$ and any m, we write $\sigma(n)$ and $X(m)$ to denote respectively the value of the n-th bit of σ and the value of the m-th bit of X (starting at position 0). For two strings σ, τ, we denote the concatenation of σ to τ by $\sigma\hat{}\tau$. Finally, for an integer n, a string σ and a sequence X, we denote by $X\upharpoonright_n$ and $\sigma\upharpoonright_n$, respectively the n first bits of X and the n first bits of σ.

Computable bijection

We will very often use computable bijections from $\mathbb{N} \times \mathbb{N}$ to $\mathbb{N}$ or more generally from $\mathbb{N}^n$ to $\mathbb{N}$. We denote such bijections by $\langle , \dots , \rangle$ and we write for example $\langle a, b\rangle$ for the result of the binary bijection on a and b. We give a first example of a the use of $\langle , \rangle$ by introducing for sequences $\{X_i\}_{i\in\mathbb{N}}$, the notation $\oplus_{i\in\mathbb{N}} X_i$, which denotes the sequence Z such that $Z(\langle i, j\rangle) = X_i(j)$. We also write $X \oplus Y$ to denote the sequence Z such that $Z(2i) = X(i)$ and $Z(2i + 1) = Y(i)$.

The topology

The Cantor space is endowed with the usual topology (the product topology). For a string σ, we call **cylinder** the set of all the sequences extending σ, and we denote it by $[\sigma]$. The topology is the one generated by the set of all the cylinders $[\sigma]$. We introduce a notation that will often be used to deal with open sets:

Definition 1.1.1. *For a set $W \subseteq 2^{<\mathbb{N}}$ we write $[W]^{\prec}$ to denote the open set $\bigcup_{\sigma\in W}[\sigma]$.* ■

Note that the cylinders form a basis of the topology, as any non-empty intersection of two cylinders is still a cylinder. We now a few basic properties of this topological space:

- The Cantor space is a compact space. In particular a subset of $2^{\mathbb{N}}$ is closed iff it is compact.
- The Cantor space is completely metrizable with the Cantor distance:

$$d(x, y) = 2^{-\min\{i\,:\,(x(i)\neq y(i))\}}$$

- The Cantor space is 0-dimensional, i.e., has a basis of clopen sets. Indeed, each cylinder is both open and closed (clopen). This is also true of finite unions of cylinders and indeed, these are exactly the clopen sets of $2^{\mathbb{N}}$.

Different interpretations of the Cantor space

There is another canonical way to give meaning to elements of the Cantor space, by viewing them as subsets of $\mathbb{N}$. So for $X \in 2^{\mathbb{N}}$, the corresponding subset of $\mathbb{N}$ is the one containing the natural number n iff $X(n) = 1$. Here again it is clear that this provides a canonical one-to-one map between elements of $2^{\mathbb{N}}$ and subsets of $\mathbb{N}$, that we will from now on consider as well, as elements of $2^{\mathbb{N}}$.

A last way we can view elements of the Cantor space, is as binary representations of the elements of $[0, 1] \subseteq \mathbb{R}$. However, one should notice that the real $0,010000\dots$ is the same as the real $0,001111\dots$, whereas the sequence $010000\dots$ is different from the sequence $001111\dots$. The topology of both spaces is indeed different, as $[0, 1]$ is not 0-dimensional. In practice, this difference won't matter for our purposes.

1.1.2 The Baire space

Basic vocabulary

As for the Cantor space, we call **string** a finite sequence of natural numbers (including the empty word ϵ), **sequence** an infinite one, and we define $\mathbb{N}^{<\mathbb{N}}$ to be the set of strings and $\mathbb{N}^{\mathbb{N}}$ to be the set of sequences. In practice it will be in general clear when strings/sequences are meant to be strings/sequences of the Baire space rather than of the Cantor space, and when it might be ambiguous, we will always give precisions.

Elements of $\mathbb{N}^{<\mathbb{N}}$ will be usually denoted by σ, τ or ρ and elements of $\mathbb{N}^{\mathbb{N}}$ will be usually denoted by f, g or h. For an integer n, a sequence f and strings σ, τ, the notions of length $|\sigma|$, extension/prefix $\sigma \prec X$, $\sigma \leq \tau$, $\sigma \prec \tau$, comparability $\sigma \perp \tau$, $\sigma \parallel \tau$, n-th value $\sigma(n)$, $f(n)$, concatenation $\sigma \hat{} \tau$, and restrictions $f \upharpoonright_n$, $\sigma \upharpoonright_n$, are as in the Cantor space.

The topology

Just as for the Cantor space, we define for each string σ the **cylinder** $[\sigma]$ as the set of all sequences extending σ. The topology is then the one generated by all the cylinders, and the cylinders form a basis for the topology. The main topological difference with the Cantor space is that the Baire space is not compact.

Different interpretations of the Baire space?

Elements of the Baire space can also be viewed as total functions from $\mathbb{N}$ to $\mathbb{N}$. So for $f \in \mathbb{N}^{\mathbb{N}}$, the corresponding function is equal to m on input n if $f(n) = m$. It is clear that this provides a canonical one-to-one map between elements of $\mathbb{N}^{\mathbb{N}}$ and total functions from $\mathbb{N}$ to $\mathbb{N}$, that we will from now on consider as well, as elements of $\mathbb{N}^{\mathbb{N}}$.

1.1.3 Trees

The trees over $\mathbb{N}^{<\mathbb{N}}$ will play a very important role in this thesis, because they are a convenient tool to work with things as different as ordinals, closed sets and analytic sets. We also briefly present a generalization of the notion of tree, over any partial order.

Basic vocabulary

A **tree** T of $\mathbb{N}^{\mathbb{N}}$ is a subset of $\mathbb{N}^{<\mathbb{N}}$ closed under the prefix relation: if $\sigma \in T$ and $\tau \leq \sigma$ then $\tau \in T$. Elements of T are called **nodes** of T. By convention, we assume that any tree contains at least the element ϵ which corresponds to the empty word. The node ϵ will be also called the **root** of the tree. An **infinite path** of T is a sequence f so that $f \upharpoonright_n \in T$ for every n. The **body** of T, denoted by $[T]$ is the set of infinite paths of T.

For σ a node of T we call **children** of σ the nodes of the form $\sigma \hat{} n$ for some $n \in \mathbb{N}$, whereas we say that σ is the **father** of its children (obviously). We call **descendants** of σ the nodes τ of T so that $\sigma \prec \tau$. A tree T is said to be **pruned** is every node σ of T has a child. A node σ having at least two distinct children σ_1, σ_2 is said to be **branching**. For a tree T with at least one branching node we denote by $\mathrm{stem}(T)$ the first branching node of T. For a tree T and a string σ, we write $T \upharpoonright_\sigma$ the subtree of T obtained by keeping only strings compatible with σ. Also for a tree T and a node σ of T, we write $T \upharpoonleft_\sigma$ to denote, informally, the shifting to the left of every string of $T \upharpoonright_\sigma$ by $|\sigma|$, which becomes, so to speak, the new root the T. Concretely we remove from $T \upharpoonright_\sigma$ every prefix of σ and we replace every other nodes $\sigma \hat{} \tau$ (they all follow this pattern) by τ. Finally for a tree T

and a string σ we denote by $\sigma\hat{\ }T$ the 'shifting to the right' of every string of T by σ, that is, we put all prefixes of σ in the tree and we replace each node τ of T by $\sigma\hat{\ }\tau$.

1.2 Basic computability notions

We assume that general notions of computability on $\mathbb{N}$ are known. Elements of $\mathbb{N}$ will be denoted by $a, b, c, d, e, i, j, k, l, m, n$, with e more specifically used for 'codes'. We just recall here some standard notation which will be used in this thesis.

1.2.1 Computability on the Cantor space

For any e we denote by $\varphi_e : \mathbb{N} \to \mathbb{N}$ the **computable function** of code e. If we allow a 'computable process' to access infinite objects as **oracle**, we then speak of **computable functional**. So for any $e \in \mathbb{N}$, we will denote by $\Phi_e : 2^{\mathbb{N}} \times \mathbb{N} \to \mathbb{N}$ the computable functional of code e. Sometimes it will happen that we want our functionals to have more than one oracle in input, with possibly some of them from the Baire space. When it is so we will always give precisions. For a given fixed oracle $X \in 2^{\mathbb{N}}$, we denote by $\Phi_e^X : \mathbb{N} \to \mathbb{N}$ the curryfication of Φ_e applied to oracle X. We write $\Phi_e^X(n)\downarrow$ or sometimes $\Phi_e(X,n)\downarrow$ if the computation converges with oracle X and input n. We write $\Phi_e^X(n)\uparrow$ or sometimes $\Phi_e(X,n)\uparrow$ otherwise. Also for any $e \in \mathbb{N}$, we denote by W_e the **computably enumerable** set of code e, that is the domain of φ_e. The notion relativizes and for $X \in \mathbb{N}$, we denote W_e^X the domain of Φ_e^X. Note that we will not make any difference between W_e and $W_e^{0^\infty}$ (where 0^∞ denotes the sequence corresponding to the empty set of natural numbers).

We will often consider functionals $\Phi_e : 2^{\mathbb{N}} \times \mathbb{N} \to \mathbb{N}$ as functions from $2^{\mathbb{N}}$ to $\mathbb{N}^{\mathbb{N}}$, or as functions from $2^{\mathbb{N}}$ to $2^{\mathbb{N}}$. In this case we write $\Phi_e : 2^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ (respectively $\Phi_e : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$) and we write $\Phi_e(X)$ to denote the image of Φ_e on the sequence X. Such a function Φ_e is defined on X when $\forall n\ \Phi_e(X,n)\downarrow$ (respectively when $\forall n\ \Phi_e(X,n)\downarrow\in\{0,1\}$).

Quite often we will have to consider the running time of a given computation. So for a functional Φ_e, an oracle X and an integer n, we denote by $\Phi_{e,t}(X,n)$ or by $\Phi_e(X,n)[t]$ the result of the computation up to time t. If a given functional Φ_e halts on some oracle X and some input n, then it necessarily uses only finitely many bits of the oracle. The smallest prefix σ of X so that the functional Φ_e does not access values bigger than $|\sigma|$ will be called the **use** of X on input n and will be denoted by $\mathrm{use}_e^X(n)$. Note that this definition is a bit non standard, as in the literature, the use often refers to the size of σ rather than to σ itself.

1.2.2 The fixed point theorem

The fixed point theorem, also called the second recursion theorem has been proved by Kleene in [37], the same paper in which the constructive ordinals are introduced (see Section 1.4.3). In some sense the theorem and the proof are quite simple, but it is not necessarily obvious to understand its implications, that we shall detail below.

Theorem 1.2.1 (Kleene's fixed point theorem):
If f is a total computable function, there exists an integer e so that $\Phi_{f(e)} = \Phi_e$.

PROOF: Let a be a code for a total function which takes n in parameter and returns a code for the function which on m returns the result of $\Phi_{f(\Phi_n(n))}(m)$. Formally: $\Phi_{\Phi_a(n)}(m) = \Phi_{f(\Phi_n(n))}(m)$. We then have that $\Phi_{\Phi_a(a)}(m) = \Phi_{f(\Phi_a(a))}(m)$ which makes $\Phi_a(a)$ the desired fixed point. ∎

Note that for a given function f, the fixed point can be obtain effectively. A first obvious interpretation of the fixed point theorem, is that for any computable function which modifies programs, there is always a program which has the same behavior before and after the modification. In practice we will always use the fixed point theorem as a tool which allows us to say that a program can 'access its own code'. So when you have a program M using some integer n, you can define the total computable function f taking n in parameter and outputting the code for the version M_n of the program that uses integer n. But by fixed point theorem, there is a version M_e of this program that uses its own code, that is e is a code for M_e.

It is no accident that Kleene gave a proof of the fixed point theorem in the same paper in which he introduced a coding system for ordinal, as we will see it with Section 1.4.3 and most particularly with Example 1.4.2.

1.2.3 Reductions

We now give the main notions of reduction between oracles X, Y. For all of them, the intuition is that when X is reduced to Y, sufficient knowledge of Y is enough to get the knowledge we need about X.

Many-one reductions

The strongest notion of reduction of this thesis is the so called **many-one reduction** introduced by Post in [75]. For two elements $X, Y \in 2^\omega$ we say that X is many-one-reducible to Y and we write $X \leq_m Y$ if there exists a total computable function $f : \mathbb{N} \to \mathbb{N}$ such that $n \in X \leftrightarrow f(n) \in Y$. So to know any bit of X we can ask only one question to Y to have the answer. Moreover we cannot change that answer, as $Y(f(n))$ has to be equal to $X(n)$. If $X \leq_m Y$ and $Y \leq_m X$ we write $X \equiv_m Y$. It is clear that $\equiv_m$ is an equivalence relation, which leads us to notion of equivalence classes for this relation, that will be called **many-one degrees**. The notion of many-one reduction is important for its connection with the arithmetical and hyperarithmetical hierarchy, developed in Section 1.6

Turing reductions

A more general reduction notion in computability theory is the so-called **Turing reduction**, introduced by Turing in his PhD thesis (see [93]). Turing reduction is also the most used and studied in the literature nowadays. We say that X is Turing reducible to Y and we write $X \leq_T Y$ if there exists a functional $\Phi_e : 2^{\mathbb{N}} \times \mathbb{N} \to \mathbb{N}$ so that Φ_e^Y is the characteristic function of X. We also define the equivalence relation $X \equiv_T Y$ which occurs if $X \leq_T Y$ and $Y \leq_T X$ and we call **Turing degrees** the equivalence classes of this relation.

Truth-table reductions

There are three equivalent ways to define truth-table reductions. Perhaps we want to give first the one making clear what such a reduction has to do with truth-tables. A truth-table reduction is a uniform infinite computable sequence of truth-tables $\{t_n\}_{n \in \mathbb{N}}$, so that each t_n associates to *every* possible Boolean combination of a given length (the length depends on n), a Boolean value 0 or 1. We then say that $X \leq_{tt} Y$ if there exists a truth table reduction so that for each n, the value of $X(n)$ is the one decided by the truth-table t_n when taking the first bits of Y as the input of the Boolean combination. We define $X \equiv_{tt} Y$ and **truth-table degrees** analogously to what we did for Turing reducibility.

In practice we will often use another equivalent and maybe simpler definition, that is $X \leq_{tt} Y$ if there exists a total computable function $\Phi_e : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ so that $\Phi_e^Y = X$. It is clear that a truth-table reduction can always be transformed into a total Turing reduction, as each truth-table t_n covers the whole Cantor space. One can also transform a total Turing reduction into a truth-table reduction, by building each truth-table t_n with the use of every oracle. More formally, for a given n we can compute the smallest length l so that for every string σ of size l we have that $\Phi_e^\sigma(n) \downarrow\in \{0,1\}$. The corresponding truth-table is then built by associating the result of the computation on each string of size l. We can argue that such a length l always exists, and can be found computably: If there were strings σ of arbitrarily long length so that $\Phi_e^\sigma(n) \uparrow$, then also we would have by compactness, a limit point X for this set of strings, for which $\Phi_e^X(n) \uparrow$, and this would contradict the totality of Φ_e.

We finally give a last equivalent definition for truth-table reduction. We say that $X \leq_{tt} Y$ if there is a computable functional $\Phi_e : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ so that $\Phi_e^Y = X$ and so that the computation time that Φ_e^Y takes to halt on n, is bounded by a total computable function of n. More formally we have a computable function $f : \mathbb{N} \to \mathbb{N}$ so that $\Phi_e^Y(n)[f(n)] \downarrow= X(n)$. Suppose so, then it is easy to make such a function Φ_e total without damaging the result of Φ_e on Y, as for each n we can simply wait $f(n)$ step of computation and then decide arbitrary values on strings σ for which $\Phi_e^\sigma(n)[f(n)] \uparrow$. Conversely, with a total computable function $\Phi_e : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$, uniformly in n, we can as before compute the smallest length l so that the computation halts on every string σ of length l. The supremum of all the computation steps used so far is computable.

Weak truth-table reductions

We say that $X \leq_{wtt} Y$ if there is a functional $\Phi_e : 2^{\mathbb{N}} \times \mathbb{N} \to \mathbb{N}$ so that $\Phi_e^Y = X$ and so that the length of the use of Y is bounded by a total computable function f, that is, for every n we have $|\mathrm{use}_e^Y(n)| \leq f(n)$. As for the truth-table reduction, the interpretation is that we have to ask *in advance* which bit of the oracle we want to use (unlike in the full Turing reduction, where new requests to the oracle may depend on the answer of previous requests). But unlike the truth-table reductions, the functional doesn't have to be total. We define $X \equiv_{wtt} Y$ and **weak-truth-table degrees** analogously to what we did for Turing reducibility and truth-table .

Relations between reductions

It is clear from the definition we gave of the different reductions that:

$$X \leq_m Y \to X \leq_{tt} Y \to X \leq_{wtt} Y \to X \leq_T Y$$

For a proof that all those implications are strict, one can refer for example to [73].

1.2.4 The arithmetical hierarchy

Within his work about first incompleteness theorem (see [26]), Gödel made a very clever use of the Chinese remainder theorem, leading to the well-known theorem saying that the domains of computable functions are exactly the sets of integer one can define by a formula of arithmetic using only existential unbounded quantifiers, called Σ_1^0 formulas. At the time the notion of computable function was yet to be introduced by Herbrand in [28], and Gödel's work only dealt with primitive recursive functions.

This equivalence led to precise definitions in order to capture 'being computable' or 'being computably enumerable' in term of definability by arithmetical formulas. Also if Σ_1^0 formulas are exactly those which define sets of the form W_e, what to say about the more complex formula, when we add for example unbounded universal quantifiers? The study of those questions has been conduct, with success, mainly by Kleene, roughly between 1940-1955 (see 'Historical remarks' sections of [65]). He introduced in 1943 (see [38]) a hierarchy (discovered independently by Mostowsky in 1946 [66]) called nowadays 'arithmetical hierarchy', or 'Kleene hierarchy'.

Definition 1.2.1. *for this definition, for any i, the formula $\psi(n_1, \dots, n_i)$ denotes a formula of arithmetic so that $n_1, \dots, n_i$ are the only free variable of ψ. For any i, a formula of arithmetic $\psi(n_1, \dots, n_i)$ is defined by induction to be:*

- Δ_0, Π_0 *or* Σ_0 *if $\psi(n_1, \dots, n_i)$ only has bounded quantifiers ('$\exists x < t$' or '$\forall x < t$', for t an arithmetical term which may involve the variables n_i but no other variable).*
- Σ_{n+1} *if it is of the form $\exists m\ \psi(n_1, \dots, n_i, m)$ for $\psi(n_1, \dots, n_i, m)$ a Π_n formula.*
- Π_{n+1} *if it is of the form $\forall m\ \psi(n_1, \dots, n_i, m)$ for $\psi(n_1, \dots, n_i, m)$ a Σ_n formula.* ■

It is clear that if a formula $\psi(n)$ is Σ_n, then the formula $\neg\psi(n)$ does not match the definition of Π_n formulas, but it is 'morally' Π_n, in the sense that it is logically equivalent to a Π_n formula. We then extend the definition:

Definition 1.2.2. *For any i, the formula $\Psi(n_1, \dots, n_i)$ is Σ_n^0 (respectively Π_n^0) if it is logically equivalent to a Σ_n (respectively Π_n) formula $\psi(n_1, \dots, n_i)$, under the axioms of Peano arithmetic. That is, the formula:*

$$\forall n_1, \dots, n_i\ \Psi(n_1, \dots, n_i) \leftrightarrow \psi(n_1, \dots, n_i)$$

is provable in Peano arithmetic. If a formula is both Σ_n^0 and Π_n^0, then it is said to be Δ_n^0. ■

Any formula of arithmetic is Σ_n^0 or Π_n^0 for some n, because any predicate calculus formula is logically equivalent to a formula in 'prenex normal form' (that is starting with a quantifier part, followed by a quantifier-free part).

It is worth saying that since Matiyasevich proved in [60] his famous theorem, generally called the MRDP theorem (in reference of the earlier work of Julia Robinson, Martin Davis and Hilary Putnam), we know that bounded quantifiers are not necessary in the previous definitions, that is, a Σ_1^0 formula is provably equivalent in Peano arithmetic to a Σ_1 formula with no bounded quantifiers.

Digression

Not only the MRDP theorem provided of solution to Hilbert 10th problem, but it also has the 'philosophical consequence' that the undecidability of a formula does not depend on its complexity. Already Gödel proved that there are undecidable Σ_1^0 statements, but one could have argued that their undecidability depends on an intensive use of bounded quantification. By the MRDP theorem, we now know that their undecidability only depends on the property of numbers, with respect to addition and multiplication.

Also we could not resist giving the following example of Verena Dyson, James Jones and John Sheperdson (see [19]), that illustrates the mystery of undecidability, which seems deeply connected to the structure of integers, and certainly in that respect, is to be meditated...

Theorem 1.2.2 (Dyson, Jones, Sheperdson [19]):
Let T be any axiomatizable ω-consistent theory containing Robinson Arithmetic. Then there is an n (different for different theories) such that the following sentence is undecidable in T:

$$\begin{aligned}
&\exists a, b\ \forall i \leq \overline{n}\ \exists s, w, p, q, j, v, e, g \\
&\quad \{(s+w)^2 + 3w + s = 2i \wedge ([j = w \wedge v = q] \vee [j = 3i \wedge v = p + q] \\
&\qquad \vee [j = s \wedge (v = p \vee (i = \overline{n} \wedge v = q + \overline{n}))] \vee [j = 3i + 1 \wedge v = pq] \\
&\qquad\quad \to a = v + e + ejb \wedge v + g = jb)\}
\end{aligned}$$

The purpose of the two previous definitions then lies in the following one, which establishes the arithmetical hierarchy as a classification of sets according to their complexity.

Definition 1.2.3. *A set $X \subseteq \omega$ is said to be Σ^0_n (respectively Π^0_n, Δ^0_n) if there is a Σ^0_n (respectively Π^0_n, Δ^0_n) formula Ψ so that $n \in X \leftrightarrow \mathbb{N} \models \Psi(n)$.* ∎

What Gödel 'essentially proved', is that the sets of the form W_e are exactly the Σ^0_1 sets. Keeping this in mind, one can also notice easily the correspondence between logical operations and set-theoretical operations: The existential quantification corresponds to union of sets, the universal one to intersection of sets and the negation to complement of sets.

One can then equivalently define the Σ^0_1 sets as the sets W_e for some e, then the Π^0_n sets as complements of Σ^0_n sets, and the Σ^0_{n+1} sets as effective unions of Π^0_n sets. So for example a set is Σ^0_4 if there exists a code e so that:

$$\bigcup_{n_1 \in W_e} \bigcap_{n_2 \in W_{n_1}} \bigcup_{n_3 \in W_{n_2}} W^c_{n_3}$$

This point of view will be necessary in particular to extend the arithmetical hierarchy to the hyperarithmetic one, where we will define sets that cannot necessarily be defined by first order formulas of arithmetic.

1.3 Ordinals

1.3.1 Well-founded relations and ordinals

The concept of well-order has been introduced by Cantor in 1883 (see [52], page 38), who defined a totally ordered set A to be well-ordered if any subset of A, bounded in A, has an immediate successor in A. It can be seen to be equivalent to the modern definition, that we shall now give:

Definition 1.3.1. *An order relation (strict or non strict) $R \subseteq A \times A$ is said to be* ***well-founded*** *if every subset of A admits a minimal element in the sense of R. Formally:*

$$\forall B \subseteq A \quad B \neq \varnothing \to \exists a \in B \quad \forall b \neq a \in B \quad \neg(b,a) \in R$$

If R is also total, that is, for any two elements x, y of A we have $(x,y) \in R$ or $(y,x) \in R$, then R is said to be a well-order. ■

This notion revealed itself to be essential in mathematical logic, as it is the backbone of both proofs and definitions by induction. Also it naturally led mathematicians to the attempt of capturing the notion of ordinals, this is to say, not just specific well-orders, but their order-types: Two orders defined on $\{1,2,3\}$ respectively by $1 \leq 2 \leq 3$ and by $2 \leq 1 \leq 3$, are 'structurally the same', they have the same order-type, that is, we can find an order isomorphism between the two. So the notion of ordinals was first a notion of equivalence classes of sets having the same order-type. This was then modernized by Von Neumann who proposed some canonical well-ordered sets as the definition of ordinals themselves. The simple definition he gave remains today the one that everyone uses, informally: "each ordinal is the well-ordered set of all smaller ordinals".

Example 1.3.1:
The set of ordinals smaller than the first of them is empty, and then the first ordinal is naturally equal to $\varnothing$. The second one is $\{\varnothing\}$, the set containing the empty set, the third one is $\{\varnothing, \{\varnothing\}\}$, and we can then continue to define all finite ordinals inductively. The first non finite ordinal is denoted by ω and is by definition the set of all finite ordinals. It is the first ordinal bigger than $\varnothing$ which does not have any predecessor. Such an ordinal is called a ***limit*** *ordinal, by opposition to the others which are called* ***successor*** *ordinals.* ◊

The trained logician certainly noticed that the informal definition of ordinals that is given in the previous example, is done by induction over... the ordinals themselves. To avoid such a loop, we provide now the official definition, which is more obscure, but necessary:

Definition 1.3.2. *A set α is an ordinal if α is well-ordered with respect to set membership and if every element of α is also a subset of α.* ■

The reader can see [52], page 52, where the author, Azriel Levy, credits Von Neumann in [94], and Zermelo (unpublished work) for this definition. One can prove that the formal definition is equivalent to the informal one. Ordinals will be denoted by α, β and γ. We now give a few basic properties of ordinals.

To denote finite ordinals, we sometimes use the notations for natural numbers, that is, 0 for the first ordinal, 1 for the second one, etc.... Also every ordinal α has a **successor** (a smallest ordinal strictly bigger than α) that we denote by α^+ or $\alpha + 1$.

Every set A of ordinals has a smallest strict upper bounded (according to the first definition of well-ordered, given by Cantor). We will denote it by $\sup^+(A)$, which is equal to $\sup\{\alpha + 1 : \alpha \in A\}$, where $\sup(A)$ denotes the smallest non-strict upper bound of a set of ordinals A, that is, the smallest ordinal bigger or equal than all the ordinals in A. If $A = \varnothing$, by convention $\sup^+(A) = 1$.

The class of all ordinals is itself well-ordered by the set membership relation. It is clear that any set (or class) of ordinals is well-ordered. Then for a class of ordinals having a given property, we can always argue that there is a smallest of them.

We shall now argue that Von Neumann's definition of ordinal really captures every possible well-order, that is, up to isomorphism, every well-order is also the set membership relation of an ordinal.

Theorem 1.3.1 (Mostowsky collapse):
Let R be a well-order on a set A. For all $a \in A$ we define:

$$|a|_o = \{|b|_o \ : \ (b,a) \in R \text{ with } b \neq a\}$$

We have that $|a|_o$ is an ordinal which is order-isomorphic to the set $\{b \neq a \ : \ (b,a) \in R\}$ endowed with the order relation R restricted to it. We also have that the supremum of $|a|_o$ for all $a \in A$ is an ordinal, order-isomorphic to R.

One can see for example Kunen's book [43], for a proof of the Mostowsky collapse theorem. For each $a \in A$, the ordinal $|a|_o$ will be called the **order-type** of a. The ordinal corresponding to the supremum of $|a|_o$ for all $a \in A$ will be called the **order-type** of A, which will be denoted by $|A|_o$.

1.3.2 Ordinal arithmetic

The main theorem of this section, that we will occasionally reuse in this thesis, is a version of the euclidean division for ordinals. But first we should make explicit a version of addition and multiplication for ordinals.

Definition 1.3.3 (Addition). *The addition is defined on the ordinals by induction over its second parameter:*

$$\begin{array}{lcl} \alpha + 0 & = & \alpha \\ \alpha + \beta^+ & = & (\alpha+\beta)^+ \\ \alpha + \sup\{\beta \ : \ \beta < \gamma\} & = & \sup\{(\alpha+\beta) \ : \ \beta < \gamma\} \end{array}$$

The following example provides an equivalent way to define addition on well-orders.

Example 1.3.2:
For two well-ordered sets A and B, we define a well-order on $A \sqcup B$, the disjoint union of A and B, by putting A 'at the left of' B, that is, elements of A are smaller than elements of B. We have that the resulting order is a well-order, with $|A \sqcup B|_o = |A|_o + |B|_o$. ◊

Definition 1.3.4 (Multiplication). *The multiplication is defined on the ordinals by induction over its second parameter:*

$$\begin{array}{lcl} \alpha \times 0 & = & 0 \\ \alpha \times \beta^+ & = & (\alpha\times\beta)+\alpha \\ \alpha \times \sup\{\beta \ : \ \beta < \gamma\} & = & \sup\{(\alpha\times\beta) \ : \ \beta < \gamma\} \end{array}$$

As for the addition, we can provide an equivalent way to define multiplication on well-orders.

Example 1.3.3:
For two well-ordered sets A and B, we define a well-order on $A \times B$, the cartesian product of A and B, by simply taking the lexicographic order, that is $(a_1, b_1) < (a_2, b_2)$ if $a_1 < a_2$ or if $a_1 = a_2$ and $b_1 < b_2$. We have that the resulting order is a well-order, with $|A \times B| = |A| \times |B|$. ◊

The reader should note that addition and multiplication over ordinal are not commutative (for example $\omega + 3$ is different from $3 + \omega$ which is equal to ω). We now state the ordinal version of the Euclidean division:

Proposition 1.3.1 (left division for ordinals):
For all $\alpha \geq \beta > 0$, there are unique $\gamma_1 \leq \alpha$ and $\gamma_2 < \beta$, such that $\alpha = \beta \times \gamma_1 + \gamma_2$.

1.4 Computable ordinals

1.4.1 Introduction to computable ordinals

In this thesis, we will exclusively be interested in countable ordinals, which are those we can represent by well-orders of $\mathbb{N}$. Among them, we will take a particular interest to those that we can computably represent, in a way we shall make precise. The computable ordinals are of great importance to study the effectively Borel sets, and the effectively analytical and co-analytical sets.

Definition 1.4.1. *A **computable ordinal** is the order-type of a well-ordered non-strict relation $R \subseteq \mathbb{N} \times \mathbb{N}$ such that there is a code e with $(n, m) \in R \leftrightarrow \langle n, m \rangle \in W_e$. For X a sequence, we define **X-computable ordinals** as the obvious relativized notion. We denote the set of codes for computable ordinals by $\boldsymbol{\mathcal{W}}$.* ■

The reason we take non-strict relation is to have a way to encode the ordinal 1. So we have that 0 is encoded by any empty enumeration and that 1 is encoded by any enumeration outputting $\langle n, n \rangle$ for a unique n.

In the context of computable ordinals, $|W_e|_o$ will denote the order-type of the relation coded by W_e. Also we will call **domain** of W_e the integers which are in an enumerated pair of W_e. The study of computable ordinals should be credited first to Kleene and Church, who conceived in the 30's a system of notation for ordinals, leading to the notion of **constructive ordinal** (see [37]). The notion of constructive ordinal provides a coding system for ordinals, in a restricted way, so we can get more information about n ordinal by just knowing its code (for example we can know from a constructive code a if a codes for a limit or a successor ordinal, whereas this requires the double jump to be decided on a computable ordinal 's code). The presentation we will give of them in Section 1.4.3 differs a bit from the one invented by Kleene, but the underlying ideas are exactly the same. It was not clear at first that the constructive ordinals were the same than the computable ordinals. It was solved by the affirmative a couple of years following Kleene and Church's definition, by Markwald in [56] (credited in [39]).

It is clear from the definition of the computable ordinals that they form an initial segment of the countable ordinal. Indeed if $e \in \mathcal{W}$ codes for α, then for each ordinal smaller than α there is an a so that the set of $\{b : \langle b, a\rangle \in W_e\}$ endowed with the order relation of W_e, codes for α. Of course, as there are uncountably many countable ordinals, this initial segment is strict. Kleene and Church then defined the supremum of the computable ordinals:

Definition 1.4.2. *The smallest non-computable ordinal will be denoted by ω_1^{ck} which stands for **Church-Kleene omega one**. For a sequence X, the smallest non X-computable ordinal will be denoted by ω_1^X.* ■

Note that any countable ordinal is computable from its own representation as an oracle, and therefore we have $\sup_{X \in 2^{\mathbb{N}}} \{\omega_1^X : X \in 2^\omega\} = \omega_1$, where ω_1 is the least non countable ordinal.

1.4.2 Computable ordinals and trees

We shall now introduce computable well-founded trees of the Baire space, as they are a convenient way to represent computable ordinals. This can be done inductively by mapping each node σ of a well-founded tree T to the smallest strict upper bound of all ordinals associated to children of σ:

Definition 1.4.3. *A tree $T \subseteq \mathbb{N}^{\mathbb{N}}$ is **well-founded** if $[T]$ is empty. For T a well-founded tree and for a node σ of T we define $|\sigma|_o$ by induction, to be $|\sigma|_o = \sup^+\{|\sigma\hat{\ }n|_o : \sigma\hat{\ }n \in T\}$. We then define $|T|_o = |\epsilon|_o$ where ϵ is the root of T.* ■

Example 1.4.1:
If $T = \varnothing$ then $|T|_o = 0$. If $T = \epsilon$ then $|T|_o = 1$. If T has only nodes of length n of shorter than $|T|_o$ is smaller than or equal to $n + 1$. ◊

By abusing notation, in what follows we can write $|T|_o$ for a tree T ill-founded, in which case we consider that $|T|_o$ is bigger than any ordinal $|T|_o$ for any well-founded c.e. tree. We now argue that the computable ordinals are exactly those that can be represented this way by a computably enumerable tree T. Consider a code $e \in \mathcal{W}$ and let us build a tree T so that $|W_e|_o = |T|_o$. For each element $\langle a, b\rangle$ enumerated in W_e with $a \neq b$, we enumerate a as a child of the root in T, and for each element a enumerated in T, we continue to enumerate recursively as the children of the node a, all the nodes $a\hat{\ }n$ for each n which is witnessed to be strictly smaller than a at some point in W_e. The tree T is well-founded because W_e codes for a well-founded relation. By induction, it is easy to prove that for any a in the domain of W_e we have $|a|_o = \sup^+\{|b|_o : \langle b, a\rangle \in W_e \text{ with } a \neq b\}$, and then that the tree $|T|_o$ is equal to $|W_e|_o$.

For the other direction, we introduce another way to encode an ordinal by a well-founded tree, known as the Kleene-Brouwer ordering. This ordering for tree T looks like the lexicographic order of its nodes, with the difference that a string prefix is bigger than the string itself. So σ is smaller than τ if σ is a suffix (a descendant in the tree) of τ or if σ is at the left of τ in the tree.

Another way to define the Kleene-Brouwer ordering is by assigning to each node the supremum for $n \in \mathbb{N}$ of the finite sums of the ordinals assigned to its n first children:

Definition 1.4.4. *For T a well-founded tree and a node σ of T we define by induction:*

$$|\sigma|_{KB} = \sup^{+}_{n\in\mathbb{N}} \left\{ \sum_{i\leq n} |\sigma\hat{\ } m_i|_{KB} \ : \ \sigma\hat{\ } m_i \text{ the } i\text{-th child of } \sigma \right\}$$

We then define $|T|_{KB} = |\epsilon|_{KB}$ where ϵ is the root of T. ■

Suppose now we have a c.e. well-founded tree T. The goal is to define a code e so that $|W_e|_o \geq |T|_o$. Then, as the computable ordinals form an initial segment of the countable ordinals, we would then also have that $|T|_o$ is a computable ordinal. If T were computable we could, with an appropriate coding for strings of the Baire space, enumerate its Kleene-Brouwer ordering and get the result, as it is clear from the previous definition, that $|T|_{KB} \geq |T|_o$. But as T is only c.e. we have to enumerate a variation of it, where a node σ_1 is smaller than its sibling node σ_2 if σ_2 appears latter than σ_1 in the enumeration. It is still clear that even with this modification we have a resulting ordinal bigger or equal to $|T|_o$.

We now define:

Definition 1.4.5. *We call $\mathcal{T}$ the set of codes e so that W_e enumerates the nodes of a well-founded tree. For any $a \in \mathcal{T}$ coding for T we write $|a|_o$ to denote $|T|_o$. For any computable ordinal α we write $\mathcal{T}_{<\alpha}$ to denote the elements $a \in \mathcal{T}$ so that $|a|_o < \alpha$, we write $\mathcal{T}_{\leq\alpha}$ to denote the elements $a \in \mathcal{T}$ so that $|a|_o \leq \alpha$ and we write $\mathcal{T}_{=\alpha}$ to denote the elements $a \in \mathcal{T}$ so that $|a|_o = \alpha$.* ■

We now give a few technical but easy lemmas that will be useful to work with well-founded trees. The first one is about the existence of a total computable function $OR : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ which we named this way because it can be seen as a Boolean "or" between trees, when "being well-founded" is interpreted as the value "true". So $OR(T_1, T_2)$ is a code for a c.e. well-founded tree iff T_1 is well-founded, or T_2 is well-founded. Furthermore, we show that it is possible to achieve $|OR(T_1, T_2)|_o = \min(|T_1|_o, |T_2|_o)$:

Lemma 1.4.1 *There is a total computable function $OR : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$, which on any two codes for c.e. trees T_1, T_2, returns the code of a c.e. tree T so that T is well-founded iff T_1 or T_2 is well-founded. Also in case T is well-founded we have $|T|_o = \min(|T_1|_o, |T_2|_o)$.*

Proof: In what follows, we use a pairing function over pairs of strings of $\mathbb{N}^{<\mathbb{N}}$ of the same size, defined by applying the integer binary pairing function on each pair of elements that are at the same position on the two strings. Formally: $\langle \sigma_1, \sigma_2 \rangle = \langle \sigma_1(0), \sigma_2(0) \rangle \hat{\ } \dots \hat{\ } \langle \sigma_1(n-1), \sigma_2(n-1) \rangle$, where $n = |\sigma_1| = |\sigma_2|$.

The definition of T is rather simple: We enumerate a node σ in T if $\sigma = \langle \sigma_1, \sigma_2 \rangle$ where for $i \in \{1, 2\}$, each string σ_i has been enumerated in T_i. We easily verify that T is a c.e tree. Furthermore it is clear that we have an infinite path in T iff we have an infinite path in both T_1 and T_2. We now prove by induction that $|T|_o = \min(|T_1|_o, |T_2|_o)$.

For any two c.e. trees T_1, T_2 let us denote $\min(|T_1|_o, |T_2|_o)$ by γ. Consider now two c.e. trees T_1, T_2 so that $\gamma = \varnothing$. It is clear that we also have $|T|_o = \varnothing$. Suppose that for any c.e. trees T_1, T_2 with $\gamma < \alpha$ we have $|T|_o = \gamma$. Consider now c.e. trees T_1, T_2 with $\gamma \leq \alpha$ and let us prove that $|T|_o = \gamma$. By definition we have that σ is a node of T iff there is a node n_1 of T_1 and a node n_2 of T_2 (with $|n_1| = |n_2| = 1$), so that σ is a node of the tree $\langle n_1, n_2\rangle \hat{} OR(T_1 \upharpoonright_{n_1}, T_2 \upharpoonright_{n_2})$. As we have for any sequence of ordinals that $\sup^+_{n,m} \min(\alpha_n, \beta_m) = \min(\sup^+_n \alpha_n, \sup^+_n \beta_n)$, then using the induction hypothesis, we also have that $|T|_o = \min(|T_1|_o, |T_2|_o) = \gamma$. ∎

We now prove a similar lemma, but this time we want an infinite Boolean OR, that is, we now have a code for a computable infinite sequence of trees $\{T_i\}_{i\in\omega}$ and we want that $OR(T_1, T_2, \dots)$ is well-founded iff for at least one i we have that T_i is well-founded. In this case, we cannot have a bound as accurate as before:

Lemma 1.4.2 *There is a total computable function $OR : \mathbb{N} \to \mathbb{N}$, which on any code for an infinite computable enumeration of c.e. trees $\{T_i\}_{i\in\mathbb{N}}$, possibly ill-founded, returns a code for a c.e. tree T, so that if at least one T_i is well-founded we have:*

$$|T|_o < \min\{|T_i|_o \ : \ T_i \ \ \textit{is a well-founded tree}\ \} + \omega$$

and if every T_i is ill-founded, we have that T is ill-founded as well.

Proof: We define a sequence of trees $\{U_i\}_{i\in\mathbb{N}}$, by $U_0 = T_0$ and if U_i is defined, U_{i+1} is obtained by putting in U_{i+1} all nodes σ of length $i+1$ that are in U_i and their prefixes, and by adding for each of those σ, the nodes that are in the tree $\sigma \hat{} OR(T_{i+1}, U_i \upharpoonright_\sigma)$, where $OR : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ is the function of the previous lemma. Our function returns a code for the tree $T = \lim_i U_i$.

It is clear by definition that $\lim_i U_i$ exists because for any i and any $j \geq i$, the nodes of length less than $i+1$ are the same in each U_j. The limit is equal to the tree described by the union over i of the set of nodes of length $i+1$ that are in U_i. As each U_i is a c.e. tree uniformly in i, we also have that the limit is a c.e. tree.

For i so that $|T_i|_o$ is minimal we have that $|U_i|_o \leq |T_i|_o + i$. Indeed, every node in U_i is in a tree $\sigma \hat{} OR(T_i, U_{i-1} \upharpoonright_\sigma)$ for some σ of length i, and $|OR(T_i, U_{i-1} \upharpoonright_\sigma)|_o \leq |T_i|_o$. Also for any i we have that $|U_{i+1}|_o \leq |U_i|_o$ and that $|T|_o = \inf_i |U_i|_o$. Therefore, we have $|T|_o < |T_i|_o + \omega$. ∎

Similarly, we can define a function $AND : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ as well as an infinite AND working with infinite computable sequence of c.e. trees. It is much easier than for the OR, as we can just put any tree of the sequence into a bigger tree to obtain the result:

Lemma 1.4.3 *There is a total computable function $AND : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$, which on any two codes for c.e. trees T_1, T_2, return the code of a c.e. tree T so that T is well-founded iff T_1 and T_2 are well-founded. Also in case T is well-founded we have $|T|_o = \sup(|T_1|_o, |T_2|_o)$.*

Proof: We simply define T to be the disjoint union of T_1 and T_2, that is, for every $n > 0$ and every σ of length n in T_1 we put $\langle 0, \sigma(0)\rangle \hat{} \dots \hat{} \langle 0, \sigma(n-1)\rangle$ in T, and for every σ of length n in T_2 we put $\langle 1, \sigma(0)\rangle \hat{} \dots \hat{} \langle 1, \sigma(n-1)\rangle$ in T. It is clear by the definition of $|T|_o$ that $|T|_o = \sup(|T_1|_o, |T_2|_o)$. ∎

Lemma 1.4.4 *There is a total computable function $AND : \mathbb{N} \to \mathbb{N}$, which on any code for an infinite computable enumeration of c.e. trees $\{T_i\}_{i\in\mathbb{N}}$, returns a code for a c.e. tree T, so that if every T_i is well-founded we have:*

$$|T|_o = \sup^+\{|T_i|_o\}$$

and if at least one T_i is ill-founded, we have that T is ill-founded as well.

Proof: We define T like in the previous lemma, but with the infinite disjoint union of every tree T_i. ∎

1.4.3 Transfinite recursion over the computable ordinals

We will need in this thesis to build several computable functions by transfinite recursion over the ordinals. To do so we use a generalization of the recursion scheme for primitive recursive functions. Informally, to define a computable function f on the value n, we can safely reuse the values of f on every $m < n$. So to compute $f(n)$ we need to compute first $f(n-1)$, $f(n-2)$, etc... until we need to compute the value $f(0)$ in a computation which should not use any other value of f.

This works along ω (for because ω is well-founded, and it can be generalized to any well-founded relation. So to define $f(n)$, we can require the knowledge the values of $f(m)$ first, for any m smaller than n in the well-order we use. Of course, to keep f computable, we need some restrictions on our well-order:

1. For an element n of the well-order, we should be able to computably enumerate elements which are smaller than n, so at least we can ask the required previous values of f.

2. We should be able to recognize 'a halting criterion', that is, we should be able to know in a computable way when nothing is smaller than the current element n.

The use of codes for computable ordinals as defined so far does not ensure that the second condition is satisfied, because a code for the smallest ordinal is a code for an empty relation, and we can never decide in finite time if something will be enumerated or not in an empty relation. Perhaps the best way to overcome this difficulty is by using Kleene's coding system for constructive ordinals, because this coding system satisfies the above two and has two more useful properties:

3. We can decide if n codes for a limit ordinal or for a successor ordinal.

4. If n codes for $\alpha + 1$, a successor ordinal, we can effectively find a code for α, and if n codes for α, a limit ordinal, we can effectively enumerate a sequence of codes for ordinals $\alpha_0 < \alpha_1 < \dots$ so that $\alpha = \sup_n \alpha_n$.

Originally, Kleene's coding system works with integers. We present here a slightly different system (all the underlying ideas are exactly the same, only the presentation differs) using well-founded computable trees.

The constructive ordinals

We now impose some restrictions on the c.e. well-founded trees we are going to use. The trees which match each condition of this restriction are called **constructive trees**, and they code for the **constructive ordinals** using the coding system introduced previously (A well-founded tree T codes for the ordinal $|T|_o$). We now give the definition:

Definition 1.4.6. *A tree is constructive if it matches the following conditions:*

- *The first condition to be a constructive tree is to be well-founded and computable, that is for a given string $\sigma \in \mathbb{N}^{<\mathbb{N}}$ we should be able to decide if σ is in the tree or not. Furthermore, it also should be decidable given a code of a constructive tree T if T equals the empty set or not.*

- *The second condition is that every node σ should be either a leaf (in which case $T\restriction_\sigma = \epsilon$ codes for 1), or have exactly one child (in which case $T\restriction_\sigma$ codes for a successor ordinal), or have countably many children (in which case $T\restriction_\sigma$ codes for a limit ordinal).*

- *For every node, we should be able to computably tell the difference between the three cases of the second condition. To do so, the third condition is that every node is 'tagged', with 0 if it is a leaf, with 1 if it has exactly one child and with 2 if it has countably many children.*

- *For every node σ tagged with 2, we should make sure that σ really codes for a limit ordinal. To ensure that, the fourth condition is that for every node σ with infinitely many children $\{\sigma_i\}_{i\in\omega}$ ordered lexicographically, we have $|\sigma_0|_o < |\sigma_1|_o < \dots$.* ■

The restriction from c.e. well-founded trees to constructive trees might seem drastic, but we shall soon see that any computable ordinal can still be coded by a constructive tree. We shall now define the codes for well-founded trees:

Definition 1.4.7. *Let us fix an element e_0 such that $W_{e_0} = \varnothing$. We call $\mathcal{O}$ the set of codes e so that $e = e_0$ or such that W_e enumerates the tagged nodes of a non-empty constructive tree, in a way that for any siblings $\sigma_1 < \sigma_2$, we have that σ_1 is enumerated before σ_2 (This way the tree is computable, and note that every constructive tree has a corresponding code).*

For any $a \in \mathcal{O}$ coding for a tree T we write $|a|_o$ to denote $|T|_o$. Also for any computable ordinal α we write $\mathcal{O}_{<\alpha}$ to denote the elements $a \in \mathcal{O}$ so that $|a|_o < \alpha$, we write $\mathcal{O}_{\leq\alpha}$ to denote the elements $a \in \mathcal{O}$ so that $|a|_o \leq \alpha$ and we write $\mathcal{O}_{=\alpha}$ to denote the elements $a \in \mathcal{O}$ so that $|a|_o = \alpha$ ■

Also we will always be interested in constructive trees up to isomorphism. So if the children of a node σ are all odd natural numbers or all even natural numbers, there is no difference for us so far. If T_1 is isomorphic to T_2 we will write $T_1 \simeq T_2$. Also for two constructive trees $T_1 \simeq T_2$, note that we can compute the isomorphism uniformly in codes for T_1 and T_2. We say that T_1 and T_2 are **computably isomorphic**.

In practice, when we describe a computable function by induction over constructive ordinals, we will describe it by induction over elements of $\mathcal{O}$. But Let us first introduce some notation.

If $a \in \mathcal{O}$ codes for a finite ordinal n, we sometimes write $a = n$ instead of $|a|_o = n$. If $a \in \mathcal{O}$ codes for a successor ordinal we can clearly obtain, in a canonical and computable way, a code b for the predecessor of a. We will write in this case $a = \mathrm{succ}(b)$. On the other hand, when we have $a \in \mathcal{O}$ coding for a tree T, if we want a code for the successor of a, there is no unique way to get it. So we decide arbitrarily that in this case, the successor is a code for the constructive tree $0\hat{\ }T$. The reader should note that as two isomorphic constructive trees are computably isomorphic, this decision does not really matter, and in this case also we will denote the successor of a by $\mathrm{succ}(a)$.

The same phenomenon happens for limit ordinals. If $a \in \mathcal{O}$ codes for a limit ordinal we can clearly enumerate a list of codes $b_0, b_1, \dots$ so that $|b_0|_o < |b_1|_o < \dots$ and so that $|a|_o = \sup_n |b_n|_o$. We will write in this case $a = \sup_n(b_n)$. On the other hand, when we have an effective enumeration $b_0, b_1, \dots$ of codes for constructive trees $T_0, T_1, \dots$ with $|b_0|_o < |b_1|_o < \dots$, we also don't have a unique way to build the limit tree. So we decide that in this case, the limit code is a code for the tree consisting of the union of $n\hat{\ }T_n$, for every n. Here again we keep the same notation and denote by $\sup_n(b_n)$ the code for this resulting tree.

To make things more concrete we should maybe give an example of a computable function that we define by induction over elements of $\mathcal{O}$:

Example 1.4.2:
We define an addition function $+_o$, which takes $a, b \in \mathcal{O}$ and returns $(a +_o b) \in \mathcal{O}$ so that $|a|_o + |b|_o = |a +_o b|_o$:

$$\begin{array}{rcll} a +_o b & = & a & \textit{If } b = 0 \\ & = & \mathrm{succ}(a +_o c) & \textit{If } b = \mathrm{succ}(c) \\ & = & \sup_n(a +_o c_n) & \textit{If } b = \sup_n(c_n) \end{array}$$

◇

The function of the previous example might seem a bit obscure at first. We try here to explicit what happens. First the reader should remember that $\mathrm{succ}(a)$ and $\sup_n(b_n)$ are notations for codes (elements of $\mathcal{O}$) and not for the constructive trees themselves. Also when we have to return the code $\mathrm{succ}(a +_o c)$, we can just wait for the computation of $a +_o c$ to return and then return $\mathrm{succ}(a +_o c)$. But when we have to return the code $\sup_n(a +_o c_n)$, we cannot wait for each computation $a +_o c_n$ to return because there are infinitely many of them. Fortunately $\sup_n(a +_o c_n)$ is merely a code and this is a typical case where Kleene's fixed point theorem is necessary: To return the code $\sup_n(a +_o c_n)$, we can use a code for the function $+_o$ inside the function $+_o$ itself.

Instead of considering that the function $+_o$ takes as input codes a, b and outputs a code for $a +_o b$, we could consider that it takes as input the corresponding enumerations W_a, W_b (as oracles that we enumerate) and outputs an enumeration of the resulting tree. Such a function might be easier to conceptualize and does not required the use of Kleene's fixed point theorem. However if at the end we want to obtain the code corresponding to the resulting enumeration, we still need the fixed point theorem, to make the function output its own code applied to the two codes it has in input.

We shall now briefly prove that the function $+_o$ works as expected:

Proposition 1.4.1:
For $a, b \in \mathcal{O}$ we have $a +_o b \in \mathcal{O}$ and $|a|_o + |b|_o = |a +_o b|_o$.

PROOF: We shall prove the proposition by induction over the second parameter of the function $+_o$. Fix any $a \in \mathcal{O}$. The proposition is clear for a and b when $b = 0$.

Suppose now that for every $b \in \mathcal{O}_{\leq\alpha}$ we have that $a +_o b \in \mathcal{O}$ and $|a|_o + |b|_o = |a +_o b|_o$. Consider $a +_o b$ for any $b = \text{succ}(c)$ for some $c \in \mathcal{O}_{=\alpha}$. By definition we have $a +_o b = \text{succ}(a +_o c)$, but by the induction hypothesis we have $a +_o c \in \mathcal{O}$ and then $\text{succ}(a +_o c) \in \mathcal{O}$. Still by the induction hypothesis we have $|a|_o + |c|_o = |a +_o c|_o$ and then by definition of 'succ' we have $\text{succ}(a +_o c) = (|a|_o + |c|_o)^+ = |a|_o + |c|_o^+ = |a|_o + |b|_o$.

Take now α limit and suppose that for every $b \in \mathcal{O}_{<\alpha}$ we have that $|a +_o b|_o \in \mathcal{O}$ and $|a|_o + |b|_o = |a +_o b|_o$. Consider $a +_o b$ for any b such that $b = \sup_n c_n$ with $|c_0|_o < |c_1|_o < \dots$ and with $\sup_n |c_n|_o = \alpha$. By definition, we have $a +_o b = \sup_n (a +_o c_n)$, but by induction hypothesis we have $a +_o c_n \in \mathcal{O}$ and $|a|_o + |c_n|_o = |a +_o c_n|_o$ for every n, and therefore $|a +_o c_0|_o < |a +_o c_1|_o < |a +_o c_2|_o < \dots$. Then $\sup_n (a +_o c_n) \in \mathcal{O}$. Still using induction hypothesis, by definition of 'sup', we have $\sup_n (a +_o c_n) = \sup_n (|a|_o + |c_n|_o) = |a|_o + \sup_n (|c_n|_o) = |a|_o + |b|_o$.■

The constructive ordinals and the computable ordinals coincide

We shall now prove as announced that any computable ordinal can be encoded by a constructive tree. From any c.e. well-founded tree T, we will build a computable well-founded tree U whose code is in $\mathcal{O}$ and so that $|T|_o \leq |U|_o$. If we can do that, it will then be enough to argue that the constructive ordinals are closed downward to see that the constructive ordinals are the same as the computable ordinals.

We first start by defining a tree T' obtained by adding to each node of T countably many children that we tag as leaves. All the other nodes of T' are (rightfully) tagged to have countably many children. The resulting tree T' is still only c.e. and furthermore, for a given node of T' with countably many children $\sigma_0 < \sigma_1 < \dots$, we probably do not have $|\sigma_0|_o < |\sigma_1|_o < \dots$. This is where the function $+_o$ will be helpful. We define a computable function G on nodes of T', in order to inductively transform T' into a constructive tree, with the help of the function $+_o$:

$G(\sigma)$	$=$	a	If σ is tagged as a leaf, where $a \in \mathcal{O}$ codes for 1.
	$=$	$\sup_n \left(\sum_{i \leq n} G(\sigma_i)\right)$	If σ is tagged to have countably many children $\{\sigma_i\}_{i \in \omega}$ (given in order on their enumeration). The finite sum that we use is of course to be understood using the function $+_o$.

One can easily prove by induction that G applied to the root of T' produces a code of $\mathcal{O}$ for a computable tree U, so that $|T'|_o \leq |U|_o$. As we surely have $|T|_o \leq |T'|_o$, this proves that as long as the constructive ordinals are closed downwards, all of them can be represented by a constructive tree.

We shall now prove that the constructive ordinals are closed downwards. We prove so in an effective way, that is, given the constructive code of an ordinal α, we can uniformly enumerate constructive codes for every ordinal $\beta < \alpha$.

Proposition 1.4.2:
There is a total computable function $q : \omega \to \omega$ so that for any computable α and any $a \in \mathcal{O}_{=\alpha}$, we have that $W_{q(a)}$ enumerates elements in $\mathcal{O}$ corresponding to all ordinals smaller than α. Formally: $W_{q(a)} \subseteq \mathcal{O}_{<\alpha}$ and $\forall \beta < \alpha$, there exists $b \in W_{q(a)}$ with $|b|_o = \beta$. Note that we can have repetitions.

Proof: On a code $a \in \mathcal{O}$ for a tree T, the function q simply creates an index which enumerates a code for 0, and for every $\sigma \in T$, a code for $T\upharpoonright_\sigma$. We shall prove by induction that the function q satisfies the proposition. If the tree only contains ϵ it enumerates nothing as expected. Suppose now that for every code $a \in \mathcal{O}$ so that $|a|_o \leq \alpha$ we have that $W_{q(a)}$ satisfies the proposition. Then surely for any $a = \text{succ}(b)$ with $|b|_o = \alpha$ we have that $W_{q(a)}$ enumerates b and also everything that $W_{q(b)}$ would enumerate. Thus by induction hypothesis we have that $W_{q(a)}$ satisfies the proposition. Suppose now for α limit and for every code $a \in \mathcal{O}_{<\alpha}$, we have that $W_{q(a)}$ satisfies the proposition. Consider $a = \sup_n b_n$ for $|b_0|_o < |b_1|_o < \dots$ with $\sup_n |b_n|_o = \alpha$. We clearly have that $W_{q(a)}$ enumerates every b_n and also everything that $W_{q(b_n)}$ would enumerate. As the sequence $|b_n|_o$ is unbounded in α, we have by induction hypothesis that $W_{q(a)}$ satisfies the proposition. ■

1.5 Descriptive complexity of sets of sequences

We give in this section basic notions on descriptive complexity of sequences. A large part of this thesis will deal with the descriptive complexity of various sets. Informally, this section deals with the general philosophical question of 'what sets can be described?'.

We provide with the Borel and the effectively Borel hierarchy a well-known framework to study the complexity of sets. The way we will present things is now standard, resulting from the work of many mathematicians during the beginning of the 20th century. Interesting historical remarks can be found in the section 1H of "Descriptive set theory" by Moschovakis (see [65]): It seems that this study first arose from the study of the complexity of functions, with the Baire classes of functions from $\mathbb{R}^n$ to $\mathbb{R}$, defined by Baire in [2]. Lebesgue then derived from the Baire hierarchy of functions (see [48]), the hierarchy of complexity of sets, known today as the Borel hierarchy.

1.5.1 The Borel hierarchy

Definitions

We give in this section a description of the Borel hierarchy in the Baire space. It will be clear that the following description can be applied to any topological space.

We say that a set $\mathcal{B} \subseteq \mathbb{N}^\mathbb{N}$ is $\mathbf{\Sigma^0_1}$ if it is open, i.e., if there exists a countable set of strings $\{\sigma_n\}_{n\in\mathbb{N}}$ so that $\mathcal{A} = \bigcup_n [\sigma_n]$. We then say that a set $\mathcal{B} \subseteq \mathbb{N}^\mathbb{N}$ is $\mathbf{\Pi^0_1}$ if it is closed, i.e., its complement is $\mathbf{\Sigma^0_1}$.

We can now iterate the definition by induction over the natural numbers. Suppose that the class of sets which are $\mathbf{\Pi^0_n}$ has been defined, we then say that a set $\mathcal{B} \subseteq \mathbb{N}^{\mathbb{N}}$ is $\mathbf{\Sigma^0_{n+1}}$ if it is the union of countably many $\mathbf{\Pi^0_n}$ sets. We then say that a set $\mathcal{B} \subseteq \mathbb{N}^{\mathbb{N}}$ is $\mathbf{\Pi^0_{n+1}}$ if its complement is $\mathbf{\Sigma^0_{n+1}}$.

We can even iterate the definition by induction over the ordinals. Suppose that the classes of sets which are $\mathbf{\Pi^0_\alpha}$ have been defined for any ordinal $\alpha < \beta$. We then say that a set $\mathcal{B} \subseteq \mathbb{N}^{\mathbb{N}}$ is $\mathbf{\Sigma^0_\beta}$ if it is the union of countably many sets which are $\mathbf{\Pi^0_\alpha}$ for $\beta < \alpha$. We then say that a set $\mathcal{B} \subseteq \mathbb{N}^{\mathbb{N}}$ is $\mathbf{\Pi^0_\beta}$ if its complement is $\mathbf{\Sigma^0_\beta}$. It can also be the case that a set is both a $\mathbf{\Sigma^0_\alpha}$ and $\mathbf{\Pi^0_\alpha}$, in which case we say that it is $\mathbf{\Delta^0_\alpha}$.

We can maybe now give a quick sum up of the previous definitions:

Definition 1.5.1. *The Borel hierarchy is defined by induction over ordinals as follow:*

- *A set is $\mathbf{\Sigma^0_1}$ if it is open.*
- *A set is $\mathbf{\Pi^0_1}$ if it is closed.*
- *A set is $\mathbf{\Sigma^0_\alpha}$ if it is a countable union of sets which are $\mathbf{\Pi^0_\beta}$ for $\beta < \alpha$.*
- *A set is $\mathbf{\Pi^0_\alpha}$ if its complement is a $\mathbf{\Sigma^0_\alpha}$ set.*
- *A set is $\mathbf{\Delta^0_\alpha}$ if it is both a $\mathbf{\Sigma^0_\alpha}$ set and a $\mathbf{\Pi^0_\alpha}$ set.*

We also say that a set is $\mathbf{\Sigma^0_{<\alpha}}$ (resp. $\mathbf{\Pi^0_{<\alpha}}$) if it is $\mathbf{\Sigma^0_{<\beta}}$ (resp. $\mathbf{\Pi^0_{<\beta}}$) for some $\beta < \alpha$. ∎

It is clear that at ordinal step ω_1, no new set is added in the hierarchy, because a countable set of countable ordinals is bounded in ω_1. More formally, a $\mathbf{\Sigma^0_{\omega_1}}$ set is also a $\mathbf{\Sigma^0_\alpha}$ set for $\alpha < \omega_1$, and we easily prove by induction that any $\mathbf{\Sigma^0_\beta}$ set for $\beta \geq \omega_1$ is also a $\mathbf{\Sigma^0_\alpha}$ set for $\alpha < \omega_1$. We will see later that the hierarchy is strict below ω_1.

Closure properties of the Borel hierarchy

We shall state here various closure properties of Borel sets, without the proofs, that can be found for example in [65]. For any countable ordinal α, for $\mathbf{\Gamma}$ meaning $\mathbf{\Pi}$ or $\mathbf{\Sigma}$, we have the three following straightforward closure properties for the Borel sets of any topological spaces:

- The class of $\mathbf{\Sigma^0_\alpha}$ sets is closed under countable union.
- The class of $\mathbf{\Pi^0_\alpha}$ sets is closed under countable intersection.
- The class of $\mathbf{\Gamma^0_\alpha}$ sets is closed under finite union and finite intersection.

The last one is to be proved by induction, starting with the fact that the class of open sets is closed by finite intersection.

The Borel sets are mainly studied in **Polish topological spaces**, that is, separable completely metrizable topological spaces. A detailed study of such spaces can be found in [65] or in [34]. In such spaces, the Borel sets have the following very nice closure property, that is easily seen to be true on the Cantor space or on the Baire space:

- A $\mathbf{\Gamma^0_\alpha}$ set is also both $\mathbf{\Pi^0_{\alpha+1}}$ and $\mathbf{\Sigma^0_{\alpha+1}}$.

It is clear by definition that a $\boldsymbol{\Sigma^0_\alpha}$ set is always a $\boldsymbol{\Pi^0_{\alpha+1}}$ set. The fact that it is also a $\boldsymbol{\Sigma^0_{\alpha+1}}$ set requires a bit of work. Also we shall see in Section 6.4 an example of hierarchy where this does not hold anymore. Finally we have a last straightforward property, which is useful to study the connections between Borel sets and logical formulas. For any two topological spaces $\mathcal{A}_1, \mathcal{A}_2$, with $\mathbb{B}_1$ the class of Borel sets of $\mathcal{A}_1$ and $\mathbb{B}_2$ the class of Borel sets of $\mathbb{B}_2$ we have:

- For a total continuous function $f : \mathcal{A}_1 \to \mathcal{A}_2$ and $\mathcal{B} \in \mathbb{B}_2$ a $\boldsymbol{\Gamma^0_\alpha}$ set of $\mathbb{B}_2$, we have that $f^{-1}(\mathcal{B})$ is a $\boldsymbol{\Gamma^0_\alpha}$ set of $\mathbb{B}_1$.

In Moschovakis' book, this property is called **being closed under continuous substitution**. We will see an example of how this closure property is helpful, with Proposition 1.6.1 and Example 1.6.1.

1.5.2 The effective Borel hierarchy

We give in this section a description of the effective Borel hierarchy in the Baire space. It will be clear that the following description can be applied to the Cantor space. First we describe the effective arithmetical Borel hierarchy. We will later iterate the definition through the ordinals.

The finite effective Borel hierarchy

We say that a set $\mathcal{B} \subseteq \mathbb{N}^\mathbb{N}$ is Σ^0_1 if it is **effectively open**, i.e., if there exists a code e such that $\mathcal{B} = \bigcup_{\sigma \in W_e} [\sigma]$. We then say that a set $\mathcal{B} \subseteq \mathbb{N}^\mathbb{N}$ is Π^0_1 if it is **effectively closed**, i.e., its complement is Σ^0_1. In the context of Σ^0_1 sets, if we have that $\mathcal{B} = \bigcup_{\sigma \in W_e} [\sigma]$ we say that e is in **index** for $\mathcal{B}$, whereas in the context of Π^0_1 sets, e will be considered to be an index for $\mathcal{B}^c$, the complement of $\mathcal{B}$.

We can now iterate the definition by induction over the natural numbers. Suppose that the class of Π^0_n sets has been defined, we then say that a set $\mathcal{B} \subseteq \mathbb{N}^\mathbb{N}$ is Σ^0_{n+1} if $\mathcal{B} = \bigcup_m \mathcal{B}_m$ for Π^0_n sets $\{\mathcal{B}_m\}_{m\in\mathbb{N}}$, where the i-th element enumerated in W_e is an index for the $\mathcal{B}_i$. Then we say that a set $\mathcal{B} \subseteq \mathbb{N}^\mathbb{N}$ is Π^0_{n+1} if its complement is Σ^0_{n+1}.

The transfinite effective Borel hierarchy

It is a bit less easy to extend the effective hierarchy in the transfinite setting, than it was in the finite setting. With the finite effective Borel hierarchy, an index does not need to encode more than the enumeration of indices at the lower level. But in the transfinite case, there is no canonical way to know at which level we are. For example an index for a Σ^0_ω set should enumerate indices of $\Pi^0_{n_i}$ sets with n_i unbounded in $\mathbb{N}$. But how do we know an index would be for a Π^0_3 set rather than a Π^0_7 set? In particular we should be able to determine when we have reached indices for Σ^0_1/Π^0_1 sets.

In order to work this out, we now decide that an index for a Σ^0_1 sets is a pair $\langle 0, e\rangle$ where e is so that W_e enumerates a set of strings describing the Σ^0_1 set. An index for a Π^0_α set is a pair $\langle 1, e\rangle$ where e is an index for a Σ^0_α set, and finally an index for a Σ^0_α set, for $\alpha > 1$, is a pair $\langle 2, e\rangle$ where W_e enumerates a set of indices for Π^0_β sets, with $\beta < \alpha$. We sum up this in the following definition:

Definition 1.5.2. *The effective Borel hierarchy is defined by induction over ordinals as follows:*

- *A* Σ^0_1***-index*** *is given by a pair* $\langle 0, e\rangle$. *The set corresponding to this* Σ^0_1*-index is given by* $\bigcup_{\sigma\in W_e}[\sigma]$.
- *A* Π^0_α***-index*** *is given by a pair* $\langle 1, e\rangle$ *where* e *is a* Σ^0_α*-index. The set corresponding to this* Π^0_α*-index is given by* $\mathcal{B}^c$ *where* $\mathcal{B}$ *is the set corresponding to the index* e.
- *A* Σ^0_α***-index*** *is given by a pair* $\langle 2, e\rangle$ *where* W_e *is not empty and enumerate only* $\Pi^0_{\beta_n}$ *indices, with* $\sup^+_n(\beta_n) = \alpha$. *The set corresponding to this* Σ^0_α*-index is given by* $\bigcup_n \mathcal{B}_n$ *where* $\mathcal{B}_n$ *is the set corresponding to the* n*-th index enumerated by* W_e.

We say that a set $\mathcal{B}$ *is* Σ^0_α *(resp.* Π^0_α*) if for some* Σ^0_α*-index (resp.* Π^0_α*-index)* e*,* $\mathcal{B}$ *is the set corresponding to* e*. We say that a set* $\mathcal{B}$ *is* Δ^0_α *if it is both* Σ^0_α *and* Π^0_α*. Finally we say that a set is* $\Sigma^0_{<\alpha}$ *(resp.* $\Pi^0_{<\alpha}$*) if it is* Σ^0_β *(resp.* Π^0_β*) for some* $\beta < \alpha$*.* ■

For the non-effective Borel hierarchy we have that no new set is added at step $\alpha \geq \omega_1$, and similarly we argue now that for the effective Borel hierarchy, no new set is added at step $\alpha \geq \omega_1^{ck}$. The reason is that an index for such a set is essentially a code for a c.e. well-founded tree. Given an index e for an effectively Borel set, we build the corresponding tree by first enumerating e as the root of the tree. Then recursively, on each node $\sigma\hat{}n$ enumerated so far in the tree we apply the following algorithm:

- If $n = \langle 2, n'\rangle$ we enumerate as children of $\sigma\hat{}n$ the nodes $\sigma\hat{}n\hat{}e$ for $e \in W_{n'}$.
- If $n = \langle 1, n'\rangle$ we enumerate $\sigma\hat{}n\hat{}n'$ as the only child of $\sigma\hat{}n$.
- If $n = \langle 0, n'\rangle$ then $\sigma\hat{}n$ is a leaf and therefore no child is enumerated.

It is clear from the definition of indices that such a tree is well-founded. We can then show by induction that for a Σ^0_α-index, we have $\alpha \leq |T|_o$ where T is the corresponding tree.

We can also define the relativized version of the effective Borel hierarchy:

Definition 1.5.3. *For an oracle* $X \in 2^{\mathbb{N}}$*, The* X*-effective Borel hierarchy is defined by induction over ordinals as follows:*

- *A* $\Sigma^0_1(X)$***-index*** *is given by a pair* $\langle 0, e\rangle$. *The set corresponding to this* $\Sigma^0_1(X)$*-index is given by* $\bigcup_{\sigma\in W_e^X}[\sigma]$.
- *A* $\Pi^0_\alpha(X)$***-index*** *is given by a pair* $\langle 1, e\rangle$ *where* e *is a* $\Sigma^0_\alpha(X)$*-index. The set corresponding to this* $\Pi^0_\alpha(X)$*-index is given by* $\mathcal{B}^c$ *where* $\mathcal{B}$ *is the set corresponding to the index* e.
- *A* $\Sigma^0_\alpha(X)$***-index*** *is given by a pair* $\langle 2, e\rangle$ *where* W_e^X *is not empty and enumerate only* $\Pi^0_{\beta_n}(X)$ *indices, with* $\sup^+_n(\beta_n) = \alpha$. *The set corresponding to this* $\Sigma^0_\alpha(X)$*-index is given by* $\bigcup_n \mathcal{B}_n$ *where* $\mathcal{B}_n$ *is the set corresponding to the* n*-th index enumerated by* W_e^X.

We say that a set $\mathcal{B}$ *is* $\Sigma^0_\alpha(X)$ *(resp.* $\Pi^0_\alpha(X)$*) if for some* $\Sigma^0_\alpha(X)$*-index (resp.* $\Pi^0_\alpha(X)$*-index)* e*,* $\mathcal{B}$ *is the set corresponding to* e*. We say that a set* $\mathcal{B}$ *is* $\Delta^0_\alpha(X)$ *if it is both* $\Sigma^0_\alpha(X)$ *and* $\Pi^0_\alpha(X)$*. Finally we say that a set is* $\Sigma^0_{<\alpha}(X)$ *(resp.* $\Pi^0_{<\alpha}(X)$*) if it is* $\Sigma^0_\beta(X)$ *(resp.* $\Pi^0_\beta(X)$*) for some* $\beta < \alpha$*.* ■

Similarly, at step $\alpha \geq \omega_1^X$, no new set is added in the X-effective Borel hierarchy.

We should emphasize that in practice, we won't consider an effective Borel set the way it is given by its index, that is, for example of the form $\bigcup(\bigcup(\bigcup W_e^c)^c)^c$; but we will rather replace the complements by intersections and work with : $\bigcup\bigcap\bigcup W_e^c$. It should be clear how to decide, given an index e and a sub-index $\langle 2, W_a\rangle$ of e, if $\langle 2, W_a\rangle$ corresponds to a union, or to an intersection.

Closure properties of the effective Borel hierarchy

The properties we gave in Section 1.5.1 are easily seen to be have effective counterparts. For any computable ordinal α and for Γ denoting Σ or Π we have:

- The class of Σ_α^0 sets is closed under effective countable union.
- The class of Π_α^0 sets is closed under effective countable intersection.
- The class of Γ_α^0 sets is closed under finite union and finite intersection.
- A Γ_α^0 set is also both $\Pi_{\alpha+1}^0$ and $\Sigma_{\alpha+1}^0$.

We said in the preamble of this section that the definition of the effective hierarchy can be converted straightforwardly to the Cantor space. It can also be converted to the space of natural numbers, which will be studied in the next section. But also we can extend the definition of those hierarchies to finite products of any of the spaces $\mathbb{N}^\mathbb{N}, 2^\mathbb{N}$ or $\mathbb{N}$, without any difficulty, as those new spaces still have a canonical countable basis that can be put in bijection with the natural numbers, and then on which we can therefore apply the notion of computable enumerability.

So we define $\mathbb{A}$ to be the smallest class of topological spaces such that $2^\mathbb{N}, \mathbb{N}^\mathbb{N}, \mathbb{N}$ are in $\mathbb{A}$ and such that for $\mathcal{A}_1, \mathcal{A}_2$ in $\mathbb{A}$ we have $\mathcal{A}_1 \times \mathcal{A}_2$ is in $\mathbb{A}$. Then for any two topological spaces $\mathcal{A}_1, \mathcal{A}_2 \in \mathbb{A}$, with $\mathbb{B}_1$ the class of effective Borel sets of $\mathcal{A}_1$ and $\mathbb{B}_2$ the class of effective Borel sets of $\mathbb{B}_2$ we have the following closure property which is also called **closure under computable substitution**:

- For a total computable function $f : \mathcal{A}_1 \to \mathcal{A}_2$ and $\mathcal{B} \subseteq \mathbb{B}_2$ a Γ_α^0 set of $\mathbb{B}_2$, we have that $f^{-1}(\mathcal{B})$ is a Γ_α^0 set of $\mathbb{B}_1$.

This last property is important in order to still be able to use the powerful counterpart between logical formulas and Borel sets, even when we work in the effective transfinite hierarchy. In particular Proposition 1.6.1 is a consequence of the closure properties of Borel sets that are stated here, including the closure under computable substitution.

1.5.3 Borel hierarchies are strict

We provide in this section a proof that both effective and non-effective hierarchies do not collapse. It seems that the argument we will give (essentially a diagonal argument) should be credited to Lusin (see [55], credited in section 1H of [65]). What we prove now is probably a bit stronger (but surely known), since at the time, effectivity aspects were not a concern:

Theorem 1.5.1:
For every $\alpha \leq \omega_1^{ck}$, there is a set $\mathcal{A} \subseteq 2^\omega$ which is Σ^0_α but not $\mathbf{\Pi^0_\alpha}$. Also the theorem relativizes, that is, for every $X \in 2^\omega$ and every $\alpha \leq \omega_1^X$, there is a set $\mathcal{A} \subseteq 2^\omega$ which is $\Sigma^0_\alpha(X)$ but not $\mathbf{\Pi^0_\alpha}$.

Proof: This theorem says that the boldface hierarchy does not collapse in a strong sense, since we can even find a lightface 'non collapsing witness'. We provide a proof for the hierarchy in the Cantor space. It will be clear that the proof works the same way in the Baire space.

In order to conduct the proof, we need to make a non-trivial use of computable ordinals, and in particular the fixed point theorem will be needed. We prove first the theorem for Σ^0_1 sets (even though it is obvious in this case, this gives the first step of the diagonalization that will be reused inductively).

Fix a computable enumeration $\{\sigma_n\}_{n\in\mathbb{N}}$ of every string of $2^{<\mathbb{N}}$. Let u_0 be a code so that for every X, the set $W^X_{u_0}$ enumerates σ_i iff $X(i) = 1$. This index u_0 has two important properties:

- For any $\mathbf{\Sigma^0_1}$ set $\mathcal{U}$, there is a X so that $\langle 0, u_0\rangle$ is a $\Sigma^0_1(X)$-index for $\mathcal{U}$
- For any X, $\langle 0, u_0\rangle$ is the $\Sigma^0_1(X)$-index.

We now consider the following set:

$$\mathcal{A} = \{X \,:\, X \text{ belongs to the } \Sigma^0_1(X) \text{ set of } \Sigma^0_1(X)\text{-index } \langle 0, u_0\rangle\}$$

First it is clear that this set is Σ^0_1, because if W^X_e enumerates a prefix of X, this is witnessed already with a finite part of X. We shall now prove that the complement of $\mathcal{A}$ cannot be $\mathbf{\Sigma^0_1}$. Suppose otherwise, then also there is a X so that $\langle 0, u_0\rangle$ is a $\Sigma^0_1(X)$-index for the complement of $\mathcal{A}$. But is X in $\mathcal{A}$ or in the complement of $\mathcal{A}$? In either case we arrive at a contradiction, because if $X \in \mathcal{A}$ then $X \notin \mathcal{A}^c$ and therefore X does not belong to the set of $\Sigma^0_1(X)$-index $\langle 0, u_0\rangle$ which contradicts that X is in $\mathcal{A}$. Also if $X \notin \mathcal{A}$ then X belongs to the set of $\Sigma^0_1(X)$-index $\langle 0, u_0\rangle$ which contradicts the fact that $X \notin \mathcal{A}$. So $\mathcal{A}^c$ is not $\mathbf{\Sigma^0_1}$ and then $\mathcal{A}$ is not $\mathbf{\Pi^0_1}$.

We shall now iterate the proof by defining indices for more and more complex sets by induction through the computable ordinals. It appears that the constructive trees are very close to the codes for effectively Borel sets, once they are expanded into a tree. First we should describe a computable function $G : 2^\mathbb{N} \times \mathcal{O} \to \mathbb{N}$, which uniformly in an oracle X and in the code $a \in \mathcal{O}_{=\alpha}$, gives the index of a $\Sigma^0_\alpha(X)$ set. The function G uses the total computable function q described in Proposition 1.4.2 (The function q is so that on $a \in \mathcal{O}$ we have that $W_{q(a)}$ enumerates a list of codes for all the ordinals smaller than $|a|_o$ and only for those ordinals).

$$
\begin{array}{rcll}
G(X,a) & = & \langle 0, u_0\rangle & \text{if } a = 1 \text{, where } W^X_{u_0} \text{ enumerates } \sigma \text{ iff } X(\sigma) = 1. \\
 & = & \langle 2, e\rangle & \text{if } a = \mathrm{succ}(b) \text{, where } W^X_e \text{ enumerates the set of pairs } \langle 1, G(X_i, b)\rangle \text{ with } X = \oplus_{i\in\mathbb{N}} X_i \\
 & = & \langle 2, e\rangle & \text{if } a \text{ is limit, where } W^X_e \text{ enumerates the set of pairs } \langle 1, G(X_{i+1}, c_i)\rangle \text{ with } X = \oplus_{i\in\mathbb{N}} X_i. \text{ The sequence } \{c_i\} \text{ is obtained the following way: Let } \{b_i\} \text{ be the sequence of codes enumerated by } W_{q(a)}. \text{ Then } \{c_i\} \text{ is the subsequence obtained by keeping the elements } b_i \text{ so that } X_0(i) = 1.
\end{array}
$$

To build the function G we should use the fixed point theorem, so we can use a code for G inside G itself. Also we should maybe say a word on what it means to split the oracle X and pass only pieces of it to recursive calls of G.

We shall now prove by induction that:

1/ For any α, any $a \in \mathcal{O}_{=\alpha}$ and any X, we have that $G(X,a)$ is a $\Sigma^0_\alpha(X)$-index.

2/ For any α, any $a \in \mathcal{O}_{=\alpha}$ and any total computable function f we have that the set $\{X : X \in G(f(X),a)\}$ is a Σ^0_α set. (We make a slight abuse of notation with $X \in G(f(X),a)$, which means that X belongs to the set of index $G(f(X),a)$).

3/ For any $\boldsymbol{\Sigma^0_\alpha}$ set, and any $a \in \mathcal{O}_{=\alpha}$, there is a X such that $G(X,a)$ returns a $\Sigma^0_\alpha(X)$-index for it.

For $\alpha = 1$ we have that 1/, 2/ and 3/ are obvious.

Successor step:

Suppose now that 1/, 2/ and 3/ are true up to ordinal α and let us prove that 1/, 2/ and 3/ are true at ordinal $\alpha+1$. First to prove 1/, take any X and any $a = \mathrm{succ}(b)$ with some $b \in \mathcal{O}_{=\alpha}$. By induction hypothesis we have for any i that $G(X_i,b)$ returns a $\Sigma^0_\alpha(X_i)$-index. Then $\langle 1, G(X_i,b)\rangle$ is a $\Pi^0_\alpha(X_i)$-index, and by definition of G we have that $G(X,a)$ returns a $\Sigma^0_{\alpha+1}(X)$-index.

Let us now prove 2/. Fix any computable function f. We have that $\{X : X \in G(f(X),a)\} = \{X : \exists i\ X \notin G(f(X)_i,b)\}$ (where $f(X) = \oplus_i f(X)_i$). By induction hypothesis we have for each i that $\{X : X \notin G(f(X)_i,b)\}$ is Π^0_α (using the computable function g which associates $f(X)_i$ to X). Thus $\{X : X \in G(f(X),a)\} = \bigcup_{i\in\mathbb{N}}\{X : X \notin G(f(X)_i,b)\}$ is a $\Sigma^0_{\alpha+1}$ set and we have 2/.

To prove 3/, consider any $\boldsymbol{\Sigma^0_{\alpha+1}}$ set $\mathcal{A}$. By definition we have that $\mathcal{A} = \bigcup_n \mathcal{A}_n$ for $\mathcal{A}_n$ some $\boldsymbol{\Pi^0_\alpha}$ sets. By induction hypothesis we have a sequence $\{X_n\}_{n\in\mathbb{N}}$ so that for some $a \in \mathcal{O}_{=\alpha}$ we have that $G(X_n,a)$ returns a $\Sigma^0_\alpha(X_n)$-index for $\mathcal{A}^c_n$. But then $\langle 1, G(X_n,a)\rangle$ is a $\Pi^0_\alpha(X_n)$-index for $\mathcal{A}_n$ and by definition of G we have that $G(X,\mathrm{succ}(a))$ returns a $\Sigma^0_{\alpha+1}(X)$-index for $\mathcal{A}$ where $X = \oplus_{n\in\mathbb{N}} X_n$.

Limit step:

Consider now α limit, suppose that 1/, 2/ and 3/ are true for every ordinal $\beta < \alpha$, and let us prove that 1/, 2/ and 3/ are true for α. First to prove 1/, consider any X, any $a \in \mathcal{O}_{=\alpha}$, the c.e. sequence $W_{q(a)} = \{b_i\}_{i\in\mathbb{N}}$ and $\{c_i\}_{i\in\mathbb{N}}$, the subsequence, c.e. in X_0, as defined above. By induction hypothesis we have for each i that $G(X_i, c_i)$ returns a $\Sigma^0_{|c_i|_o}(X_i)$-index. Then $\langle 1, G(X_i, c_i)\rangle$ is a $\Pi^0_{|c_i|_o}(X_i)$-index, and by definition of G we have that $G(X,a)$ is a $\Sigma^0_\alpha(X)$-index.

Let us now prove 2/. Fix any computable function f. We have that $\{X : X \in G(f(X), a)\}$ is equal to the union over all strings σ and every code $c_i \in W_{q(a)}$ that are selected by σ, of the sets $[\sigma] \cap \{X : X \notin G(f(X)_{i+1}, c_i)\}$, with $f(X) = \oplus_i f(X)_i$. By induction hypothesis we have for each i that $\{X : X \notin G(f(X)_{i+1}, c_i)\}$ is $\Pi^0_{|c_i|_o}$ (using the computable function g which associates $f(X)_{i+1}$ to X). Thus $\{X : X \in G(f(X), a)\}$ is a Σ^0_α set and we have 2/.

To prove 3/, consider any $\boldsymbol{\Sigma^0_\alpha}$ set $\mathcal{A}$. By definition we have that $\mathcal{A} = \bigcup_n \mathcal{A}_n$ with each $\mathcal{A}_n$ a $\boldsymbol{\Pi^0_{\beta_n}}$ sets with $\beta_n < \alpha$ for each n. By induction hypothesis we have a sequence $\{X_n\}_{1\le n\in\mathbb{N}}$ and a sequence $\{a_n\}_{1\le n\in\mathbb{N}}$ with $a_n \in \mathcal{O}_{=\beta_n}$, such that $G(X_{n+1}, a_{n+1})$ is a $\Sigma^0_{\beta_{n+1}}(X_{n+1})$-index for $\mathcal{A}^c_n$. But then $\langle 1, G(X_{n+1}, a_{n+1})\rangle$ is a $\Pi^0_{\beta_{n+1}}(X_{n+1})$-index for $\mathcal{A}_n$. We can now take any code $a \in \mathcal{O}_{=\alpha}$ and use the reserved space X_0 to select codes for the ordinals β_i in the c.e. sequence $W_{q(a)}$. We then have by definition of G that $G(X,a)$ returns a $\Sigma^0_\alpha(X)$-index for $\mathcal{A}$, where $X = \oplus_{n\in\mathbb{N}} X_n$.

It is now easy to prove that for each α and any $a \in \mathcal{O}_{=\alpha}$, the set $\{X : X \in G(X,a)\}$ is not $\boldsymbol{\Pi^0_\alpha}$ (but is Σ^0_α, as already proved). The proof works exactly as for the Σ^0_1 case. ∎

Corollary 1.5.1:
For every $X \in 2^{\mathbb{N}}$ and every $\alpha < \omega_1^X$:

1. *There is a $\Pi^0_\alpha(X)$ set which is not $\boldsymbol{\Sigma^0_\alpha}$ (and a $\Sigma^0_\alpha(X)$ set which is not $\boldsymbol{\Pi^0_\alpha}$).*
2. *There is a $\Delta^0_\alpha(X)$ set which is neither $\boldsymbol{\Sigma^0_{<\alpha}}$ nor $\boldsymbol{\Pi^0_{<\alpha}}$.*

Proof: The first item is a direct consequence of the previous theorem. We prove the second one without any oracle. It is then easy to see that the proof can be relativized.

Consider first $\alpha = \beta + 1$, and a Σ^0_β set $\mathcal{A}$ which is not $\boldsymbol{\Pi^0_\beta}$. We define $\mathcal{B}_0 = \{X : X = 0\hat{\ }Y \wedge Y \in \mathcal{A}\}$ and $\mathcal{B}_1 = \{X : X = 1\hat{\ }Y \wedge Y \notin \mathcal{A}\}$. The function $f : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ which on $i\hat{\ }X$ returns X is computable and then $\mathcal{B}_0 = f^{-1}(\mathcal{A}) \cap \{X : X(0) = 0\}$ is Σ^0_β (by the computable substitution closure property, followed with the finite intersection closure property). Also it is easily seen not to be $\boldsymbol{\Pi^0_\beta}$, as otherwise $\mathcal{A} = g^{-1}(\mathcal{B}_0)$, where $g : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ is the computable function which on X returns $0\hat{\ }X$, would be $\boldsymbol{\Pi^0_\beta}$.

Symmetrically the set $\mathcal{B}_1$ is Π^0_β and not $\boldsymbol{\Sigma^0_\beta}$. Therefore $\mathcal{B}_0 \cup \mathcal{B}_1$ is a Δ^0_α set, as both $\mathcal{B}_0$ and $\mathcal{B}_1$ are $\Sigma^0_{\beta+1}$ and $\Pi^0_{\beta+1}$. Also it is clear that if this set were $\boldsymbol{\Pi^0_\beta}$ then also $\mathcal{B}_0$ would be $\boldsymbol{\Pi^0_\beta}$. Symmetrically if it was $\boldsymbol{\Sigma^0_\beta}$ then also $\mathcal{B}_1$ would be $\boldsymbol{\Sigma^0_\beta}$.

For α limit, consider a code $a \in \mathcal{O}_{=\alpha}$ with $a = \sup_n b_n$. Let $\beta_n = |b_n|_o$. In the proof of Theorem 1.5.1 we saw that uniformly in b_n, we can define a $\Sigma^0_{\beta_n}$-index for a set $\mathcal{B}_n$ which is not $\boldsymbol{\Pi^0_{\beta_n}}$. We define the set:

$$\mathcal{B} = \{X \ : \ \exists n \ X = 0^n \hat{} 1 \hat{} Y \wedge Y \in \mathcal{B}_n\}$$

Just like before we can easily prove that $\mathcal{B}$ is a Σ^0_α set which is neither $\boldsymbol{\Sigma^0_{<\alpha}}$ nor $\boldsymbol{\Pi^0_{<\alpha}}$. But the set $2^{\mathbb{N}} - \mathcal{B}$ also has a Σ^0_α-index, because every $X \neq 0^\infty$ is actually of the form $0^n \hat{} 1 \hat{} Y$ for some n and some Y, and because $\{0^\infty\}$ is Π^0_1:

$$2^{\mathbb{N}} - \mathcal{B} = \{X \ : \ \exists n \ X = 0^n \hat{} 1 \hat{} Y \wedge Y \notin \mathcal{B}_n\} \cup \{0^\infty\}$$

and then $\mathcal{B}$ is Δ^0_α, and neither $\boldsymbol{\Sigma^0_{<\alpha}}$ nor $\boldsymbol{\Pi^0_{<\alpha}}$. ∎

1.5.4 Effectively closed and open sets

We will often deal in this thesis with open or closed sets which have some degree of definability. Generally the open sets we deal with are merely Σ^0_1 subsets of $2^{\mathbb{N}}$, and the closed sets Π^0_1 subsets of $2^{\mathbb{N}}$, but this is not always the case. Also we make the following definition:

Definition 1.5.4. *A Σ^0_α**-open set** (resp. Π^0_α**-open set**) is a an open set which can be described by a Σ^0_α (resp. Π^0_α) set of strings. A Π^0_α**-closed set** (resp. Σ^0_α**-closed set**) is a closed set whose complement is a Σ^0_α-open set (resp. Π^0_α-open set).* ∎

We give a proposition establishing a connection between Σ^0_α-open sets and the effective Borel hierarchy:

Proposition 1.5.1:
A Σ^0_α-open set $\mathcal{U}$ is also a Σ^0_α set uniformly in an index for $\mathcal{U}$. A Π^0_α-open set $\mathcal{U}$ is also a $\Sigma^0_{\alpha+1}$ set uniformly in an index for $\mathcal{U}$.

Proof: We actually prove the two following statements:

1. One can find uniformly in any string σ and any Σ^0_α set of strings U, an index for a Σ^0_α set $\mathcal{U}_\sigma$ which is equal to $[\sigma]$ if $\sigma \in U$ and equal to the empty set otherwise.
2. One can find uniformly in any string σ and any Π^0_α set of strings U, an index for a Π^0_α set $\mathcal{U}_\sigma$ which is equal to $[\sigma]$ is $\sigma \in U$ and equal to the empty set otherwise.

If U is a Σ^0_1 set of strings it is obvious. If U is a Π^0_1 set of strings we return the Π^0_1 set equal to $[\sigma]$ as long as σ is in $U[t]$, and equal to the empty set if σ gets out of U at some stage. Note that everything is uniform.

If U is a Σ^0_α set of strings $\bigcup_n U_n$ where each U_n is a $\Pi^0_{<\alpha}$ set of strings uniformly in α, by induction hypothesis, for each U_n we can find a $\Pi^0_{<\alpha}$ set $\mathcal{U}_{n,\sigma}$ uniformly in n and σ such that $\sigma \in U_n$ implies $\mathcal{U}_{n,\sigma} = [\sigma]$ and $\sigma \notin U_n$ implies $\mathcal{U}_{n,\sigma} = \varnothing$. Therefore $\bigcup_n \mathcal{U}_{n,\sigma}$ is a Σ^0_α set such that $\sigma \in U_n$ implies $\mathcal{U}_{n,\sigma} = [\sigma]$ and $\sigma \notin U_n$ implies $\mathcal{U}_{n,\sigma} = \varnothing$.

If U is a Π^0_α set of strings $\bigcap_n U_n$ where each U_n is a $\Sigma^0_{<\alpha}$ set of strings uniformly in α, by induction hypothesis, for each U_n we can find a $\Sigma^0_{<\alpha}$ set $\mathcal{U}_{n,\sigma}$ uniformly in n and σ such that $\sigma \in U_n$ implies $\mathcal{U}_{n,\sigma} = [\sigma]$ and $\sigma \notin U_n$ implies $\mathcal{U}_{n,\sigma} = \varnothing$. Therefore $\bigcap_n \mathcal{U}_{n,\sigma}$ is a Π^0_α set such that $\sigma \in U_n$ implies $\mathcal{U}_{n,\sigma} = [\sigma]$ and $\sigma \notin U_n$ implies $\mathcal{U}_{n,\sigma} = \varnothing$.

Now given any Σ^0_α set of strings U, using (1) it is clear that the union of $\mathcal{U}_\sigma$ over σ is a Σ^0_α set equal to $[U]^\prec$, and given any Π^0_α set of strings U, using (2) it is clear that the union of $\mathcal{U}_\sigma$ over σ is a $\Sigma^0_{\alpha+1}$ set equal to $[U]^\prec$. ■

We shall now see a small proposition which will be useful for Proposition 4.5.1:

Proposition 1.5.2:
If a sequence X is the only element of a Π^0_1 subset of $2^\mathbb{N}$, then X is Turing computable.

PROOF: Given a Π^0_1 subset of $2^\mathbb{N}$ that contains only one element X, we can enumerate a set of string W describing its complement. Then by compactness, for any n there is necessarily a stage at which the finite set of strings $W[s]$ covers every string of length n but one, which is then necessarily a prefix of X. ■

We will see an analogue of this proposition for the Baire space with Example 3.4.1.

1.6 Effective complexity of sets of integers

1.6.1 Definition and closure properties

In this section we generalize Definition 1.2.3 for Σ^0_n sets of integers, where n can now be a countable ordinal. The definition is similar to the one of effectively Borel sets of the Baire space:

Definition 1.6.1. *The effective Kleene's hierarchy is defined by induction over the ordinals as follows:*

- *A Σ^0_1-index is given by a pair $\langle 0, e\rangle$. The set A corresponding to $\langle 0, e\rangle$ is given by $A = W_e$.*
- *A Π^0_α-index is given by a pair $\langle 1, e\rangle$ where e is a Σ^0_α-index. The set A corresponding to $\langle 1, e\rangle$ is given by $A = \mathbb{N} - B$ where B is the set corresponding to e.*
- *A Σ^0_α-index is given by a pair $\langle 2, e\rangle$ where W_e is not empty and enumerates only $\Pi^0_{\beta_n}$-indices for $\beta_n < \alpha$, with $\sup^+_n(\beta_n) = \alpha$. The set A corresponding to $\langle 2, e\rangle$ is given by $\bigcup_n A_n$, where A_n is the set corresponding to the n-th index enumerated by W_e.*

We say that a set A is Σ^0_α (resp. Π^0_α) if for some Σ^0_α-index (resp. Π^0_α-index) e, A is the set corresponding to e. We say that a set A is Δ^0_α if it is both Σ^0_α and Π^0_α. Finally we say that a set is $\Sigma^0_{<\alpha}$ (resp. $\Pi^0_{<\alpha}$) if it is Σ^0_β (resp. Π^0_β) for some $\beta < \alpha$. ■

As for the effective Borel hierarchy, it is clear that no new set is added at step ω_1^{ck}. We saw in Section 1.5.2 some closure properties for the effective Borel hierarchy and we argued that they apply also to the effective Kleene's hierarchy. We now state one of their consequences, that will be used a lot in this thesis without explicit reference. We consider the language of arithmetic, with a range of *element* variables $v_1, v_2, \dots$ and a range of *set* variables $V_1, V_2, \dots$. We also add the binary symbol $\in$ which can only be used between elements and sets in logical formulas.

Proposition 1.6.1:
Let $\Psi(v, V)$ be a Σ^0_1 formula of arithmetic which contains no instance of $\neg(x \in V)$ in Ψ for x any variable. Then for any Σ^0_α set $X \subseteq \mathbb{N}$, the set $\{n \in \mathbb{N} \mid \mathbb{N} \models \Psi(n, X)\}$ is also a Σ^0_α set.

In particular, if such a formula contains both instances of $\neg(x \in V)$ and $x \in V$, we can always consider the disjoint union of X with its complement and modify Ψ into a formula Ψ' that contains no instance of $\neg(x \in V)$, and such that $\{n \in \mathbb{N} \mid \mathbb{N} \models \Psi(n, X)\} = \{n \in \mathbb{N} \mid \mathbb{N} \models \Psi'(n, X \oplus X^c)\}$. As for X a Σ^0_α set, the set $X \oplus X^c$ is $\Sigma^0_{\alpha+1}$, the set defined this way is also a $\Sigma^0_{\alpha+1}$ set. We can use this and the previous proposition to prove that Π^0_n such formulas for $n \geq 1$ gives us $\Pi^0_{\alpha+n}$ sets for X a Σ^0_α set and Σ^0_n such formulas for $n \geq 2$ give us $\Sigma^0_{\alpha+n}$ sets for X a Σ^0_α set.

We don't give a proof of Proposition 1.6.1 and the reader can refer to [65] to see how this works. We however give an example of how to build a $\Sigma^0_{\alpha+1}$-index for $A' = \{e \ : \ e \in W^A_e\}$ assuming A has a Σ^0_α-index. This shows what kind of techniques we would need to prove Proposition 1.6.1:

Example 1.6.1:
We have:

$$n \in A' \text{ iff } \exists \sigma \ (\forall i < |\sigma| \ (\sigma(i) = 0 \wedge i \notin A) \vee (\sigma(i) = 1 \wedge i \in A)) \wedge n \in W^\sigma_n$$

As A has a Σ^0_α-index, then also for any σ the set $\{i < |\sigma| : \ \sigma(i) = 1 \wedge i \in A\}$ has a Σ^0_α-index uniformly in σ: It is the index corresponding to the intersection of $\{i < |\sigma| : \ \sigma(i) = 1\}$ with A. Similarly the set $\{i < |\sigma| : \ \sigma(i) = 0 \wedge i \notin A\}$ has a Π^0_α-index uniformly in σ. It follows that the set:

$$B_\sigma = \{i \ : \ (\sigma(i) = 0 \wedge i \notin A) \vee (\sigma(i) = 1 \wedge i \in A)\}$$

has a $\Sigma^0_{\alpha+1}$-index uniformly in σ. Now uniformly in σ and in $i < |\sigma|$ we can define the computable function $f_i : \mathbb{N} \to \mathbb{N}$ which on any n returns i. Also for any $i < n$ the set $f_i^{-1}(B_\sigma)$ has a $\Sigma^0_{\alpha+1}$-index (by the computable substitution closure property). Also if $i \in B_\sigma$ the set $f_i^{-1}(B_\sigma) = \mathbb{N}$ and if $i \notin B_\sigma$ the set $f_i^{-1}(B_\sigma) = \varnothing$. It follows that the set

$$C_\sigma = \bigcap_{i<|\sigma|} f_i^{-1}(B_\sigma)$$

is equal to $\mathbb{N}$ if $\forall i < |\sigma| \ (\sigma(i) = 0 \wedge i \notin A) \vee (\sigma(i) = 1 \wedge i \in A)$ and is equal to $\varnothing$ otherwise. Also by the finite intersection closure property, it has a $\Sigma^0_{\alpha+1}$-index uniformly in σ.

We can now intersect C_σ with the set $J_\sigma = \{n \ : \ n \in W^\sigma_n\}$ which has a Σ^0_1-index uniformly in σ. We then have that $\bigcup_\sigma C_\sigma \cap J_\sigma$ has a $\Sigma^0_{\alpha+1}$-index, by the effective countable union closure property. We easily verify that this set is equal to A'. $\diamond$

1.6.2 The Kleene hierarchy and the computable trees

As in Theorem 1.5.1, which says that the Borel hierarchy is strict, we now should show that the Kleene hierarchy is strict. Recall what we did in the proof of Theorem 1.5.1. What would be an analogous diagonal argument for the Kleene hierarchy? A natural candidate for a Σ^0_α set which is not Π^0_α could be:

$$\{e \;:\; e \text{ is a } \Sigma^0_{\leq\alpha}\text{-index of corresponding set } P \text{ with } e \in P\}$$

Unfortunately, we will show in Section 1.6.5 that for some ordinal α, the set of $\Sigma^0_{\leq\alpha}$-indices is not itself a Σ^0_α set. We will see for example that for $\alpha = \omega^\omega$, the set of corresponding indices is $\Pi^0_{\omega^\omega+1}$ but not $\Sigma^0_{\omega^\omega+1}$. We now start a detailed analysis of this, by first giving the precise complexity of various sets of codes for c.e. well-founded trees.

Proposition 1.6.2:
For any ordinal $\alpha = 0$ or α limit and for any $k, p \in \omega$ we have:

- *The set $\mathcal{T}_{<\omega(\alpha+k)}$ is $\Sigma^0_{\alpha+2k}$ uniformly in k and in any code of $\mathcal{O}_{=\alpha}$.*
- *The set $\mathcal{T}_{\leq\omega(\alpha+k)+p}$ is $\Pi^0_{\alpha+2k+1}$ uniformly in k, p and in any code of $\mathcal{O}_{=\alpha}$.*

Proof: For any $p \in \omega$, the set $\mathcal{T}_{\leq p}$ is Π^0_1 uniformly in p, because $\mathcal{T}_{\leq p}$ is the set of codes which enumerates trees of height less than p, which is a Π^0_1 condition, uniformly in p.

Suppose that for $\alpha = 0$ or α limit, for some $k \in \omega$ and for any $p \in \omega$ we have that $\mathcal{T}_{\leq\omega(\alpha+k)+p}$ is $\Pi^0_{\alpha+2k+1}$ uniformly in p, in k and in any code of $\mathcal{O}_{=\alpha}$. Then it is clear by induction hypothesis that $\mathcal{T}_{<\omega(\alpha+k+1)}$ is a $\Sigma^0_{\alpha+2(k+1)}$ set, uniformly in $k+1$ and in any code of $\mathcal{O}_{=\alpha}$, as $\mathcal{T}_{<\omega(\alpha+k+1)} = \bigcup_p \mathcal{T}_{\leq\omega(\alpha+k)+p}$.

Suppose that for $\alpha = 0$ or α limit, for every $k \in \omega$, we have that $\mathcal{T}_{\leq\omega(\alpha+k)}$ is $\Pi^0_{\alpha+2k+1}$ uniformly in k and in any code of $\mathcal{O}_{=\alpha}$. Then consider any code of $a \in \mathcal{O}_{=\alpha+\omega}$ with $|a|_o = \sup_n \alpha_n$. One can uniformly obtain for each α_n some codes for ordinal β_n and $k_n \in \omega$ so that $\beta_n = 0$ or β_n limit, and so that $\alpha_n = \beta_n + k_n$. As we have $\mathcal{T}_{<\omega(\alpha+\omega)} = \bigcup_k \mathcal{T}_{\leq\omega(\beta_n+k_n)}$, we have that $\mathcal{T}_{<\omega(\alpha+\omega)}$ is a $\Sigma^0_{\alpha+\omega}$ set, uniformly in a code of $\mathcal{O}_{\alpha+\omega}$. We can conduct a similar induction to prove that $\mathcal{T}_{<\omega(\alpha)}$ is a Σ^0_α set uniformly in a code of $\mathcal{O}_{=\alpha}$, for α a limit of limit ordinals.

Suppose now that for $\alpha = 0$ or α limit, for $k \in \omega$ we have that $\mathcal{T}_{<\omega(\alpha+k)}$ is $\Sigma^0_{\alpha+2k}$ uniformly in k and in any code of $\mathcal{O}_{=\alpha}$. For $p = 0$ it is clear that $\mathcal{T}_{\leq\omega(\alpha+k)+p}$ is a $\Pi^0_{\alpha+2k+1}$ set, because $\mathcal{T}_{\leq\omega(\alpha+k)+p}$ is the set of codes for c.e. trees T so that for every node n enumerated in T, a code for $T\upharpoonleft_n$ belongs to $\mathcal{T}_{<\omega(\alpha+k)}$, which is by induction hypothesis a $\Pi^0_{\alpha+2k+1}$ condition, uniformly in p, k and in any code of $\mathcal{O}_{=\alpha}$. Then we can iterate to $p+1$ and say that $\mathcal{T}_{\leq\omega(\alpha+k)+p+1}$ is the set of codes for c.e. trees T so that for every node n enumerated in T, a code for $T\upharpoonleft_n$ belongs to $\mathcal{T}_{\leq\omega(\alpha+k+p)}$, which is by induction hypothesis a $\Pi^0_{\alpha+2k+1}$ condition uniformly in p, k and in any code of $\mathcal{O}_{=\alpha}$. ■

In practice we will often use rougher bounds, that cannot be improved in the general case, as we will argue in Section 1.6.5:

Porism 1.6.1:
For any computable ordinal α we have:

1. *The set $\mathcal{T}_{<\alpha}$ is $\Sigma^0_{\alpha+1}$ uniformly in any code of $\mathcal{O}_{=\alpha}$.*
2. *The set $\mathcal{T}_{\leq\alpha}$ is $\Pi^0_{\alpha+1}$ uniformly in any code of $\mathcal{O}_{=\alpha}$.*
3. *The set $\mathcal{O}_{<\alpha}$ is $\Sigma^0_{\alpha+1}$ uniformly in any code of $\mathcal{O}_{=\alpha}$.*
4. *The set $\mathcal{O}_{\leq\alpha}$ is $\Pi^0_{\alpha+1}$ uniformly in any code of $\mathcal{O}_{=\alpha}$.*

PROOF: Without uniformity, (1) and (2) are direct consequences of the previous proposition. To get uniformity in (1) and (2) we can simply perform the same proof but with rougher bounds.

To get (3) and (4) (formally proved by induction by induction), we have more things to verify. It is Π^0_1 to check that a node σ tagged as a leaf is really a leaf, Δ^0_2 to check that a node σ tagged to have exactly one child really has exactly one child. Also it is Π^0_2 to check if a node σ tagged to have countably many children, really has countably many children. In this last case, if $\{\sigma_i\}_{i\in\mathbb{N}}$ is an enumeration of the children of σ, we shall also check that $|\sigma_i|_o < |\sigma_{i+1}|_o$. Let a_i be an index for the tree $T\upharpoonright_{\sigma_i}$. We can simply ask for each a_i to be a member of $\mathcal{O}_{<|a_{i+1}|_o}$, which is a Π^0_1 condition over sets which are all simple enough to keep the proposition true. ∎

The sets of the form $\mathcal{T}_{<\omega(\alpha+k)}$ and $\mathcal{T}_{\leq\omega(\alpha+k)}$ for α limit or 0 and $k\in\omega$ will be used a lot in this section. We will prove that there are universal sets, at every level of the hierarchy, in a sense that will be made precise. So for each α we create the set $\emptyset^{(\alpha)}$ in such a way that it should be a Σ^0_α set, according to Proposition 1.6.2.

Definition 1.6.2. *In the following, β is 0 or limit, and $k\in\omega$:*

- *For $\alpha=\beta+2k$ we define $\emptyset^{(\alpha)}$ to be $\mathcal{T}_{<\omega(\beta+k)}$.*
- *For $\alpha=\beta+2k+1$ we define $\emptyset^{(\alpha)}$ to be $\mathbb{N}-\mathcal{T}_{\leq\omega(\beta+k)}$.*

Also for any $X\in 2^\omega$:

- *For $\alpha=\beta+2k$ we define $X^{(\alpha)}$ to be $\mathcal{T}^X_{<\omega(\beta+k)}$.*
- *For $\alpha=\beta+2k+1$ we define $X^{(\alpha)}$ to be $\mathbb{N}-\mathcal{T}^X_{\leq\omega(\beta+k)}$.* ∎

For α a limit ordinal, we also need to define sets which are universal for all $\Sigma^0_{<\alpha}$ sets. Unfortunately we don't have here a coding-independent definition, but in practice this will not matter.

Definition 1.6.3. *For any ordinal α, we define $\emptyset^{(<\alpha)}$ to be $\emptyset^{(\beta)}$ if $\alpha=\beta+1$. If α is limit, the set $\emptyset^{(<\alpha)}$ is defined up to a coding $a\in\mathcal{O}_{=\alpha}$. For a given such code with $a=\sup_n b_n$, we can suppose with loss of generality that $b_n=c_n+2k$ for c_n limit or 0 and $k\in\omega$. Let $\beta_n=|b_n|_o$. We then define $\emptyset^{(<\alpha)}$ with respect to the coding a to be $\oplus_n\emptyset^{(\beta_n)}$. For $X\in 2^\omega$, we define $X^{(<\alpha)}$ similarly.* ∎

In practice every use we will make of $\emptyset^{(<\alpha)}$ for α limit will be independent of the corresponding code of $a \in \mathcal{O}_{=\alpha}$. So we will make a slight abuse of notation and only write $\emptyset^{(<\alpha)}$ without specifying which code the set corresponds to.

Also we emphasize that for $\emptyset^{(<\alpha)} = \oplus_n \emptyset^{(\beta_n)}$ we have $\emptyset^{(\beta_n)} \subseteq \emptyset^{(\beta_{n+1})}$ for every n, because each β_n is equal to $\gamma_n + 2k$ for $\gamma_n = 0$ or γ_n limit, and $k \in \omega$; and then each $\emptyset^{(\beta_n)}$ is equal to $\mathcal{T}_{<\omega(\gamma_n+k)}$. However the set $\emptyset^{(<\alpha)}$ is very different from the set $\emptyset^{(\alpha)}$, because $\emptyset^{(<\alpha)}$ is a disjoint union of the sets $\mathcal{T}_{<\omega(\gamma_n+k)}$, whereas $\emptyset^{(\alpha)}$ is a non disjoint union of those sets. This will be clear with the next proposition, together with the later proof that $\emptyset^{(\alpha)}$ is not a Δ^0_α set.

Proposition 1.6.3:
For each α, the set $\emptyset^{(<\alpha)}$ is Δ^0_α.

Proof: If $\alpha = \beta + 1$, the set $\emptyset^{(\beta)}$ is certainly $\Delta^0_{\beta+1}$. For α limit with $\alpha = \sup_n \beta_n$, the set $\emptyset^{(<\alpha)} = \oplus_n \emptyset^{(\beta_n)}$ is a uniform union of $\Pi^0_{\beta_n+1}$ sets, but as the union is disjoint, the set $\mathbb{N} - \emptyset^{(<\alpha)} = \oplus_n \mathbb{N} - \emptyset^{(\beta_n)}$ is also a uniform union of $\Pi^0_{\beta_n+1}$ sets. Therefore $\emptyset^{(<\alpha)}$ is a Δ^0_α set.■

1.6.3 Complete sets and many-one reductions

We shall now prove that each set $\emptyset^{(\alpha)}$ is a universal Σ^0_α set. To do so we introduce the well-known notion of α-completeness. Informally, a Σ^0_α set if α-complete if it is powerful enough to "know" in a strong way every other Σ^0_α set:

Definition 1.6.4. *A subset $B \subseteq \mathbb{N}$ is Σ^0_α-complete or α-complete, if it is Σ^0_α, and if for each Σ^0_α set A, we have that A is many-one reducible to B, that is, there exists a total computable function f depending on A so that $n \in A \leftrightarrow f(n) \in B$.* ■

The notion of many-one reduction, and then the notion of completeness, are strongly linked to the effective complexity of sets. It is easily seen that for two sets A, B, if $A \leq_m B$, then B is at least as complex as A. Indeed, the set A is then equal to $f^{-1}(B)$, and we can then easily transform any Σ^0_α-index (resp. Π^0_α-index) for B into a Σ^0_α-index (resp. Π^0_α-index) for A.

We shall now prove that each set $\emptyset^{(\alpha)}$ is Σ^0_α-complete in a strong sense:

Theorem 1.6.1:
There exists a computable function $f : \mathbb{N} \times \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ such that for any computable ordinal α, any $a \in \mathcal{O}_{=\alpha}$, any $\beta \leq \alpha$ and any Σ^0_β-index e of a set S_e, or any $\beta < \alpha$ and any Π^0_β-index e of a set S_e, the function $n \mapsto f(a, e, n)$ is total, and for any n we have $n \in S_e$ iff $f(a, e, n) \in \emptyset^{(\alpha)}$.

PROOF: We will actually prove that there exists a computable function $f_1 : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ so that for any ordinal $\alpha = \varnothing$ or α limit and for any $k \in \omega$ we have:

- For any $\Sigma^0_{\alpha+2k}$-index e of a set S_e, the function $n \to f_1(e,n)$ is total and for any n, we have $n \in S_e$ iff $f_1(e,n) \in \mathcal{T}_{<\omega(\alpha+k)} = \emptyset^{(\alpha+2k)}$.
- For any $\Sigma^0_{\alpha+2k+1}$-index e of a set S_e, the function $n \to f_1(e,n)$ is total and for any n, we have $n \in S_e$ iff $f_1(e,n) \in \mathcal{T}_{<\omega(\alpha+k+1)} = \emptyset^{(\alpha+2k+2)}$.
- For any $\Pi^0_{\alpha+2k}$-index e of a set S_e, the function $n \to f_1(e,n)$ is total and for any n, we have $n \in S_e$ iff $f_1(e,n) \in \mathcal{T}_{\leq\omega(\alpha+k)} = \mathbb{N} - \emptyset^{(\alpha+2k+1)}$.
- For any $\Pi^0_{\alpha+2k+1}$-index e of a set S_e, the function $n \to f_1(e,n)$ is total and for any n, we have $n \in S_e$ iff $f_1(e,n) \in \mathcal{T}_{\leq\omega(\alpha+k)} = \mathbb{N} - \emptyset^{(\alpha+2k+1)}$.

and a computable function $f_2 : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ so that for any ordinal $\alpha = \varnothing$ or α limit and for any $k \in \omega$ we have:

- For any $\Sigma^0_{\alpha+2k}$-index e of a set S_e, the function $n \to f_2(e,n)$ is total and for any n, we have $n \in S_e$ iff $f_2(e,n) \in \mathbb{N} - \mathcal{T}_{\leq\omega(\alpha+k)} = \emptyset^{(\alpha+2k+1)}$.
- For any $\Sigma^0_{\alpha+2k+1}$-index e of a set S_e, the function $n \to f_2(e,n)$ is total and for any n, we have $n \in S_e$ iff $f_2(e,n) \in \mathbb{N} - \mathcal{T}_{\leq\omega(\alpha+k)} = \emptyset^{(\alpha+2k+1)}$.
- For any $\Pi^0_{\alpha+2k}$-index e of a set S_e, the function $n \to f_2(e,n)$ is total and for any n, we have $n \in S_e$ iff $f_2(e,n) \in \mathbb{N} - \mathcal{T}_{<\omega(\alpha+k)} = \mathbb{N} - \emptyset^{(\alpha+2k)}$.
- For any $\Pi^0_{\alpha+2k+1}$-index e of a set S_e, the function $n \to f_2(e,n)$ is total and for any n, we have $n \in S_e$ iff $f_2(e,n) \in \mathbb{N} - \mathcal{T}_{<\omega(\alpha+k+1)} = \mathbb{N} - \emptyset^{(\alpha+2k+2)}$.

The function $f(a,e,n)$ then returns $f_1(e,n)$ if a codes for an ordinal of the form $\alpha+2k$ and returns $f_2(e,n)$ otherwise. We only prove the existence of the function f_1, the proof of the existence of the function f_2 being similar.

Let us first note that according to the definition of Σ^0_α sets, if $\langle 2,e\rangle$ is a Σ^0_α-index, W_e needs to be non-empty, but not necessarily infinite. However, to do this proof, due to technical reasons related to the function OR of Lemma 1.4.2, we need such sets W_e to be infinite. This can always be achieved by adding to the enumeration of W_e infinitely many indices coding for the empty set. We can continue to inductively transform each index $\langle 2, W_a\rangle$, resulting from a previous enumeration, by adding to W_a infinitely many indices for the empty set, or infinitely many indices for $2^{\mathbb{N}}$, depending on whether $\langle 2, W_a\rangle$ corresponds to a union or an intersection of the set whose indices are enumerated by W_a.

The function f_1 is defined on its first parameter by induction over indices of effective Borel sets. First if e is the code of the Π^0_1 set $W_a^c \subseteq \mathbb{N}$, then $f_1(e,n)$ returns the code for a c.e. tree which is empty as long as $n \notin W_a$ and which becomes ill-founded if n is witnessed to be in W_a at some point.

If e is the code of the Σ^0_1 set $W_a \subseteq \mathbb{N}$, then $f_1(e,n)$ returns the value of $f_1(e',n)$, where e' is a Σ^0_2-index describing the same set that e describes.

If e is a code for the Σ^0_α set $\bigcup_i F_{a_i}$ where each F_{a_i} is a $\Pi^0_{<\alpha}$ set of code a_i, the function $f_1(e,n)$ returns the value of the function OR of Lemma 1.4.2, applied to the computable sequence of trees $\{f_1(a_i,n)\}_{i\in\mathbb{N}}$.

If e is a code for the Π^0_α set $\bigcap_i F_{a_i}$ where each F_{a_i} is a $\Sigma^0_{<\alpha}$ set of code a_i, the function $f_1(e,n)$ returns the value of the function AND of Lemma 1.4.4, applied to the computable sequence of trees $\{f_1(a_i,n)\}_{i\in\omega}$.

We now verify by induction that such a function f_1 satisfies the proposition. Let us start the induction with $\alpha = 0$ and $k = 1$. Consider W^c_a a Π^0_1 set. We clearly have that $n \in W^c_a$ implies $f_1(e,n) \in \mathcal{T}_{\leq\emptyset}$ and $n \notin W^c_a$ implies $f_1(e,n) \notin \mathcal{T}$. Consider now W_a a Σ^0_1 set. The function f_1 returns a recursive call on an index for an equivalent Σ^0_2 set. So this case will be handled in the case $\alpha = 0, k = 2$, that we deal with now.

Suppose now that the theorem is true up to $\alpha + 2k + 1$ and let us show that it is true for $\alpha + 2k + 2$. Consider a $\Sigma^0_{\alpha+2k+2}$-index e coding for $S_e = \bigcup_i F_{a_i}$ where a_i are $\Pi^0_{\beta_i}$-indices with $\sup_i(\beta_i + 1) = \alpha + 2k + 2$. By induction hypothesis, for each a_i we have $n \in F_{a_i}$ implies $f_1(a_i,n) \in \mathcal{T}_{\leq\omega(\alpha+k)}$ and $n \notin F_{a_i}$ implies $f_1(a_i,n) \notin \mathcal{T}$. Therefore by the properties of the function OR of Lemma 1.4.2, we have that $n \in \bigcup_i F_{a_i}$ implies $f_1(e,n) \in \mathcal{T}_{<\omega(\alpha+k+1)}$ and $n \notin \bigcup_i F_{a_i}$ implies $f_1(e,n) \notin \mathcal{T}$.

Consider a $\Pi^0_{\alpha+2k+2}$-index e coding for $S_e = \bigcap_i F_{a_i}$ where a_i are $\Sigma^0_{\beta_i}$-indices with $\sup_i(\beta_i + 1) = \alpha + 2k + 2$. By induction hypothesis, for each a_i we have $n \in F_{a_i}$ implies $f_1(a_i,n) \in \mathcal{T}_{<\omega(\alpha+k+1)}$ and $n \notin F_{a_i}$ implies $f_1(a_i,n) \notin \mathcal{T}$. Therefore by the properties of the function AND of Lemma 1.4.4, we have that $n \in \bigcap_i F_{a_i}$ implies $f_1(e,n) \in \mathcal{T}_{\leq\omega(\alpha+k+1)}$ and $n \notin \bigcap_i F_{a_i}$ implies $f_1(e,n) \notin \mathcal{T}$.

Suppose now that the theorem is true up to ordinals smaller than $\alpha = \sup_i \alpha_i$ and let us show it is true for α. Consider a Σ^0_α-index e coding for $S_e = \bigcup_i F_{a_i}$ where a_i are $\Pi^0_{\beta_i}$-indices with $\sup_i(\beta_i + 1) = \alpha$. By induction hypothesis, for each a_i we have $n \in F_{a_i}$ implies $f_1(a_i,n) \in \mathcal{T}_{\leq\omega(\beta_i)}$ and $n \notin F_{a_i}$ implies $f_1(a_i,n) \notin \mathcal{T}$. Therefore by the properties of the function OR of Lemma 1.4.2, we have that $n \in \bigcup_i F_{a_i}$ implies $f_1(e,n) \in \mathcal{T}_{<\omega(\alpha)}$ and $n \notin \bigcup_i F_{a_i}$ implies $f_1(e,n) \notin \mathcal{T}$.

Consider a Π^0_α-index e coding for $S_e = \bigcap_i F_{a_i}$ where a_i are $\Sigma^0_{\beta_i}$-indices with $\sup_i(\beta_i + 1) = \alpha$. By induction hypothesis, for each a_i we have $n \in F_{a_i}$ implies $f_1(a_i,n) \in \mathcal{T}_{<\omega(\alpha)}$ and $n \notin F_{a_i}$ implies $f_1(a_i,n) \notin \mathcal{T}$. Therefore by the properties of the function AND of Lemma 1.4.4, we have that $n \in \bigcap_i F_{a_i}$ implies $f_1(e,n) \in \mathcal{T}_{\leq\omega(\alpha)}$ and $n \notin \bigcap_i F_{a_i}$ implies $f_1(e,n) \notin \mathcal{T}$. ∎

Corollary 1.6.1:
For each α, any Σ^0_α set A is $\Sigma^0_1(\emptyset^{(<\alpha)})$, uniformly in an index for A and a code of $\mathcal{O}_{=\alpha}$.

Proof: We have to decompose into two cases, the first one, when $\alpha = \beta + 1$ and the second one, when α is limit. Suppose first $\alpha = \beta + 1$. The set A is a union of Π^0_β sets $\bigcup_n A_n$. Also using Theorem 1.6.1, each set A_n is many-one reducible to $\mathbb{N} - \emptyset^{(\beta)}$ uniformly in n and in a code of $\mathcal{O}_{=\beta}$. Therefore A is $\Sigma^0_1(\emptyset^{(\beta)})$.

Suppose now α is limit with $\alpha = \sup_m \beta_m$, $\emptyset^{(<\alpha)} = \oplus_m \emptyset^{(\beta_m)}$ and $\emptyset^{(\beta_m)} \subseteq \emptyset^{(\beta_{m+1})}$ for every m. The set $\mathcal{A}$ is a union of $\Pi^0_{<\alpha}$ sets $\bigcup_n \mathcal{A}_n$. Also using Theorem 1.6.1, for each n, the set A_n is many-one reducible to $\emptyset^{(\beta_m)}$ for m large enough, uniformly in n and in a code of $\mathcal{O}_{=\beta_m}$. But by the definition of $\emptyset^{(<\alpha)}$ and by the proof the previous theorem, there is actually a function $f_n : \mathbb{N} \to \mathbb{N}$ which reduces each A_n to $\emptyset^{(\beta_m)}$ for m large enough, and independently of any code of $\mathcal{O}_{=\beta_m}$ (the code is actually only useful to decide if we want ill-founded or well-founded tree, but each $\emptyset^{(\beta_m)}$ is a set of codes for well-founded trees).

As we have $\emptyset^{(\beta_m)} \subseteq \emptyset^{(\beta_{m+1})}$ for any m, it follows that $e \in A_n$ iff $\exists m\ f_n(e) \in \emptyset^{(\beta_m)}$, which is c.e. in $\emptyset^{(<\alpha)}$ uniformly in n. Then A is $\Sigma^0_1(\emptyset^{(<\alpha)})$. ■

We shall now see how the α-complete sets behave with Turing reductions. Shoenfield proved in [81] that a set is Δ^0_2 iff it is Turing reducible to $\emptyset'$. He also proved a lemma, known as the Shoenfield's limit lemma, which is very useful in computability theory. We will study in this thesis (in particular in Section 5.4) a lot of different possible counterpart of this lemma, for higher computability.

Theorem 1.6.2 (Shoenfield's limit lemma):
A set X is Δ^0_α iff as a sequence, X is Turing computable from $\emptyset^{(<\alpha)}$. In particular we have the three following statements are equivalent for a $X \in 2^{\mathbb{N}}$:

1. *As a set, X is Δ^0_2.*
2. *As a sequence, X is Turing computable from $\emptyset'$.*
3. *There is a computable sequence (of sequences) $\{X_s\}_{s\in\mathbb{N}}$ such that $X = \lim_{s\in\mathbb{N}} X_s$.*

Proof: Suppose a set X is Δ^0_α. Then it is Σ^0_α and therefore $\Sigma^0_1(\emptyset^{(<\alpha)})$. Also $\mathbb{N} - X$ is Σ^0_α and therefore $\Sigma^0_1(\emptyset^{(<\alpha)})$. It follows that X is Turing computable from $\emptyset^{(<\alpha)}$.

Now suppose that a sequence X is Turing computable from $\emptyset^{(<\alpha)}$. In particular, as a set, X is $\Sigma^0_1(\emptyset^{(<\alpha)})$ and then (using Proposition 1.6.1) it is Σ^0_α, as $\emptyset^{(<\alpha)}$ is Δ^0_α. Similarly, $\mathbb{N} - X$ is $\Sigma^0_1(\emptyset^{(<\alpha)})$ and then it is Σ^0_α. Therefore X is Δ^0_α.

We now prove that (1) $\leftrightarrow$ (2) $\leftrightarrow$ (3). We already proved (1) $\to$ (2). Let us prove (2) $\to$ (3). Suppose a sequence X is Turing computable from $\emptyset'$ via the functional Φ. We have that $\emptyset'$ is a computably enumerable set. In particular $\emptyset' = \lim_{s\in\mathbb{N}} \emptyset'_s$ where $\emptyset'_s$ is the enumeration of $\emptyset'$ up to stage s. We define for each s the sequence X_s, by defining for each n the bit $X_s(n) = \Phi(\emptyset'_s, n)[s]$ if $\Phi(\emptyset'_s, n)[s]$ halts and $X_s(n) = 0$ otherwise. We shall prove that $X = \lim_{s\in\mathbb{N}} X_s$. For every n there exists a m such that $X \upharpoonright_n = \Phi(\emptyset' \upharpoonright_m)$. Also there is a stage t such that $\emptyset'_s \upharpoonright_m = \emptyset' \upharpoonright_m$ for every $s \geq t$ and then for a stage s large enough we have $X \upharpoonright_n = \Phi(\emptyset'_s)[s]$.

Finally let us prove (3) $\to$ (1). If there is a computable sequence $\{X_s\}$ such that $X = \lim_{s\in\mathbb{N}} X_s$, the set X can be defined by the predicate $n \in X$ iff $\exists t\ \forall s \geq t\quad n \in X_t$, which makes X a Σ^0_2 set. Also the set $\mathbb{N} - X$ can be defined by the predicate $n \in \mathbb{N} - X$ iff $\exists t\ \forall s \geq t\quad n \notin X_t$, which makes $\mathbb{N} - X$ a Σ^0_2 set. Then the set X is a Δ^0_2 set. ■

We now give a definition for a restriction of being Δ^0_2, which is interesting for its counterpart in higher computability, that we will study in Section 4.4.3:

Definition 1.6.5. *A sequence X is ω-computably approximable if there is a computable sequence (of sequences) $\{X_s\}_{s\in\mathbb{N}}$ such that $X = \lim_{s\in\mathbb{N}} X_s$ and if there is a computable function $f : \mathbb{N} \to \mathbb{N}$ such that for any n, the cardinality of the set $\{s : X_s(n) \neq X_{s+1}(n)\}$ is bounded by $f(n)$.* ∎

We don't prove now the following proposition, as a proof of its higher counterpart, which works similarly, will be given in Section 4.4.3.

Proposition 1.6.4:
For a sequence X the following are equivalent:

1. *X is ω-computably approximable.*
2. *X is wtt-reducible to $\emptyset'$.*
3. *X is tt-reducible to $\emptyset'$.*

And we finally give a restriction of being ω-computably approximable which will sometimes be useful:

Definition 1.6.6. *A sequence X is* ***approximable from below****, or* ***left-c.e.*** *if there is a computable sequence (of sequences) $\{X_s\}_{s\in\mathbb{N}}$ such that for each s we have $X_s \leq X_{s+1}$, when X_s, X_{s+1} are seen as real numbers, and such that $X = \lim_{s\in\mathbb{N}} X_s$.* ∎

A left-c.e. sequence is always ω-computably approximable because for any $n > 0$ when $X_s \upharpoonright_n \neq X_{s+1} \upharpoonright_n$ then $X_{s+1} \upharpoonright_n$ if bigger than $X_s \upharpoonright_n$ in the lexicographic order. Also as there are only 2^n strings of length n, the cardinality of the set $\{s : X_s \upharpoonright_n \neq X_{s+1} \upharpoonright_n\}$ is bounded by 2^n and then also the cardinality of the set $\{s : X_s(n-1) \neq X_{s+1}(n-1)\}$ is bounded by 2^n

Example 1.6.2:
Any non-empty Π^0_1 set $\mathcal{F}$ contains a sequence, which as a real is smaller than any other sequence of $\mathcal{F}$. This sequence is generally called the ***leftmost path*** *of $\mathcal{F}$ and is left-c.e.* ◇

1.6.4 The jump and the H-sets

In the literature, see for example [73] or [11], the canonical Σ^0_n-complete sets for n finite are denoted by $\emptyset^n$ and have a rather different definition:

Definition 1.6.7. *The set $\emptyset'$ is defined to be $\{e : e \in W_e\}$. We relativize this to any set X by defining $X' = \{e : e \in W_e^X\}$. Then the set $\emptyset^1$ is defined to be $\emptyset'$ and the set $\emptyset^{n+1}$ is inductively defined to be $(\emptyset^n)'$. For an oracle X, the set X' is called the* ***jump*** *of X.* ∎

One can easily prove that $\emptyset^n$ is Σ^0_n-complete. Following this idea, this notion of iterated jump has been extended through the computable ordinal. The successor step is the same as for the jump in the natural numbers, but what should be for example $\emptyset^{\omega}$? An idea, to make it more powerful than any $\emptyset^n$, is to define it as the disjoint union of every $\emptyset^n$. To continue through the computable ordinals, we should rely on some coding for ordinals. Also in the literature, only a definition along coding for constructive ordinal has been made:

Definition 1.6.8. *For any $a \in \mathcal{O}$ with $a = 1$, the set H_a is defined to be the empty set. For any $a \in \mathcal{O}$ with $a = \operatorname{succ}(b)$ we define H_a to be the jump of H_b. Finally for any $a \in \mathcal{O}$ with $a = \sup_n b_n$, we define H_a to be the disjoint union of H_{b_n}, that is, $\langle k, n\rangle \in H_a$ iff $k \in H_{b_n}$. The sets H_a for $a \in \mathcal{O}$ are called the H-sets.* ■

We shall now see that for each $\alpha > 0$ and each $a \in \mathcal{O}_{=\alpha}$, the set $H_{\operatorname{succ}(a)}$ is a Σ^0_α-complete set.

Proposition 1.6.5:
For any $\alpha > 0$ and any $a \in \mathcal{O}_{=\alpha}$, the set $H_{\operatorname{succ}(a)}$ is a Σ^0_α set.

PROOF: It is done by induction. For a start if $a = 1$ we have H_a is the empty set and then that $H_{\operatorname{succ}(a)} = \emptyset'$ is a Σ^0_1 set.

If a code for a successor ordinal we have $n \in H_{\operatorname{succ}(a)}$ iff $n \in W_n^{H_a}$ which is a $\Sigma^0_1(H_a)$ condition. By induction hypothesis, the set H_a together with its complement, is a $\Delta^0_{|a|_o}$ set, and then the set $H_{\operatorname{succ}(a)}$ is a $\Sigma^0_{|a|_o}$ set.

If a is limit, by induction hypothesis the set H_a is the disjoint union of $\Sigma^0_{<|a|_o}$ sets. But as the union is disjoint, then $\mathbb{N} - H_a$ is a disjoint union of $\Pi^0_{<|a|_o}$ sets. Therefore H_a is a $\Delta^0_{|a|_o}$ set. Then like in the previous paragraph we have that $H_{\operatorname{succ}(a)}$ is a $\Sigma^0_{|a|_o}$ set. ■

We now prove that any H-set is also a complete set, for its class of complexity:

Theorem 1.6.3:
For each computable α and any $a \in \mathcal{O}_{=\alpha}$, the set $\emptyset^{(\alpha)}$ is many-one reducible to $H_{\operatorname{succ}(a)}$ uniformly in a.

PROOF: We prove that there exists a computable function $f : \mathbb{N} \times \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ so that for any ordinal $\alpha = \varnothing$ or α limit and for any $k, p \in \omega$ we have:

- For any code $a \in \mathcal{O}_{=\alpha+2k}$, the function $n \mapsto f(a, n, p)$ is total and for any n, we have $n \in \mathcal{T}_{<\omega(\alpha+k)}$ iff $f(a, n, p) \in H_{\operatorname{succ}(a)}$.
- For any code $a \in \mathcal{O}_{=\alpha+2k+1}$, the function $n \mapsto f(a, n, p)$ is total and for any n, we have $n \in \mathbb{N} - \mathcal{T}_{\leq\omega(\alpha+k)+p}$ iff $f(a, n, p) \in H_{\operatorname{succ}(a)}$.

The reduction, uniform in a, is then given by $n \mapsto f(a, n, 0)$. Let us define the function f.

First it is easy to check whether $a = \alpha + 2k$ or whether $a = \alpha + 2k + 1$, for α limit or 0, and $k \in \omega$, as well as to determine what is α and what is k.

In case $a = 1$, the function $f(a, n, p)$ returns some e such that $e \in W_e$ iff n enumerates a tree of height strictly bigger than p. We clearly have $n \in \mathbb{N} - \boldsymbol{\mathcal{T}}_{\leq p}$ iff $f(a, n, p) \in H_{\mathrm{succ}(a)} = \emptyset'$.

In case $a = \alpha + 2k$ with $a = \mathrm{succ}(b)$ (and then $k \neq 0$), we want the function $f(a, n, p)$ to return an element of $H_{\mathrm{succ}(a)}$ iff there is some p for which $n \in \boldsymbol{\mathcal{T}}_{\leq \omega(\alpha+k-1)+p}$, which is true by induction iff $\exists p\ f(b, n, p) \notin H_{\mathrm{succ}(b)}$. As $H_{\mathrm{succ}(a)} = H'_{\mathrm{succ}(b)}$, one can easily find in an effective way such a value for $f(a, n, p)$.

In case $a = \alpha + 2k$ with $a = \sup_m b_m$ (and then $k = 0$), we can suppose without loss of generality that each b_m is of the form $c_m + 2k_m$ with $|c_m|_o$ limit and $|k_m|_o$ finite (If not then $H'_{c_m+2k_m} = H_{b_m}$ and $H_{b_m} >_m H_{c_m+2k_m}$). Then we want the function $f(a, n, p)$ to return an element of $H_{\mathrm{succ}(a)}$ iff there is some m for which $n \in \boldsymbol{\mathcal{T}}_{<\omega(|c_m|_o+|k|_o)}$, which is true by induction iff $\exists m\ f(b_m, n, 0) \in H_{\mathrm{succ}(b_m)}$. Also $H_{\mathrm{succ}(b_m)}$ is $\Sigma^0_1(H_{b_m})$ and as $H_a = \oplus_m H_{b_m}$, it is then $\Sigma^0_1(H_a)$. As $H_{\mathrm{succ}(a)} = H'_a$, one can easily find in an effective way a value for $f(a, n, p)$ such that $n \in \boldsymbol{\mathcal{T}}_{<\omega(\alpha)}$ iff $f(a, n, p) \in H_{\mathrm{succ}(a)}$.

In case $a = \alpha + 2k + 1$ with $a = \mathrm{succ}(b)$ and any p, we want the function $f(a, n, p)$ to return an element of $H_{\mathrm{succ}(a)}$ iff there exists a node σ of length $p+1$ enumerated in the tree T described by n, such that $T \upharpoonleft_\sigma$ does not belong to $\boldsymbol{\mathcal{T}}_{<\omega(\alpha+k)}$, which is true by induction iff $\exists \sigma$ of length $p+1$ such that $f(b, n(\sigma), 0) \notin H_{\mathrm{succ}(b)}$, where $n(\sigma)$ is a code for $T \upharpoonleft_\sigma$. As $H_{\mathrm{succ}(a)} = H'_{\mathrm{succ}(b)}$, one can easily find in an effective way such a value for $f(a, n, p)$. ∎

Corollary 1.6.2:
For any $a \in \mathcal{O}_{=\alpha}$, the set $H_{\mathrm{succ}(a)}$ is Σ^0_α-complete.

It is interesting to note that for α a limit ordinal, we do not always have $H_a \equiv_m H_b$ for $a, b \in \mathcal{O}_{=\alpha}$. In fact Moschovakis proved in [64] that either α is successor or of the form $\beta + \omega$, in which case for $a, b \in \mathcal{O}_{=\alpha}$ we have $H_a \equiv_m H_b$, or that α is not of this form, in which case the partial ordering of the many one degrees of the sets H_a for $a \in \mathcal{O}_{=\alpha}$ contains a well-ordered chain of length ω_1, as well as incomparable elements.

1.6.5 Kleene's hierarchy is strict

Theorem 1.6.4:
For any α, there is a Σ^0_α set which is not Π^0_α.

Proof: For each α and each $a \in \mathcal{O}_{=\alpha}$, the set $H_{\mathrm{succ}(a)}$ is a candidate. Suppose for contradiction that $H_{\mathrm{succ}(a)}$ is a Π^0_α set. We then want to prove that $H_{\mathrm{succ}(a)}$ is a $\Pi^0_1(H_a)$

set. From this we can easily derive a contradiction, because $\mathbb{N}-H_{\mathrm{succ}(a)}$ is then a $\Sigma^0_1(H_a)$ set and there is then some e such that $W_e^{H_a}=\mathbb{N}-H_{\mathrm{succ}(a)}$. But then $e\in\mathbb{N}-H_{\mathrm{succ}(a)}$ iff $e\in H_{\mathrm{succ}(a)}$ which is a contradiction.

So suppose $H_{\mathrm{succ}(a)}$ has a Π^0_α-index with $H_{\mathrm{succ}(a)}=\bigcap_m A_m$. We have by definition that $H_{\mathrm{succ}(a)}$ is $\Sigma^0_1(H_a)$. Also $\emptyset^{(\alpha)}$ is many-one reducible to $H_{\mathrm{succ}(a)}$ and then $\omega-\emptyset^{(\alpha)}$ is $\Pi^0_1(H_a)$. From Theorem 1.6.1, each set A_m is many-one reducible to $\mathbb{N}-\emptyset^{(\alpha)}$ uniformly in an index for A_m. So there is a function $f:\mathbb{N}\times\mathbb{N}\to\mathbb{N}$ such that $n\in A_m$ iff $f(m,n)\in\mathbb{N}-\emptyset^{(\alpha)}$. Then $n\in\bigcap_m A_m$ iff $\forall m\ f(m,n)\in\mathbb{N}-\emptyset^{(\alpha)}$, and as $\mathbb{N}-\emptyset^{(\alpha)}$ is $\Pi^0_1(H_a)$, then also $H_{\mathrm{succ}(a)}$ is $\Pi^0_1(H_a)$. ■

Corollary 1.6.3:
For any computable α:

1. *There is a Σ^0_α set which is not Π^0_α.*
2. *There is a Π^0_α set which is not Σ^0_α.*
3. *There is a Δ^0_α set which is neither $\Sigma^0_{<\alpha}$ nor $\Pi^0_{<\alpha}$.*

Proof: For the third one, the argument is similar to the proof of Corollary 1.5.1. For $\alpha=\beta+1$, consider any $a\in\mathcal{O}_{=\beta}$. Then $H_{\mathrm{succ}(a)}\oplus(\omega-H_{\mathrm{succ}(a)})$ is easily seen to be Δ^0_α but neither Σ^0_β nor Π^0_β. For α limit, each set H_a for $a\in\mathcal{O}_{=\alpha}$ is easily seen to be Δ^0_α but neither Σ^0_β nor Π^0_β. ■

We shall now argue as promised that the set of $\Sigma^0_{\le\alpha}$-indices is not always a Σ^0_α set. In particular, the set of $\Sigma^0_{\le\omega^\omega}$-indices is a $\Pi^0_{\omega^\omega+1}$ set which is not $\Sigma^0_{\omega^\omega+1}$. To do so, we simply argue that the set $\mathcal{T}_{\le\alpha}$ is many-one equivalent to the set of $\Sigma^0_{\le\alpha}$-indices.

Proposition 1.6.6:
For any $\alpha\ge\omega$, the set of $\Pi^0_{\le\alpha}$-indices is many-one equivalent to the set of $\Sigma^0_{\le\alpha}$-indices, which is itself many-one equivalent to $\mathcal{T}_{\le\alpha}$.

Proof: First to reduce the set of $\Pi^0_{\le\alpha}$-indices to the set of $\Sigma^0_{\le\alpha}$-indices, we define the total computable function f which on $e=\langle 1,n\rangle$ for some n, outputs n, and on $e\neq\langle 1,n\rangle$ for any n, outputs $\langle 1,e\rangle$. We have that e is a $\Pi^0_{\le\alpha}$-index iff $f(e)$ is a $\Sigma^0_{\le\alpha}$-index. The reduction of the set of $\Sigma^0_{\le\alpha}$-indices to the set of $\Pi^0_{\le\alpha}$-indices is similar.

We now reduce the set of $\Sigma^0_{\le\alpha}$-indices to $\mathcal{T}_{\le\alpha}$. For this purpose we describe two total computable functions $h_1:\mathbb{N}\to\mathbb{N}$ and $h_2:\mathbb{N}\to\mathbb{N}$, obtained by fixed point.

If $e=\langle 2,n\rangle$, the function $h_1(e)$ returns the tree which enumerates all the nodes that are in the tree $a\,\hat{}\,h_2(a)$ for any a enumerated in W_n, and if e is of a different form, $h_1(e)$

returns the code of an ill-founded tree. If $e = \langle 1, n\rangle$ then $h_2(e)$ returns $h_1(n)$, if $e = \langle 0, n\rangle$ then $h_2(e)$ returns a tree with only one node, and if e is of a different from then $h_2(e)$ returns an ill-founded tree.

We also need to check that for any index of the form $\langle 2, n\rangle$, the set W_n is not empty. This is a Π^0_2 condition which is then reducible to $T_{\leq\omega}$. The reduction is then given by an index for the tree corresponding to the disjoint union of the tree coded by the result of h_1, together with the tree resulting from checking this Π^0_2 condition. One can prove by induction that for $\alpha \geq \omega$, an integer e is a $\Sigma^0_{\leq\alpha}$-index iff $f(e) \in \mathcal{T}_{\leq\alpha}$.

The idea to many-one reduce $\mathcal{T}_{\leq\alpha}$ to the set of $\Sigma^0_{\leq\alpha}$-indices is similar. The only problem is that for a given c.e. tree T we cannot decide if a node of T is a leaf. However, we should be able to transform any leaf into an index of the form $\langle 0, e\rangle$. To overcome this problem, we add a leaf to every node in T, and perform the reduction from this new tree. In case T was coding for an ordinal $\alpha \geq \omega$, we easily see that the new tree will code for the same ordinal. ∎

Then as $\mathcal{T}_{\leq\omega\omega^\omega}$ is $\Pi^0_{\omega^\omega+1}$-complete, and as $\omega\omega^\omega = \omega^\omega$, also using the previous proposition, the set of $\Sigma^0_{\leq\omega^\omega}$-indices is $\Pi^0_{\omega^\omega+1}$-complete, and then as there is a $\Pi^0_{\omega^\omega+1}$ set which is not $\Sigma^0_{\omega^\omega+1}$, also the set of $\Sigma^0_{\leq\omega^\omega}$-indices is not $\Sigma^0_{\omega^\omega+1}$.

1.7 Connection between the effective Kleene's and Borel's hierarchies

We now make a connection between the effective Borel hierarchy and the effective Kleene's hierarchy.

Theorem 1.7.1:
A set $\mathcal{A} \subseteq 2^{\mathbb{N}}$ is Σ^0_α iff there exists an integer e such that $\mathcal{A} = \{X \in 2^{\mathbb{N}} : e \in X^{(\alpha)}\}$. Furthermore if $\mathcal{A}$ is Σ^0_α, such an integer e can be found uniformly in a Σ^0_α-index for $\mathcal{A}$.

Proof: Let us prove that for any e the set $\{X \in 2^{\mathbb{N}} : e \in X^{(\alpha)}\}$ is Σ^0_α. Proposition 1.6.2 is easily seen to relatize, that is, the set $\mathcal{T}^X_{<\omega(\alpha+k)}$ is $\Sigma^0_{\alpha+2k}(X)$ uniformly in X, in k and in any code of $\mathcal{O}_{=\alpha}$, and the set $\mathcal{T}^X_{\leq\omega(\alpha+k)}$ is $\Pi^0_{\alpha+2k+1}(X)$ uniformly in X, in k and in any code of $\mathcal{O}_{=\alpha}$. Then one can prove by induction that for any set $\mathcal{B}^X \subseteq \mathbb{N}$ which is $\Sigma^0_\alpha(X)$ uniformly in X and for any e, the set $\{X : e \in \mathcal{B}^X\}$ is Σ^0_α.

Let us prove that for any $\mathcal{A}$ a Σ^0_α set, we can find uniformly some e with $\mathcal{A} = \{X \in 2^{\mathbb{N}} : e \in X^{(\alpha)}\}$. To do it, we should first prove that can uniformly find some a, a $\Sigma^0_\alpha(X)$-index uniformly in X, such that 0 belongs to the $\Sigma^0_\alpha(X)$ set coded by a iff $X \in \mathcal{A}$. This is easily achieved by transforming, uniformly in X, the index for the tree corresponding to $\mathcal{A}$ into the $\Sigma^0_\alpha(X)$-index of the same tree, except that every leaf corresponding to a Σ^0_1 set W_b is replaced by a leaf which enumerates 0 if a prefix of X is enumerated in W_b, and which enumerates nothing otherwise; and every leaf corresponding to a Π^0_1 set W_b^c is replaced by a leaf corresponding to a Π^0_1 set equals to $\{0\}$ if no prefix of X is enumerated in W_b, and equal to the empty set otherwise.

We can then use a relativized version of Theorem 1.6.1 to have that $X \in \mathcal{A}$ iff $f(0) \in X^{(\alpha)}$ for a computable function f that we can find uniformly in a. The integer e is then given by $f(0)$. ∎

We finally give a version of Porism 1.6.1, but for sets of sequences:

Theorem 1.7.2:
For a computable ordinal α, and any e:

1. *The set $\{X \in 2^{\mathbb{N}} : e \in \mathcal{T}^X_{<\alpha}\}$ is $\Sigma^0_{\alpha+1}$ uniformly in e and in any code of $\mathcal{O}_{=\alpha}$.*
2. *The set $\{X \in 2^{\mathbb{N}} : e \in \mathcal{T}^X_{\leq\alpha}\}$ is $\Pi^0_{\alpha+1}$ uniformly in e and in any code of $\mathcal{O}_{=\alpha}$.*
3. *The set $\{X \in 2^{\mathbb{N}} : e \in \mathcal{O}^X_{<\alpha}\}$ is $\Sigma^0_{\alpha+1}$ uniformly in e and in any code of $\mathcal{O}_{=\alpha}$.*
4. *The set $\{X \in 2^{\mathbb{N}} : e \in \mathcal{O}^X_{\leq\alpha}\}$ is $\Pi^0_{\alpha+1}$ uniformly in e and in any code of $\mathcal{O}_{=\alpha}$.*

PROOF: We use a uniform relativization of Porism 1.6.1. Then we can prove by induction that for any set $\mathcal{B}^X \subseteq \mathbb{N}$ which is $\Sigma^0_\alpha(X)$ uniformly in X and for any e, the set $\{X : e \in \mathcal{B}^X\}$ is Σ^0_α. ∎

1.8 Background on measures

1.8.1 Classical facts on measures

Measures an probability measures

If $\mathcal{X}$ is a space, we say the a set $\mathbb{B} \subseteq P(\mathcal{X})$ (the set of subset of $\mathcal{X}$) is a **σ-algebra** on $\mathcal{X}$ if:

- $\varnothing \in \mathbb{B}$
- $\mathbb{B}$ is closed under countable union
- $\mathbb{B}$ is closed under complementation

Example 1.8.1:
For $\mathcal{X}$ a topological space, the Borel hierarchy on $\mathcal{X}$ is a σ-algebra. It is clear that it is the smallest σ-algebra containing the open sets of $\mathcal{X}$. We will only use such σ-algebras in this thesis. ◇

We can now give the formal definition of a measure, concept introduced by Borel in the last years of the 19th century, and then developed by Lebesgue.

Definition 1.8.1. *Let $\mathcal{X}$ be a set and $\mathbb{B}$ a σ-algebra on $\mathcal{X}$. Then a function $\mu : \mathbb{B} \to \overline{\mathbb{R}}$ is a* ***measure*** *if*

- *$\mu(\mathcal{B}) \geq 0$ for all $\mathcal{B} \in \mathbb{B}$*

- $\mu(\varnothing) = 0$
- *For all countable family* $(\mathcal{B}_i)_{i\in\mathcal{N}}$ *of pairwise disjoint sets, we have* $\mu\left(\bigcup_i \mathcal{B}_i\right) = \sum_i \mu(\mathcal{B}_i)$

If additionally we have that $\mu(\mathcal{X}) = 1$*, then the measure is called a* ***probability measure****, and if the* σ*-algebra is the Borel sets of* $\mathcal{X}$*, the measure is called a* ***Borel measure****, and a Borel set is also said to be a* ***Borel measurable set****.* ■

All the measures we use in this thesis are probability measures defined on the Borel sets of the Cantor space, so we will simply call them measures, and denote them by μ, ν or ξ. The third property in the definition of measures is called **countable additivity**. It is clear that for a measure μ and a family $(\mathcal{B}_i)_{i\in\mathcal{N}}$ of sets which are not necessarily pairwise disjoints, we also have $\mu\left(\bigcup_i \mathcal{B}_i\right) \leq \sum_i \mu(\mathcal{B}_i)$. This property is called **countable subadditivity**. So every measure satisfies countable subadditivity for any countable family of sets, but if this family is formed of pairwise disjoint sets, we have more, that is, countable additivity.

Complete measures

Sometimes it will not be enough for the purpose of this thesis to have a measure only defined on the Borel sets of the Cantor space. We will need a little bit more than that to study randomness on analytical and co-analytical sets (see Section 3.7). In fact we will need what is called a complete measure.

One can notice that once a measure is well-defined on the σ-algebra $\mathbb{B}$ of a set $\mathcal{X}$, it is 'morally' possible to extend it to any set $\mathcal{A}$, with $\mathcal{A}$ not necessarily in the σ-algebra, but at least with $\mathcal{A}$ included in a set $\mathcal{B}$ of the σ-algebra, that has measure 0. In this case the measure of $\mathcal{A}$ can safely be assigned to 0 as well, and such a set $\mathcal{A}$ is said to be **negligible**. Formally we have:

Definition 1.8.2. *Let* $\mathcal{X}$ *be a set,* $\mathbb{B}$ *a* σ*-algebra on* $\mathcal{X}$ *and* μ *a measure on* $\mathbb{B}$*. Then the set* $\overline{\mathbb{B}}$ *defined by*

$$\overline{\mathbb{B}} = \{\mathcal{B} \cup \mathcal{A} \;:\; \mathcal{B} \in \mathbb{B} \text{ and } \mathcal{A} \text{ is negligible }\}$$

is still a σ*-algebra on* $\mathcal{X}$*, called* ***the completed*** σ***-algebra****. Moreover the measure* μ *can be extended into a measure* $\overline{\mu} : \overline{\mathbb{B}} \to \mathbb{R}$ *by setting* $\overline{\mu}(\mathcal{B} \cup \mathcal{A}) = \mu(\mathcal{B})$*. The measure* $\overline{\mu}$ *is then said to be* ***complete*** *and any set in the completed* σ*-algebra is said to be a* ***Lebesgue-measurable set****.* ■

Probability measure descriptions

We now describe more concretely how to build a measure. First, if we want to define a measure μ on the Borel sets of the Cantor space, we should define it on every cylinder $[\sigma]$ (or at least on sufficiently many cylinders so that the measure is uniquely defined on all of them). And actually, that is it. A theorem of Carathéodory says that if a function, defined from the cylinders to the reals, does not violate yet the definition of a probability measure, then it can be uniquely extended to a probability measure on all the Borel sets[1].

[1] The result is more general, one can see for example [1] for a statement of the theorem and its proof

In particular, the only way to extend the probability measure, once it is defined on every cylinder, is by doing so inductively on Borel sets, by defining $\mu(\bigcup_n \mathcal{A}_n) = \sup_n \mu(\bigcup_{m\leq n} \mathcal{A}_m)$ and $\mu(\bigcap_n \mathcal{A}_n) = \inf_n \mu(\bigcap_{m\leq n} \mathcal{A}_m)$, for any Borel sets $\{A_n\}_{n\in\mathbb{N}}$.

There is a measure on the Cantor space, whose uniformity makes it canonical. This is the measure denoted by λ and defined by $\lambda([\sigma]) = 2^{-|\sigma|}$ for every cylinder $[\sigma]$, and known as the **Lebesgue measure**. One convenient way to consider this measure is to see $\lambda([\sigma])$, as the probability that one obtains exactly the sequence σ by tossing a fair coin $|\sigma|$ times (with head corresponding to 0 and tail to 1).

Measures in product spaces

We discuss here a common way to create a measure on a product space of two spaces endowed with σ-algebras and measures on their σ-algebras. First for $\mathbb{B}_1$ a σ-algebra on $\mathcal{X}_1$ and $\mathbb{B}_2$ a σ-algebra on $\mathcal{X}_2$, let us denote by $\mathbb{B}_1 \otimes \mathbb{B}_2$ the σ-algebra on $\mathcal{X}_1 \times \mathcal{X}_2$ which is generated by sets $\mathcal{A}_1 \times \mathcal{A}_2$ for $\mathcal{A}_1 \in \mathbb{B}_1$ and $\mathcal{A}_2 \in \mathbb{B}_2$.

Definition 1.8.3. *Let $\mathcal{X}_1$, $\mathcal{X}_2$ be sets, $\mathbb{B}_1$, $\mathbb{B}_2$ some σ-algebras on respectively $\mathcal{X}_1$ and $\mathcal{X}_2$ and let μ_1, μ_2 be probability measures respectively on $\mathbb{B}_1$, $\mathbb{B}_2$. The **product measure** ν on $\mathbb{B}_1 \otimes \mathbb{B}_2$ is defined to be the unique measure generated by $\nu(\mathcal{A}_1 \times \mathcal{A}_2) = \mu_1(\mathcal{A}_1) \times \mu_2(\mathcal{A}_2)$, where $\mathcal{A}_1$ and $\mathcal{A}_2$ are elements of respectively $\mathbb{B}_1$ and $\mathbb{B}_2$.* ■

Existence and unicity in the previous definition is again given by the Carathéodory theorem.

Example 1.8.2:
We extend the Lesbegue measure to $2^{\mathbb{N}} \times 2^{\mathbb{N}}$ by defining $\lambda([\sigma_1] \times [\sigma_2])$ to be $\lambda([\sigma_1])\lambda([\sigma_2])$ for any strings σ_1, σ_2. Again we can prove that λ is then uniquely defined on every Borel set of $2^{\mathbb{N}} \times 2^{\mathbb{N}}$, and we can then prove that for any Borel set $\mathcal{B}$ of $2^{\mathbb{N}} \times 2^{\mathbb{N}}$ we have:

$$\lambda(\mathcal{B}) = \lambda(\{X \oplus Y \ : \ (X, Y) \in \mathcal{B}\})$$

1.8.2 Measures and computability

The Lebesgue measure has the interesting property of being computable, that is, we can compute the value of $\lambda([\sigma])$ uniformly in any string σ. The theory of algorithmic randomness is generally studied with respect to the Lebesgue measure, λ. Also the computability of λ is intensively used in the field, to obtain a large variety of theorem, like for example the 'first of them' in some sense, that is, the existence of a universal Martin-Löf test (see Theorem 2.1.1). The study of algorithmic randomness with respect to different measures has been done, and the results are quite different depending on whether or not the used measure is computable. This is why we now introduce this notion:

Definition 1.8.4. *A Borel probability measure μ on the Borel sets of $2^{\mathbb{N}}$ is said to be computable if we have a total computable function $f : 2^{<\mathbb{N}} \to [0, 1]$ which returns $\mu([\sigma])$ on σ.* ■

So given a computable measure μ, the measure of each clopen set is computable, and the more complex the description of a set, the more complex the description of its measure.

We will now see that the measure of Σ^0_α sets is a Σ^0_α real number, uniformly in the measure and in an index for the Σ^0_α set.

Proposition 1.8.1:
For μ a computable Borel probability measure and $\mathcal{A} \subseteq 2^{\mathbb{N}}$ a Σ^0_α set, the predicate $\mu(\mathcal{A}) > q$ is Σ^0_α, uniformly in μ, in an index for $\mathcal{A}$, and in q a positive rational number.

PROOF: The proof goes by induction on computable ordinals. If $\mathcal{A}$ is a Σ^0_1 set, the predicate $\mu(\mathcal{A}) > q$ is equivalent to $\exists t\ \mu(\mathcal{A}[t]) > q$, which is Σ^0_1 as $\mathcal{A}[t]$ is a clopen set with then a computable measure. Everything is clearly uniform.

Suppose that for an ordinal α, any $\Sigma^0_{<\alpha}$ set $\mathcal{A}$ and any q, the predicate $\mu(\mathcal{A}) > q$ is $\Sigma^0_{<\alpha}$ uniformly in an index for $\mathcal{A}$ and in q. Consider the Σ^0_α set $\mathcal{A} = \bigcup_n \mathcal{B}_n$ where each $\mathcal{B}_n$ is $\Pi^0_{<\alpha}$ uniformly in n. The predicate $\mu(\mathcal{A}) > q$ is equivalent to $\exists m\ \mu(\bigcup_{n \leq m} \mathcal{B}_n) > q$.

By induction hypothesis, we have that $\mu(2^\omega - \bigcup_{n\leq m} \mathcal{B}_n) > 1-q$ is a $\Sigma^0_{<\alpha}$ predicate, which is equivalent to the predicate $\mu(\bigcup_{n\leq m} \mathcal{B}_n) \leq q$. But then the negation of this is the predicate $\mu(\bigcup_{n\leq m} \mathcal{B}_n) > q$ which is then $\Pi^0_{<\alpha}$ and which makes the predicate $\exists m\ \mu(\bigcup_{n\leq m} \mathcal{B}_n) > q$ a Σ^0_α predicate. ■

In particular, the measure of a Σ^0_1 set is a left-c.e. real, and more generally, the measure of a Δ^0_2 set is a Δ^0_2 real, which, by the Shoenfiled limit lemma, can then be approximated. We then introduce the following notation:

Definition 1.8.5. *For a computable measure μ and a Δ^0_2 set $\mathcal{A}$, we write $\mu(\mathcal{A})[s]$ to denote the approximation of $\mu(\mathcal{A})$ at stage s.* ■

We now should see an important tool, for the purpose of algorithmic randomness. Following the work of Lebesgue, it was well-known that any Borel set of arbitrary complexity was approximable from above by $\boldsymbol{\Pi^0_2}$ sets of the same measure, and from below by $\boldsymbol{\Sigma^0_2}$ sets of the same measure. This was effectivized later in the thesis of Kurtz [44] and Kautz [32], for the arithmetical hierarchy and it is well-known that the effectivization can be extended to the whole effective hyperarithmetical hierarchy.

Theorem 1.8.1:
For any Σ^0_α set $\mathcal{A} \subseteq 2^{\mathbb{N}}$, any positive rational q and any computable Borel probability measure μ, there are:

- *A Σ^0_α-open set $\mathcal{U}$ with $\mathcal{A} \subseteq \mathcal{U}$ such that $\mu(\mathcal{U} - \mathcal{A}) \leq q$*
- *A $\Pi^0_{<\alpha}$-closed set $\mathcal{F}$ with $\mathcal{F} \subseteq \mathcal{A}$ such that $\mu(\mathcal{A} - \mathcal{F}) \leq q$*

Moreover an index for $\mathcal{U}$ can be found uniformly in q and in an index for $\mathcal{A}$, and an index for $\mathcal{F}$ can be found uniformly in q, in an index for $\mathcal{A}$ and in $\emptyset^{(\alpha)}$.

PROOF: The proof goes by induction on computable ordinals. For a Σ^0_1 set $\mathcal{A}$, the Σ^0_1 set $\mathcal{U}$ is trivially $\mathcal{A}$ itself for any q. The Π^0_0 set $\mathcal{F}$ is $\mathcal{U}[t]$ for t the smallest integer such that $\mu(\mathcal{U}-\mathcal{U}[t]) \leq q$. As $\mathcal{U}-\mathcal{U}[t]$ is a Σ^0_1 set, from Proposition 1.8.1 we have that $\mu(\mathcal{U}-\mathcal{U}[t]) \leq q$ is a Π^0_1 predicate, making t computable in $\emptyset^{(1)}$, in function of q and an index for $\mathcal{U}$. This makes $\mathcal{U}[t]$ a Π^0_0 set whose index can be uniformly obtained in an index for $\mathcal{A}$, in q and in $\emptyset^{(1)}$.

Suppose that the theorem is true below ordinal α and let us prove that it is true at ordinal α. Let $\mathcal{A} = \bigcup_n \mathcal{B}_n$ be a Σ^0_α set, with each $\mathcal{B}_n$ a $\Pi^0_{<\alpha}$ set. By induction hypothesis, for each $\mathcal{B}_n$ and any positive rational q, we can find a $\Sigma^0_{<\alpha}$-open set $\mathcal{U}_n \supseteq \mathcal{B}_n$ uniformly in q, in an index for $\mathcal{A}$ and in $\emptyset^{(<\alpha)}$ such that $\mu(\mathcal{U}_n - \mathcal{B}_n) \leq q$. Still by induction hypothesis, for each $\mathcal{B}_n$ and any positive rational q, we can find a $\Pi^0_{<\alpha}$-closed set $\mathcal{F}_n \subseteq \mathcal{B}_n$ uniformly in q, and in an index for $\mathcal{A}$, such that $\mu(\mathcal{B}_n - \mathcal{F}_n) \leq q$.

For any q, fix a computable sequence $\{q_n\}_{n<\omega}$ such that $\sum_n q_n \leq q$. The desired Σ^0_α open set $\mathcal{U}$ is then the union of $\Sigma^0_{<\alpha}$-open sets $\mathcal{U}_n \supseteq \mathcal{B}_n$ such that $\mu(\mathcal{U}_n - \mathcal{B}_n) \leq q_n$. Each open set $\mathcal{U}_n$ is $\Sigma^0_1(\emptyset^{(<\alpha)})$ uniformly in an index for $\mathcal{B}_n$, in q_n and in $\emptyset^{(<\alpha)}$, making their union a $\Sigma^0_1(\emptyset^{(<\alpha)})$ set and then a Σ^0_α set, uniformly in an index for $\mathcal{B}_n$ and in q_n.

Still using the computable sequence $\{q_n\}_{n<\omega}$ such that $\sum_n q_n \leq q$, the desired $\Pi^0_{<\alpha}$-closed set $\mathcal{F}$ is equal to $\bigcup_{n<m} \mathcal{F}_n$ where m is the smallest integer such that $\mu(\mathcal{A} - \bigcup_{n\leq m} \mathcal{B}_n) \leq q_0$ and with $\mathcal{F}_n \subseteq \mathcal{B}_n$ and $\mu(\mathcal{B}_n - \mathcal{F}_n) \leq q_{n+1}$. As each closed set $\mathcal{F}_n$ is $\Pi^0_{<\alpha}$ and as there are only finitely many of them, then their union is still a $\Pi^0_{<\alpha}$ closed set. Besides $\mathcal{A} - \bigcup_{n\leq m} \mathcal{B}_n$ is a Σ^0_α set uniformly in m and therefore, using Proposition 1.8.1, the integer m can be found uniformly in $\emptyset^{(\alpha)}$, in q and in an index for $\mathcal{A}$. We also have that $\mathcal{A} - \mathcal{F} \subseteq \bigcup_{n<m}(\mathcal{B}_n - \mathcal{F}_n) \cup (\mathcal{A} - \bigcup_{n\leq m} \mathcal{B}_n)$ and therefore $\mu(\mathcal{A} - \mathcal{F}) \leq \sum_{n<m} \mu(\mathcal{B}_n - \mathcal{F}_n) + \mu(\mathcal{A} - \bigcup_{n\leq m} \mathcal{B}_n) \leq q$. ∎

We can deduce an interesting corollary from this, which is a particular case of what is known as the Lebesgue density theorem. Before proving it, we need the following definition:

Definition 1.8.6. *For a Borel set $\mathcal{A} \subseteq 2^{\mathbb{N}}$, a Borel probability measure μ and a cylinder $[\sigma]$ such that $\mu([\sigma]) > 0$, we write $\mu(\mathcal{A} \mid [\sigma])$ to denote the measure of $\mathcal{A}$ inside $[\sigma]$, and relatively to $[\sigma]$:*

$$\mu(\mathcal{A} \mid [\sigma]) = \mu(\mathcal{A} \cap [\sigma])/\mu([\sigma])$$

We now prove a weak version of the Lebesgue density theorem

Corollary 1.8.1 (Lebesgue density theorem, weak version):
For any Borel set $\mathcal{A} \subseteq 2^{\mathbb{N}}$ and any Borel probability measure μ, if $\mu(\mathcal{A}) > 0$, then the measure of $\mathcal{A}$ can be made as close as we want to 1, inside a cylinder. Formally, for every $\varepsilon > 0$, there exists a cylinder σ such that $\mu(\mathcal{A} \mid [\sigma]) \geq 1 - \varepsilon$.

PROOF: Let $\mathcal{A}$ be a Borel set and fix $\varepsilon > 0$. From the previous theorem, for any δ, there exists an open set $\mathcal{U} \supseteq \mathcal{A}$ such that $\mu(\mathcal{U} - \mathcal{A}) < \delta$. Picki $\delta = \varepsilon\mu(\mathcal{A})$ and let $\mathcal{U}$ be such an open set. Let W be a set of strings $\sigma_0, \sigma_1, \ldots$ all pairwise incomparable and such that $[W]^{\prec} = \mathcal{U}$.

Suppose that for every string $\sigma \in W$ we have $\mu(\mathcal{A} \mid [\sigma]) < 1 - \varepsilon$. Then we have $\mu(\mathcal{A} \cap [\sigma]) < (1-\varepsilon)\mu([\sigma])$ and therefore:

$$\mu(\mathcal{A}) = \sum_{\sigma \in W} \mu(\mathcal{A} \cap [\sigma]) \leq (1-\varepsilon) \sum_{\sigma \in W} \mu([\sigma]) = (1-\varepsilon)\mu(\mathcal{U})$$

Also $\mu(\mathcal{U}) - \mu(\mathcal{A}) < \varepsilon\mu(\mathcal{A}) \leq \varepsilon\mu(\mathcal{U})$ and then $\mu(\mathcal{A}) > \mu(\mathcal{U}) - \varepsilon\mu(\mathcal{U}) = (1-\varepsilon)\mu(\mathcal{U})$ which is a contradiction with $\mu(\mathcal{A}) \leq (1-\varepsilon)\mu(\mathcal{U})$ obtained above. Therefore there is a cylinder such that $\mu(\mathcal{A} \mid [\sigma]) \geq 1 - \varepsilon$. ■

Lebesgue proved a stronger version, roughly saying that for any Borel set $\mathcal{A}$ of positive measure, the set of elements $X \in \mathcal{A}$ such that $\lim_n \mu(\mathcal{A} \mid [X\upharpoonright_n]) = 1$ is a (Borel) set of measure $\mu(A)$.

1.8.3 Fubini's theorem

Fubini's theorem says something about the behavior of a measure in a product space, with respect to the two underlying measures it is built with.

Theorem 1.8.2 (Fubini's theorem):
For any integrable function $f : 2^{\mathbb{N}} \times 2^{\mathbb{N}} \to \mathbb{R}$ we have:

$$\begin{aligned}\int_{2^{\mathbb{N}} \times 2^{\mathbb{N}}} f(X,Y)\, d_{\lambda(X,Y)} &= \int_{2^{\mathbb{N}}} \left(\int_{2^{\mathbb{N}}} f(X,Y)\, d_{\lambda(X)}\right) d_{\lambda(Y)} \\ &= \int_{2^{\mathbb{N}}} \left(\int_{2^{\mathbb{N}}} f(X,Y)\, d_{\lambda(Y)}\right) d_{\lambda(X)}\end{aligned}$$

We prove here a corollary of Fubini's theorem, that we are going to use later in the context of algorithmic randomness (see Theorem 4.3.3). If $\mathcal{A}$ is a subset of $2^{\mathbb{N}} \times 2^{\mathbb{N}}$, we write

$$\mathcal{A}_X = \{Y \ : \ (X,Y) \in \mathcal{A}\}$$

so $\mathcal{A}_X$ is a 'section' of $\mathcal{A}$ along X. We first mention something which should be intuitive. Suppose $\lambda(\mathcal{A}) = 0$. Then maybe for some X's we have $\lambda(\mathcal{A}_X) > 0$, but the set of X's such that this is so, should be small, and in fact it should be of measure 0. We generalize this in the following theorem (which could directly be proved with Fubini's theorem):

Theorem 1.8.3:
For any Borel set $\mathcal{A} \subseteq 2^{\mathbb{N}} \times 2^{\mathbb{N}}$ and any Borel probability measure μ we have:

$$\mu(\{X \mid \mu(\mathcal{A}_X) > \sqrt{\mu(\mathcal{A})}\}) \leq \sqrt{\mu(\mathcal{A})}$$

PROOF: We first prove the case where $\mathcal{A} \subseteq 2^{\mathbb{N}} \times 2^{\mathbb{N}}$ is an open set. For more clarity we denote $\sqrt{\mu(\mathcal{A})}$ by m_A. We have that $\mathcal{A} = \bigcup_{(\sigma_1,\sigma_2) \in W} [\sigma_2] \times [\sigma_2]$ for some set of pairs of strings W.

Also the set $\{X \mid \mu(\mathcal{A}_X) > m_A\}$ is an open set and it can then be described by a pairwise disjoint set of strings A. We then have $\mu(\mathcal{A}) \geq \sum_{\sigma \in A} \mu([\sigma])\mu(\mathcal{A}_\sigma)$, where

$\mathcal{A}_\sigma = \bigcup\{[\tau] : [\sigma]\times[\tau] \subseteq \mathcal{A}\}$, and as $\mu(\mathcal{A}_\sigma) > m_A$ for $\sigma \in A$ we have $\mu(\mathcal{A}) \geq m_A \sum_{\sigma\in A} \mu([\sigma])$. But if we suppose now that $\sum_{\sigma\in A} \mu([\sigma]) > m_A$ we then have $\mu(\mathcal{A}) > m_A^2 = \mu(\mathcal{A})$ which is a contradiction.

Now suppose that $\mathcal{A}$ is any Borel set, then by Theorem 1.8.1 (modified for the product space), for any n there exists an open set $\mathcal{U} \supseteq \mathcal{A}$ such that $\mu(\mathcal{U} - \mathcal{A}) \leq 2^{-n-2}$. For more clarity we denote $\sqrt{\mu(\mathcal{U})}$ by m_U. We already proved that $\mu(\{X \mid \mu(\mathcal{U}_X) > m_U\}) \leq m_U$, and certainly we have:

$$\{X \mid \mu(\mathcal{A}_X) > m_U\} \subseteq \{X \mid \mu(\mathcal{U}_X) > m_U\}$$

But then $\mu(\{X \mid \mu(\mathcal{A}_X) > m_U\}) \leq \sqrt{\mu(\mathcal{U})}$. Also $\{X \mid \mu(\mathcal{A}_X) > m_A + 2^{-n}\} \subseteq \{X \mid \mu(\mathcal{A}_X) > m_U\}$ and then $\mu(\{X \mid \mu(\mathcal{A}_X) > m_A + 2^{-n}\}) \leq m_U \leq m_A + 2^{-n}$. Therefore, for any n we have $\mu(\{X \mid \mu(\mathcal{A}_X) > m_A + 2^{-n}\}) \leq m_A + 2^{-n}$, and as

$$\{X \mid \mu(\mathcal{A}_X) > m_A\} = \bigcap_n \{X \mid \mu(\mathcal{A}_X) > m_A + 2^{-n}\}$$

we then have $\mu(\{X \mid \mu(\mathcal{A}_X) > m_A\}) = \inf_n \mu(\{X \mid \mu(\mathcal{A}_X) > m_A + 2^{-n}\}) \leq \inf_n m_A + 2^{-n} \leq m_A$. ■

1.9 Category

In his doctoral thesis ([2]), Baire introduced in 1899, the notion of 'Baire category'. This provides a powerful tool in general topology and functional analysis, but also constitutes the premises of the celebrated Cohen's technique of forcing, who could achieve, using in a very clever way some ideas behind the notions of Baire category, a proof that both the negation of continuum hypothesis and a negation of the axiom of choice are both consistent with ZF (see [9]). This major breakthrough in set theory completed the work initiated by Gödel, who proved earlier that the continuum hypothesis and the axiom of choice are also both consistent with ZF (see [27]), making the two statements independent from ZF.

We briefly introduce in this section the basics about Baire category, and especially the effective version of the fact that every Borel set has the Baire property.

Definition 1.9.1. *A* ***Baire space*** *is a topological space such that any countable intersection of dense open sets is dense.* ■

Recall that the space $\mathbb{N}^\mathbb{N}$ is called 'the Baire space', rightfully, as it is straightforward to verify that it is indeed a Baire space. We examine here an effective version of that:

Proposition 1.9.1 (effective Baire theorem):
In $\mathbb{N}^\mathbb{N}$, any dense Π^0_2 set $\mathcal{A}$ contains densely many computable points, that is, for any interval σ one can find uniformly in σ a computable point in $\mathcal{A} \cap [\sigma]$.

Proof: With $\mathcal{A} = \bigcap_n \mathcal{U}_n$ where each $\mathcal{U}_n$ is dense, given σ, we can search for the first extension σ_0 in $\mathcal{U}_0$, then for the first extension of σ_0 in $\mathcal{U}_1$, and so on. By construction $\bigcap_n [\sigma_n] \subseteq \bigcap_n \mathcal{U}_n$ contains only one computable point. ■

Similarly we can verify that Proposition 1.9.1 also works in the Cantor space. Conversely we then have that the countable union of closed set with empty interior also has empty interior.

Definition 1.9.2. *In a Baire space, a set is said to be* ***meager****, or of* ***first category****, if it is contained in a* $\mathbf{\Sigma^0_2}$ *set of empty interior. A set is said to be* ***co-meager****, if it contains a* $\mathbf{\Pi^0_2}$ *dense set. A set is of* ***second category*** *is it is not meager.* ■

One can view the notion of being meager as 'being small', and the notion of being co-meager as 'being big'.

Fact 1.9.1

The notion of being meager is closed under subset and by countable union. The notion of being co-meager is closed under superset and countable intersection. This follows from the definition of a Baire space.

We now define the most important notion for this section, the notion of having the Baire property. It is the notion which is behind the idea of forcing, that Cohen developed later. The very general idea is that we do not want to deal with sets which are too complex to describe. Also up to a set that is considered 'small' (here meager), we would like any 'complex set' to be equal to a simple set, here, an open set. Therefore, if we can find a way so that 'small set' do not matter in some sense, it will be much easier to deal with complex sets, as they can be considered as open sets.

Definition 1.9.3. *In a Baire space, a set $\mathcal{A}$ has the* ***Baire property*** *if there is an open set $\mathcal{U}$ and a meager set $\mathcal{M}$ such that $\mathcal{A} = \mathcal{U} \triangle \mathcal{M}$, where $\mathcal{U} \triangle \mathcal{M}$ is the symmetric difference of $\mathcal{A}$ and $\mathcal{M}$, equals to $(\mathcal{U} - \mathcal{M}) \cup (\mathcal{M} - \mathcal{U})$.* ■

We now show that any Borel set has the Baire property. As usual, we will prove an effective version of it and before we do so we prove a small proposition. Recall Definition 1.5.4 of Σ^0_α or Π^0_α-open sets and of their complement.

Proposition 1.9.2:
For any Π^0_α-closed set $\mathcal{A} \subseteq 2^{\mathbb{N}}$, the interior of $\mathcal{A}$ is a Π^0_α-open set and the boundary of $\mathcal{A}$ is a $\Pi^0_{\alpha+1}$-closed set. Also they can be obtained uniformly.

For any Σ^0_α-closed set $\mathcal{A} \subseteq 2^{\mathbb{N}}$, the interior of $\mathcal{A}$ a $\Pi^0_{\alpha+1}$-open set and the boundary of $\mathcal{A}$ is a $\Sigma^0_{\alpha+1}$-closed set. Also they can be obtained uniformly.

Proof: Recall that a Π^0_1-closed set is the complement of a Σ^0_1-open set. Following this definition, consider any set of strings A intended to describe a Π^0_1-closed set $\mathcal{A}$ (whose complement is then equal to $[A^c]^{\prec}$), the interior $\mathcal{U}$ of $\mathcal{A}$ is described by the set $\{\tau \mid \forall \sigma \geq \tau \ \ \sigma \in A\}$, and the boundary $\mathcal{F}$ of $\mathcal{A}$, is equal to $\mathcal{A} - \mathcal{U}$.

If $\mathcal{A}$ is a Π^0_α-closed set, $\mathcal{U}$ is clearly a Π^0_α-open set, and as the complement of $\mathcal{F}$, described by $\mathcal{A}^c \cup \mathcal{U}$ is a $\Delta^0_{\alpha+1}$-open set, the set $\mathcal{F}$ is then a $\Delta^0_{\alpha+1}$-closed set, and then a $\Pi^0_{\alpha+1}$-closed set.

Similarly, if $\mathcal{A}$ is a Σ^0_α-closed set, $\mathcal{U}$ is clearly a $\Pi^0_{\alpha+1}$-open set and $\mathcal{F}$ clearly a $\Sigma^0_{\alpha+1}$-closed set. ■

We recall here a few equalities about symmetric difference, that are needed below, and are easy to verify. First $\mathcal{U} \vartriangle \mathcal{F} \subseteq \mathcal{U} \cup \mathcal{F}$. Also if $\mathcal{A} = \mathcal{U} \vartriangle \mathcal{F}$ then $\mathcal{A}^c = \mathcal{U}^c \vartriangle \mathcal{F}$. Then we have that $\mathcal{A} \vartriangle (\mathcal{B} \vartriangle \mathcal{A}) = \mathcal{B}$, and we have that $(\bigcup_n \mathcal{A}_n) \vartriangle (\bigcup_n \mathcal{B}_n) \subseteq \bigcup_n (\mathcal{A}_n \vartriangle \mathcal{B}_n)$. Finally, if $\mathcal{U}$ and $\mathcal{F}$ are disjoint we have $(\mathcal{U} \sqcup \mathcal{F}) \vartriangle \mathcal{B} = \mathcal{U} \vartriangle (\mathcal{F} \vartriangle \mathcal{B})$.

Theorem 1.9.1 (Effective Baire property theorem):
For any Σ^0_α set $\mathcal{A} \subseteq 2^{\mathbb{N}}$, one can find uniformly in an index for $\mathcal{A}$ a Σ^0_α-open set $\mathcal{U}$ and uniformly in n a $\Pi^0_{<\alpha}$-closed set $\mathcal{F}_n$ such that $\mathcal{A} = \mathcal{U} \vartriangle \mathcal{B}$ for some set $\mathcal{B}$ included in $\bigcup_n \partial\mathcal{F}_n$, where $\partial\mathcal{F}_n$ is the boundary of $\mathcal{F}_n$.

For any Π^0_α set $\mathcal{A} \subseteq 2^{\mathbb{N}}$, one can find uniformly in an index for $\mathcal{A}$ a Π^0_α-open set $\mathcal{U}$ and a uniformly in n a Π^0_α-closed set $\mathcal{F}_n$ such that $\mathcal{A} = \mathcal{U} \vartriangle \mathcal{B}$ for some set $\mathcal{B}$ included in $\bigcup_n \partial\mathcal{F}_n$, where $\partial\mathcal{F}_n$ is the boundary of $\mathcal{F}_n$.

Proof: We show the result by induction over computable ordinals. If $\mathcal{A}$ is Σ^0_1 take $\mathcal{U} = \mathcal{A}$ and each $\mathcal{F}_n = \varnothing$. Suppose the theorem is true for every Σ^0_α set and let us prove it is true for every Π^0_α set. Consider a Π^0_α set $\mathcal{A}$. By induction we have a Σ^0_α set $\mathcal{U}$ and a sequence of $\Pi^0_{<\alpha}$ closed sets $\mathcal{F}_n$ such that $\mathcal{A}^c = \mathcal{U} \vartriangle \mathcal{B}$ for some set $\mathcal{B} \subseteq \bigcup_n \partial\mathcal{F}_n$.

Then we have $\mathcal{A} = \mathcal{U}^c \vartriangle \mathcal{B}$. Let us denote the Π^0_α closed set $\mathcal{U}^c$ by $\mathcal{F}$. By Proposition 1.9.2 the interior of $\mathcal{F}$ is a Π^0_α open set $\mathcal{V}$ and we have $\mathcal{F} = \mathcal{V} \sqcup \partial\mathcal{F}$. Therefore as $\mathcal{A} = (\mathcal{V} \sqcup \partial\mathcal{F}) \vartriangle \mathcal{B}$ we also have $\mathcal{A} = \mathcal{V} \vartriangle (\partial\mathcal{F} \vartriangle \mathcal{B})$. Furthermore $\partial\mathcal{F} \vartriangle \mathcal{B} \subseteq \partial\mathcal{F} \cup \mathcal{B} \subseteq \partial\mathcal{F} \cup \bigcup_n \partial\mathcal{F}_n$.

Suppose now that the theorem is true for any Π^0_β set with $\beta < \alpha$ and let us prove it is true for any Σ^0_α set. Consider the Σ^0_α set $\mathcal{A} = \bigcup_n \mathcal{A}_n$ where each $\mathcal{A}_n$ is $\Pi^0_{\beta_n}$ for $\beta_n < \alpha$. By induction hypothesis, for each n there is a $\Pi^0_{\beta_n}$ open set $\mathcal{U}_n$ and uniformly in m a $\Pi^0_{\beta_n}$ closed set $\mathcal{F}_m$ such that $\mathcal{A}_n = \mathcal{U}_n \vartriangle \mathcal{B}_n$ for some set $\mathcal{B}_n \subseteq \bigcup_m \partial\mathcal{F}_{m,n}$.

Using the fact that $\mathcal{A} \vartriangle (\mathcal{B} \vartriangle \mathcal{A}) = \mathcal{B}$ for any sets $\mathcal{A}$ and $\mathcal{B}$, we have that $\bigcup_n \mathcal{A}_n = \bigcup_n \mathcal{U}_n \vartriangle \mathcal{B}$ where $\mathcal{B}$ is equal to $(\bigcup_n \mathcal{A}_n \vartriangle \bigcup_n \mathcal{U}_n)$. We also have $\mathcal{B} = (\bigcup_n \mathcal{A}_n \vartriangle \bigcup_n \mathcal{U}_n) \subseteq \bigcup_n (\mathcal{A}_n \vartriangle \mathcal{U}_n) = \bigcup_n \mathcal{B}_n$. Then $\mathcal{B}$ is included in $\bigcup_n \bigcup_m \partial\mathcal{F}_{n,m}$. Also note that $\bigcup_n \mathcal{U}_n$ is a Σ^0_α open set and that uniformly in n and m, the set $\mathcal{F}_{m,n}$ is $\Pi^0_{<\alpha}$. ■

Chapter 2

Algorithmic randomness and Cohen genericity

> Toutes les théories qui se rattachent à la mesure des ensembles peuvent donc être considérées comme une contribution à la théorie des nombres inaccessibles ; si nous ne pouvons étudier individuellement aucun de ces nombres nous pouvons étudier des problèmes de probabilité qui sont relatifs, soit à l'ensemble de ces nombres, soit à certains sous-ensembles. La réponse à certaines questions se trouve être ainsi un coefficient de probabilité. Une telle réponse peut avoir souvent un grand intérêt dans bien des questions scientifiques.
>
> Les nombres inaccessibles, Émile Borel

In this chapter, we present the basic notions of algorithmic randomness and Cohen genericity. Each of those field provides a way to study an aspect of the general notion of 'being typical', for an element of the Cantor space. Algorithmic randomness is the study of the sequences which are typical with respect to measure theory, whereas Cohen genericity is the study of the sequences which are typical with respect to Baire categoricity.

2.1 Algorithmic randomness

Let us start with algorithmic randomness. Intuitively a random sequence of 0's and 1's should not have any atypical property. Here, a property is considered atypical if the set of sequences having it is of measure 0. First we have to specify what measure to consider. Unless we explicitly say otherwise, we always consider the Lebesgue measure, denoted by λ.

Then we have to make a selection among all the possible sets of measure 0. Indeed, any X has the property of being in the set $\{X\}$, thus if we consider every set of measure 0, nothing would be random. Actually, if we want the definition to make sense, we should select up to countably many sets of measure 0. By the countable subadditivity property of measures, the corresponding set of random sequences is then of measure 1. But what set should we select? Effective descriptive set theory provides a hierarchy of natural answers to that question: We can select some α and decide that something is random if it is in no Π^0_α set of measure 0[1].

[1] We could also adopt a stronger definition and say that something is random if it belongs to no Π^0_α set of measure 0 for every computable α. We then obtain Δ^1_1-randomness, a notion studied in Section 3.7

2.1.1 Martin-Löf randomness

Definitions

It appears that the Π_2^0 sets give us enough description power to capture most of the 'natural' atypical properties, as illustrated in the next example:

Example 2.1.1:
One would expect that the average frequency of 0's and 1's among the first bits of a random sequence, has a limit and that this limit is 1/2. Also the set $\mathcal{A}$ of sequences so that the superior limit of the frequency of $0'$s is above $1/2+\varepsilon$ is a Π_2^0 set of measure 0 for any ε. Formally we can decompose $\mathcal{A}$ the following way:

$$\mathcal{A} = \bigcap_n \mathcal{U}_n \text{ with } \mathcal{U}_n = \bigcup_{m\geq n} \mathcal{C}_m \text{ and } \mathcal{C}_m = \left\{\sigma \in 2^m \;:\; \frac{\#\{i \leq m \;:\; \sigma(i) = 0\}}{m} - \frac{1}{2} > \varepsilon\right\}$$

where $\#X$ denotes the cardinality of the set X. It is clear that $\mathcal{A}$ is Π_2^0. We should now prove it is of measure 0, by sketching a proof of a particular case of the law of large numbers: Using Hoeffding's inequality (see [31]) we have for each m that $\lambda(C_m) \leq e^{-2m\varepsilon^2}$. For the reading clarity we now set $a_m = e^{-2m\varepsilon^2}$. Then for each n, using measures' subadditivity we have $\lambda(\mathcal{U}_n) \leq \sum_{m\geq n} a_m = \sum_{m\geq n} a_m = a_n \times (1 + a_1^1 + a_1^2 + \dots)$. The geometric series convergence gives us $\lambda(\mathcal{U}_n) \leq a_n/(1-a_1)$. As n goes to infinity, the sequence $a_n/(1-a_1)$ clearly converges to 0. ◇

In the above example, we have that the Π_2^0 set is proved to be of measure 0 in a strong sense, that is the function which to each n associates the measure of $\mathcal{U}_n$ is bounded by a computable function converging to 0. We will see later that this is not possible for every Π_2^0 set of measure 0. It was Martin-Löf, in 1966 who had the brilliant idea of making the distinction in [58]:

Definition 2.1.1. *An intersection of measurable sets $\bigcap_n \mathcal{A}_n$ is said to be **effectively of measure** 0 if the function which to n associates the measure of $\mathcal{A}_n$ is bounded by a decreasing computable function whose limit is 0. A **Martin-Löf test** is a Π_2^0 set $\bigcap_n \mathcal{U}_n$ effectively of measure 0. We say that $Z \in 2^{\mathbb{N}}$ is **Martin-Löf random** if it is in no Martin-Löf test.* ■

Why did Martin-Löf make the distinction between Π_2^0 sets effectively of measure 0 and just Π_2^0 sets of measure 0? The reason is described in the 1966 paper: There exists a universal Martin-Löf test, i.e., a test containing all the others. Before proving this, we should give a few general facts. First we should argue that we can require without loss of generality that a set $\bigcap \mathcal{A}_n$ is effectively of measure 0 if $\lambda(\mathcal{A}_n) \leq 2^{-n}$.

Fact 2.1.1

If $\bigcap_n \mathcal{A}_n$ is a set so that $\lambda(\mathcal{A}_n) \leq f(n)$ with $f : \mathbb{N} \to \mathbb{N}$ a computable function such that f goes to 0, one can always find in a computable way, for every n, the first index m so that $\lambda(\mathcal{A}_m) \leq 2^{-n}$. Formally there is a total computable function $g : \omega \to \omega$ which to n associate the first value m so that $f(m) \leq 2^{-n}$. As we then have $\bigcap_n \mathcal{A}_n = \bigcap_n \mathcal{A}_{g(n)}$ and $\lambda(\mathcal{A}_g(n)) \leq 2^{-n}$, we can require without loss of generality that a set $\bigcap \mathcal{A}_n$ is effectively of measure 0 if $\lambda(\mathcal{A}_n) \leq 2^{-n}$.

We now make an easy but important remark. When enumerating the set of strings W describing a Σ^0_1 set $\mathcal{U}$, at each time t of the enumeration, $[W_t]^{\prec}$ is a clopen set. In particular a Σ^0_1 set can always be described with an enumeration of pairwise incomparable strings (such a set of strings is also said to be prefix-free, as we will see with Definition 3.7.8): At each enumeration step $t+1$, instead of enumerating τ in W at step $t+1$, we enumerate a pairwise incomparable finite set of strings describing the clopen set $[\tau] - W[t]$.

This will not be the case anymore when dealing with higher randomness. We will see with Theorem 7.1.1 that an open set described by a Π^1_1 set of strings cannot necessarily be described by a Π^1_1 set pairwise incomparable of strings. We now give a fact about Π^0_2 sets in general.

Fact 2.1.2

Given a Π^0_2 set $\bigcap_n \mathcal{U}_n$, we can always suppose that the strings we enumerate to describe $\mathcal{U}_n$ are pairwise incomparable. Furthermore we can always suppose that the Π^0_2 set is **decreasing**, that is $\mathcal{U}_{n+1} \subseteq \mathcal{U}_n$. Indeed, we can simply consider that $\mathcal{U}_n$ is $\bigcap_{m \leq n \mathcal{U}_m}$. Similarly a Σ^0_2 set can always be considered **increasing**, and more generally any Π^0_α set can be considered decreasing and any Σ^0_α set can be considered increasing.

Universal Martin-Löf randomness test

We now prove that there is a universal Martin-Löf test:

Theorem 2.1.1:
There is a universal Martin-Löf test, i.e., a Martin-Löf test $(\mathcal{V}_n)_{n \in \mathbb{N}}$ such that for any sequence X, we have that X is not Martin-Löf random iff $X \in \bigcap_n \mathcal{V}_n$.

Proof: Let $\{\mathcal{P}_n\}_{n \in \omega}$ be the canonical computable enumeration of the Π^0_2 sets, where n is the index of the set $\mathcal{P}_n$. To build the universal Martin-Löf test, we simply diagonalize against all possible tests. To do so, we should first argue that we can enumerate all Martin-Löf tests. It is of course not possible to determine in advance in a computable way if an open set has measure smaller than some ε. However it is always possible to transform a Π^0_2 set $\mathcal{A}$ into a Martin-Löf test $\mathcal{A}'$, in a way that keeps $\mathcal{A}$ unchanged if it is already a Martin-Löf test.

Formally, we have a total computable function $f : \mathbb{N} \to \mathbb{N}$ so that for every n, first $\mathcal{P}_{f(n)}$ is always a Martin-Löf test, and then if n is already the index of a Martin-Löf test, we have $\mathcal{P}_n = \mathcal{P}_{f(n)}$. To do so, given a Π^0_2 set $\bigcap_n \mathcal{U}_n$, we simply enumerate for each $\mathcal{U}_n$, its corresponding set of strings U_n as long as $\lambda([U_n]^{\prec})[t] \leq 2^{-n}$. If there is a first stage t such that $\lambda([U_n]^{\prec})[t] > 2^{-n}$, we stop the enumeration at stage $t-1$. It is clear that applying this technique we can have a computable enumeration of the Martin-löf tests, that contains them all.

So let $U_{n,m}$ be the m-th Σ^0_1 set of strings corresponding to the m-th component of the n-th Martin-Löf test. We simply define $\mathcal{V}_n$ to be the open set described by the set of strings $\bigcup_k U_{k,k+n+1}$. It is clear that each $\mathcal{V}_n$ is a Σ^0_1 set, uniformly in n. Also by countable subadditivity we have $\lambda(\mathcal{V}_n) \leq \sum_k \lambda([U_{k,k+n+1}]^{\prec}) \leq \sum_k 2^{-n-k-1} \leq 2^{-n}$. Thus $\bigcap_n \mathcal{V}_n$ is a Martin-Löf test. We then already have that $X \in \bigcap_n \mathcal{V}_n$ implies that X is not Martin-Löf random. All we have to prove is the converse. But we clearly have for any e that $\bigcap_m [U_{e,m}]^{\prec} \subseteq \bigcap_n \bigcup_k [U_{k,k+n+1}]^{\prec}$. Also if X is not Martin-Löf random it belongs to $\bigcap_m [U_{e,m}]^{\prec}$ for some e and then it belongs to $\bigcap_n \mathcal{V}_n$. ∎

The fact that there is a universal Martin-Löf test can be used to provide a canonical example of a definable Martin-Löf random sequence.

Example 2.1.2:
Let $\bigcap_n \mathcal{U}_n$ be a universal Martin-Löf test. In particular the complement of each $\mathcal{U}_n$ is a Π^0_1 set containing only Martin-Löf random sequences. Also the leftmost path of such a Π^0_1 set is a left-c.e. Martin-Löf random sequence. Initially, the first example of a definable Martin-Löf random sequence was made by Chaitin in [5], who proved that the probability that a computer program halts (in a sense that we don't make precise here) is both Martin-Löf random and Turing complete (it can Turing compute $\emptyset'$). Such a number is called a Chaitin's Ω number.

Later, Kučera and Slaman proved in [46] that a Martin-Löf random sequence is a Chaitin's Ω number iff it is left-c.e. iff it is the leftmost path of a Π^0_1 set containing only Martin-Löf random sequences. ◊

Generally, dealing with Martin-Löf randomness, we assumed that a universal Martin-Löf test is fixed, and we refer to it as *the* universal Martin-Löf test.

Martin-Löf randomness relatively to some oracle

One can easily relatizive the notion of Martin-Löf randomness to any oracle X, by defining that a sequence is Martin-Löf random relatively to X if it is in no $\Pi^0_2(X)$ set effectively of measure 0. Similarly we can prove the existence of an X-universal Martin-Löf test, and the use of the oracle for those universal tests can actually be made continuous in a sense that we now make precise:

Definition 2.1.2. *An* ***oracle*** $\boldsymbol{\Sigma^0_1}$ ***set*** $\mathcal{U}$ *is a* Σ^0_1 *subset of* $2^{<\mathbb{N}} \times 2^{<\mathbb{N}}$. *Given any oracle* X *we then write* $\mathcal{U}^X$ *to denote the* $\Sigma^0_1(X)$ *set described by the set of strings* $\{\sigma : \exists \tau \prec X\ (\tau, \sigma) \in \mathcal{U}_n\}$. ∎

Definition 2.1.3. *An* ***X-Martin-Löf test*** *is a uniform sequence of oracle* Σ^0_1 *sets* $\{\mathcal{U}_n\}_{n\in\mathbb{N}}$ *such that for any* n *we have* $\lambda(\mathcal{U}^X_n) \leq 2^{-n}$. *A sequence* Z *is* X*-Martin-Löf random if it is in no* X*-Martin-Löf test. An* ***oracle Martin-Löf test*** *is a uniform sequence of oracle* Σ^0_1 *sets* $\{\mathcal{U}_n\}_{n\in\mathbb{N}}$ *which is an* X*-Martin-Löf test for every oracle* X. ∎

Theorem 2.1.2:
There is a universal oracle Martin-Löf test $\{\mathcal{U}_n\}_{n\in\mathbb{N}}$*, that is, for every oracle* X *and every* X*-Martin-Löf test* $\{\mathcal{V}_n\}_{n\in\mathbb{N}}$*, we have* $\bigcap_n \mathcal{V}^X_n \subseteq \bigcap_n \mathcal{U}^X_n$.

We only sketch a proof of the theorem: The delicate part is to make sure that given an oracle Σ^0_1 set $\mathcal{U}$ and an integer n, we can uniformly transform $\mathcal{U}$ in a way that $\lambda(\mathcal{U}^X_n)$ is bounded by 2^{-n} for every X, without damaging $\mathcal{U}^X$ on oracles X for which we already had $\lambda(\mathcal{U}^X_n)$ bounded by 2^{-n} in the first place. This being mentioned, there is then no difficulty to prove the existence of a universal oracle Martin-Löf test. We shall see that this does not hold anymore using continuous relativization with Π^1_1-Martin-Löf randomness.

2.1.2 Notions of n-randomness and α-randomness

One can iterate this idea of Martin-Löf randomness by considering Π^0_n sets effectively of measure 0 for any $n \geq 2$. Martin-Löf randomness is also called **1-randomness**, the use of Π^0_3 sets effectively of measure 0 gives us **2-randomness**, Π^0_4 sets give us **3-randomness**, and so on:

Definition 2.1.4. *A **n-Martin-Löf** test is a Π^0_{n+1} set $\bigcap_n \mathcal{A}_n$ effectively of measure 0. We say that $Z \in 2^{\mathbb{N}}$ is **n-random** if it is in no **n-Martin-Löf** test.* ∎

It is of course possible to extend the notion of n-randomness through the computable ordinals. In the following definition, ω-randomness corresponds to $\Pi^0_{\omega+1}$ sets effectively of measure 0, $(\omega+1)$-randomness to $\Pi^0_{\omega+2}$ set effectively of measure 0, and so on. Note that we have no name for the notion of being in no Π^0_n set effectively of measure 0 for any n, or for the notion of being in no Π^0_ω set effectively of measure 0. The reason is that we do not have universal tests for those notions. Consider for example $\bigcup_n \mathcal{A}_n$, a Σ^0_ω set effectively of measure 1. For some n we have that $\mathcal{A}_n$ contains a $\Pi^0_1(\emptyset^{(m)})$ set of positive measure for some $m \geq n$. Also the leftmost-path of such a set is Δ^0_{m+2} and can then be captured by a Π^0_ω test.

Definition 2.1.5. *For $\alpha \geq \omega$, a **α-Martin-Löf test** is a $\Pi^0_{\alpha+1}$ set $\bigcap_n \mathcal{A}_n$ effectively of measure 0. We say that $Z \in 2^{\mathbb{N}}$ is **α-random** if it is in no α-Martin-Löf test.* ∎

The descriptive set theorist might find the two previous definitions rather strange, because more description power can be used for each set in the intersection, but in the mean time we keep the same notion for 'being effectively of measure 0'. Instead it could seem normal, for example for 2-randomness, to only require for the function which to n associates the measure of $\mathcal{A}_n$ to be bounded by a decreasing Δ^0_2 function converging to 0 (instead of Δ^0_1). We shall actually see now that this does not matter.

Lemma 2.1.1 *The following are equivalent for any computable ordinal α and any $X \in 2^{\mathbb{N}}$:*

1. *X is in no $\Pi^0_2(\emptyset^{(<\alpha)})$ set of measure 0, effectively in $\emptyset^{(<\alpha)}$.*
2. *X is in no $\Pi^0_2(\emptyset^{(<\alpha)})$ set effectively of measure 0.*
3. *X is in no set $\bigcap_n \mathcal{U}_n$ where each $\mathcal{U}_n$ is a $\Sigma^0_1(\emptyset^{(<\alpha)})$ set uniformly in n, of measure smaller then 2^{-n}.*
4. *X is in no $\Pi^0_{\alpha+1}$ set effectively of measure 0.*
5. *X is in no $\Pi^0_{\alpha+1}$ set of measure 0, effectively in $\emptyset^{(<\alpha)}$.*

PROOF: (1) $\Longrightarrow$ (2) : Let $\bigcap_n \mathcal{U}_n$ be a $\Pi^0_2(\emptyset^{(<\alpha)})$ set of measure 0, effectively in $\emptyset^{(<\alpha)}$, that is with a $\emptyset^{(<\alpha)}$-computable function $f : \mathbb{N} \to \mathbb{N}$ whose limit is 0, and such that $\lambda(\mathcal{U}_n) \leq f(n)$. We can define the $\emptyset^{(<\alpha)}$-computable function $g : \mathbb{N} \to \mathbb{N}$ which to n associates the smallest m such that $f(m) \leq 2^{-n}$. We can build the new test $\bigcap_n \mathcal{U}_{g(n)}$, which is a $\Pi^0_2(\emptyset^{(<\alpha)})$ set effectively of measure 0 and equal to $\bigcap_n \mathcal{U}_n$.

(2) $\Longrightarrow$ (3) : Let $\bigcap_n \mathcal{U}_n$ be a $\Pi^0_2(\emptyset^{(<\alpha)})$ set effectively of measure 0. We now have to make an equivalent test $\bigcap_n \mathcal{V}_n$ where each $\mathcal{V}_n$ is $\Sigma^0_1(\emptyset^{(<\alpha)})$ uniformly in n, that is we cannot use $\emptyset^{(<\alpha)}$ anymore to find the index of the n-th open set. We simply define $\mathcal{V}_n$ to be the $\Sigma^0_1(\emptyset^{(<\alpha)})$ open set which first uses $\emptyset^{(<\alpha)}$ to compute the index of $\mathcal{U}_n$, and then is equal to $\mathcal{U}_n$.

(3) $\Longrightarrow$ (4) : Let $\bigcap_n \mathcal{U}_n$ be such that each $\mathcal{U}_n$ is a $\Sigma^0_1(\emptyset^{(<\alpha)})$ set uniformly in n, of measure smaller then 2^{-n}. As $\emptyset^{(<\alpha)}$ is Δ^0_α, the set of strings W_n that describes $\mathcal{U}_n$ is a Σ^0_α set of strings. Also by Proposition 1.5.1, each $\mathcal{U}_n$ is then a Σ^0_α set, and then their intersection is a $\Pi^0_{\alpha+1}$ set.

(4) $\Longrightarrow$ (5) : Trivial.

(5) $\Longrightarrow$ (1) : Consider a $\Pi^0_{\alpha+1}$ set $\bigcap \mathcal{A}_n$, of measure 0 effectively in $\emptyset^{(<\alpha)}$, where each $\mathcal{A}_n$ is a Σ^0_α set. From Theorem 1.8.1 one can find uniformly in q and in an index for $\mathcal{A}_n$ a Σ^0_α-open set $\mathcal{U}_n \supseteq \mathcal{A}_n$ with $\lambda(\mathcal{U}_n - \mathcal{A}_n) \leq q$. Also using Corollary 1.6.1, the Σ^0_α set of strings describing $\mathcal{U}_n$ is $\Sigma^0_1(\emptyset^{(<\alpha)})$, uniformly in q and in an index for $\mathcal{A}_n$. But then we can easily build a $\Pi^0_2(\emptyset^{(<\alpha)})$ set of measure 0, effectively in $\emptyset^{(<\alpha)}$, which contains $\bigcap \mathcal{A}_n$. For each n we simply find $\mathcal{U}_n \supseteq \mathcal{A}_n$ such that $\lambda(\mathcal{U}_n - \mathcal{A}_n) \leq 2^{-n}$ and we then have $\lambda(\mathcal{U}_n) \leq \lambda(\mathcal{A}_n) + 2^{-n}$. Then the measure of $\mathcal{U}_n$ also goes to 0, effectively in $\emptyset^{(<\alpha)}$. ■

The previous lemma is interesting also because it shows that the notion of n-randomness or α-randomness actually corresponds to the notion of Martin-Löf randomness, but relatively to some oracle. We extract this important part of Lemma 2.1.1 into the following theorem:

Theorem 2.1.3:
The following are equivalent for any computable ordinal α and any $Z \in 2^{\mathbb{N}}$.

1. *Z is in no $\Pi^0_2(\emptyset^{(<\alpha)})$ set effectively of measure 0.*
2. *Z is in no $\Pi^0_{\alpha+1}$ set effectively of measure 0 (i.e. Z is α-random).*

Corollary 2.1.1:
For any computable α, there is a universal α-Martin-Löf test, that is, a $\Pi^0_2(\emptyset^{(<\alpha)})$ set effectively of measure 0, that contains every α-Martin-Löf test.

PROOF: From Theorem 2.1.2 there is a universal oracle Martin-Löf test, which can then be used with an appropriate oracle as a universal α-test for any α. ∎

Most of the time, the α-Martin-Löf tests will be considered to be $\Pi^0_2(\emptyset^{(<\alpha)})$ sets effectively of measure 0 instead of $\Pi^0_{\alpha+1}$ sets effectively of measure 0.

2.1.3 Notions of weak-n-randomness

The case of what happens if we drop the 'effectively of measure 0' condition is also interesting. Indeed, the corresponding notion of randomness is more natural in the sense that it is simpler to describe. However, many nice properties of Martin-Löf randomness, such as the existence of a universal test, disappear when we drop the 'effectively of measure 0' condition.

Definition 2.1.6. *A Π^0_2 nullset is also called a **weak-2-test**. We say that $X \in 2^\omega$ is **weakly-2-random** if it is in no weak-2-test.* ∎

We shall now see an equivalent way to define weak-2-tests, that will help us to get a better understanding of weakly-2-randomness. As we saw in the previous section with Theorem 2.1.3, the notion of 2-randomness is equivalent the notion of 1-randomness, where $\emptyset^{(1)}$ can be used to both pick the index of the n-th component of the Martin-Löf test and to enumerate this n-th component. We shall now prove that weakly-2-randomness is equivalent to the notion of Martin-Löf randomness, where $\emptyset^{(1)}$ can be used to pick the index of the n-th component of the Martin-Löf test, but where each of these components is still Σ^0_1 and not $\Sigma^0_1(\emptyset^{(1)})$. A higher randomness analogue of this notion of randomness, characterized by this special use of $\emptyset^{(1)}$ will be studied in Section 5.2.

Theorem 2.1.4:
Let $\{\mathcal{U}_n\}_{n\in\omega}$ be a canonical enumeration of the Σ^0_1 sets. A sequence X is weakly-2-random iff X is in no test $\bigcap_n \mathcal{U}_{f(n)}$ where $\lambda(\mathcal{U}_{f(n)}) \leq 2^{-n}$ and where $f : \mathbb{N} \to \mathbb{N}$ is a total $\emptyset^{(1)}$-computable function.

PROOF: If X is in a Π^0_2 nullset $\bigcap_n \mathcal{U}_n$, then one can use $\emptyset^{(1)}$ to find uniformly in n the first m so that $\lambda(\mathcal{U}_m) \leq 2^{-n}$. For the converse, notice that if f is a $\emptyset^{(1)}$-computable function, then $\bigcap_n \mathcal{U}_{f(n)} = \bigcap_{n,t} \bigcup_{s\geq t} \mathcal{U}_{f_s(n)}$. ∎

We saw with Example 2.1.2 that there is a left-c.e. Martin-Löf random sequence. We shall see that this does not hold anymore for weak-2-randomness.

Proposition 2.1.1:
No Δ^0_2 sequence is weakly-2-random.

PROOF: Using the equivalent test notion of Theorem 2.1.4, one can easily put a weak-2-test on any $\emptyset^{(1)}$-computable sequence X. We simply build the $\emptyset^{(1)}$-computable function f which to n associates an index of the Σ^0_1 set $[X\upharpoonright_n]$. ∎

Corollary 2.1.2:
There is no universal weakly-2-test.

PROOF: For any weak-2-test $\bigcap_n \mathcal{U}_n$, the leftmost path of the complement of each $\mathcal{U}_n$ is a left-c.e. sequence (hence Δ^0_2) and therefore not weakly-2-random. ∎

Liang Yu actually proved in [96] much more that the non existence of a universal weakly-2-test: The set of weakly-2-randoms is not even a $\mathbf{\Sigma^0_3}$ set (and therefore not a $\mathbf{\Sigma^0_2}$ set). This implies that the exact Borel complexity of the weakly-2-randoms is $\mathbf{\Pi^0_3}$, as the complement of the union of all weak-2-tests.

From Proposition 2.1.1 we have that the set of weakly-2-randoms is strictly included in the set of 1-randoms. It is also clear from Theorem 2.1.4 that the set of 2-randoms is included in the set of weakly-2-randoms. Also Liang Yu's theorem imply that this inclusion is strict, as the set of 2-randoms is $\mathbf{\Sigma^0_2}$. We shall see in Section 2.2.2 a direct proof of that.

We shall now mention a theorem of Downey, Nies, Weber and Yu (see [16]) which will have some interesting counterpart in Higher randomness:

Theorem 2.1.5:
For a Martin-Löf random sequence Z, the following are equivalent:

1. *Z is weakly-2-random.*
2. *Z does not compute any non-computable Δ^0_2 sequence.*
3. *Z does not compute any non-computable c.e. sequence.*

In order to prove that no sequence Z which Turing computes a non computable Δ^0_2 sequence is weakly-2-random, we use a theorem from Sacks which appears first in [76], but which is also a direct consequence of a similar theorem from de Leeuw, Moore, Shannon, and Shapiro [14]:

Theorem 2.1.6 (de Leeuw, Moore, Shannon, and Shapiro):
Given a set $X \subseteq \mathbb{N}$ which is not Σ^0_1, the set of oracles Y such that X is $\Sigma^0_1(Y)$ has measure 0.

Proof: Suppose that for some X, we have $\lambda(\{Y : \exists e\ X = W_e^Y\}) > 0$. Then by the countable additivity of a measure, already for some e we have $\lambda(\{Y : X = W_e^Y\}) > 0$. Also by the Lebesgue density theorem, there exists a cylinder $[\sigma]$ such that $\lambda(\{Y : X = W_e^Y\} \mid [\sigma]) > 1/2$.

We then claim that X is actually already Σ^0_1. For any integer n, we can enumerate the open set of oracles Y such that $n \in W_e^Y$. Also the measure inside $[\sigma]$, of this open set, goes above $1/2$ iff $n \in X$, in which case when we witness it (which always happens), we can actually enumerate n in X. ∎

From this, we can deduce Sacks theorem:

Corollary 2.1.3 (Sacks):
If a set X is not computable, then the set of oracles which computes X has measure 0.

With a bit of work, we can then prove that the set of sequences which Turing compute a Δ^0_2 set via a given Turing functional is a Π^0_2 set. Also this Π^0_2 set has measure 0, by Sacks theorem, and then no sequence Z which Turing computes a non computable Δ^0_2 sequence is weakly-2-random.

For the converse, one should prove that if Z is Martin-Löf random, but not weakly-2-random, it can Turing compute a non computable c.e. set. The proof is similar to its higher analogue that we will give with Theorem 6.3.1, but using Π^1_1-randomness instead of weakly-Π^1_1-random (the higher analogue of weak-2-randomness).

2.1.4 More on Martin-Löf randomness

Solovay tests

We now give an equivalent notion of test for Martin-Löf randomness, which will reveal itself to be often useful:

Definition 2.1.7. *A **Solovay test** is a computable sequence of effectively open sets $(\mathcal{S}_n)_{n\in\mathbb{N}}$ such that $\sum_{n\in\mathbb{N}} \lambda(\mathcal{S}_n) < +\infty$. We say that $Z \in 2^{\mathbb{N}}$ **passes the test** if Z belongs only to finitely many $\mathcal{S}_n$. We say that a Solovay test $(\mathcal{S}_n)_{n\in\mathcal{N}}$ is **c-bounded** if $\sum_{n\in\mathbb{N}} \lambda(\mathcal{S}_n) \leq c$.* ∎

We shall now see how to turn Solovay tests into Martin-Löf tests, along with the notion of 'being captured' by a Solovay test.

Theorem 2.1.7 ([86];[80]):
Let $Z \in 2^\omega$. The following statements are equivalent:

1. *Z passes each Solovay test*
2. *Z passes each 1-bounded Solovay test*
3. *Z is Martin-Löf random*

PROOF: (1) $\implies$ (2): Trivial.

(2) $\implies$ (3): Suppose that there is a Martin-Löf test $(\mathcal{U}_n)_{n\in\mathbb{N}}$ such that $Z \in \bigcap_n \mathcal{U}_n$. Then $(\mathcal{U}_n)_{n\in\mathbb{N}}$ is also a 1-bounded Solovay test and so Z fails this Solovay test.

(3) $\implies$ (1): Suppose that there is a Solovay test $(\mathcal{U}_n)_{n\in\mathbb{N}}$ such that Z belongs to infinitely many $\mathcal{U}_n$. As $\sum_{n\in\mathbb{N}} \lambda(\mathcal{U}_n)$ is finite, there exists m such that $\sum_{n\geq m} \lambda(\mathcal{U}_n) \leq 1$. Without loss of generality we can remove the m first Σ^0_1 sets from the enumeration and still have that Z is in infinitely many of them. So we now consider that $\sum_n \lambda(\mathcal{U}_n) \leq 1$.

Let G_m be the Σ^0_1 set of strings defined by $\{\sigma \ : \ [\sigma] \subseteq [\mathcal{U}_n]$ for at least 2^m many $n\}$. Since Z belongs to infinitely many $\mathcal{U}_n$, then also there is a prefix of Z in each G_m. Therefore Z is in the Π^0_2 set $\bigcap_n \mathcal{G}_n$ where each $\mathcal{G}_n$ is the Σ^0_1 set corresponding to the set of strings G_n. Let us now prove that $\lambda(\mathcal{G}_n) \leq 2^{-n}$. Suppose otherwise, as $\mathcal{G}_n$ is included in 2^n distinct sets $\mathcal{U}_k$, we have $\sum_n \lambda(\mathcal{U}_n) > 2^{-n}2^n \geq 1$, which is a contradiction. ■

The randomness deficiency

We can easily prove the following fact:

Fact 2.1.3

The Martin-Löf randomness of a sequence remains unchanged by adding, removing or switching finitely many bits of the sequence.

Also given a Martin-Löf random sequence Z, the sequence obtained by adding ten billions of 0's in front of Z is still Martin-Löf random. However it is in some sense less random. Also there is a way to formalize this, by assigning to each random sequence Z an integer value, that measures its randomness deficiency.

Definition 2.1.8. *The* ***randomness deficiency*** *of a sequence X is given by smallest integer c such that $X \notin \mathcal{U}_c$, where $\bigcap_n \mathcal{U}_n$ is a the universal Martin-Löf test. For a given computable ordinal α, the* ***α-randomness deficiency*** *of a sequence X is given by smallest integer c such that $X \notin \mathcal{U}_c$, where $\bigcap_n \mathcal{U}_n$ is the universal α-Martin-Löf test.* ■

Proposition 2.1.2:
For any computable α, any α-random sequence Z and any α-Martin-Löf test $\bigcap_n \mathcal{U}_n$ of index e, one can find uniformly in an upper bound c for the randomness deficiency of Z and in e, an integer m such that $Z \notin \mathcal{U}_m$.

PROOF: Recall that the universal α-Martin-Löf test is given by a $\Pi^0_2(\emptyset^{(<\alpha)})$ set $\bigcap_n \mathcal{V}_n$, effectively of measure 0, which is equal to $\bigcup_k \mathcal{U}_{k,k+n+1}$, where $\bigcap_n \mathcal{U}_{k,n}$ is the α-Martin-Löf test of index k. Also given an upper bound c for the randomness deficiency of Z and an index e for the α-Martin-Löf test $\bigcap_n \mathcal{U}_n$, we easily see that Z is not in $\mathcal{U}_{e+c+1}$. ■

2.2 Genericity

2.2.1 Cohen genericity

Cohen introduced in [9] his general technique of forcing, starting with the simple example of forcing with all dense open sets of the Cantor space in a countable model of ZFC from which he proved the independence of the continuum hypothesis and the independence of the axiom of choice. Forcing revealed itself to be an extraordinary powerful tool of set theory to prove a large variety of independence results. In addition, the study of the effectivization of various forcing notions also appeared to be a powerful tool in computability theory, and also algorithmic randomness.

We discuss here various effective versions of 'being Cohen-generic'. In some sense, Cohen genericity is to categoricity what algorithmic randomness is to measure theory. Roughly, something is random if it belongs to every set of measure 1, whereas something is Cohen generic if it belongs to every dense open set. In [9] Cohen generalized this notion by considering elements which are in every dense open set of a topological space generated by a given partial order. However, most of the time we write 'generic' instead of 'Cohen generic' and if there might be a confusion, we will always precise. In particular, we will see that randomness can be considered to be a type of genericity for some topological space.

Generic sequences have been introduced mainly to be able to speak of their properties without requiring a full knowledge of them. More precisely, if G is generic there is then a way to write "$\Phi(G)$ is true" without fully using G. The "essence" of forcing lies in this stunning property that generic sets have, and which is already described in Theorem 1.9.1 : Every Borel set is equal to an open set, up to a meager set. Also we can study sequences which are in every Σ^0_n-open set, for a given n:

Definition 2.2.1 (Kurtz). *For $n \in \mathbb{N}$, we say that G is **weakly-n-generic** if it belongs to all dense Σ^0_n-open sets.* ∎

Let n be fixed. Theorem 1.9.1 says that any Σ^0_n set $\mathcal{A}$ is equal to a Σ^0_n open set $\mathcal{U}$, up to a meager set $\mathcal{B}$ included in $\bigcup_m \partial\mathcal{F}_m$, where each $\mathcal{F}_m$ is Π^0_{n-1} uniformly in m. Also using Proposition 1.9.2 we have that $2^{\mathbb{N}} - \partial\mathcal{F}_m$ is a dense Σ^0_n-open set. Therefore any weakly-n-generic set G is in $\mathcal{A}$ iff it is in $\mathcal{U}$ and there is a prefix σ of G with $[\sigma] \subseteq \mathcal{U}$ such that every weakly-n-generic sequence extending σ is in $\mathcal{A}$, which is in the language of forcing is : 'σ forces $\mathcal{A}$'. Note that the previous definition can be generalized to any computable α.

Definition 2.2.2. *For α computable, we say that G is **weakly-α-generic** if it belongs to all dense Σ^0_α-open sets.* ∎

Jockusch introduced earlier the notion of n-genericity, in order to force not only every Σ^0_n statement, but also every Π^0_n statement:

Definition 2.2.3 (Jockusch). *For $n \in \omega$, We say that G is **n-generic** if for any Σ^0_n-open set $\mathcal{U}$, either G belongs to $\mathcal{U}$ or G belongs to a cylinder $[\sigma]$ disjoint from $\mathcal{U}$. We generalize this to any computable ordinal α, and we say that G is **α-generic** if for any Σ^0_α-open set $\mathcal{U}$, either G belongs to $\mathcal{U}$ or G belongs to a cylinder $[\sigma]$ disjoint from $\mathcal{U}$.* ∎

Another way to say that G is α-generic, is to say that for any Σ^0_α-open set $\mathcal{U}$, the sequence G belongs to $\mathcal{U}$ or to the interior of the complement of $\mathcal{U}$. It is clear that any weakly-α-generic is also α-generic. We now prove the following theorem:

Theorem 2.2.1:
If G is weakly-α-generic then for any Σ^0_α set $\mathcal{A}$ we have G is in $\mathcal{A}$ iff there is $\sigma \prec G$ such that any weakly-α-generic extending σ is in $\mathcal{A}$.

If G is α-generic then for any Π^0_α set $\mathcal{A}$ we have G is in $\mathcal{A}$ iff there is $\sigma \prec G$ such that any α-generic extending σ is in $\mathcal{A}$.

Proof: Fix G a weakly-α-generic sequence. Let $\mathcal{A}$ be a Σ^0_α set. From Theorem 1.9.1 we have a Σ^0_α open set $\mathcal{U}$ and uniformly in n we have $\Pi^0_{<\alpha}$-closed set $\mathcal{F}_n$ such that $\mathcal{A} = \mathcal{U} \triangle \mathcal{B}$ with $\mathcal{B} \subseteq \bigcup_n \partial\mathcal{F}_n$. Using Proposition 1.9.2 we have that each $\partial\mathcal{F}_n$ is a Π^0_α closed set and then no weakly-α-generic is in $\mathcal{B}$.

Suppose now that G is in $\mathcal{A}$. Then as it is weakly-α-generic it is in $\mathcal{U}$ and then there is a prefix σ of G such that $[\sigma] \subseteq \mathcal{U}$, but then also any weakly-α-generic extending σ is in $\mathcal{A}$. Conversely if we have a prefix σ of G such that any α-generic extending σ is in $\mathcal{A}$, then in particular we have G in $\mathcal{A}$ since G is α-generic.

Let $\mathcal{A}$ now be a Π^0_α set. From Theorem 1.9.1 we have a Π^0_α open set $\mathcal{U}$ and uniformly in n we have Π^0_α closed set $\mathcal{F}_n$ such that $\mathcal{A} = \mathcal{U} \triangle \mathcal{B}$ with $\mathcal{B} \subseteq \bigcup_n \partial\mathcal{F}_n$. But then any α-generic is in $\mathcal{F}_n$ iff it is in the interior of $\mathcal{F}_n$. Therefore any α-generic is in $\mathcal{A}$ iff it is in $\mathcal{U}$ and we can continue the proof like in the previous case. ■

Digression

There is for categoricity, a theorem which is similar to Fubini's Theorem (Theorem 1.8.3) and known as the Kuratowski-Ulam theorem. It roughly says that a Borel set $\mathcal{A} \subseteq 2^{\mathbb{N}} \times 2^{\mathbb{N}}$ is co-meager iff $\{X \mid \mathcal{A}_X \text{ is co-meager}\}$ is co-meager. One can refer for example to Kechris' book (see [34]) for the general statement of Kuratowski-Ulam theorem, and its proof.

Effective versions of the Kuratowski-Ulam theorem can be used, for example to prove that $X \oplus Y$ is 1-generic iff X is 1-generic and Y is 1-generic relatively to X (proved by Liang Yu in [95]), which is an analogue of van Lambalgen's theorem (see Theorem 4.3.2), but for genericity. This can be made much more general, and extended to various notions of set theoretical forcing. One can see for example Theorem 1.4 in Chapter VIII of Kunen's book (see [43]) for a general such theorem, that is then useful to study iterated forcing.

2.2.2 Randomness as a genericity notion

The Cantor space, endowed with the topology generated by Π^0_1 sets is clearly a Baire space, because a decreasing intersection of non-empty closed sets is also not empty. Several notions of genericity related to this topology have been studied in [63], and in particular their connection with domination properties. We give here a genericity notion related to this topology in order to prove a separation of weak-2-randomness from 2-randomness.

Definition 2.2.4. *Let $\{\mathcal{G}_i\}_{i\in\omega}$ be the collection of all Σ^0_2 sets $\mathcal{G}$ such that for any Π^0_1 set $\mathcal{F}$ of positive measure we have $\lambda(\mathcal{G}\cap\mathcal{F})>0$. Then we say that a sequence X is **weakly-Π^0_1-Solovay-generic** if it belongs to $\bigcap_i \mathcal{G}_i$.* ■

This notion is called weak-Π^0_1-Solovay-genericity by analogy with Cohen weak-1-genericity, whereas the analogue of Cohen 1-genericity can be defined to be the sequences X such that for any Σ^0_2 set $\mathcal{G}$, either X is in $\mathcal{G}$, or there exists a Π^0_1 set $\mathcal{F}$ of positive measure and disjoint from $\mathcal{G}$ such that X is in $\mathcal{F}$.

It is clear that the set of weakly-Π^0_1-Solovay-generic is not empty and contained in the set of weakly-2-random. We shall however prove now that this is a set of measure 0, and in particular none of those sequences is 2-random:

Theorem 2.2.2:
No weakly-Π^0_1-Solovay-generic sequence is 2-random.

Proof: We construct uniformly in n a Σ^0_2 set intersecting with positive measure all Π^0_1 sets of positive measure, and with measure smaller than 2^{-n}. Let $\{\mathcal{F}_e\}_{e\in\omega}$ be an enumeration of the Π^0_1 sets. For each e we initialize σ_e to the first string (using lexicographic order) of length $n+e+1$. Our Σ^0_2 set will consist of a computably enumerable set A of indices of Π^0_1 sets. We now describe the algorithm to enumerate elements of A: At stage t, for each substage $e<t$ in increasing order, if the index of $\mathcal{F}_e\cap[\sigma_e]$ has not already been enumerated into A, then enumerate it. After that, if $\lambda(\mathcal{F}_e\cap[\sigma_e])[t]=0$ then reset σ_e to be the string of length $n+e+1$ following σ_e in the lexicographic order. If σ_e is already the last such string, leave it unchanged.

Let us prove that the measure of the Σ^0_2 set represented by A is smaller than 2^{-n}. For each e, if $\lambda(\mathcal{F}_e\cap[\sigma_e])=0$ then by compactness $\lambda(\mathcal{F}_e\cap[\sigma_e])[t]=0$ for some t. Thus at most one string σ_e of length $n+e+1$ such that $\lambda(\mathcal{F}_e\cap[\sigma_e])>0$ has been enumerated into A, and then the measure of the set represented by A is bounded by $\sum_e 2^{-n-e-1}\leq 2^{-n}$. Now our Σ^0_2 set intersects with positive measure every Π^0_1 set of positive measure, because if $\lambda(\mathcal{F}_e)>0$ then there exists a string σ_e of length $n+e+1$ such that $\lambda(\mathcal{F}_e\cap[\sigma_e])>0$ and then the set represented by A will intersect $\mathcal{F}_e$ with positive measure.

From this we can then construct a Π^0_3 set effectively of measure 0 and containing all the weakly-Π^0_1-Solovay-generic sequences. ■

Corollary 2.2.1:
We have 1-randomness $\leftarrow$ weak-2-randomness $\leftarrow$ 2-randomness $\leftarrow$ weakly-3-randomness $\leftarrow\dots$, and all those implications are strict.

Proof: The fact that 1-randomness is strictly implied by weakly-2-randomness is a consequence of Proposition 2.1.1, whereas the fact that weakly-2-randomness is strictly implied by 2-randomness is a consequence of Theorem 2.2.2. Then those proofs relativize to the oracle $\emptyset^{(\alpha)}$ for any α. ∎

Chapter 3

Beyond the Borel hierarchy

> Higher recursion theory (HRT) has been one of my two major obsessions for the last twenty years. Nonetheless my interest has not waned. Perhaps because, as Browning claimed:
>
> "The best is yet to be."
>
> I was talked into the subject, skittish all the way, by G. Kreisel. The old devil insisted, in several conversations beginning in 1961, on the existence of golden generalizations of recursion theory in which infinitely long computations converged. I listened for hours, without understanding a word, to his tales of the mother lode of recursion theory hidden far below the peaks of effective descriptive set theory.
>
> Higher recursion theory, Gerald Sacks

3.1 The complexity of sets

We should now go beyond the Borel hierarchy and study the Π^1_1 and Σ^1_1 sets. We said previously that the arithmetical sets of integers or reals are the one that can be defined using first-order formulas of arithmetic. We slightly abuse here of the use of the word 'real', by which we mean either elements of the Baire space or of the Cantor space.

We also define hyperarithmetical sets of integers, and we define hyperarithmetical sets of reals, as an effective version of the Borel sets. Also, starting from the Σ^0_ω or Π^0_ω sets, there is no longer a way to define them with first-order formulas of arithmetic. They are however definable (since we defined them...). But to do so, still with formulas of arithmetic, we need second order quantification, that is, quantification over infinite objects, such as functions or sequences of 0's and 1's.

We shall see that allowing second order quantification in formulas of arithmetic gives us much more power than what we need to just define sets of the hyperarithmetical hierarchy. We actually will use a very small part of the power that second order quantification could give us, and we will use in this thesis only universal second order quantification, or only existential second order quantification without ever mixing the two.

Definition 3.1.1. *A subset of $\mathbb{N}^\mathbb{N}$ or of $\mathbb{N}$ is Σ^1_1 if it is definable by a formula of arithmetic with quantification over integers or elements of $\mathbb{N}^\mathbb{N}$, such that the quantifications over* *elements of $\mathbb{N}^\mathbb{N}$ are only existential (and not preceded by a negation). Similarly we define the Π^1_1 sets as those corresponding to formulas containing only universal quantifications over elements of $\mathbb{N}^\mathbb{N}$.* ■

Example 3.1.1:
We can easily see that the set $\mathcal{W}$ of codes of computable ordinals is Π^1_1. Indeed, we can define the set of codes of linear orders by a Π^0_2 formula: for all distinct integers n_1, n_2, we have that $n_1 < n_2$ is enumerated in the order described by W_e and $n_2 < n_1$ is never enumerated, or $n_2 < n_1$ is enumerated and $n_1 < n_2$ is never enumerated. The only thing that remains to check is the absence of infinite backward sequence in the order described by W_e, which can be expressed with a universal quantification over functions: $\forall f\ \exists n\ \neg f(n+1) <_e f(n)$, where $<_e$ denotes the order described by e. ◊

We shall see that the Π^1_1 set of computable ordinals, described in the previous example, cannot be Σ^0_α for any α. Similarly we will see that some Π^1_1 subsets of $\mathbb{N}^\mathbb{N}$ cannot be Borel (and therefore neither their Σ^1_1 complement). We will actually see that a set is effectively Borel iff it is both Π^1_1 and Σ^1_1.

As explained in the introduction of 'Descriptive set theory' by Moschovakis (see [65]), the study of Σ^1_1 sets probably starts with Suslin, who spotted a mistake in a proof of Lebesgue, who was wrongly assuming that the image of a Borel set of the Baire space by a continuous function is also a Borel set.

Suslin proved that this was not necessarily the case and called **analytic sets** the sets that could be described as images of Borel sets by continuous functions. We shall see that Σ^1_1 sets are actually an effective version of the notion of analytic sets. For this reason we define:

Definition 3.1.2. *A subset of $\mathbb{N}^\mathbb{N}$ is $\boldsymbol{\Sigma^1_1}$ or **analytic** if it is the range of a total continuous function $f : \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$. A subset $\mathbb{N}^\mathbb{N}$ is $\boldsymbol{\Pi^1_1}$ or **co-analytic** if its complement is $\boldsymbol{\Sigma^1_1}$.* ■

We shall see that a set is $\boldsymbol{\Sigma^1_1}$ iff it is $\Sigma^1_1(X)$ for some X, and therefore $\boldsymbol{\Pi^1_1}$ iff it is $\Pi^1_1(X)$ for some X. But first, let us show that there exists a convenient normal form for Π^1_1 and Σ^1_1 sets. To deal at the same time with both subsets of $\mathbb{N}^\mathbb{N}$ and subsets of $\mathbb{N}$, we directly consider subsets of $\mathbb{N}^\mathbb{N} \times \mathbb{N}$.

Theorem 3.1.1 (Kleene, [39]):
A subset $\mathcal{A}$ of $\mathbb{N}^\mathbb{N} \times \mathbb{N}$ is Σ^1_1 iff there is a computable functional Φ such that

$$(f, m) \in \mathcal{A} \leftrightarrow \exists g\ \Phi(g, f, m) \uparrow$$

Similarly a subset $\mathcal{A}$ of $\mathbb{N}^\mathbb{N} \times \mathbb{N}$ is Π^1_1 iff there is a computable functional Φ such that

$$(f, m) \in \mathcal{A} \leftrightarrow \forall g\ \Phi(g, f, m) \downarrow$$

PROOF: Consider any Σ_1^1 predicate which we can suppose to be in prenex normal form (starting with only quantifiers, followed by a quantifier-free part). For any existential quantifier over an integer variable n, we can obtain an equivalent formula by replacing it with an existential quantifier over a function f, and by modifying any other instance of n by $f(0)$ in the formula. Once this is done for any existential quantification over integers, we now transform this new formula into an equivalent one where existential quantifiers over functions come first, then universal quantifiers over integers, followed by a quantifier-free part.

To do so, we should see that for any $\mathbf{\Sigma_1^1}$ predicate $Q(n, f)$, where n and f are free variables, we have:

$$\forall n \; \exists f \; Q(n, f) \leftrightarrow \exists f \; \forall n \; Q(n, \langle f \rangle_n)$$

Where $\langle f \rangle_n$ is the n-th inverse of a pairing function from $(\mathbb{N}^{\mathbb{N}})^{\mathbb{N}}$ to $\mathbb{N}^{\mathbb{N}}$. We can first easily prove this with the axiom of choice. Indeed if $\forall n \; \exists f \; Q(n, f)$ then in every set $\mathcal{A}_n = \{f \mid Q(n, f)\}$, we can pick one function and then prove the existence of the function resulting of the infinite pairing between each of them. It follows also that $\exists f \; \forall n \; Q(n, \langle f \rangle_n)$ is true (the converse being obvious).

Now, using the fact that each set $\mathcal{A}_n = \{f \mid Q(n, f)\}$ is $\mathbf{\Sigma_1^1}$, it is possible to remove the use of the axiom of choice. This is a consequence the developments of Section 3.2, roughly saying that any non-empty $\mathbf{\Sigma_1^1}$ contains in some sense a 'leftmost path', that we can then pick uniformly in each $\mathcal{A}_n$, without the use of choice.

We can then transform the formula into an equivalent formula (equivalent over ZF) where existential quantifiers over functions come first, then universal quantifiers over integers, followed by a quantifier-free part. All that remains to do is to merge all the existential quantifications over functions into one existential quantification, and all the universal quantifications over integers into one universal quantification, and this can be done using the fact that for any predicate Q we have:

$$\exists f_1, \dots, f_2 \; Q(f_1, \dots, f_n) \leftrightarrow \exists f \; Q(\langle f \rangle_1, \dots, \langle f \rangle_n)$$

and

$$\forall x_1, \dots, x_n \; Q(x_1, \dots, x_n) \leftrightarrow \forall x \; Q(\langle x \rangle_1, \dots, \langle x \rangle_n)$$

Finally the function Φ is created from the universal quantifier over the integers, from the quantifier-free part of the final formula. The normal form theorem for Π_1^1 sets is then a consequence of the one for Σ_1^1 sets. ∎

It follows that there is a canonical enumeration of the Σ_1^1 and Π_1^1 sets, given by the indices of their corresponding functionals:

Definition 3.1.3. *In the context of Π_1^1 subsets of $\mathbb{N}^{\mathbb{N}} \times \mathbb{N}$, any $e \in \mathbb{N}$ is a* ***$\mathbf{\Pi_1^1}$-index*** *whose corresponding set is given by $\{(f, m) \; : \; \forall g \; \Phi_e(g, f, m) \downarrow\}$. Similarly, in the context of Σ_1^1 subsets of $\mathbb{N}^{\mathbb{N}} \times \mathbb{N}$, any $e \in \mathbb{N}$ is a* ***$\mathbf{\Sigma_1^1}$-index*** *whose corresponding set is given by $\{(f, m) \; : \; \exists g \; \Phi_e(g, f, m) \uparrow\}$.* ∎

We shall now separate the notions of Π_1^1 and Σ_1^1. We start with a separation for sets of integers. Later, the proof that a set is Σ_α^0 for some computable α iff it is both Π_1^1 and Σ_1^1, will imply that also some sets of integer are Π_1^1 or Σ_1^1, but not Σ_α^0 for any α.

Proposition 3.1.1:
There is a set of integers which is Π^1_1 but not Σ^1_1.

PROOF: We proceed by a standard diagonalization. Let $\{P_e\}_{e\in\omega}$ be an enumeration of the Π^1_1 sets. Let A be the Π^1_1 set $\{e : e \in P_e\}$. Suppose $\mathbb{N} - A$ is Π^1_1. Then for some e we have $\mathbb{N} - A = P_e$ and $e \in A \leftrightarrow e \in P_e \leftrightarrow e \in \mathbb{N} - A$ which is a contradiction. ∎

We now separate Π^1_1 from $\boldsymbol{\Sigma^1_1}$, for sets of reals. As for sets of integers, we will prove later that a set of reals is Borel iff it is both $\boldsymbol{\Pi^1_1}$ and $\boldsymbol{\Sigma^1_1}$. This will then imply that some Π^1_1 set is not Borel.

Proposition 3.1.2:
There is a set of reals which is Π^1_1 but not $\boldsymbol{\Sigma^1_1}$.

PROOF: We proceed by a standard diagonalization. Let e be an integer such that on oracle $1^n\hat{\ }0\hat{\ }Y$, it becomes a $1^n\hat{\ }0\hat{\ }Y$-index for the set $\{X : \forall g\ \Phi_n(Y,g,X)\downarrow\}$. Proposition 3.2.1 will make clear that a set is $\boldsymbol{\Pi^1_1}$ iff it is $\Pi^1_1(Y)$ for some oracle Y. Therefore, for any $\boldsymbol{\Pi^1_1}$ set $\mathcal{A}$, there exists an oracle Y such that e is a Y-index for $\mathcal{A}$.

Consider the Π^1_1 set $\mathcal{A} = \{1^n\hat{\ }0\hat{\ }X : \forall g\ \Phi_n(X,g,1^n\hat{\ }0\hat{\ }X)\downarrow\}$. Suppose that $2^\omega - \mathcal{A}$ is $\boldsymbol{\Pi^1_1}$. Then there is an oracle $1^n\hat{\ }0\hat{\ }Y$ such that $2^\omega - \mathcal{A} = \{X : \forall g\ \Phi_n(Y,g,X)\downarrow\}$. But then $1^n\hat{\ }0\hat{\ }Y \in \mathcal{A} \leftrightarrow \forall g\ \Phi_n(Y,g,1^n\hat{\ }0\hat{\ }Y)\downarrow \leftrightarrow 1^n\hat{\ }0\hat{\ }Y \in 2^\omega - \mathcal{A}$ which is a contradiction.∎

3.2 The Σ^1_1 sets

There is a convenient way to represent Σ^1_1 subsets of $\mathbb{N}^\mathbb{N}$ (or of $2^\mathbb{N}$). Consider $\mathcal{A} \subseteq \mathbb{N}^\mathbb{N}$ defined by $\mathcal{A}(f) \leftrightarrow \exists g\ \Phi(f,g)\uparrow$. Let us define the computable tree:

$$T = \{\sigma_1 \oplus \sigma_2 \text{ with } t = |\sigma_1| = |\sigma_2| : \Phi(\sigma_1,\sigma_2)[t]\uparrow\}$$

We have that $\exists g\ g \oplus f \in [T]$ iff $f \in \mathcal{A}$: If for some function g we have $\Phi(g,f)\uparrow$ then clearly $g \oplus f \in [T]$. For the converse, if for every function g we have $\Phi(g,f)\downarrow$, then also for all g there is some t large enough to have $\Phi(g\upharpoonright_t, f\upharpoonright_t)[t]\downarrow$ and thus such that $g\upharpoonright_t \oplus f\upharpoonright_t \notin T$.

One can similarly represent any Σ^1_1 subset of $2^\mathbb{N}$ as the set of infinite paths of a computable tree T of the Baire space, where for $f \in T$, the corresponding element X is coded by $X(n) = 0$ if $f(n)$ is even and $X(n) = 1$ otherwise.

We can now fix the missing part of the proof of Theorem 3.1.1: Using those trees, it follows that the axiom of choice is not needed to pick elements in a sequence of Σ^1_1 sets, as we can pick in each of them the element coded by the leftmost path of its corresponding tree.

We shall now see why the $\Sigma^1_1(X)$ predicates for some oracle X, are exactly those that can be defined as the range of a continuous function.

Proposition 3.2.1:
A subset of $\mathbb{N}^\mathbb{N}$ is the range of a total continuous function $F : \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ iff it is $\Sigma^1_1(X)$ for some oracle X.

PROOF: Let $F : \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ be a total continuous function. We have that $g \in F(\mathbb{N}^\mathbb{N})$ iff $\exists h\ g = F(h)$, which is a $\Sigma^1_1(X)$ predicate, where X is an oracle coding for F.

Consider now the non-empty $\Sigma^1_1(X)$ set $\mathcal{A}$ defined by $h \in \mathcal{A} \leftrightarrow \exists g\ \Phi(X, g, h) \uparrow$, and its corresponding X-computable tree T, as described above. Let T' be the pruned tree obtained by removing all dead nodes from T. We still have that $f \in \mathcal{A}$ iff there exists a function g such that $g \oplus f \in [T']$.

Let $G : \mathbb{N}^\mathbb{N} \to [T']$ be the continuous function which to f associates the element g of $[T']$ the following way. First, $g(0)$ is the $f(0)$-th node of T' of length 1 if it exists, or the last node of T' of length 1 otherwise. Then inductively, $g(n+1)$ is the $f(n+1)$-th node of T' of length $n+2$ that extends $g(n)$, if it exists, or the last node of T' of length $n+2$ that extends $g(n)$ otherwise.

The function G is clearly continuous with range $[T']$. We then define the continuous function F by $F(h)$ to be the second half of $G(h)$. We clearly have that F is continuous and that $F(\mathbb{N}^\mathbb{N}) = \mathcal{A}$. ∎

3.3 The Π^1_1 sets

The Π^1_1 sets are strongly connected to the notion of being well-founded, in the sense that, informally, the set of well-founded objects is a universal Π^1_1 set. This is made precise with the following proposition:

Proposition 3.3.1:
For $\mathcal{A} \subseteq \mathbb{N}^\mathbb{N} \times \mathbb{N}$ a Π^1_1 set, one can define uniformly in an index for $\mathcal{A}$ a total computable function $h : \mathbb{N} \to \mathbb{N}$ such that $\mathcal{A}(f, n) \leftrightarrow h(n) \in \mathcal{T}^f$. Recall that $\mathcal{T}^f$ is the set of codes for c.e. well-founded trees relatively to the oracle f.

PROOF: Consider $\mathcal{A} \subseteq \mathbb{N}^\mathbb{N} \times \mathbb{N}$ defined by $\mathcal{A}(f, n) \leftrightarrow \forall g\ \Phi(g, f, n) \downarrow$. We can define uniformly in n and f the f-computable tree:

$$T^f_n = \{\sigma \text{ with } t = |\sigma| \ :\ \Phi_t(\sigma, f\upharpoonright_t, n) \uparrow\}$$

We define the function h by associating to n a code for the 'oracle tree', which on oracle f becomes a code for T^f_n. We should now prove that $(f, n) \in \mathcal{A}$ iff T^f_n is well-founded. But this is clear because we have that $(f, n) \in \mathcal{A}$ iff $\forall g\ \exists t\ \Phi_t(g\upharpoonright_t, f\upharpoonright_t, n) \downarrow$ iff $\forall g\ \exists t\ g\upharpoonright_t \notin T^f_n$ iff T^f_n is well-founded. ∎

Corollary 3.3.1:
For $\mathcal{A} \subseteq \mathbb{N}^{\mathbb{N}}$ a Π^1_1 set, One can find uniformly in an index for $\mathcal{A}$ and in any representation X of a countable ordinal α, a code for a $\Pi^0_{\alpha+1}(X)$ set $\mathcal{A}_\alpha$, such that $\mathcal{A} = \bigcup_{\alpha<\omega_1} \mathcal{A}_\alpha$. If α is a computable ordinal, then the representation X can simply be a code of $\mathcal{O}_{=\alpha}$, which makes $\mathcal{A}_\alpha$ a $\Pi^0_{\alpha+1}$ set.

Proof: From the previous proposition we have an integer e such that $f \in \mathcal{A} \leftrightarrow e \in \mathcal{T}^f$. Consider the set $\mathcal{A}_\alpha = \{f \ : \ e \in \mathcal{T}^f_{\leq\alpha}\}$. From Theorem 1.7.2, we have that $\mathcal{A}_\alpha$ is a $\Pi^0_{\alpha+1}(X)$ set uniformly in any X representing the ordinal α; and a $\Pi^0_{\alpha+1}$ set uniformly in a code of $\mathcal{O}_{=\alpha}$ if α is computable. Then by design we have $\mathcal{A} = \bigcup_\alpha \mathcal{A}_\alpha$. ∎

We now give an example of a Π^1_1 set of reals which will be important in order to study higher randomness, and especially the different notions of randomness that lie between Δ^1_1-randomness and Π^1_1-randomness.

Proposition 3.3.2:
The set $\{X \mid \omega_1^X > \omega_1^{ck}\}$ is a Π^1_1 set.

Proof: The set $\{X \mid \omega_1^X > \omega_1^{ck}\}$ can be defined by the following predicate: "There exists a code $e \in \mathcal{W}^X$ such that for every integer n coding for a linear order, for every $f : \omega \to \omega$, the function f is not an isomorphism from the order coded by e into the one coded by n". It is easy to check that this is a Π^1_1 predicate.

If $\omega_1^X > \omega_1^{ck}$ then there exists an integer $e \in \mathcal{W}^X$ such that $|e|_o^X \geq \omega_1^{ck}$. Therefore, for any n coding for a linear order, either n is well-founded in which case $|e|_o^X > |n|_o$ and we do not have an isomorphism, or n is ill-founded in which case we also do not have an isomorphism.

Now if $\omega_1^X = \omega_1^{ck}$, then for every integer $e \in \mathcal{W}^X$, there is an integer $n \in \mathcal{W}$ such that $|e|_o^X = |n|_o$, and therefore such that we have an isomorphism between the two. ∎

An important example of Π^1_1 set of integers is the one described in Example 3.1.1. One can equivalently consider the Π^1_1 set $\mathcal{W}$ of codes for computable ordinals, the set $\mathcal{T}$ of codes for c.e. well-founded trees, or the set $\mathcal{O}$ of codes for constructive ordinals. All those sets can play for Π^1_1 sets the same role $\emptyset^{(1)}$ plays for Σ^0_1 sets, that is, they are Π^1_1-complete sets.

Theorem 3.3.1:
The sets $\mathcal{O}$, $\mathcal{W}$, $\mathcal{T}$ are Π^1_1-complete sets. In particular, they are not Σ^1_1.

PROOF: We argued in Example 3.1.1 that $\mathcal{W}$ is Π^1_1. The set $\mathcal{T}$ is also easily seen to be Π^1_1, as a c.e. tree T is well-founded iff "for all f, the function f is not an element of $[T]$". It is a bit more difficult for the set $\mathcal{O}$. The non-trivial part is to check that for every limit node, the sequence of ordinals coded by its children is strictly increasing. For a sequence of nodes $\{\sigma_n\}_{n\in\omega}$ of a tree T, this can be expressed by the predicate : "for all f, the function f is not an injective morphism from the tree $T\upharpoonright_{\sigma_{n+1}}$ into the tree $T\upharpoonright_{\sigma_n}$".

Now for completeness, for any Π^1_1 set $A \subseteq \omega$, using Proposition 3.3.1, we can find uniformly in an index for A, a total computable function f such that $n \in A \leftrightarrow f(n) \in \mathcal{T}$. This makes $\mathcal{T}$ a Π^1_1-complete set.

Then using the Kleene-Brouwer ordering (see Section 1.4.2), the set of $\mathcal{T}$ is many-one reducible to the set $\mathcal{W}$ which makes $\mathcal{W}$ a Π^1_1-complete set.

Also using the technique described in Section 1.4.3, the set $\mathcal{T}$ is many-one reducible to the set $\mathcal{O}$, which then makes $\mathcal{O}$ a Π^1_1-complete set.

It then follows from the existence of a Π^1_1 set which is not Σ^1_1 (Proposition 3.1.1) that $\mathcal{O}$, $\mathcal{W}$ and $\mathcal{T}$ are not Σ^1_1 sets. ∎

We can deduce an important corollary from this, that will be referred to as Spector's **Σ^1_1-boundedness principle**, whose consequences will be used a lot in this thesis.

Corollary 3.3.2 (Spector, Σ^1_1-boundedness principle):
Any Σ^1_1 set of codes for computable ordinals is bounded below ω_1^{ck}. Formally for $A \subseteq \mathcal{T}$ or $A \subseteq \mathcal{W}$ or $A \subseteq \mathcal{O}$ a Σ^1_1 set, there exists a computable ordinal α such that for any $e \in A$ we have $|e|_o < \alpha$. Furthermore, one can find uniformly in an index for A a code $a \in \mathcal{O}$ for such an ordinal α.

PROOF: Consider a Σ^1_1 set of integers A in either $\mathcal{T}, \mathcal{W}$ or $\mathcal{O}$. Using the technique described in Section 1.4.2 we can suppose without loss of generality $A \subseteq \mathcal{T}$. It is interesting to see first a non constructive argument that there exists some computable α such that $A \subseteq \mathcal{T}_{<\alpha}$. Suppose otherwise, then the predicate $n \in \mathcal{T}$ can be expressed by the Σ^1_1 formula:

"There exists $e \in A$ and a function f which is an injective morphism from the tree coded by n into the tree coded by e".

But by Theorem 3.3.1, the set $\mathcal{T}$ is not Σ^1_1 and we then have a contradiction. So there is some α such that $A \subseteq \mathcal{T}_{<\alpha}$.

Now to get uniformity we need a constructive argument. Suppose a Σ^1_1 set A is included in $\mathcal{T}$. In order to find an ordinal α such that $A \subseteq \mathcal{T}_{<\alpha}$, we can use the function OR of Lemma 1.4.1 that combines two trees T_1, T_2 into one tree with order-type $\min(|T_1|_o, |T_2|_o)$, together with the function AND of Lemma 1.4.4, that combines a sequence of trees $\{T_n\}_{n\in\mathbb{N}}$ into one tree with order-type $\sup^+_n(|T_n|_o)$.

As A is Σ^1_1, we have $a \in A$ iff some tree T_a, computable uniformly in a, is ill-founded. All we have to do is apply the function AND to the sequence of trees $\{OR(a, t_a)\}_{a\in\mathbb{N}}$

where t_a is a code for the tree T_a. Either $a \in A$ and then a codes for a well-founded tree, in which case $OR(a, t_a)$ is well-founded with order-type $|a|_o$, or $a \notin A$ and then T_a codes for a well-founded tree, in which case $OR(a, t_a)$ is well-founded with order-type $|t_a|_o$. It is clear that the result e of the function AND is such that $A \subseteq \mathcal{T}_{<|e|_o}$. We can then transform e into an element of $\mathcal{O}$. ∎

Corollary 3.3.3 (Spector, $\boldsymbol{\Sigma^1_1}$-boundedness principle for sets of reals): *For any Σ^1_1 set $\mathcal{A} \subseteq 2^{\mathbb{N}}$ such that for some $e \in \mathbb{N}$, either $X \in \mathcal{A} \rightarrow e \in \mathcal{T}^X$ or $X \in \mathcal{A} \rightarrow e \in \mathcal{W}^X$ or $X \in \mathcal{A} \rightarrow e \in \mathcal{O}^X$, there is a computable ordinal α such that for any X in $\mathcal{A}$ we have $|e|^X_o < \alpha$. Furthermore, one can find uniformly in an index of $\mathcal{A}$, a code $a \in \mathcal{O}$ for such an ordinal α.*

Proof: As in the previous corollary, we can suppose that for any X in $\mathcal{A}$ we have $e \in \mathcal{T}^X$. We directly give here a constructive argument. As $\mathcal{A}$ is Σ^1_1, one can find uniformly in an index for $\mathcal{A}$, a code a such that $X \notin \mathcal{A}$ iff $a \in \mathcal{T}^X$. In particular, using the function OR of Lemma 1.4.1 that combines two trees T_1, T_2 into one tree with order-type $\min(|T_1|_o, |T_2|_o)$, we have, arguing like in the previous corollary, that $OR(a, e)$ is a code of $\mathcal{T}^X$ for any $X \in 2^{\mathbb{N}}$.

All we have to do is combine all those trees into a single one. Let T^X be the X computable tree coded by $OR(a, e)$. We define the c.e. tree T to be the downward closure of the following c.e. set of nodes:

$$\{\sigma \oplus \tau \;:\; |\tau| = |\sigma| \text{ and } \tau \text{ is a node of } T^\sigma\}$$

The tree T is clearly well-founded. Suppose otherwise, then there would be an infinite path f in T^X for some X. Also it is clear that for any X, there is an injective morphism f from T^X into T: Given $\sigma \in T^X$, the function $f(\sigma)$ returns $\sigma \oplus X \upharpoonright_{|\sigma|}$. It implies that $|T|_o \geq |T^X|_o$ for any X. We can then transform a code of T into a code of $\mathcal{O}$. ∎

3.4 The Δ^1_1 sets

Definition 3.4.1. *A set A of reals or of integers is Δ^1_1 if it is both Π^1_1 and Σ^1_1. A set A of reals or of integers is $\boldsymbol{\Delta^1_1}$ if it is both $\boldsymbol{\Pi^1_1}$ and $\boldsymbol{\Sigma^1_1}$. A Δ^1_1-**index** is given by a pair of integers $\langle e_1, e_2\rangle$, such that e_1 is a Σ^1_1-index for A and e_2 a Π^1_1-index for A.* ∎

We now give a small example of where to find Δ^1_1 set of integers, that can be considered as a higher counterpart of Proposition 1.5.2 saying that if a Π^0_1 set contains only one element, then this element is computable:

Example 3.4.1:
If a Σ^1_1 set $\mathcal{A} \subseteq 2^{\mathbb{N}}$ contains only one element X, then X is Δ^1_1. Indeed, we can define X with a Σ^1_1 formula:

$$n \in X \text{ iff } \exists Y\ Y \in \mathcal{A} \wedge n \in Y$$

But we can also define X with a Π^1_1 formula:

$$n \in X \text{ iff } \forall Y\ Y \notin \mathcal{A} \vee n \in Y$$

Proposition 3.4.1:
For any computable α and any m, the set $\emptyset^{(\alpha)}$ and the set $\{X : m \in X^{(\alpha)}\}$ are Δ^1_1. Furthermore their Δ^1_1-index can be obtained uniformly in a code of $\mathcal{O}_{=\alpha}$.

Proof: Let us prove that for any computable ordinal α and any m, the sets $\mathcal{T}_{<\alpha}$, $\mathcal{T}_{\leq\alpha}$, $\{X : m \in \mathcal{T}^X_{<\alpha}\}$ and $\{X : m \in \mathcal{T}^X_{\leq\alpha}\}$ are Δ^1_1 and that their Δ^1_1-indices can be obtained uniformly in a code of $\mathcal{O}_{=\alpha}$.

Consider any $e \in \mathcal{O}_{=\alpha}$. The predicate $n \in \mathcal{T}_{<\alpha}$ can be described by the Σ^1_1 formula: "There exists an injective morphism between the tree coded by n and a strict subtree of the tree coded by e."

The predicate $n \in \mathcal{T}_{\leq\alpha}$ can be described by the Σ^1_1 formula: "There exists an injective morphism between the tree coded by n and the tree coded by e."

The predicate $n \in \mathcal{T}_{<\alpha}$ can also be described by the Π^1_1 formula: "There exists no infinite path in the tree coded by n and no injective morphism from the tree coded by e into the one coded by n."

The predicate $n \in \mathcal{T}_{\leq\alpha}$ can be described by the Π^1_1 formula: "There exists no infinite path in the tree coded by n and no injective morphism from the tree coded by $\mathrm{succ}(e)$ into the one coded by n."

Now given a code $a \in \mathcal{O}_{=\alpha}$, we can identify b and k such that $|b|_o$ is 0 or limit, $|k|_o \in \omega$, and such that $|a|_o = |b|_o + 2|k|_o$ or $|a|_o = |b|_o + 2|k|_o + 1$. In the first case $\emptyset^{(\alpha)} = \mathcal{T}_{<\omega(|b|_o+|k|_o)}$ and in the second case $\emptyset^{(\alpha)} = \mathbb{N} - \mathcal{T}_{\leq\omega(|b|_o+|k|_o)}$. In any case we can easily compute a code of $\mathcal{O}_{=\omega(|b|_o+|k|_o)}$ from a and then compute the appropriate index for $\emptyset^{(\alpha)}$.

The proof for sets of reals is similar. ■

We now prove the famous equivalence between Δ^1_1 and hyperarithmetic. Due to Kleene, it is an effective version of the equivalence between $\mathbf{\Delta^1_1}$ and Borel, proved earlier by Suslin [91].

Theorem 3.4.1 (Kleene):
A set of reals or of integers is Δ^1_1 iff it is Σ^0_α for some computable ordinal α. Furthermore, uniformly in a Δ^1_1-index for it, one can obtain a Σ^0_α-index for the same set, and uniformly in a Σ^0_α-index for it, and in a code of $\mathcal{O}_{=\alpha}$, one can obtain a Δ^1_1-index for the same set.

PROOF: Suppose first that a set of integers A is Σ^0_α for some computable ordinal α. Then by Theorem 1.6.1 the set A is many-one reducible to $\emptyset^{(\alpha)}$ uniformly in an index for A and in a code of $\mathcal{O}_{=\alpha}$. We can conclude using the previous proposition.

For a Σ^0_α set of reals $\mathcal{A}$, by Theorem 1.7.1 we can find uniformly in a Σ^0_α-index for $\mathcal{A}$ and in a code of $\mathcal{O}_{=\alpha}$, an index e such that $\mathcal{A} = \{X \ : e \in X^{(\alpha)}\}$. Again, we can use the previous proposition to conclude that such a set is Δ^1_1.

Suppose now that we have a Δ^1_1 set of integers $\mathcal{A}$. In particular it is Π^1_1 and by Proposition 3.3.1, uniformly in a Δ^1_1-index for $\mathcal{A}$, one can find a total computable function f such that $n \in \mathcal{A}$ iff $f(n) \in \mathcal{T}$. Also, as A is Σ^1_1, the range of f is a Σ^1_1 subset of $\mathcal{T}$, but the Spector Σ^1_1-boundedness principle, we can uniformly find a code of $\mathcal{O}_{=\alpha}$ for some α such that the set of ordinals coded by elements of this set is bounded by α. Therefore we have $n \in \mathcal{A}$ iff $f(n) \in \mathcal{T}_{<\alpha}$, which makes $\mathcal{A}$ a $\Sigma^0_{\alpha+1}$ set uniformly in a Δ^1_1-index for $\mathcal{A}$.

Finally, suppose that we have a Δ^1_1 set of reals $\mathcal{A}$. In particular, it is Π^1_1 and by Proposition 3.3.1 we can find uniformly in a Δ^1_1-index for $\mathcal{A}$, an integer e such that $X \in \mathcal{A}$ iff $e \in \mathcal{T}^X$. Also as $\mathcal{A}$ is Σ^1_1, by the Σ^1_1-boundedness principle for sets of reals, we can find uniformly in an index for $\mathcal{A}$ and in e, a code $a \in \mathcal{O}$ such that $X \in \mathcal{A}$ iff $e \in \mathcal{T}^X_{<\alpha}$ where $\alpha = |a|_o$, which makes $\mathcal{A}$ a $\Sigma^0_{\alpha+1}$ set, whose index can be obtained uniformly in its Δ^1_1-index. ■

The previous theorem is easily seen to relativize, and we then have:

Corollary 3.4.1:
For any $X \in 2^{\mathbb{N}}$, a set of reals is $\Delta^1_1(X)$ iff it is $\Sigma^0_\alpha(X)$ for some $\alpha < \omega_1^X$. In particular, a set of reals is $\boldsymbol{\Delta}^1_1$ iff it is $\boldsymbol{\Sigma}^0_\alpha$ for some $\alpha < \omega_1$.

Finally, we can define a hyperarithmetic analogue of the notion of Turing reduction:

Definition 3.4.2. *For two sequences $X, Y \in 2^{\mathbb{N}}$ we say that Y is* ***hyperarithmetically reducible*** *to X, and we write $X \geq_h Y$ if Y is $\Delta^1_1(X)$.* ■

We emphasize that from a topological point of view, hyperarithmetic reductions behave much differently than Turing reductions. In particular, we no longer have anymore the finite use property, and hyperarithmetic reductions are not continuous on their domain of definition. We will study in Section 4.1 a version of hyperarithmetic reduction for which we force continuity. For now we give the complexity of the hyperarithmetical reduction:

Proposition 3.4.2:
The set $\{X \oplus Y \ : \ X \geq_h Y\}$ is a Π^1_1 set.

PROOF: Using a relativized version of Theorem 3.3.1 we have for any X that the set $\mathcal{T}^X$ is $\Pi^1_1(X)$-complete uniformly in X. Also using a relativized version of Porism 1.6.1 and

Theorem 1.6.1 we have for any $\alpha < \omega_1^X$, that $\mathcal{T}^X_{<\alpha}$ and $\mathcal{O}^X_{<\alpha}$ are $\Sigma^0_{\alpha+1}(X)$-complete sets uniformly in X.

For any X, and any Y, we then have that Y is $\Delta^1_1(X)$ iff there exists $\alpha < \omega_1^X$ such that Y is many-one reducible to $\mathcal{T}^X_{<\alpha}$. Therefore the set $\{X \oplus Y \ : \ X \geq_h Y\}$ is equal to:

$$\{X \oplus Y \ : \ \exists a \in \mathcal{O}^X \ \mathcal{T}^X_{<|a|} \geq_m Y\}$$

which is then clearly a Π^1_1 set. ∎

We now prove a theorem from Sacks, which is an analogue to Corollary 2.1.3 (also from Sacks) with respect to hyperarithmetical reducibility.

Theorem 3.4.2 (Sacks):
If X is not Δ^1_1, then $\lambda(\{Y \ : \ Y \geq_h X\}) = 0$.

Proof: Suppose $\lambda(\{Y \ : \ Y \geq_h X\}) > 0$. We shall see later with Theorem 4.2.3 that for a Π^1_1-random sequence Y, we have $Y \geq_h X$ iff there exists a computable α such that $Y \oplus \emptyset^{(\alpha)} \geq_T X$. Also as the set of Π^1_1-randoms has measure 1, and by countable measure subadditivity, we have for some computable α that the measure of the set $\{Y \ : \ Y \oplus \emptyset^{(\alpha)} \geq_T X\}$ is positive. But then relativizing Corollary 2.1.3 we have that X is already Turing computable in $\emptyset^{(\alpha)}$ and then that it is Δ^1_1. ∎

3.5 Further study of Kleene's $\mathcal{O}$

In this section we shall say a little bit more about Π^1_1-complete sets. We pick Kleene's $\mathcal{O}$ for this study, which is in the literature the Π^1_1 complete set of reference. Note however that we everything we say here about Kleene's $\mathcal{O}$ is valid for any other Π^1_1-complete set, such as $\mathcal{T}$ or $\mathcal{W}$.

Theorem 3.5.1 (Spector [87]):
For a sequence X we have $X \geq_h \mathcal{O}$ iff $\omega_1^X > \omega_1^{ck}$.

Proof: Let us suppose that $\omega_1^X > \omega_1^{ck}$. We have that $\mathcal{O} = \bigcup_{\alpha<\omega_1^{ck}} \mathcal{O}_{\leq\alpha}$ and that $\mathcal{O}_{\leq\alpha}$ is $\Pi^0_{\alpha+1}$ uniformly in an code of $\mathcal{O}_{=\alpha}$. A mere relativization of Porism 1.6.1 gives us also that for any X, the set $\mathcal{O}^X_{\leq\alpha}$ is $\Pi^0_{\alpha+1}(X)$ uniformly in an code of $\mathcal{O}_{=\alpha}$. Also, if there is a code $a \in \mathcal{O}^X$ such that $|a|^X_o = \omega_1^{ck}$, it is then clear that $\mathcal{O}$ is a $\Sigma^0_{\omega_1^{ck}}(X)$ set and then a $\Delta^1_1(X)$ set.

For the converse, we first prove that for any two sequences X, Y, if $X \geq_h Y$ then $\omega_1^X \geq \omega_1^Y$. Suppose for contradiction that $X \geq_h Y$ but that $\omega_1^X < \omega_1^Y$. Then given any code $a \in \mathcal{O}^Y$ with $|a|^Y_o = \omega_1^X$, the set $\mathcal{O}^X$ can be defined by the following $\Sigma^1_1(X \oplus Y)$ formula:

"$n \in \mathcal{O}^X$ iff there exists a function f which is an injective morphism from the order X-coded by n into the order Y-coded by a"

Also as $X \geq_h Y$ the set $\mathcal{O}^X$ would actually be $\Sigma^1_1(X)$, which is a contradiction with the fact that $\mathcal{O}^X$ is not $\Sigma^1_1(X)$ (the relativized version of the fact that $\mathcal{O}$ is not Σ^1_1).

Also we easily prove $\omega_1^{\mathcal{O}} > \omega_1^{ck}$. Indeed, one can easily build a well-founded tree, computably enumerable in $\mathcal{O}$, which codes for the ordinal ω_1^{ck}: Simply take the union of each well-founded tree coded by elements of $\mathcal{O}$. It then follows that $X \geq_h \mathcal{O}$ implies $\omega_1^X \geq \omega_1^{\mathcal{O}} > \omega_1^{ck}$. ∎

We have a somehow puzzling corollary of the previous theorem:

Corollary 3.5.1:
For any Δ^1_1 well-founded tree T, there exists a computable well-founded tree T' with $|T|_o = |T'|_o$.

Proof: As T does not hyperarithmetically compute $\mathcal{O}$ we then have $\omega_1^T = \omega_1^{ck}$ and then there must exists a computable tree T' with $|T|_o = |T'|_o$. ∎

So for ordinals, having the whole Δ^1_1 definability power is useless to get anything else that what could be defined with simply Δ^0_1 formulas. We shall now prove two very useful basis theorems for Σ^1_1 sets. The second one will strengthen the first one:

Theorem 3.5.2:
A non-empty Σ^1_1 set of reals contains a sequence X such that $\mathcal{O} \geq_{\mathrm{T}} X$.

Proof: Recall the representation of a Σ^1_1 set as the infinite paths of a computable tree of the Baire space. We claim that the leftmost path of such a tree T is Turing computable by Kleene's $\mathcal{O}$. With the help of $\mathcal{O}$, a Π^1_1-complete set, we simply search for the first node σ_1 of length 1 such that $T \upharpoonright_{\sigma_1}$ is an ill-founded tree. Then we search for the first node σ_2 of length 2 and extending σ_1 such that $T \upharpoonright_{\sigma_2}$ is an ill-founded tree. We continue and at convergence we have found the leftmost path of T, which encodes an element of our Σ^1_1 set. ∎

We now give a theorem known as the **Gandy basis theorem**, which is a very useful tool to find some example of sets preserving ω_1^{ck}.

Theorem 3.5.3 (The Gandy Basis theorem):
A non-empty Σ^1_1 set of reals contains a sequence X such that $\mathcal{O} \geq_{\mathrm{T}} X$ and such that $\omega_1^X = \omega_1^{ck}$.

PROOF: Given a non-empty Σ^1_1 set $\mathcal{A}$, we create the set $\mathcal{A}'$ such that $X \oplus Y$ is in $\mathcal{A}'$ iff X is in $\mathcal{A}$ and X does not hyperarithmetically computes Y. Using Proposition 3.4.2 the set $\mathcal{A}'$ is Σ^1_1. We know that for every $X \in \mathcal{A}$ there are uncountably many sequences Y such that $X \oplus Y$ is in $\mathcal{A}'$, because for any X, only countable many sequences are $\Delta^1_1(X)$.

Then we perform the computation of an element $X \oplus Y$ of $\mathcal{A}'$ by $\mathcal{O}$, as described in the previous theorem. We clearly have that $\mathcal{O}$ Turing computes both X and Y. However X cannot hyperarithmetically compute $\mathcal{O}$, as otherwise it would also hyperarithmetically compute Y. ■

Corollary 3.5.2:
The set $\{X : \omega_1^X > \omega_1^{ck}\}$ is not Σ^1_1, and actually contains no non-empty Σ^1_1 subset.

However, the set $\{X : \omega_1^X > \omega_1^{ck}\}$ is easily seen to be a Borel set. We shall prove its exact Borel complexity in Section 6.7.

3.6 Π^1_1 as a higher analogue of c.e.

3.6.1 Motivation

We should start this section by citing a section of Sack's book ([78] V.3.3) that we could not write in a better way:

"Post in a celebrated paper ([75]) liberated classical recursion theory from formal arguments by presenting recursive enumerability as a natural mathematical notion safely handled by informal mathematical procedures. He also stressed what may be called a dynamic view of recursion theory. For example, he proves the existence of a simple set S by giving instructions in ordinary language for the enumeration of S and then verifying that the instructions do in fact produce a simple set. A formal approach to S would refer to formulas or equations from some formal system. A static approach would attempt to define S by some explicit formula. The advantages of Post's informal, dynamic method are considerable. Without it arguments in classical recursion theory would be lengthy and hard to devise. His method, and its advantages, lift to metarecursion theory."

Metarecursion theory attacks the problem of transposing notions of classical recursion theory, that takes place in the world of integers, into the world of computable ordinals, where elements of the Cantor space are now replaced by functions from ω_1^{ck} to $\{0,1\}$ (sequences of "length" ω_1^{ck}) and where a computational time is now a computable ordinal; and we shall see that such a computational time provides a way to naturally deal with the power of Π^1_1 predicates.

We will not deal with Metarecursion theory in this thesis, as we still want to work with sequences of the Cantor space, however we will do it with ordinal computational time just like in Metarecursion theory. It might seem rather odd to mix things that at first glance seem not meant to be mixed. By this we mean the mixing of computational time bounded by ω_1^{ck}, and of sequences of 'length' ω rather than sequences of 'length' ω_1^{ck}, as it is done in Metarecursion theory. But this actually arises naturally in the study of higher randomness (see Section 3.7), where we consider various kinds of Δ^1_1 or Π^1_1 nullsets of the Cantor space.

We will now make precise a general method to deal with Π^1_1, Σ^1_1 or Δ^1_1 sets, in a natural way, safely handled by informal mathematical procedures, just like Post did for classical recursion theory. This will help us to conduct several proofs, that would be otherwise very difficult to deal with, in a more formal (but not more rigorous) way.

The construction of a c.e. set A is often done step by step, by describing A_s at computational step s, where A_s possibly depends on the values of A_t for $t < s$, and by then defining $A = \bigcup_{s<\omega} A_s$. A formal description of A can then be given by $n \in A \leftrightarrow \exists s\ n \in A_s$. As each set A_s is Δ^0_1 uniformly in s, the description can then be formally written as a Σ^0_1 predicate.

We argue that we can similarly build a Π^1_1 set A by describing A_s for each ordinal computational step $s < \omega_1^{ck}$, where A_s possibly depends on the values of A_t for $t < s$, and then by defining $A = \bigcup_{s<\omega_1^{ck}} A_s$. But for A to be Π^1_1, we need to use codes for ordinals and not ordinals themselves. Also infinitely many codes corresponds to a given ordinal. A solution which is commonly used to overcome this difficulty, is to use a unique set of codes for ordinals:

Theorem 3.6.1 (Feferman and Spector [21]):
There is a Π^1_1 set $\mathcal{O}_1 \subseteq \mathcal{O}$ of codes for ordinals, such that for every computable α there is a unique $e \in \mathcal{O}_1$ such that $|e|_o = \alpha$. Furthermore, if $e = \mathrm{succ}(n) \in \mathcal{O}_1$ then $n \in \mathcal{O}_1$ and if $e = \sup_n(e_n) \in \mathcal{O}_1$ then each $e_n \in \mathcal{O}_1$.

Using this, one can define, inductively over codes of $\mathcal{O}_1$, a Δ^1_1 function F mapping elements of $\mathcal{O}_1$ to Δ^1_1 sets. If for every $a_1, a_2 \in \mathcal{O}_1$ with $|a_1|_o < |a_2|_o$ we have $F(a_1) \subseteq F(a_2)$, the sequence $A = \lim_{a\in\mathcal{O}_1} F(a) = A_s$ can be defined by the predicate $n \in A \leftrightarrow \exists a \in \mathcal{O}_1\ n \in A_s$, which makes X a Π^1_1 set.

However, we would like to stress that as the informal definition we can make of A_s do not depend on specific code for s, then the formal one also should not depend on such a code. Sometimes, codes can be used to encode information that is obviously not meant to be encoded. We give here an example which is slightly beyond of the scope of this thesis, but interesting:

Digression

For a given c.e. theory T containing PA, that we suppose consistent, we can define the c.e. theory T' to be T together with the axiom $CONS(T)$, saying that T is consistent. This way we can define $T_0 = T$ for some theorey T and then $T_{n+1} = T'_n$. We can then continue such a definition through the computable ordinals, by defining $T_{\alpha+1} = T'_\alpha$ and $T_\alpha = \bigcup_{\beta<\alpha} T_\beta$ if α is limit. But of course, if one want to keep each T_α computably enumerable, one should use codes for each ordinal α. But this introduces undesired side effects. Indeed, Turing proved the following theorem (the original theorem is more general) :

Theorem 3.6.2 (Turing completness theorem, [93]):
For $T_0 = PA$ and any true Π^0_1 sentence Φ, there is a code $a \in \mathcal{O}_{=\omega+1}$ such that T_a proves Φ.

Actually the fact that T_a proves Φ for $a \in \mathcal{O}_{=\omega+1}$ does not rely at all on the new power that we gained from the knowledge of the inductive consistencies of previous theories. It is simply done by a trick in the encoding of ω by a: Given $\Phi \equiv \forall n\ \Psi(n)$, we define a to be the code $\sup_n a_n$ where $a_n \in \mathcal{O}_{=n}$ if $\forall m \leq n\ \Psi(m)$ and $a_n = \mathrm{succ}(a)$ otherwise (this definition uses the fixed point theorem).

Now because Φ is true, we have that $a \in \mathcal{O}_{=\omega}$. But also PA can prove that if Φ is false, then for n large enough we have $a_n = \mathrm{succ}(a)$, and therefore that $T_{a_n} = T_{\mathrm{succ}(a)}$. But then already T_{a_n} contains the statement $CONS(T_a)$ which implies that T_a can prove $CONS(T_a)$, which by Gödel's theorem implies that T_a is inconsistent. Therefore also PA can prove that $CONS(T_a)$ implies that Φ is true, but then as $T_{\mathrm{succ}(a)}$ contains the statement $CONS(T_a)$, the theory $T_{\mathrm{succ}(a)}$ can prove that Φ is true...

3.6.2 Enumerating Π^1_1 sets

In this thesis, the inductive definition that will be made over computable ordinals, will be independent from ordinal notations. We rely on the following theorem to make those inductive definitions. We simply have to give an effective version of the basic set-theoretical argument that definitions by induction over ordinals can be made.

The main theorem

In the next theorem, we make a slight abuse of notation by writing things like $f : \mathcal{O}_{<\alpha} \to \mathbb{N}$. The underlying representation for such a function f is made by a function $g : \mathbb{N} \to \mathbb{N}$ such that on $\mathcal{O}_{<\alpha}$, we have $g = f$, and on $\mathbb{N} - \mathcal{O}_{<\alpha}$, we have $g = 0$ (or any other value, the goal is not to have two distinct representations for the same function). Also for a function $f : \mathcal{O}_{<\alpha} \to \mathbb{N}$ and $\beta < \alpha$ we write $f\restriction_\beta$ to denote the restriction of f to $\mathcal{O}_{<\beta}$.

Theorem 3.6.3:
Let $F \subseteq \mathbb{N} \times \mathbb{N}^{\mathbb{N}} \times \mathbb{N}$ be a Σ^1_1 set such that for any computable α, any function $f : \mathcal{O}_{<\alpha} \to \mathbb{N}$, any $a \in \mathcal{O}_{=\alpha}$, there exists a unique n such that $F(a, f, n)$, which is also the same n for any $a \in \mathcal{O}_{=\alpha}$. Then there exists a unique Π^1_1 function $f : \mathcal{O} \to \omega$ such that for any $a \in \mathcal{O}$ we have $f(a) = n$ iff $F(a, f\restriction_{|a|_o}, n)$. Furthermore for $a_1, a_2 \in \mathcal{O}_{=\alpha}$ we have $f(a_1) = f(a_2)$, and each restriction $f\restriction_\alpha$ is Δ^1_1 uniformly in a code of $\mathcal{O}_{=\alpha}$.

Proof: For any α and any $a \in \mathcal{O}_{=\alpha}$, let $\mathcal{B}_a$ be the following Σ^1_1 set, uniformly in a:

$$\left\{ f : \mathcal{O}_{\leq\alpha} \to \mathbb{N} \;\middle|\; \forall b \in \mathcal{O}_{\leq\alpha}\ F\big(b, f\restriction_{|b|_o}, f(b)\big) \right\}$$

Let us first argue that this set is a Σ^1_1 set, uniformly in a. We have that $\mathcal{O}_{\leq\alpha}$ is Σ^1_1 uniformly in a, and $f\restriction_{|b|_o}$, the restriction of f to $\mathcal{O}_{<|b|_o}$, is also Σ^1_1 uniformly in b.

We should prove that for every $a \in \mathcal{O}$, the set $\mathcal{B}_a$ contains exactly one element, and for $a_1, a_2 \in \mathcal{O}_{=\alpha}$, we have $\mathcal{B}_{a_1} = \mathcal{B}_{a_2}$. For each $a \in \mathcal{O}_{=\alpha}$, the set $\mathcal{B}_a$ is then denoted by $\mathcal{B}_\alpha$ and its element by f_α.

For any $a \in \mathcal{O}_{=\varnothing}$, the set $\mathcal{B}_a$ obviously contains only the unique function $f_\varnothing$, which to $a \in \mathcal{O}_{=\varnothing}$ associates the unique n such that $F(a, \varnothing, n)$, where $\varnothing$ is the unique empty function. We easily see that by hypothesis, the function $f_\varnothing$ is independent from $a \in \mathcal{O}_{=\varnothing}$.

Suppose now that for any $\beta < \alpha$ and any $a_1, a_2 \in \mathcal{O}_{=\beta}$, the set $\mathcal{B}_{a_1}$ is equal to the set $\mathcal{B}_{a_2}$, now denoted by $\mathcal{B}_\beta$. Suppose also that for any $\beta < \alpha$ the set $\mathcal{B}_\beta$ contains exactly one element f_β. Let us prove that for any $a \in \mathcal{O}_{=\alpha}$, the set $\mathcal{B}_a$ contains exactly one element, independent from the code of $\mathcal{O}_{=\alpha}$.

Consider any two ordinals $\beta_1 < \beta_2 < \alpha$. By definition of $\mathcal{B}_{\beta_1}$ and $\mathcal{B}_{\beta_2}$ we clearly have that $f \upharpoonright_{\beta_2}$ extends $f \upharpoonright_{\beta_1}$. We can then define $f_{<\alpha}$ to be $\bigcup_{\beta<\alpha} f_\beta$. Finally we can define f_α to be $f_{<\alpha}$, to which we add the mapping of any $a \in \mathcal{O}_{=\alpha}$ to the unique value n such that $F(a, f_{<\alpha}, n)$. By definition of f_α and B_a, we have that f_α is an element of $\mathcal{B}_a$ for any $a \in \mathcal{O}_{=\alpha}$. Also by induction hypothesis it is the only such element for any a, as otherwise we would have $n_1 \neq n_2$ such that $F(a, f_{<\alpha}, n_1)$ and $F(a, f_{<\alpha}, n_2)$, which contradicts the hypothesis.

As a Σ^1_1 singleton, each element of $\mathcal{B}_a$ is a Δ^1_1 function, uniformly in a. Furthermore, those functions extend each other, and we can then define $f : \mathcal{O} \to \omega$ by $f(a) = n$ to be $a \in \mathcal{O} \ \wedge \forall g \in \mathcal{B}_a \ g(a) = n$. ∎

The previous theorem not only justifies definitions by induction over codes for computable ordinals, but it also justifies that we don't need to worry about which code we use for a given ordinal, as long as our inductive definition appears to be coding independent, which is normally the case if the inductive definition is written in a set theoretical fashion, using ordinals rather than codes.

Then we will think of each ordinal to be a computational time, just like in classical recursion theory. When ordinals are viewed as computation time, we will write s, t or r to denote them. Inside our 'higher algorithm', where computational times are now ordinals, it should be clear that any Δ^1_1 operation is now allowed, as long as this operation is done uniformly in the current computational stage. Also things like 'at stage s, if Φ_e is a total function then ...' can be written safely.

Substages

In practice we will often use the previous theorem to define functions from $\mathcal{O}$ into 2^ω rather than ω. This can be done easily still using the previous theorem, by dividing each stage s into ω substages, so that at each substage n we output the first n bits of our sequence.

Quite often in classical computability theory we see things like 'at stage s, at substage $n \leq s$'. We should quickly argue that dealing with ordinals, it is now natural to have at a given computation step, infinitely many substages. Recall Proposition 1.3.1 of left division for ordinal: For any ordinal s there are unique ordinal t and $n < \omega$ such that $s = \omega \times t + n$. Also when we will write things like, for example, 'At stage t, and substage $n < \omega$...', it is an informal way to say 'at stage s, find t, n such that $s = \omega \times t + n$...'. Finding such t, n uniformly in s is certainly a Δ^1_1 operation and then can be done safely.

The projectum function

Quite often, we will need to use what is called in admissibility theory, a projectum function, in our case, an injection $p:\omega_1^{ck}\to\omega$. Also recall the construction of our Σ^1_1 set $F\subseteq\mathbb{N}\times\mathbb{N}^{\mathbb{N}}\times\mathbb{N}$ of Theorem 3.6.3. We should prove that there is an injection p, for which given any α, we can use $p(\alpha)$ into the definition of F and keep it meanwhile Σ^1_1 and coding independent. The following definition propose a natural candidate for the projectum function:

Definition 3.6.1. *The* ***projectum function*** *denoted by* $p:\omega_1^{ck}\to\omega$ *is given by*

$$p(\alpha)=\min\{a\ :\ a\in\mathcal{O}_{=\alpha}\}$$

We easily verify that it defines an injection from ω_1^{ck} to ω, which is coding independent. Also as $\mathcal{O}_{=\alpha}$ is Δ^1_1 uniformly in any code of $\mathcal{O}_{=\alpha}$, then also F remains Σ^1_1.

An example

We give an example of the construction of a higher simple set, that is a co-infinite Π^1_1 set of integers which intersects every infinite Π^1_1 set of integers.

Example 3.6.1:
Let $\{P_e\}_{e\in\omega}$ be an enumeration of the Π^1_1 sets of integers. We build a simple Π^1_1 set A by describing for each step s a Δ^1_1 set A_s that should be considered as the enumeration of the set A up to computational step s. During the enumeration, we keep track of a Boolean value R_e for each e, initialized to false, and that will be set to true at stage s when an element of P_e is enumerated into A at stage s.

At stage 0, set $A_0=\varnothing$. At successor stage $s+1$, for each substage e, if R_e is marked as false at stage s and if some element n bigger than $2e$ is into $P_e[s]$ then we put n into A at stage s, that is, we define A_{s+1} to be $A_s\cup\{n\}$, and we mark R_e as true at stage $s+1$. At limit stage s, we define A_s to be $\bigcup_{t<s}A_t$. We then define $A=\bigcup_{s<\omega_1^{ck}}A_s$. The verification that A is a higher simple set is like in the lower case. ◇

In the previous example, everything happens as in the lower case, and the proof is transfered without any particular problem. This is however not always the case, and there is indeed something which differs greatly: the existence of limit computational steps. In the previous example, this does not pose any problem, but we will see many examples in which it does.

Getting a nice enumeration of Π^1_1 sets

We argued that we can define a Π^1_1 set by enumerating it along the computable ordinals. We shall argue here that the converse is true, that is, if we are given a Π^1_1 set, we can consider that it is enumerated along computable ordinals. Also it is often convenient to consider that at each successor stage at most one new element is enumerated, and nothing new is enumerated at a limit stage.

We already have with Corollary 3.3.1 that a Π^1_1 set A is equal to $\bigcup_{s<\omega_1^{ck}}A_s$ where each A_s is Δ^1_1 uniformly in a code of $\mathcal{O}_{=s}$. We can simply define a Σ^1_1 predicate $F\subseteq\mathbb{N}\times\mathbb{N}^{\mathbb{N}}\times\mathbb{N}$ in the style of Theorem 3.6.3 which performs the following:

"At limit stage we enumerate nothing, and at stage $\omega \times s + n$ for $n > 0$ we enumerate the n-th element of $A_s - A_{s-1}$ if it exists."

Note that F is supposed to define only total functions. In practice we can always decide for a value corresponding to 'nothing is enumerated'.

Vocabulary for Π^1_1 approximations

We finish by introducing a bit of vocabulary that we will use in our different Π^1_1 approximations

Definition 3.6.2. *For s_1, s_2 some stages (computable ordinals), we say that a sequence of objects $\{a_t\}_{s_1 \leq t < s_2}$ is **stable** is for any stage t with $s_1 \leq t < s_2$ we have $a_t = a_{s_1}$.* ■

Definition 3.6.3. *For s_1, s_2 some stages (computable ordinals), we say that a sequence of objects $\{a_t\}_{s_1 \leq t < s_2}$ **changes finitely often** if for some n, there are at most n stages $s_1 \leq t_1 < t_2 < \cdots < t_n < s_2$ such that $a_{t_i} \neq a_{t_{i+1}}$. Otherwise we say that $\{a_t\}_{s_1 \leq t < s_2}$ **changes infinitely often**.* ■

3.7 Higher randomness

3.7.1 Overview of the different classes

We now would like to extend the usual randomness notions, but using now the descriptive power that the restricted use of second order predicates (only Π^1_1 and Σ^1_1) gives us. Maybe the simplest thing we can do is first to say that a sequence is random iff it is in no Δ^1_1 set of measure 0. The equivalence between hyperarithmetic sets and Δ^1_1 sets makes it clear that any Δ^1_1 set is measurable, as it is a Borel set.

Definition 3.7.1 (Sacks). *We say that $Z \in 2^\omega$ is Δ^1_1**-random** if it is in no Δ^1_1 nullset.* ■

Martin-Löf was actually the first to promote this notion (see [59]), suggesting that it was the appropriate mathematical concept of randomness. Even if his first definition undoubtedly became the most successful over the years, this other definition got a second wind recently on the initiative of Hjorth and Nies who started to study the analogy between the usual notions of randomness and theirs higher counterparts. In order to do so they created in [30] a higher analogue of Martin-Löf randomness.

Definition 3.7.2. *An open set $\mathcal{U}$ is a Π^1_1**-open** set if there is a Π^1_1 set of strings W such that $\mathcal{U} = [W]^<$. A closed set $\mathcal{F}$ is a Σ^1_1**-closed set** if it is the complement of a Π^1_1-open set.* ■

Definition 3.7.3 (Hjorth, Nies). *A Π^1_1**-Martin-Löf test** is given by an effectively null intersection of open sets $\bigcap_n \mathcal{U}_n$ (with $\lambda(\mathcal{U}_n) \leq 2^{-n}$), each $\mathcal{U}_n$ being Π^1_1 uniformly in n. A sequence X is Π^1_1**-ML-random** if it is in no Π^1_1-Martin-Löf test.* ■

Here also, as our tests are $\mathbf{\Pi}^0_2$ sets, there are all measurable and the previous definition makes sense. It will be sometimes convenient to use a higher version of Solovay tests:

Definition 3.7.4. *A* Π^1_1-***Solovay test*** *is given by a sequence* $\{\mathcal{U}_n\}_{n\in\mathbb{N}}$ *of uniformly (in* n*)* Π^1_1*-open sets such that* $\sum_{n\in\mathbb{N}}\lambda(\mathcal{U}_n)$ *is finite. A sequence* X ***passes*** *the* Π^1_1*-Solovay test if it belongs to only finitely many* $\mathcal{U}_n$*.* ■

The proof that X is Π^1_1-Martin-Löf random iff it passes all the Solovay tests works as in the lower setting. An interesting possibility with Π^1_1-Solovay test, that will be used sometimes, is that we can index each open set with a computable ordinal instead of indexing it with an integer. Formally, given a sequence of Π^1_1-open sets $\{\mathcal{U}_s\}_{s<\omega_1^{ck}}$, we can build the Π^1_1-Solovay test $\mathcal{V}_n$ where each $\mathcal{V}_n$ starts with an empty enumeration, until n is witnessed to be a code for the ordinal s, in which case $\mathcal{V}_n$ becomes equal to $\mathcal{U}_s$. It is clear that the notion of being captured in unchanged between $\{\mathcal{U}_s\}_{s<\omega_1^{ck}}$ and $\{\mathcal{V}_n\}_{n\in\mathbb{N}}$.

We now discuss the relationship between Π^1_1-Martin-Löf randomness and Δ^1_1 randomness. Theorem 1.8.1 implies that the set of Π^1_1-Martin-Löf randoms is included in the set of Δ^1_1 randoms. In other words, the notion of Π^1_1-Martin-Löf randomness is stronger or equal than the notion of Δ^1_1 randomness.

Proposition 3.7.1:
If Z is Π^1_1-Martin-Löf random, then Z is Δ^1_1-random.

Proof: Suppose Z is in a Δ^1_1 nullset $\mathcal{A}$. This nullset is Σ^0_α for some computable α. Now using Theorem 1.8.1, we can find uniformly in n a Σ^0_α-open set of measure less than 2^{-n}, and containing $\mathcal{A}$. Also a Σ^0_α-open set is clearly a Π^1_1-open set and we can then build a Π^1_1-Martin-Löf test capturing Z. ■

We shall see now that Π^1_1-Martin-Löf randomness is strictly stronger than Δ^1_1-randomness. This was proved by Chong, Nies and Yu in [7] using the notion of higher Kolmogorov complexity that we will introduce in Section 3.7.2. The proof they gave can be seen as a higher analogue of the separation between computable randomness and Martin-Löf randomness. We give here a similar proof, without using higher Kolmogorov complexity, but rather a mix between higher priority method and forcing with closed sets of positive measure. A similar technique will be reused in this thesis, for both Theorem 5.3.3 and Theorem 6.4.4.

Theorem 3.7.1:
There is a sequence X which is Δ^1_1-random and not Π^1_1-Martin-Löf random.

Proof: Let $\{\mathcal{A}_s\}_{s<\omega_1^{ck}}$ be an enumeration of the Δ^1_1 sets of measure 1. To get this enumeration, recall that the Δ^1_1 sets are the Σ^0_α sets, and that the measure of a Σ^0_α set is Δ^1_1, uniformly in α. Recall that $p:\omega_1^{ck}\to\omega$ is the projectum function, let $\mathcal{O}^1_{\leq s}=\{p(t)\ :\ t\leq s\}$, and for $m\in\mathcal{O}^1_{\leq s}$ let $\mathcal{O}^1_{\leq s}\restriction_m=\{n\in\mathcal{O}^1_{\leq s}\ :\ n<m\}$.

The construction:

We can suppose without loss of generality that $\mathcal{A}_0 = 2^{\mathbb{N}}$. At stage 0 we define for each n the set $\mathcal{F}_0^n$ to be $2^{\mathbb{N}}$ and the string σ_0^n to be the string consisting of $2n$ 0's.

Suppose that at every stage $t < s$ we have defined for each $n \in \mathbb{N}$ a Δ_1^1 closed set $\mathcal{F}_t^n$ and a string σ_t^n such that $\sigma_t^n \prec \sigma_t^{n+1}$ and with $|\sigma_t^n| = 2n$. Suppose also that for each m we have $\lambda(\bigcap_{n\leq m} \mathcal{F}_t^n \cap [\sigma_t^m]) > 0$ and that if $m \in \mathcal{O}_{\leq t}^1$ we have $\mathcal{F}_t^m \subseteq \mathcal{A}_{p^{-1}(m)}$.

Suppose first that s is successor and let us define $\mathcal{F}_s^n$ and σ_s^n for each $n \in \mathbb{N}$. For each $n < p(s)$ we define $\sigma_s^n = \sigma_{s-1}^n$ and $\mathcal{F}_s^n = \mathcal{F}_{s-1}^n$.

For each $m \in \mathbb{N}$ in increasing order, and starting with $m = p(s)$, if $m \in \mathcal{O}_{\leq s}^1$ with $t = p^{-1}(m)$, let us compute an increasing union of Δ_1^1 closed sets $\bigcup_n \mathcal{F}_n \subseteq \mathcal{A}_t$ with $\lambda(\mathcal{A}_t - \bigcup_n \mathcal{F}_n) = 0$. Let $\mathcal{F}_s^m$ be the first closed set of the union $\bigcup_n \mathcal{F}_n$ such that $\lambda(\bigcap_{n<m} \mathcal{F}_s^n \cap \mathcal{F}_s^m \cap [\sigma_s^{m-1}]) > 0$. If $m \notin \mathcal{O}_{\leq s}^1$, let $\mathcal{F}_s^m = 2^{\mathbb{N}}$.

Then we let σ_s^m be the first string of length $2m$ which extends σ_s^{m-1}, such that $\lambda(\bigcap_{n\leq m} \mathcal{F}_s^n \cap [\sigma_s^m]) > 0$.

Finally, for a stage s limit we define for each n the string σ_s^n to be the limit of the sequence $\{\sigma_s^n\}_{t<s}$ and the closed set $\mathcal{F}_s^n$ to be the limit of the sequence $\{\mathcal{F}_s^n\}_{t<s}$. We shall argue that later that such a limit always exists.

The verification:

For every m there is a stage s such that $\{\mathcal{O}_{\leq t}^1 \upharpoonright_m\}_{s\leq t<\omega_1^{ck}}$ is stable. Furthermore, for each m, the sequence $\{\mathcal{O}_{\leq t}^1 \upharpoonright_m\}_{t<\omega_1^{ck}}$ can change at most m times, because at most m values can be enumerated in $\mathcal{O} \upharpoonright_m$. It follows that at every limit stage s and for every m, the sequences $\{\sigma_s^m\}_{t<s}$ and $\{\mathcal{F}_s^m\}_{t<s}$ also can change at most m times, and then converges.

Also by design for every $s \leq \omega_1^{ck}$, the unique limit point X_s of $\{[\sigma_s^n]\}_{n\in\mathcal{O}_{\leq s}^1}$ belongs to $\bigcap_{t\leq s} \mathcal{A}_t$. In particular the limit X of the sequence $\{X_s\}_{s<\omega_1^{ck}}$ belongs to $\bigcap_{t\leq\omega_1^{ck}} \mathcal{A}_t$ and is then Δ_1^1-random.

We should now prove that it is not Π_1^1-Martin-Löf random. We argued already that $\{\sigma_t^m\}_{t<\omega_1^{ck}}$ can change at most m times. Then we can put each string σ_s^m of length $2m$, into the m-th component of a Π_1^1-Martin-Löf test which has measure smaller than $m \times 2^{-2m} \leq 2^{-m}$. ∎

The higher analogue of weak-2-randomness has also been studied by Chong and Yu in [8]. We call this notion weak-Π_1^1-randomness, because as we will prove it in Section 6.1, this notion can be seen as a weak form of Π_1^1-randomness; it is however not obvious from the definition that we now give:

Definition 3.7.5. *We say that Z is **weakly-Π_1^1-random** if it belongs to no uniform intersection of Π_1^1-open sets $\bigcap_n \mathcal{U}_n$, with $\lambda(\bigcap_n \mathcal{U}_n) = 0$.* ∎

It is clear that the notion of weak-Π_1^1-randomness is stronger than the notion of Π_1^1-Martin-Löf randomness. We shall see later that it is a strictly stronger notion.

So far, the full descriptive power of Π^1_1 or Σ^1_1 predicates has not been used, because all our tests are still only $\mathbf{\Pi^0_2}$ sets. Also Sacks gave earlier an even stronger definition, where tests are now Π^1_1 nullsets. This definition is made possible by a theorem of Lusin saying that even though Π^1_1 sets are not necessarily Borel, they remain all Lebesgue-measurable, that is, measurable for a complete measure. Recall that from Corollary 3.3.1, any Π^1_1 set $\mathcal{A}$ is a uniform union of Borel sets $\mathcal{A}_\alpha$, over $\alpha < \omega_1$, with each $\mathcal{A}_\alpha = \{X : e \in \mathcal{T}^X_{<\alpha}\}$ for some e.

Theorem 3.7.2 (Lusin):
There is an ordinal γ and a Borel set $\mathcal{B}$ of measure 0 such that for any Π^1_1 set $\mathcal{A} = \bigcup_{\alpha<\omega_1} \mathcal{A}_\alpha$, the set $\mathcal{A} - \mathcal{A}_\gamma$ is contained in $\mathcal{B}$. In particular any Π^1_1 set is measurable.

Proof: Fix some integer e. For any α, consider the set of sequences X for which e is an X-code of a well-founded X-c.e. tree coding for α. Formally $\mathcal{B}_{e,\alpha} = \{X : e \in \mathcal{T}^X_{|=\alpha|}\}$. For each rational $q > 0$ we can only have finitely many β such that $\lambda(\mathcal{B}_{e,\beta}) > q$, otherwise, as $\mathcal{B}_{e,\alpha_1} \cap \mathcal{B}_{e,\alpha_2} = \varnothing$ for $\alpha_1 \neq \alpha_2$, we would have $\lambda(2^{\mathbb{N}}) > 1$, by countable additivity of measures. Therefore, there is a smallest countable ordinal γ_e such that $\alpha \geq \gamma_e \to \lambda(\mathcal{B}_{e,\alpha}) = 0$.

Let $\gamma = \sup_e \gamma_e$ and $\mathcal{B} = \bigcup_e \mathcal{B}_{e,\gamma}$. By the definition of γ and countable additivity of measures, we have that $\lambda(\mathcal{B}) = 0$. Consider now any Π^1_1 predicate $\mathcal{A} = \bigcup_{\alpha<\omega_1} \mathcal{A}_\alpha$. It is clear that $\mathcal{A} - \mathcal{A}_\gamma \subseteq \mathcal{B}$, because if $X \in \mathcal{A} - \mathcal{A}_\gamma$, it means that $e \in \mathcal{T}^X_{=\beta}$ for some e and some $\beta \geq \gamma$; but then also, some other X-code can enumerate a well-founded tree coding for γ. Therefore $\mathcal{A} \setminus \mathcal{A}_\gamma \subseteq \mathcal{B}$. ∎

Sacks proved later that the ordinal γ of the previous theorem can actually be equal to ω_1^{ck}, making the set $\{X : \omega_1^X > \omega_1^{ck}\}$ a Π^1_1 set of measure 0:

Theorem 3.7.3 (Sacks):
The set $\{X : \omega_1^X > \omega_1^{ck}\}$ has measure 0. This set is in fact a Borel set $\mathcal{B}$ of measure 0 such that for any Π^1_1 set $\mathcal{A} = \bigcup_{\alpha<\omega_1} \mathcal{A}_\alpha$, we have that $\mathcal{A} - \mathcal{A}_{\omega_1^{ck}}$ is contained in $\mathcal{B}$.

We do not yet give a proof of $\lambda(\{X : \omega_1^X > \omega_1^{ck}\}) = 0$, as we will later prove something slightly stronger later, in Theorem 6.1.1. Also we argued already that the set of sequences which do not preserve ω_1^{ck} is not a Σ^1_1 set. It is however not hard to prove that it is $\Sigma^1_1(\mathcal{O})$ and then Borel. The exact Borel complexity of this set will be studied in Section 6.7. For now we simply prove an interesting corollary of this theorem, proved independently by Tanaka [92] and Sacks [77].

Corollary 3.7.1:
The Δ^1_1 sequences are a basis for the Π^1_1 sets of positive measure.

PROOF: If a Π^1_1 set $\mathcal{A} = \bigcup_{\alpha<\omega_1} \mathcal{A}_\alpha$ is of positive measure. From the previous theorem and by countable additivity of the Lebesgue measure, already for some computable ordinal α we should have that the set $\bigcup_{\beta<\alpha} \mathcal{A}_\beta$ is of positive measure. Also this set is a Δ^1_1 set of positive measure, and contains then a Δ^1_1 closed set of positive measure. Also the leftmost path of such a closed set is a Δ^1_1 sequence. ■

We now give the definition of randomness corresponding to Π^1_1 nullsets.

Definition 3.7.6 (Sacks). *We say that $Z \in 2^{\mathbb{N}}$ is Π^1_1-**random** if it is in no Π^1_1 nullset.*■

This last notion is very interesting for many reasons. One of them is that no X such that $\omega_1^X > \omega_1^{ck}$ is Π^1_1-random, and we shall see now that this is the best we can do, as any randomness notion weaker than Π^1_1-randomness contains elements that make ω_1^{ck} a computable ordinal. This is achieved through the following beautiful theorem of Chong, Nies and Yu (see [7]):

Theorem 3.7.4 (Chong, Yu, Nies):
A sequence Z is Π^1_1-random iff it is Δ^1_1-random and $\omega_1^Z = \omega_1^{ck}$.

PROOF: Suppose that Z is Δ^1_1-random. If $\omega_1^Z > \omega_1^{ck}$ then by Theorem 3.7.3, Z is not Π^1_1-random.

Suppose now that Z is not Π^1_1-random and then captured by a Π^1_1 set $\mathcal{A} = \bigcup_{\alpha<\omega_1} \mathcal{A}_\alpha$ of measure 0. If there is a computable α such that $Z \in \mathcal{A}_\alpha$ then Z is not Δ^1_1-random as $\mathcal{A}_\alpha$ is a Δ^1_1 set of measure 0. Otherwise $Z \in \mathcal{A} - \bigcup_{\alpha<\omega_1^{ck}} \mathcal{A}_\alpha$ and then $\omega_1^Z > \omega_1^{ck}$. ■

Another important property of Π^1_1-randomness is certainly the existence of a universal Π^1_1 nullset, in the sense that it contains all the others. Kechris was the first to prove this, in [33], and he actually proved a more general result, implying for example also the existence of a largest Π^1_1 thin set (a largest Π^1_1 set which contains no perfect subset). Later, Hjorth and Nies gave in [30] an explicit construction of this Π^1_1 nullset.

Theorem 3.7.5 (Kechris, Hjorth, Nies):
There is a largest Π^1_1 nullset.

PROOF: Let $\{P_e\}_{e\in\omega} = \bigcup_{\alpha<\omega_1} P_{e,\alpha}$ be an enumeration of the Π^1_1 sets. Recall from above that each set $P_e - \bigcup_{\alpha<\omega_1^{ck}} P_{e,\alpha}$ is always null and contained in the nullset $\{X \mid \omega_1^X > \omega_1^{ck}\}$. Let us argue that uniformly in e, one can transform the set $\bigcup_{\alpha<\omega_1^{ck}} \mathcal{P}_{e,\alpha}$ into a set $\bigcup_{\alpha<\omega_1^{ck}} \mathcal{Q}_{e,\alpha}$ (where each $\mathcal{Q}_{e,\alpha}$ is Δ^1_1 uniformly in e and a code of $\mathcal{O}_{=\alpha}$) such that $\lambda(\bigcup_{\alpha<\omega_1^{ck}} \mathcal{Q}_{e,\alpha}) = 0$, and such that if $\lambda(\bigcup_{\alpha<\omega_1^{ck}} \mathcal{P}_{e,\alpha}) = 0$ then $\bigcup_{\alpha<\omega_1^{ck}} \mathcal{Q}_{e,\alpha} = \bigcup_{\alpha<\omega_1^{ck}} \mathcal{P}_{e,\alpha}$.

To do so we simply set $\mathcal{Q}_{e,\alpha} = \mathcal{P}_{e,\alpha}$ if $\lambda(\mathcal{P}_{e,\alpha}) = 0$ (recall that the measure of a Δ^1_1 set is uniformly Δ^1_1) and $\mathcal{Q}_{e,\alpha} = \varnothing$ otherwise. Then we define $\mathcal{Q}$ to be $\bigcup_e \bigcup_{\alpha<\omega_1^{ck}} \mathcal{Q}_{e,\alpha}$ together with the set $\{X \mid \omega_1^X > \omega_1^{ck}\}$. The set $\mathcal{Q}$ is clearly Π^1_1, and by construction it is a nullset containing every Π^1_1 nullset. ■

Chong and Yu proved in [8] that weak-Π^1_1-randomness is strictly stronger than Π^1_1-Martin-Löf-randomness. We will prove later that Π^1_1-randomness is strictly stronger than weak-Π^1_1-randomness.

One could also define the randomness notion obtained by considering Σ^1_1 nullsets, but this turns out to be equivalent to Δ^1_1-randomness.

Theorem 3.7.6 (Sacks):
A Δ^1_1-random sequence is in no Σ^1_1 nullset. Therefore Σ^1_1-randomness coincides with Δ^1_1-randomness.

PROOF: Let $\mathcal{A} = \bigcap_{\alpha<\omega_1} \mathcal{A}_\alpha$ be a Σ^1_1 nullset. Note that we can suppose that the intersection is decreasing. By Theorem 3.7.3 we have that $\bigcap_{\alpha<\omega_1^{ck}} \mathcal{A}_\alpha$ is already of measure 0. Then we can define the Π^1_1 function $f : \omega \to \omega_1^{ck}$ which associates to n the smallest ordinal such that $\lambda(\mathcal{A}_\alpha) \leq 2^{-n}$. As f is total, it is actually a Δ^1_1 function, and then its range is a Δ^1_1 set of computable ordinals, which is then bounded by some computable ordinal β, by the Σ^1_1-boundedness principle. Therefore we have $\lambda(\bigcap_{\alpha<\beta} \mathcal{A}_\alpha) = 0$ and then $\mathcal{A}$ is contained in a Δ^1_1 set of measure 0. ∎

3.7.2 Higher Kolmogorov complexity

In this section, we introduce a higher version of the notion of Kolmogorov complexity, which is a fundamental notion of classical randomness. For a very complete survey on the subject of lower Kolmorogov complexity, the reader can refer to Li and Vitany's book (see [54]), which also provides some interesting historical overview of the different notions of algorithmic complexity.

Background on Kolmogorov complexity

Informally, the Kolmogorov complexity of a finite object, in our case strings, is a measure of the computability resources needed to specify that object. It is named after Andrey Kolmogorov, who first published on the subject in [40], following earlier work of Solomonoff (see [84]).

Definition 3.7.7. *A **machine** is a partial computable function $M : 2^{<\mathbb{N}} \to 2^{<\mathbb{N}}$. The Kolmorogov complexity of a string σ with respect to the machine M, denoted by $C_M(\sigma)$, is the length of the smallest string τ such that $M(\tau) = \sigma$ if such a string τ exists, and by convention ∞ otherwise.* ∎

As credited in [54], it is Solomonoff who first defined, in [85], the notion of 'Kolmogorov complexity' as we know it today, and he proved in particular the universal machine theorem, which was soon after, also proved independently by Kolmorogov in [41].

Theorem 3.7.7 (Universal machine theorem):
There exists a universal Machine U, that is, for any machine M there exists a constant c_M such that $C_U(\sigma) \leq C_M(\sigma) + c_M$ for every string σ.

It then makes sense to speak of the Kolmogorov complexity of a string, independently of a specific machine, as we can then consider the Kolmorogov complexity with respect to a universal machine U. Already in [41], Kolmogorov saw the connection between this new notion of algorithmic complexity and algorithmic randomness. Also he emphasized that the complexity of a string could be used as the measure of its 'degree of randomness'. This gave rise to the vast field of algorithmic information theory, amusingly described by Chaitin to be "the result of putting Shannon's information theory and Turing's computability theory into a cocktail shaker and shaking vigorously."

However, a satisfactory notion of randomness for infinite objects using the notion of Kolmogorov complexity was still missing. And it could only be done after was designed the notion of prefix-free Kolmorogov complexity, which is said in [54] to have appeared independently and almost simultaneously in the work of Levin ([51]), Gács ([24]) and Chaitin ([5]).

Definition 3.7.8. *A set of strings W is said to be prefix-free if any two strings σ_1, σ_2 in W are incomparable, that is, $\sigma_1 \perp \sigma_2$.* ■

Definition 3.7.9. *A **prefix-free machine** is a partial computable function $M : 2^{<\mathbb{N}} \to 2^{<\mathbb{N}}$ whose domain of definition is a prefix-free set of strings. The Kolmorogov complexity of a string σ with respect to the machine M, denoted by $K_M(\sigma)$, is the length of the smallest string τ such that $M(\tau) = \sigma$, and is by convention ∞ is no such string exists.* ■

Just like for usual Kolmogorov complexity, there exists a universal prefix-free machine, and it then makes sense to speak of the prefix-free Kolmogorov complexity of a string.

Theorem 3.7.8 (Universal prefix-free machine theorem):
There exists a universal prefix-free machine U, that is, for any prefix-free machine M there exists a constant c_M such that $K_U(\sigma) \leq K_M(\sigma) + c_M$ for every string σ.

This new notion could then be used to define randomness for infinite objects, here binary sequences X, by saying that X is random if it has maximal prefix-free Kolmorogov complexity on each of its prefixes. So a sequence X is random if no prefix of X can be compressed, up to a constant. We could roughly say that the only way for a program to output a prefix σ of X is by explicitly writing σ. We then have the following theorem for which slightly different versions, using different notions of complexity, have been first proved simultaneously an independently by Levin in [50] and Schnorr in [79] (credited in [54]):

Theorem 3.7.9:
The following are equivalent for a sequence Z:

1. *The sequence Z is Martin-Löf random.*
2. *There exists a constant c such that for every n, the prefix-free Kolmorogov complexity of $Z\upharpoonright_n$ is bigger than $n - c$.*

We emphasize that a similar definition of randomness using plain (non prefix-free) Kolmogorov complexity does not work, as there is no sequence X such that all of its prefixes are incompressible, with respect to plain Kolmogorov complexity. Informally, the length of a prefix can be used to encode some information to compress prefixes.

Also the requirement of prefix-freeness is justifiable: Given a universal prefix-free machine U, one can consider that each string on which U is defined is a program, that is then executed on U. In this context, Chaitin calls those strings *self-delimiting programs*. Also this point of view matches what happens in the real world of computer programming: Any binary file executed by a computer comes with a 'end-of-file' tag, indicating where the file ends. Also seen as a binary string σ, no string $\tau > \sigma$ corresponds to a valid file, as there can be nothing after an 'end-of-file' tag.

Higher Kolmogorov complexity

While defining the notion of Π^1_1-Martin-Löf randomness in [30], Hjorth and Nies also defined the notion of Π^1_1-Kolmorogov complexity, in order to study higher analogies of theorems occurring in classical randomness. Here we don't make the distinction anymore between prefix-free Kolmogorov complexity and plain Kolmogorov complexity, as only the prefix-free version will be used. Also we simply call it Kolmorogov complexity.

Definition 3.7.10. *A* Π^1_1***-machine*** M *is a* Π^1_1 *partial function* $M : 2^{<\mathbb{N}} \to 2^{<\mathbb{N}}$*. A* Π^1_1***-prefix-free machine*** M *is a* Π^1_1 *partial function* $M : 2^{<\mathbb{N}} \to 2^{<\mathbb{N}}$ *whose domain of definition is a prefix-free set of strings. We denote by* $\mathrm{hK}_M(\sigma)$ *the* Π^1_1*-Kolmorogov complexity of a string* σ *with respect to the* Π^1_1*-machine* M*, defined to be the length of the smallest string* τ *such that* $M(\tau) = \sigma$*, if such a string exists, and by convention,* ∞ *otherwise.* ∎

We now prove a universal Π^1_1-prefix-free machine theorem:

Theorem 3.7.10 (Universal Π^1_1-p.-f. machine theorem, Hjorth and Nies): *There is a universal* Π^1_1*-prefix-free machine* U*, that is, for each* Π^1_1*-prefix-free machine* M*, there exists a constant* c_M *such that* $\mathrm{hK}_U(\sigma) \leq \mathrm{hK}_M(\sigma) + c_m$ *for any string* σ*.*

Proof: We first have to make sure that we can enumerate the Π^1_1-prefix-free machines: we have a total computable function such that for any e, the integer $f(e)$ is always an index for a Π^1_1-prefix-free machine, and if e is already an index for a Π^1_1-prefix-free machine, then $f(e)$ is an index for the same machine.

Recall that we can suppose without loss of generality that a Π^1_1-machine M_e enumerates new pairs at successor stages only, and at most one pair per stage. Given the machine M_e, suppose that (σ, τ) is enumerated in M_e at successor stage s. If $M_{f(e),s-1}$ contains (σ', τ') such that σ' is compatible with σ, then we enumerate nothing in $M_{f(e)}$ at stage s. Otherwise we enumerate (σ, τ) in $M_{f(e)}$ at stage s. At limit stage s, we define $M_{f(e),s}$ to be the union of $\{M_{f(e),t}\}_{t<s}$.

Then we simply define U to be the machine which enumerates $(0^e\hat{\ }1\hat{\ }\sigma, \tau)$ for each e, σ and τ such that (σ, τ) is enumerated in $M_{f(e)}$.

For each machine M of index $f(e)$, the constant c_M is given by $e + 1$. ∎

Definition 3.7.11. *For a string σ, we define* $\mathrm{hK}(\sigma)$ *to be* $\mathrm{hK}_U(\sigma)$ *for a universal* Π^1_1-*prefix-free machine* U, *fixed in advance.* ■

We now give a general technique, used to build prefix-free machines. For this purpose we need the following definitions.

Definition 3.7.12. *Given a set* $A \subseteq \mathbb{N} \times 2^{<\mathbb{N}}$, *the* ***weight*** *of* A, *denoted by* $\mathrm{wg}(A)$, *refers to the quantity* $\sum_{(l,\sigma)\in A} 2^{-l}$ *if this quantity is finite, and refers to* ∞ *otherwise. A set* $A \subseteq \mathbb{N} \times 2^{<\mathbb{N}}$ *such that* $\mathrm{wg}(A) \leq 1$ *is called a* ***bounded request set****.* ■

In classical randomness, given a computably enumerable bounded request set A, we can effectively build a prefix-free machine M such that as long as $(l,\sigma) \in A$, then also $M(\tau) = \sigma$ for some string τ of length l. We include here an extract of the Downey and Hirschfeldt's book [17] about the credits for the next theorem:

"This result is usually known as the Kraft-Chaitin Theorem, as it appears in Chaitin [5], but it appeared earlier in Levin's dissertation [49], as stated in Levin [51], where it is proved using Shannon-Fano codes (giving slightly weaker constants). There is also a version of it in Schnorr [79], Lemma 1, p. 380. In Chaitin [5], where the first proof explicitly done for prefix-free complexity seems to appear, the key idea of that proof is attributed to Nick Pippinger. Thus perhaps we should refer to the theorem by the rather unwieldy name of Kraft-Levin-Schnorr-Pippinger-Chaitin Theorem. Instead, we will refer to it as the KC Theorem. Since it is an effectivization of Kraft's inequality, one should feel free if one wishes to regard the initials as coming from ***Kraft's inequality (Computable version)****."*

A higher version of the KC theorem has then been proved by Hjorth and Nies in [30]:

Theorem 3.7.11 (Higher KC Theorem, Nies and Hjorth):
For any Π^1_1-*bounded request set* A, *there is a* Π^1_1-*prefix-free machine* M *such that for any string* σ, *if* $(l,\sigma) \in A$, *then for a string* τ *of length* l *we have* $M(\tau) = \sigma$.

Proof: The prefix-free machine M can be found uniformly in A. However, handling the case where A is a finite set such that $\mathrm{wg}(A) = 1$ makes the proof slightly more complicated. To keep things as simple as possible, we assume $\mathrm{wg}(A) < 1$ (see below how this hypothesis is used). Except for the sake of uniformity (which again can be achieved with a bit more work), such an assumption is harmless, because if $\mathrm{wg}(A) = 1$, by the Σ^1_1-boundedness principle, there exists a computable stage s at which $\mathrm{wg}(A_s) = 1$ already, and we can then directly define a Π^1_1-prefix-free machine M that matches the conditions of the theorem with respect to the Δ^1_1 bounded request set A_s.

At each stage s, for each length $l \geq 1$ we define some strings σ^l_s either of length l or equal to ϵ, and a sequence $r_s \in 2^{\mathbb{N}}$. The strings σ^l_s that will be different from the empty word, will correspond to the strings available for a mapping at stage $s+1$. The role of r_s is double. First, the real number represented by r_s in a binary form, will be equal to the weight of A_s, which is also the measure of the set of strings that is mapped to something in M_s. Then, if the $(n-1)$-th bit of r_s is 0 (starting at position 0), it will also mean that the string σ_s^n is different from ϵ and available for a future mapping. We need to ensure at each stage s that:

1. The set of strings currently mapped in M_s, together with each σ_s^l different from the empty word, forms a prefix free set of strings.
2. r_s is a binary representation of the weight of A_s, which is also the measure of the set of strings mapped to something in M_s.
3. If $r_s(n-1) = 0$, the string σ_s^n is a string of length n. Otherwise it is the empty word.

At stage 0, we define $\sigma_0^l = 0^{l-1} \hat{} 1$ and r_0 to be only 0's. We have that (1), (2) and (3) are verified at stage 0.

At successor stage s suppose (l, τ) enters A_s. If $r_{s-1}(l-1) = 0$ we put (σ_{s-1}^l, τ) into M_s, we set σ_s^l to the empty word and $r_s(l-1)$ to 1. For $i \neq l$ and $i \geq 1$ we set $r_s(i-1) = r_{s-1}(i-1)$ and $\sigma_s^i = \sigma_{s-1}^i$. We can easily verify by induction that (1), (2) and (3) are true at stage s.

Otherwise, if $r_{s-1}(l-1) = 1$, let n be the largest integer bigger than 0 and smaller than l such that $r_{s-1}(n-1) = 0$. We should argue that such an integer always exists. Suppose otherwise, then either $r_{s-1} = 1000\dots$, $l = 1$ and $\mathrm{wg}(A_{s-1}) + 2^{-l} = 1$, which is not possible by our special assumption, or $\mathrm{wg}(A_{s-1}) + 2^{-l} > 1$, which is not possible because A is a bounded request set. Thus such an integer n exists. We then set σ_s^n to be the empty string and $r_s(n-1) = 1$. Then for every $n < i \leq l$, we set σ_s^i to $\sigma_{s-1}^n \hat{} 0^{i-n-1} \hat{} 1$ and $r_s(i-1) = 0$. Then we map $\sigma_{s-1}^n \hat{} 0^{l-n-1} \hat{} 0$ to τ in M_s. For $1 \leq i < n$ and $i > l$ we set $r_s(i-1) = r_{s-1}(i-1)$ and $\sigma_s^i = \sigma_{s-1}^i$. We can easily verify by induction that (1), (2) and (3) are true at stage s.

At limit stage s we set r_s to the pointwise limit of $\{r_t\}_{t<s}$. Then we set each σ_s^n to the convergence value of the sequence $\{\sigma_s^n\}_{t<s}$. We shall argue that those convergence values always exist. When for some n and some stage s we have $r_s \upharpoonright_n \neq r_{s+1} \upharpoonright_n$, then $r_{s+1} \upharpoonright_n$ is bigger than $r_s \upharpoonright_n$ in the lexicographic order, but as there are at most 2^{n-1} strings of length $n-1$, the sequence $\{r_s \upharpoonright_n\}_{s<\omega_1^{ck}}$ can change at most 2^{n-1} time. Then for any s, a convergence value for $\{r_t\}_{t<s}$ always exists.

Also when for some n and some s we have $\sigma_{s+1}^n \neq \sigma_s^n$, then also $r_{s+1} \upharpoonright_n \neq r_s \upharpoonright_n$. But by the previous paragraph, we then also have that $\{\sigma_s^n\}_{s<\omega_1^{ck}}$ can change at most 2^n times. We can easily verify by induction that (1), (2) and (3) are true at stage s.

Because (1) is true at every stage s, we then have that M is a Π_1^1-prefix-free machine, also by construction we clearly have that if $(l, \sigma) \in A$, then $M(\tau) = \sigma$ for a string τ of length l. ∎

3.7.3 Higher discrete semi-measures

The Π_1^1-prefix-free machines will be used to characterize Π_1^1-Martin-Löf randomness. We shall consider here another notion that can also be used to characterize Π_1^1-Martin-Löf randomness:

Definition 3.7.13. *A* ***left-Π_1^1 function*** $M : \mathbb{N} \to \overline{\mathbb{R}}$ *is a* Π_1^1 *subset of* $\mathbb{N} \times \mathbb{Q}$*, where* $M(\sigma)$ *is defined to be* $\sup\{q \mid (\sigma, q) \in M\}$ *(with* $\sup\{\varnothing\} = 0$*). The* ***weight*** *of* M*, denoted by* $\mathrm{wg}(M)$*, is defined by* $\sum_{n\in\mathbb{N}} M(n)$ *if it is finite, and* ∞ *otherwise.* ∎

Definition 3.7.14. *A* ***Π^1_1-discrete semi-measure*** *M is a left-Π^1_1 function $M : 2^{<\mathbb{N}} \to \overline{\mathbb{R}}$, such that $\mathrm{wg}(M) \leq 1$. A Π^1_1-discrete semi-measure U is universal if for any other Π^1_1-discrete semi-measure M, there is a constant c_M such that $U(\sigma) \geq M(\sigma) \times c_M$ for any σ.* ■

Proposition 3.7.2:
For any Π^1_1-discrete semi-measure M, there is a constant c_M such that $M(\sigma) \leq 2^{-\mathrm{hK}(\sigma)} \times c_M$ for any string σ.

Proof: We build a Π^1_1-bounded request set A from our semi-measure M. At successor stage s, for every string σ such that $M_s(\sigma) \neq 0$, we simply put into A the pair (m, σ) for $m = \lceil -\log(M_s(\sigma)) \rceil + 1$ (as long as (m, σ) is not already in A_s). At limit stage s, we define as usual A_s to be $\bigcup_{t<s} A_t$.

For a given σ suppose that $M(\sigma) = r$ for r a real number and let n be the smallest integer such that $2^{-n} \leq r$. By construction the weight corresponding to σ in A is of at most of $\sum_{m \geq n} 2^{-m-1} = 2^{-n} \leq r$. Also because $\sum_\sigma M(\sigma) \leq 1$ we have that A is a bounded request set for which we can build a prefix-free machine N. Also for each string σ with $M(\sigma) = r$ and 2^{-n} the greatest power of 2 such that $2^{-n} \leq r$, we have that $(n+1, \sigma)$ is enumerated in A and then that $M(\sigma) \leq 2^{-n+1} \leq 2^{-n-1} \times 4 = 2^{-\mathrm{hK}_N(\sigma)} \times 4 \leq 2^{-\mathrm{hK}(\sigma)} \times c_M$ for c_M a constant depending on M. ■

Corollary 3.7.2:
There is a universal Π^1_1-discrete semi-measure.

Proof: We easily verify that $M(\sigma) = 2^{-\mathrm{hK}(\sigma)}$ is a Π^1_1-discrete semi-measure, thus is universal by Proposition 3.7.2. ■

For a given Π^1_1 prefix-free machine M, we can consider the probability that M outputs a given string σ. One can imagine the following process : We flip a fair coin to get a bit, either 0 or 1, and we repeat the process endlessly. So we get bigger and bigger strings $\sigma_1 \prec \sigma_2 \prec \sigma_3 \prec \ldots$. In the meantime we test each of our strings σ_i available so far, as an input for our machine M. If at some point $M(\sigma_i)$ halts for one i (and it can be at most one i), then we stop the process.

It is clear that following the previous protocol, the probability that we output a given string τ is given by $\sum\{2^{-|\sigma|} : M(\sigma) = \tau\}$. Note that this all make sense, thanks to the prefix-free requirement we have for our machine.

Definition 3.7.15. *For a Π^1_1 prefix-free machine M, we denote by $P_M(\sigma)$ the probability that M outputs σ, that is, $\sum\{2^{-|\tau|} : M(\tau) = \sigma\}$. Note that P_M is a Π^1_1-discrete semi-measure.* ■

Theorem 3.7.12 (Coding theorem):
For any Π^1_1-prefix-free machine M, we have a constant c_M such that $P_M(\sigma) \leq 2^{-\mathrm{hK}(\sigma)} \times c_M$ for any σ.

PROOF: This follows directly from Proposition 3.7.2. ∎

3.7.4 Higher continuous semi-measures

Finally we consider one last notion, that will also be used to characterize Π^1_1-Martin-Löf randomness.

Definition 3.7.16. *A Π^1_1-continuous semi-measure μ is a left-Π^1_1 function $\mu : 2^{<\mathbb{N}} \to \overline{\mathbb{R}}$, such that $\mu(\sigma\hat{\ }0) + \mu(\sigma\hat{\ }1) \leq \mu(\sigma)$ and such that $\mu(\epsilon) \leq 1$.* ∎

Proposition 3.7.3:
Uniformly in any Π^1_1-discrete semi-measure M, we can define a Π^1_1-continuous semi-measure μ such that on any σ we have $\mu(\sigma) \geq M(\sigma)$.

PROOF: We simply define $\mu(\sigma)$ to be $\sum_{\tau \geq \sigma} M(\tau)$. We clearly have that $\mu : 2^{<\mathbb{N}} \times \mathbb{R}$ is a left-Π^1_1 function. Also for any σ we have $\sum_{\tau \geq \sigma} M(\tau) \geq \sum_{\tau \geq \sigma\hat{\ }0} M(\tau) + \sum_{\tau \geq \sigma\hat{\ }1} M(\tau)$ which imply also that $\mu(\sigma) \geq \mu(\sigma\hat{\ }0) + \mu(\sigma\hat{\ }1)$. Finally as M is a discrete semi-measure we have $\mu(\epsilon) \leq 1$. ∎

Digression

The randomness literature often deals with continuous semi-measures by considering the dual notion of martingale. A martingale is, in our context, a function $M : 2^{<\mathbb{N}} \to \mathbb{R}$ such that $2M(\sigma) = M(\sigma\hat{\ }0) + M(\sigma\hat{\ }1)$ for any σ. The interesting intuition behind this is to consider M as a the betting strategy that a gambler, say John, might adopt at a casino's roulette table, betting each turn some money on either red (that we denote 0) or black (that we denote 1). John starts with a capital of $M(\epsilon)$. If $M(1) = M(0)$ it means that John decides not to bet for the first turn. If $M(1) > M(0)$ it means John decide to bet $(M(1) - M(0))/2$ on 1, and reversely if $M(0) > M(1)$. Then $M(1)$ corresponds to the new capital John has if the first outcome is 1, after winning twice his bet or loosing it; and similarly for $M(0)$. The game then continues and John can decide what to bet next with the knowledge of the previous outcomes.

A sequence X is then considered to not be random with respect to a martingale M, if $M(\sigma)$ is unbounded when σ ranges over the prefixes of X. Also we easily verify that for any martingale M, the function defined by $\mu(\sigma) = M(\sigma) \times 2^{-|\sigma|}$ is a measure on the Cantor space (Recall Section 1.8.1 in which we argued that the measure μ can then be uniquely extends to all the Borel sets). Furthermore if $M(\epsilon) = 1$, the function

μ is then a probability measure. Similarly, for any measure μ, the function defined by $M(\sigma) = \mu(\sigma) \times 2^{|\sigma|}$ is a martingale. Continuous semi-measures only give us what is called supermartingales, that is, functions $M : 2^{<\mathbb{N}} \to \mathbb{R}$ such that $2M(\sigma) \geq M(\sigma \hat{\ } 0) + M(\sigma \hat{\ } 1)$.

We can then consider the randomness notion defined by saying that X is random if it is random with respect to any computable martingale (provably equivalent to be random with respect to any computable supermartingale). This definition of randomness, referred in the literature as **computable randomness**, follows the paradigm that a random sequence, when seen as the sequence of outcomes in a fair game, should be chaotic enough to make impossible the design of a strategy that makes money with it.

An overview on martingales and computable randomness can be found for example in Nies' book [70], Chapter 7, where it is in particular proved that computable randomness is strictly weaker than Martin-Löf randomness, the correct analogue of Martin-Löf randomness being obtained by considering left-c.e. martingale instead of computable martingale.

It is of interest to notice that betting strategies in a real casino, are much more restrictive, as there is both a minimal and a maximal bet. The corresponding randomness notions have been studied by Bienvenu, Stephan, and Teutsch in [4].

Proposition 3.7.4:
There is a universal Π^1_1-continuous semi-measure μ, that is, for any Π^1_1-continuous semi-measure ν, there is a constant c_ν such that we have $\mu(\sigma) \geq \nu(\sigma) \times c_\nu$ for any σ.

Proof: The proof is similar to the one of the existence of a universal Martin-Löf test, or to the one of the existence of a universal prefix-free machine. It is enough to prove that for any left-Π^1_1 function $M : 2^{<\mathbb{N}} \times \mathbb{R}$, one can define uniformly in M a left-Π^1_1 function $M' : 2^{<\mathbb{N}} \times \mathbb{R}$ such that M' always describes a Π^1_1-continuous semi-measure, and if M describes a Π^1_1-continuous semi-measure then $M = M'$. There is no particular difficulty to conduct this.

Then let $f : \mathbb{N} \to \mathbb{N}$ be the computable function performing the operation described above, on indices of left-Π^1_1 functions, and let $\{M_e\}_{e \in \mathbb{N}}$ be an enumeration of all the left-Π^1_1 functions. We then define $\mu = \sum_e M_{f(e)} \times 2^{-e}$, and we verify easily that μ is a universal Π^1_1-continuous semi-measure. ∎

3.7.5 Equivalent characterizations of Π^1_1-Martin-Löf randomness

We shall now see an important lemma. It is clear that any Σ^0_1 set can be described by a Σ^0_1 prefix-free set of strings. We shall see with Theorem 7.1.1 that this does not hold anymore in the higher setting. For now we simply prove that from a measure theoretical point of view, a Π^1_1 open set can be described by a set of strings which is as close as we want from being prefix-free.

Definition 3.7.17. *We say that a set of strings W is **ε-prefix-free** if $\sum_{\sigma \in W} \lambda([\sigma]) \leq \lambda([W]^{\prec}) + \varepsilon$.* ∎

Lemma 3.7.1 *For any Π^1_1-open set $\mathcal{U}$, one can obtain uniformly in ε and in an index for $\mathcal{U}$, a ε-prefix-free Π^1_1 set of strings W with $[W]^{\prec} = \mathcal{U}$.*

Proof: We use here the projectum function $p : \omega_1^{ck} \to \omega$. Let U be a Π^1_1 set of strings describing $\mathcal{U}$. At successor stage s, if σ enters U, we find a finite prefix-free set of strings C_s, each of them extending σ, such that $[\sigma] \subseteq [W_{s-1}]^{\prec} \cup [C_s]^{\prec}$ and such that $\lambda([W_{s-1}]^{\prec} \cap [C_s]^{\prec}) \leq 2^{-p(s)} \times \varepsilon$ (and if nothing enters U we define $C_s = \emptyset$). To find C_s we can search for the first finite set (in some pre-defined order) which satisfies the Δ^1_1 condition stated above. We then add each string of C_s to W_s. At limit stage s we define W_s to be $\bigcup_{t<s} W_t$.

It is clear by construction that we have $\mathcal{U} = [W]^{\prec}$. Moreover, we have $\sum_{\sigma \in W} \lambda([\sigma]) \leq \lambda(\mathcal{U}) + \sum_{s<\omega_1^{ck}} [W_{s-1}]^{\prec} \cap [C_s]^{\prec} \leq \lambda(\mathcal{U}) + \varepsilon \sum_{s<\omega_1^{ck}} 2^{-p(s)} \leq \lambda(\mathcal{U}) + \varepsilon$. ∎

Theorem 3.7.13:
Let M be a universal Π^1_1-discrete semi-measure and μ a universal Π^1_1-continuous semi-measure. Given a sequence Z, the four following statements are equivalent.

1. *The sequence Z is Π^1_1-Martin-Löf-random.*
2. *There is a constant c such that for every n we have $\mu(Z\upharpoonright_n) \leq 2^{-n} \times 2^c$.*
3. *There is a constant c such that for every n we have $M(Z\upharpoonright_n) \leq 2^{-n} \times 2^c$.*
4. *There is a constant c such that for every n we have $\mathrm{hK}(Z\upharpoonright_n) \geq n - c$.*

Proof: (1) $\implies$ (2): Uniformly in $c \in \mathbb{N}$ we define $\mathcal{U}_c = \{X \mid \exists n\ \mu(X\upharpoonright_n) > 2^{-n}2^c\}$. Each $\mathcal{U}_c$ is a Π^1_1-open set and $\bigcap_c \mathcal{U}_c$ contains all the sequences that do not verify (2). It remains to prove $\lambda(\mathcal{U}_c) \leq 2^{-c}$ to deduce that none of them is Π^1_1-Martin-Löf random. Suppose for contradiction that $\lambda(\mathcal{U}_c) > 2^{-c}$ and let W be the (non effective) prefix-free set of strings which describes $\mathcal{U}_c$ and which is minimal under the prefix ordering. We have $\mu(\epsilon) \geq \sum_{\sigma \in W} \mu(\sigma) \geq \sum_{\sigma \in W} 2^{-|\sigma|}2^c \geq \lambda(\mathcal{U}_c)2^c > 1$, which contradicts that μ is a Π^1_1-continuous semi-measure.

(2) $\implies$ (3): It is clear as from Proposition 3.7.3, any Π^1_1-discrete semi-measure is bounded by a Π^1_1-continuous semi-measure.

(3) $\implies$ (4): It is clear as well, as $2^{-\mathrm{hK}}$ is a Π^1_1-discrete semi-measure and M is universal.

(4) $\implies$ (1): Consider now a Π^1_1-Martin-Löf-test $\bigcap_n \mathcal{U}_n$ and let us build a Π^1_1-prefix-free machine M such that for every $X \in \bigcap_n \mathcal{U}_n$ and every c we have some n with $\mathrm{hK}_M(X\upharpoonright_n) < n - c$. Using Lemma 3.7.1, we can get a Π^1_1 set of strings W_n, uniformly in n, such that $\mathcal{U}_n = [W_n]^{\prec}$ and such that $\sum_{\sigma \in W_n} \lambda([\sigma]) \leq \lambda(\mathcal{U}_n) + 2^{-n}$.

Then to define M, we first define the Π^1_1-bounded request set A by enumerating $(|\sigma| - n, \sigma)$ for each n and each $\sigma \in W_{2n+2}$. We have that A is a bounded request set because $\mathrm{wg}(A) \leq \sum_n \sum_{\sigma \in W_{2n+2}} 2^{-|\sigma|+n} \leq \sum_n 2^n \sum_{\sigma \in W_{2n+2}} 2^{-|\sigma|} \leq \sum_n 2^n(\lambda(\mathcal{U}_{2n+2}) + 2^{-2n-2}) \leq \sum_n 2^n 2^{-2n-1} \leq \sum_n 2^{-n-1} \leq 1$. Also we have for any $X \in \bigcap_n \mathcal{U}_n$ and any n, a prefix of X in W_{2n+2} which is compressed by at least n, with the Π^1_1 prefix-free machine defined from A. Therefore for every c there is an n such that $\mathrm{hK}(X\upharpoonright_n) < n - c$. ∎

Chapter 4

Continuity and higher randomness

> Let us suppose that we are supplied with some unspecified means of solving number-theoretic problems: a kind of oracle as it were. We shall not go any further into the nature of this oracle apart from saying that it cannot be a machine. With the help of the oracle we could form a new kind of machine (call them o-machine), having as one of its fundamental processes that of solving a given number-theoretic problem.
>
> Systems of logic based on ordinals, Alan Turing

Joint work with Noam Greenberg and Laurent Bienvenu.

In this chapter we will deal with the use of continuous reduction and continuous relativization in the theory of higher randomness. As a first motivating example, let us consider the fact that strong randomness notions are downward closed in the Turing degrees of Martin-Löf random sequences. For example, Miller and Yu [62] showed that if X Turing computes Y for two Martin-Löf-randoms X and Y, and if in addition X is weakly-2-random, then Y too is weakly-2-random (a full version of the theorem, which is much more general, will be given in Section 4.3.4).

If we want to study a higher analogue of this theorem, we should first define a higher analogue of the Turing reduction, and the hyperarithmetic reduction seems to be a natural first candidate, but then a higher version of Miller and Yu's theorem does not hold anymore. Indeed, we will prove with Theorem 5.3.3 that there exists a weakly-Π^1_1-random X which is not Π^1_1-random, also using Theorem 3.7.4, we then have $\omega_1^X > \omega_1^{ck}$ and therefore $X \geq_h \mathcal{O}$. Also, Ω, the leftmost path of a Σ^1_1-closed set containing only Π^1_1-Martin-Löf randoms, is Turing reducible to $\mathcal{O}$ and then hyperarithmetically reducible to X. But we will see in Theorem 5.3.1 that Ω is not weakly-Π^1_1-random.

The insight that randomness and traditional relative hyperarithmetic reducibility do not interact well goes back to Hjorth and Nies [30], who defined another notion of reduction, in order to study a higher analogue of the notion of 'base for randomness', that will be studied in Section 4.5.3. A celebrated result of Nies (together with Hirschfeldt) in [68], following some work by Downey, Hirschfeldt, Nies, and Stephan in [15] and together with some other work of Hirschfeldt, Nies, and Stephan in [29], is the coincidence of a number of classes, each formalizing a notion of distance from randomness, or a notion of weakness as oracle in detecting randomness: the K-trivials. We give here a non-exhaustive list of the known characterizations of this class:

- The class of K-trivial sequences, that is, the class of sequences which are in some sense the opposite of randoms: A sequence X is K-trivial if for every n we have $K(X\upharpoonright_n)$ smaller than $K(n)$, up to a constant.
- The class of low-for-K sequences, that is, the class of sequences which when used as oracle in a universal prefix-free machine, do not help to get any better compression on any string, up to a constant.
- The class of low for Martin-Löf randomness sequences, that is, the class of sequences which when used as oracle in a Martin-Löf test, do not help to capture any Martin-Löf random.
- The class of base for randomness sequences, that is, the class of sequences X which are Turing reducible to an X-Martin-Löf random sequence.

All those class coincide, and contain strictly the set of computable sequences. As we saw in Section 3.7.2, a higher notion of Kolmogorov complexity has been defined by Hjorth and Nies in [30], who also proved the existence of non Δ^1_1 hK-trivial sets. Also if we want to compare the notion of hK-trivial with higher versions of the notion of Low-for-K or of the notion of Low for Martin-Löf randomness, we need to define what it means to use an oracle to help with a Π^1_1 enumeration.

Again, the obvious way to use an oracle X in a Π^1_1 description of an open set, is to allow a $\Pi^1_1(X)$ description of this open set. However, we shall see that with this relativization, only Δ^1_1 sets are Low-for-hK or Low-for-Π^1_1-Martin-Löf randomness. Therefore the equivalences that we have in classical randomness would not hold anymore in the higher world, with this notion of relativization.

To overcome those problems we introduce the notion of both higher continuous reductions and higher continuous relativization. We will see that forcing continuity in both higher reduction and higher relativization, makes everything works similarly in the higher world and in the lower world.

4.1 The higher Turing reduction

Beyond the inherent interest in higher notions, the study of generalizations of computability sheds light on familiar notions by separating concepts which "accidentally" coincide in usual computability. An example of such a phenomenon will appear in the definition of higher Turing reducibility.

The goal in defining higher Turing reducibility, is to keep the descriptional power of Π^1_1 predicate, but in the meantime to keep continuous reductions, that is, to get finitely many bits of the output, we should require only finitely many bits of the input. One way of defining standard Turing reductions is to consider them as c.e. mappings of strings to strings, with certain restrictions. Also we can consider a similar definition, but replacing c.e. by Π^1_1.

In the definition of Turing reductions via a c.e. map Φ, one restriction is usually to require the *consistency* of the set Φ: If $(\tau,\sigma)\in\Phi$ and $(\tau',\sigma')\in\Phi$, and if τ and τ' are comparable, then σ and σ' should be comparable. Of course, if X computes Y with a functional Φ, it is normal to require that Φ is consistent on the prefixes of X, as those should only be mapped to prefixes of Y. However, the compatibility is generally required everywhere.

Another requirement, sometimes made in the definition of Turing reduction via a c.e. map Φ, is that the mapping should be closed by prefixes, that is, if $(\sigma, \tau) \in \Phi$, then also any $\sigma' < \sigma$ should be mapped to some string τ'. It is well-known that making those assumptions in the definition of Turing reductions via a c.e. mapping is harmless:

Proposition 4.1.1:
For two sequences X, Y the following are equivalent:

1. *There is a c.e. partial map $\Phi : 2^{<\mathbb{N}} \to 2^{<\mathbb{N}}$, consistent on prefixes of X, such that $\Phi(X) = Y$.*
2. *There is a c.e. partial map $\Phi : 2^{<\mathbb{N}} \to 2^{<\mathbb{N}}$, consistent everywhere, such that $\Phi(X) = Y$.*
3. *There is a c.e. partial map $\Phi : 2^{<\mathbb{N}} \to 2^{<\mathbb{N}}$, consistent everywhere and closed under prefixes, such that $\Phi(X) = Y$.*

4.1.1 The fin-h reduction

In [30] Hjorth and Nies introduced a continuous higher reducibility notion, which corresponds to the most restrictive of the three notions of Proposition 4.1.1:

Definition 4.1.1. *A **fin-h** reduction Φ is a Π^1_1 partial map from $2^{<\mathbb{N}}$ to $2^{<\mathbb{N}}$ which is:*

- *Consistent: If $(X\upharpoonright_{n_1}, \tau_1) \in \Phi$ and $(X\upharpoonright_{n_2}, \tau_2) \in \Phi$, then τ_1 is compatible with τ_2.*
- *Closed under prefixes: If $(\sigma, \tau) \in \Phi$ then also for every $\sigma' < \sigma$ we have $(\sigma', \tau') \in \Phi$ for some τ'.*

We write $\Phi(\sigma) = \tau$ for τ the longest string prefixes of σ are mapped to in Φ. Also for a given sequence X, if we the set:

$$\bigcap\{[\sigma] \;:\; \exists n\ \Phi(X\upharpoonright_n) = \sigma\}$$

contains exactly one sequence Y, we write $\Phi(X) = Y$. Otherwise the functional Φ is said to be undefined on X. If $\Phi(X) = Y$ for some fin-h reduction Φ we write $X \geq_{fin-h} Y$. ■

A first application of the fin-h reduction is a higher version of a theorem proved independently by Kučera [42] and a bit later by Gács [25].

Theorem 4.1.1 (Kučera-Gács):
For any sequence X and any Π^0_1 set $\mathcal{F} \subseteq 2^{\mathbb{N}}$ of positive measure, there exists $Z \in \mathcal{F}$ which Turing computes X. In particular any sequence X can be Turing computed by a Martin-Löf random.

For any perfect closed set $\mathcal{F}$, there is a canonical bijection between $\mathcal{F}$ and the Cantor space. But for such a bijection to be computable, we need to identify the splitting points of $\mathcal{F}$, seen as a tree, or at least sufficiently many of those splitting points. Also for a Π^0_1 perfect set $\mathcal{F}$, it is not always possible to do so. For example Cole and Simpson proved in [10] (credited in [35] by Kent and Lewis) that for any Π^0_1 set $\mathcal{F}$ with no computable point, there is a perfect Π^0_1 set such that none of its members can Turing compute any member of $\mathcal{F}$. For example there is a perfect Π^0_1 set such that none of its members Turing computes a Martin-Löf random sequence.

However when in addition the set $\mathcal{F}$ has positive measure, there is a way to handle this. We only prove here the higher version of the Kučera-Gács theorem, which works in a similar way than its lower counterpart. Also we emphasize that the reduction we obtain is a fin-h reduction, and not just a higher Turing reduction, as it will be defined later. First we need to prove an interesting lemma, which will also be useful to separate weak-Π^1_1-randomness from Π^1_1-randomness:

Lemma 4.1.1 *let σ be a string and $\mathcal{F}$ a closed set so that $\lambda(\mathcal{F} \mid [\sigma]) \geq 2^{-n}$. Then there are at least two extensions τ_1, τ_2 of σ of length $|\sigma| + n + 1$ so that for $i \in \{1, 2\}$ we have $\lambda(\mathcal{F} \mid [\tau_i]) \geq 2^{-n-1}$.*

Proof: Let C be the set of strings of length $|\sigma| + n + 1$ that extend σ. We have that $\lambda(\mathcal{F} \cap [\sigma]) = \sum_{\tau \in C} \lambda(\mathcal{F} \cap [\tau])$. Suppose that for strictly less than two extensions of length $|\sigma| + n + 1$ we have $\lambda(\mathcal{F} \cap [\tau_i]) \geq 2^{-|\tau_i|-n-1}$. Then we have:

$$\begin{array}{rcl} \sum_{\tau \in C} \lambda(\mathcal{F} \cap [\tau]) & \leq & 2^{-|\sigma|-n-1} + (2^{n+1} - 1)2^{-|\tau_i|-n-1} \\ & \leq & 2^{-|\sigma|-n-1} + 2^{n+1}2^{-|\sigma|-2n-2} - 2^{-|\sigma|-2n-2} \\ & \leq & 2^{-|\sigma|-n-1} + 2^{-|\sigma|-n-1} - 2^{-|\sigma|-2(n+1)} \\ & < & 2^{-|\sigma|-n} \end{array}$$

which contradicts $\lambda(\mathcal{F} \mid [\sigma]) \geq 2^{-n}$. ∎

We now prove the higher Kučera-Gács theorem:

Theorem 4.1.2 (higher Kučera-Gács):
For any sequence X and any Σ^1_1 closed set $\mathcal{F} \subseteq 2^{\mathbb{N}}$ of positive measure, there exists $Z \in \mathcal{F}$ which fin-h computes X. In particular any sequence X can be fin-h computed by a Π^1_1-Martin-Löf random.

Proof: Consider a Σ^1_1 closed set $\mathcal{F} \subseteq 2^{\mathbb{N}}$ with $\lambda(\mathcal{F}) \geq 2^{-c}$ and a sequence X. According to what Lemma 4.1.1 tells us, we define some length $m_0 = 0$ and inductively $m_{n+1} = m_n + c + n + 1$.

We define $\sigma_0 = \epsilon$. Assuming σ_n of length m_n is defined with $\lambda(\mathcal{F} \mid [\sigma_n]) \geq 2^{-c-n}$, we will define an extension σ_{n+1} of σ_n with the same property. From Lemma 4.1.1 there are at least two extensions τ of σ_n of length $m_n + c + n + 1 = m_{n+1}$ such that $\lambda(\mathcal{F} \mid [\tau]) \geq 2^{-c-(n+1)}$. Also if $X(n) = 0$ let σ_{n+1} be the leftmost of those extensions and if $X(n) = 1$ let σ_{n+1} be the rightmost of those extensions.

The unique limit point Z of $\{[\sigma_n]\}_{n \in \mathbb{N}}$ is our candidate. We shall now show how we use it to fin-h compute X, by describing the fin-h reduction $\Phi \subseteq 2^{<\mathbb{N}} \times 2^{<\mathbb{N}}$.

At stage 0, we enumerate (ϵ, ϵ) in Φ. Then at successor stage s, and substage $n+1$, for each string σ of length m_n which is mapped to τ in Φ_{s-1}, if there are distinct leftmost and a rightmost extensions σ_1, σ_2 of σ of length m_{n+1} such that $\lambda(\mathcal{F} \mid [\sigma_i])[s] \geq 2^{-c-(n+1)}$ for $i \in \{0,1\}$, we map the leftmost one and all of its unmapped prefixes to $\tau\hat{\ }0$ in Φ at stage s; then we map the rightmost one and all of its unmapped prefixes to $\tau\hat{\ }1$ in Φ at stage s. At limit stage s we let Φ_s to be the union of Φ_t for $t < s$.

By design, the functional Φ is consistent everywhere because for any two strings $\sigma_2 \succ \sigma_1$ which are mapped to something in Φ, the string σ_2 is always mapped to an extension of what the string σ_1 is mapped to. Also by design the mapping is closed by prefixes.

Now we clearly have $\Phi(Z) = X$, because for any prefix σ_1 of Z of length m_n which is mapped to $X\upharpoonright_n$, there is always a stage at which the prefix σ_2 of Z of length m_{n+1} will be witnessed to be either the leftmost or the rightmost path of $\mathcal{F}$ that extends σ_1 and such that $\lambda(\mathcal{F} \mid [\sigma_2])[s] \geq 2^{-c-(n+1)}$, in which case it will be mapped to $X\upharpoonright_{n+1}$. ∎

4.1.2 The higher Turing reduction

Unlike in the classical world, it is not true anymore that the higher analogues of the three notions of reduction coincide. Also we will show in Chapter 7 that if the two last coincide, the two first are distinct. Also we now make the following reduction definition, corresponding to the less restrictive of the three notions of Proposition 4.1.1:

Definition 4.1.2. *A* ***higher Turing reduction*** Φ *is a* Π^1_1 *partial map from* $2^{<\mathbb{N}}$ *to* $2^{<\mathbb{N}}$*. For a string* σ*, if* Φ *is consistent on prefixes of* σ*, we write* $\Phi(\sigma) = \tau$ *for* τ *is the longest string prefixes of* σ *are mapped to in* Φ*; otherwise* $\Phi(\sigma)$ *is said to be undefined. Also in case for a given sequence* X *we have that the set:*

$$\bigcap\{[\sigma] \ : \ \exists n \ \Phi(X\upharpoonright_n) = \sigma\}$$

contains exactly one sequence Y*, we write* $\Phi(X) = Y$*. Otherwise the functional* Φ *is said to be undefined on* X*. If* $\Phi(X) = Y$ *for some higher Turing reduction* Φ *we write* $X \geq_{\mathrm{hT}} Y$*.* ∎

In this thesis we will only use the higher Turing reducibility, because it is the one which fits the best with the general theory of higher computability and randomness. We will see for example in Chapter 7 that the higher analogue of:

'Y is computable in X iff both Y and its complements are c.e. in X'

does not hold with fin-h reducibility, but hold with higher Turing reducibility, as we will see, right after introducing a continuous higher version of being c.e.:

Definition 4.1.3. *An* ***oracle-continuous*** $\boldsymbol{\Pi^1_1}$ ***set*** *of integers is given by a set* $W \subseteq 2^{<\mathbb{N}} \times \mathbb{N}$*. For a string* σ *we write* W^σ *to denote the set* $\{n \ : \ \exists \tau \leq \sigma \ (\tau, n) \in W\}$*. For a sequence* X *we write* W^X *to denote the set* $\{n \ : \ \exists \tau \prec X \ (\tau, n) \in W\}$*. The set* W^X *is then called an* X*-continuous* Π^1_1 *set of integer. We denote by* $\{\mathrm{hW}_e\}_{e \in \mathbb{N}}$ *a canonical enumeration of the oracle-continuous* Π^1_1 *sets of integers.* ∎

Definition 4.1.4. *We say that a set* X *is* Y*-continuously* Π^1_1 *if* $X = \mathrm{hW}^Y_e$ *for some* e*.* ∎

Proposition 4.1.2:
The following are equivalent for $X, Y \in 2^{\mathbb{N}}$:

1. $X \leq_{\mathrm{hT}} Y$
2. *Both X and its complement are Y-continuously Π^1_1.*

PROOF: Suppose that $\Phi(Y) = X$ for a higher Turing reduction Φ, then we define the Π^1_1 set $W \subseteq 2^{<\mathbb{N}} \times \mathbb{N}$ by enumerating (σ, n) in W_s when $(\sigma, \tau) \in \Phi_s$ with $\tau(n) = 1$. We have that $n \in X$ iff $n \in W^Y$. We can do the same for $2^{\mathbb{N}} - X$.

Now suppose that $X = \mathrm{hW}^Y_{e_1}$ and $2^{\mathbb{N}} - X = \mathrm{hW}^Y_{e_2}$. At stage s, and at substage n, for every string σ of length n, if σ is not already mapped in Φ, but if Φ is consistent on σ so far, let τ be the longest string with $|\tau| \leq |\sigma|$ and such that for every $i < |\tau|$, $\tau(i) = 1$ implies $i \in \mathrm{hW}^\sigma_{e_1}[s]$ and $\tau(i) = 0$ implies $i \in \mathrm{hW}^\sigma_{e_2}[s]$. If $|\tau|$ is bigger than the length of the longest string a prefix of σ is mapped to so far in Φ, we then map σ to τ in Φ at stage s and substage n. Otherwise we go to the next substage. ∎

We will see in Chapter 7 that the previous proposition fails if we replace higher Turing reducibility by either of its two stronger versions. What goes wrong in the higher setting? In the lower setting, to turn a functional, not necessarily consistent everywhere, into a functional consistent everywhere, without damaging the good computations, we proceed as follow : when a mapping (σ, τ) enters the functional at some stage s, we consider all extensions of σ of length s, and we map to τ those among them for which this mapping does not introduce an inconsistency.

This argument uses what we call a **time trick**: the fact that the number of stages is the same as the length of the oracle, namely ω. This equality fails in the higher setting, in which we still use oracles of length ω but effective constructions have ω_1^{ck} many stages. Thus any argument that relies on a time trick cannot be simply copied in the higher setting.

In some cases, a proof uses a time trick because it is convenient to do so, but a time trick is actually not essential. Perhaps a good example is the proof that no Δ^0_2 set is weakly-2-random. The proof of this (see Proposition 2.1.1) uses a time trick. Also it is possible to remove the time trick, like it is done in a higher version of this proof (where a convenient higher analogue of the notion of Δ^0_2 has to be picked carefully), in Section 5.3.

In other cases, such as the equivalence of the three definitions of Turing reducibility, the higher analogue of the theorem fails.

4.1.3 The continuous higher jump

The sets hW_e give rise to a higher Turing jump operator:

Definition 4.1.5. *We define the operator $Y \to \mathrm{hJ}^Y$ by $\mathrm{hJ}^Y = \{e \ : \ e \in \mathrm{hW}^Y_e\}$.* ∎

We verify easily that $\mathrm{hJ}^{\varnothing}$ is Π^1_1 complete and thus that $\mathcal{O}$ is many-one equivalent to $\mathrm{hJ}^{\varnothing}$. We also easily verify that, like for any 'jump' notion, the higher Turing jump has no fixed point in the higher Turing degrees:

Proposition 4.1.3:
For any X, we have $\mathrm{hJ}^X >_{\mathrm{hT}} X$.

Proof: The proof works just like for the regular Turing jump. To show that $\mathrm{hJ}^X \geq_{\mathrm{hT}} X$ we can find uniformly in n, an index e such that hW^X_e enumerate everything if $X(n) = 1$ and nothing if $X(n) = 0$. We then have $e \in \mathrm{hJ}^X$ iff $n \in X$.

Now suppose that $\Phi(X) = \mathrm{hJ}^X$ for some Π^1_1 functional Φ and some X. Then in particular $\{n : n \notin \mathrm{hJ}^X\}$ is X-continuously Π^1_1 and is equal to hW^X_e for some e. But then $e \notin \mathrm{hJ}^X$ iff $e \in \mathrm{hW}^X_e$ iff $e \in \mathrm{hJ}^X$ which is a contradiction. ∎

4.2 higher Turing and continuously Π^1_1 on weak and strong oracles

Before we discuss randomness we investigate the notions of oracle continuous reducibility and enumeration, in particular when they coincide with familiar notions. With strong oracles they collapse to the familiar notions of Turing reducibility and relative computable enumerability. With weak oracles they coincide with relative Δ^1_1 and relative Π^1_1.

4.2.1 On strong oracles

We saw with Proposition 4.1.3 that we have $\mathrm{hJ}^X >_{\mathrm{hT}} X$ for any X. It is also not hard to see that the proof actually also gives us $\mathrm{hJ}^X >_{\mathrm{T}} X$. It is also easy to prove that the higher jump operator is uniformly Turing degree invariant, that is, if $X \equiv_{\mathrm{T}} Y$, then $\mathrm{hJ}^X \equiv_{\mathrm{T}} \mathrm{hJ}^Y$, and furthermore, a Turing reduction from hJ^X to hJ^Y (resp. from hJ^Y to hJ^X) can be obtained uniformly from a Turing reduction from X to Y (resp. from Y to X). Also Slaman and Steel [83] and Steel [89] proved that such operators should coincide with the Turing jump, or with iterations of the Turing jump, on a cone of Turing degrees (i.e., on every degree above one specific degree). We will soon provide more details on this. For now we show that it is indeed the case for the higher Turing jump, on a code above $\mathcal{O}$:

Proposition 4.2.1:
A set is $\mathcal{O}$-continuously Π^1_1 if and only if it is $\Sigma^0_1(\mathcal{O})$; thus a set is higher Turing reducible to $\mathcal{O}$ if and only if it is Turing reducible to $\mathcal{O}$. Furthermore, these equivalences are uniform and holds when $\mathcal{O}$ is replaced by any oracle $Y \geq_{\mathrm{T}} \mathcal{O}$.

Proof: For a given Π^1_1 set $A \subseteq 2^{<\mathbb{N}} \times \mathbb{N}$, the point is that A is many-one reducible to $\mathcal{O}$. Therefore the predicate $\exists \sigma < \mathcal{O}\ (\sigma, n) \in A$ is $\Sigma^0_1(\mathcal{O})$, but then also $\Sigma^0_1(X)$ for any $X \geq_{\mathrm{T}} \mathcal{O}$. Conversely, it is clear by definition that for any oracle X, a $\Sigma^0_1(X)$ set is also a X-continuously Π^1_1 set.

By Proposition 4.1.2 we then have that being higher Turing computable in $\mathcal{O}$ is equivalent to being Turing computable in $\mathcal{O}$. ∎

Proposition 4.2.2:
The set $\mathrm{hJ}^{\mathcal{O}}$ is many-one equivalent to $\mathcal{O}'$. Also for any $X \geq_{\mathrm{T}} \mathcal{O}$, the set hJ^X is many-one equivalent to X'

Proof: It is clear as by the previous proposition, as a $\mathcal{O}$-continuously Π^1_1 set is also a $\Sigma^0_1(\mathcal{O})$ set, uniformly. ∎

It would be too bad to mention Slaman and Steel's work without placing it into its context, which is out of the scope of this thesis, but which is also linked to one of the oldest and deepest open problems on the global structure of the Turing degrees: Martin's conjecture.

Digression

We cite here the introduction of Downey and Shore's paper [18] which constitute a good sum up of the history behind Martin's conjecture:

"A striking phenomena in the early days of computability theory was that every decision problem for axiomatizable theories turned out to be either decidable or of the same Turing degree as the halting problem $\emptyset'$ the complete computably enumerable set). Perhaps the most influential problem in computability theory over the past fifty years has been Post's problem [75] of finding an exception to this rule, i.e., a noncomputable incomplete computably enumerable degree. The problem has been solved many times in various setting and disguises but the solutions always involve specific constructions of strange sets, usually by the priority method that was first developed (Friedberg [23] and Muchnik [67]) to solve this problem. No natural decision problems or sets of any kind have been found that are neither computable nor complete. The question then becomes how to define what characterizes the natural computably enumerable degrees and show that none of them can supply a solution to Post's problem. Steel [89] suggests that a natural degree should be definable and its definition should relativize to an arbitrary degree (and so, in particular, be defined on degrees independently of the choice of representative).

Along these lines an old question of Sacks' [76] asks whether there is a degree invariant solution to Post's problem, i.e., a computably enumerable degree invariant operator W such that $A <_{\mathrm{T}} W(A) <_{\mathrm{T}} A'$. ([...] Any function $f : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ is degree invariant if for every A and B, $A \equiv_{\mathrm{T}} B$ implies that $f(A) \equiv_{\mathrm{T}} f(B)$). Such an operator would clearly be a candidate for a natural solution to Post's problem. Lachlan [47] proved that if we require the degree invariance to be uniform in the sense that there is a function h that takes the (pairs of) indices of reductions between A and B to (pairs of) indices of reductions between $W(A)$ and $W(B)$ then there is no such operator. [...]

In the setting of the Axiom of Determinacy (where, by Martin [57] there is a 0-1 valued countably complete measure defined on all the subsets of the Turing degrees for *which being of measure 1 is equivalent to containing a cone), Martin made a sweeping conjecture [...] that would entirely characterize the natural degree operators:*

Conjecture 4.2.1 (Martin) *Assume ZF + AD + DC. Then*

I If $f: 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ is degree invariant then either f is degree increasing on a cone or is constant (up to degree) on a cone.

II The relation $\leq$ on degree invariant functions defined by $f \leq g$ iff $f(X) \leq_{\mathrm{T}} g(X)$ on a cone is a prewellordering in which the immediate successor of any f is its jump f' defined by $f'(A) = f(A)'$.

Thus Martin's Conjecture can be seen as asserting that the only natural operators on degrees are the jump operators and their iterates."

Slaman and Steel proved Martin's conjecture in case the operator is uniformly Turing degree invariant, which is the case of the higher Turing jump operator.

We will see later, after defining Π^1_1-Martin-Löf randomness continuously relatively to some oracle, that Proposition 4.2.1 implies that $\mathcal{O}$-continuous Π^1_1-Martin-Löf randomness coincides with mere Martin-Löf randomness relatively to $\mathcal{O}$.

4.2.2 On weak oracles

Proposition 4.2.3:
Suppose that Y preserves ω_1^{ck}, that is, $\omega_1^Y = \omega_1^{ck}$. Then for all X, Y we have that $X \leq_{\mathrm{hT}} Y$ if and only if $X \leq_{\mathrm{T}} Y \oplus H$ for some hyperarithmetic sequence H.

Proof: If H is hyperarithmetic and $X \leq_{\mathrm{T}} Y \oplus H$ then we can easily devise a hyperarithmetic functional Φ such that $\Phi(Y) = X$, and then $X \leq_{\mathrm{hT}} Y$.

For the other direction, suppose that $\Phi(Y) = X$ with some Π^1_1 functional Φ. Suppose also that $\omega_1^Y = \omega_1^{ck}$. Let $\{\Phi_s\}_{s<\omega_1^{ck}}$ be an effective enumeration of Φ. We define the $\Pi^1_1(Y)$ function $f{:}\,\omega \to \omega_1^{ck}$ by letting $f(n)$ be the smallest stage $s < \omega_1^{ck}$ such that $\Phi_s(Y\restriction_m) \succeq X\restriction_n$ for some m. As f is total it is also $\Delta^1_1(Y)$ and then by the Σ^1_1-boundedness principle, its range is bounded by some computable ordinal $s < \omega_1^Y = \omega_1^{ck}$. Then already at stage s we have $\Phi_s(Y) = X$ and then $X \leq_{\mathrm{T}} Y + \Phi_s$, where Φ_s can be represented by a Δ^1_1 sequence.■

For $X \in 2^{\mathbb{N}}$ we let $\Delta^1_1 \oplus X$ be the set of sequences Turing reducible to $H \oplus X$ where H is any hyperarithmetic sequence. Thus Proposition 4.2.3 says that if X preserves ω_1^{ck} then $\Delta^1_1 \oplus X$ is the set of sequences higher Turing reducible to X. We argue now, by a descriptive set-theoretical argument that the converse is not true.

Let $\mathcal{A}$ be the set of sequences X for which something is in $\Delta^1_1 \oplus X$ iff it is higher Turing reducible to X. We can easily see that $\mathcal{A}$ has a fairly low Borel rank: It is arithmetic in X and a higher Turing functional Φ, to decide whether $\Phi(X)$ is defined, and it is also arithmetic in X and in $\mathcal{O}$ to decide whether there exists some ordinal α such that $X \oplus \emptyset^{(\alpha)}$ Turing computes $\Phi(X)$. Thus $\mathcal{A}$ is $\Sigma^0_\omega(\mathcal{O})$. Also we shall see in Section 6.7 that the set of sequences which preserves ω_1^{ck} is not $\boldsymbol{\Sigma^0_{\omega+2}}$. Therefore, as the set $\mathcal{A}$ contains all the sequences which preserve ω_1^{ck}, it also contains some sequences which do not preserve ω_1^{ck}.

4.2.3 On generic oracles for various forcing notions

Higher computability and relative Π^1_1

For Y sufficiently Cohen generic or sufficiently random, we will see that $\Delta^1_1(Y) = \Delta^1_1 \oplus Y$. Thus, using Proposition 4.2.3 we have $X \leq_h Y$ if and only if $X \leq_{\mathrm{hT}} Y$ for any X, Y sufficiently random or sufficiently generic.

Definition 4.2.1. *Let $Y \in 2^{\mathbb{N}}$. We say that $\Delta^1_1(Y) = \Delta^1_1 \oplus Y$ uniformly in Y if there is a Turing functional Φ and a higher Y-continuously higher Turing computable function $g : \omega_1^{ck} \to \omega_1^{ck}$ such that for all $\alpha < \omega_1^{ck}$ we have $Y^{(\alpha)} = \Phi(Y, \emptyset^{(g(\alpha))}, \alpha)$.* ∎

Proposition 4.2.4:
The following are equivalent for $Y \in 2^{\mathbb{N}}$:

1. *A set is Y-continuously Π^1_1 if and only if it is $\Pi^1_1(Y)$.*
2. *$\Delta^1_1(Y) = \Delta^1_1 \oplus Y$ uniformly in Y.*

Proof: Assume (1). Note first that if every $\Pi^1_1(Y)$ set is Y-continuously Π^1_1 then $\omega_1^Y = \omega_1^{ck}$. Suppose otherwise, then $Y^{(\omega_1^{ck}+1)}$ is certainly $\Pi^1_1(Y)$ and even $\Delta^1_1(Y)$. Also the set hJ^Y is easily seen to be $\Sigma^0_1(Y^{(\omega_1^{ck})})$ and then both hJ^Y and its complement are many-one reducible to $Y^{(\omega_1^{ck}+1)}$. Also any enumeration of $Y^{(\omega_1^{ck}+1)}$ can then Turing compute hJ^Y and it particular if $Y^{(\omega_1^{ck}+1)}$ was Y-continuously Π^1_1 then also Y could higher Turing compute hJ^Y, which is impossible by Proposition 4.1.3.

Note now that since there are universal $\Pi^1_1(Y)$ and Y-continuously Π^1_1 sets, the equivalence is uniform: there are computable functions translating between indices of $\Pi^1_1(Y)$ sets and indices of Y-continuous Π^1_1 sets. Given this, for any $\alpha < \omega_1^{ck}$ the set $Y^{(\alpha)}$ and its complement are both $\Pi^1_1(Y)$ and then both Y-continuously Π^1_1, uniformly in α. Then we have $\Phi(Y) = Y^{(\alpha)}$ for some higher Turing functional Φ, which is obtained uniformly in α. Also just like in the proof of Proposition 4.2.3 we can find a stage s such that $\Phi_s(Y)$ is defined already and equal to $Y^{(\alpha)}$. Then $\Phi_s \oplus Y$ already Turing compute $Y^{(\alpha)}$, which means $\emptyset^{(s)} \oplus Y$ already Turing compute $Y^{(\alpha)}$.

Assume (2), and let g witness the uniformity. Here again we should notice that (2) implies that Y preserves ω_1^{ck}. Suppose otherwise, then $Y^{(\omega_1^{ck}+1)}$ is certainly $\Delta^1_1(Y)$ and it can Turing compute hJ^Y (even many-one compute hJ^Y, as argued in the previous paragraph). Therefore using Proposition 4.1.3 we have that Y cannot higher Turing compute $Y^{(\omega_1^{ck}+1)}$ and in particular that $Y \oplus \emptyset^{(\alpha)}$ cannot higher Turing compute $Y^{(\omega_1^{ck}+1)}$ for any computable α.

Uniformly in $\alpha < \omega_1^{ck}$ we can get a $\Delta^1_1(Y)$-index for the set $\mathcal{O}^Y_{<\alpha}$. Also $\mathcal{O}^Y = \bigcup_{\alpha<\omega_1^{ck}} O^Y_\alpha$ because Y preserves ω_1^{ck}. Using g and varying over $\alpha < \omega_1^{ck}$ we see how to enumerate $\mathcal{O}^Y$ in a Y-continuously Π^1_1 fashion. ∎

We now discuss Proposition 4.2.4 in the context of Cohen genericity and randomness.

Cohen generics

Theorem 4.2.1:
For a given computable ordinal α, if G is α-generic, then $G^{(\alpha)} \equiv_T G \oplus \emptyset^{(\alpha)}$, and the equivalence is uniform in α.

PROOF: We have for any n that the sets $\mathcal{A}_1 = \{X \ : n \in X^{(\alpha)}\}$ and $\mathcal{A}_2 = \{X \ : n \notin X^{(\alpha)}\}$ are respectively Σ^0_α and Π^0_α sets, uniformly in α and n. Also using Theorem 1.9.1 there is a Σ^0_α open set $\mathcal{U}_1$ and a Π^0_α open set $\mathcal{U}_2$, together with $\Pi^0_{<\alpha}$ closed sets $\mathcal{F}_{1,m}$ and Π^0_α closed sets $\mathcal{F}_{2,m}$ such that $\mathcal{A}_1 = \mathcal{U}_1 \triangle \mathcal{B}_1$ and $\mathcal{A}_2 = \mathcal{U}_2 \triangle \mathcal{B}_2$ with $\mathcal{B}_1 \subseteq \bigcup_m \partial\mathcal{F}_{1,m}$ and $\mathcal{B}_2 \subseteq \bigcup_m \partial\mathcal{F}_{2,m}$.

If X is α-generic it belongs to each $\mathcal{F}_{1,m}$ or each $\mathcal{F}_{2,m}$ iff it belongs to their interior. Therefore it belongs to $\mathcal{A}_1$ or $\mathcal{A}_2$ iff it belongs respectively to $\mathcal{U}_1$ and $\mathcal{U}_2$. Then using $\emptyset^{(\alpha)}$ we search for a prefix σ of X that is included in either to $\mathcal{U}_1$ or $\mathcal{U}_2$. If $[\sigma] \subseteq \mathcal{U}_1$ then $n \in X^{(\alpha)}$ and if $[\sigma] \subseteq \mathcal{U}_2$ then $n \notin X^{(\alpha)}$ ■

We will see in Theorem 6.6.2 that G is Σ^1_1-generic iff it is Δ^1_1-generic and $\omega_1^G = \omega_1^{ck}$. For G a Σ^1_1-generic sequence we then have $\Delta^1_1(G) = \Delta^1_1 \oplus G$ uniformly in G and therefore from Proposition 4.2.4 we have the following theorem:

Theorem 4.2.2:
If G is Σ^1_1-generic then a set is $\Pi^1_1(G)$ if and only if it is G-continuously Π^1_1.

Randoms

In [8], Chong and Yu observed that $\Delta^1_1(Z) = \Delta^1_1 \oplus Z$ uniformly for any Δ^1_1-random sequence Z which preserves ω_1^{ck}. In what follows we calculate precise bounds.

Proposition 4.2.5:
For any α, if Z is $(\alpha+1)$-random, we have $Z^{(\alpha)} \equiv_T Z \oplus \emptyset^{(\alpha)}$. Moreover, an index for the reduction can be found effectively from α and an upper bound on the α-randomness deficiency of Z.

PROOF: For any e the set $\mathcal{A} = \{X \ : \ e \in X^{(\alpha)}\}$ is Σ^0_α uniformly in e. Also from Theorem 1.8.1 we can find uniformly in n and in $\emptyset^{(\alpha)}$ a $\Pi^0_{<\alpha}$-closed set $\mathcal{F}_n \subseteq \mathcal{A}$ and a Σ^0_α-open set $\mathcal{U}_n \supseteq \mathcal{A}$ such that $\lambda(\mathcal{U}_n - \mathcal{F}_n) \leq 2^{-n}$. Each set $\mathcal{U}_n - \mathcal{F}_n$ is a $\Sigma^0_1(\emptyset^{(<\alpha)})$ open set uniformly in n and in $\emptyset^{(\alpha)}$, and their intersection is therefore a $\Pi^0_2(\emptyset^{(\alpha)})$ set effectively of measure 0. Also if Z is $(\alpha+1)$-random, uniformly in a $(\alpha+1)$-randomness deficiency for Z one can find some n such that $Z \in \mathcal{A}$ iff $Z \in \mathcal{F}_n$.

We now repeat the operation for the $\Pi^0_{<\alpha}$-closed set $\mathcal{F}_n$. Again, one can find uniformly in $\emptyset^{(\alpha)}$ a sequence of clopen set $\mathcal{C}_m \supseteq \mathcal{F}_n$ such that $\lambda(\mathcal{C}_m - \mathcal{F}_n) \leq 2^{-n}$, making each set

$\mathcal{C}_m - \mathcal{F}_n$ a uniformly $\Sigma^0_1(\emptyset^{(\alpha)})$ set and therefore making their intersection a $\Pi^0_2(\emptyset^{(\alpha)})$ set effectively of measure 0. Now still using the $(\alpha+1)$-randomness deficiency of the $(\alpha+1)$-random sequence Z, we can find some m such that $Z \in \mathcal{F}_n$ iff $Z \in \mathcal{C}_m$, which implies that $Z \in \mathcal{A}$ iff $Z \in \mathcal{C}_m$. But now, once we have $\mathcal{C}_m$, it is computable in Z to decide whether $Z \in \mathcal{C}_m$. Also we have $e \in Z^{(\alpha)}$ iff $Z \in \mathcal{C}_m$ where $\mathcal{C}_m$ can be found uniformly in e and in $\emptyset^{(\alpha)}$, therefore $Z \oplus \emptyset^{(\alpha)} \geq_T Z^{(\alpha)}$. ∎

The previous theorem is tight. Lewis, Montalbán and Nies [53] showed that there is a weakly-2-random sequence Z which fails $Z \oplus \emptyset^{(1)} \geq_T Z^{(1)}$.

Theorem 4.2.3:
If Z is Π^1_1-random, we have $\Delta^1_1(Z) = \Delta^1_1 \oplus Z$ uniformly (and therefore a set is $\Pi^1_1(Z)$ if and only if it is Z-continuously Π^1_1).

Proof: Suppose the a set Y is $\Delta^1_1(Z)$ for a Π^1_1-random sequence Z. In particular we saw in Theorem 3.7.4 that if Z is Π^1_1-random then $\omega_1^Z = \omega_1^{ck}$. Also the set Y is then Turing computable in $Z^{(\alpha)}$ for some computable α.

Now also Z is Π^1_1-Martin-Löf random. From the hyperarithmetic index of a Martin-Löf test relative to some hyperarithmetic oracle we can effectively find an index for this test as a sequence of uniformly Π^1_1 open sets. Hence from the randomness deficiency for Z as a Π^1_1-Martin-Löf random sequence, we can uniformly in $\alpha < \omega_1^{ck}$ find an upper bound on the α-randomness deficiency for Z. Consequently, $\Delta^1_1(Z) = \Delta^1_1 \oplus Z$ uniformly. Hence by Proposition 4.2.4, if Z is Π^1_1-random, then a set is $\Pi^1_1(Z)$ if and only if it is Z-continuously Π^1_1. ∎

4.3 Continuous relativization and randomness

4.3.1 Continuous relativization for open sets

We now define the notion of higher continuous relativization and we show that various theorems of classical randomness hold in this setting. On the other hand, we will emphasize here and in Chapter 7 that various other theorems, like the existence of a universal X-Martin-Löf test, fail with continuous relativization.

Definition 4.3.1. *An open set $\mathcal{U}$ is **X-continuously Π^1_1** if there is an X-continuous Π^1_1 set of strings W such that $\mathcal{U} = [W^X]^\prec$. An **oracle-continuous Π^1_1-open set** $\mathcal{U}$, is a family of open sets $\{\mathcal{U}^X\}_{X\in 2^{\mathbb{N}}}$ such that for any X, the open set $\mathcal{U}^X$ is X-continuously Π^1_1 uniformly in X. Formally there is an oracle-continuous Π^1_1 set of strings W such that $\mathcal{U}^X = [W^X]^\prec$ for every oracle X.* ∎

When we don't need to know a specific set of pairs of strings $W \subseteq 2^{<\mathbb{N}} \times 2^{<\mathbb{N}}$ describing an oracle-continuous open set $\mathcal{U}$, we sometimes blur the distinction between the two, using $\mathcal{U}$ as if it was W.

Definition 4.3.2. *An* ***X-continuous $\mathbf{\Pi^1_1}$-Martin-Löf test*** *is given by a uniform sequence of oracle-continuous Π^1_1 open sets $\{\mathcal{U}_n\}_{n\in\mathbb{N}}$, such that for any n we have $\lambda(\mathcal{U}_n^X) \leq 2^{-n}$. If in addition, for every oracle Y and every n we have $\lambda(\mathcal{U}_n^Y) \leq 2^{-n}$, then $\{\mathcal{U}_n\}_{n\in\mathbb{N}}$ is said to be an* ***oracle-continuous $\mathbf{\Pi^1_1}$-Martin-Löf test****. Finally, a sequence Z is said to be* ***X-continuously $\mathbf{\Pi^1_1}$-Martin-Löf random*** *if it is in no X-continuous Π^1_1-Martin-Löf test.* ∎

Note that with any X-continuous Π^1_1-open sets $\mathcal{U}^X$ with $\lambda(\mathcal{U}^X) \leq 2^{-n}$, implicitly comes a Y-continuous Π^1_1-open set $\mathcal{U}^Y$ for any sequence Y. However $\mathcal{U}^Y$ need not to have its measure bounded by 2^{-n}. We have here a big difference with the notion of relativization in the classical case, where it is always possible to trim a X-Martin-Löf test $\bigcap \mathcal{U}_n^X$, in such a way that $\bigcap \mathcal{U}_n^Y$ becomes an Y-Martin-Löf test for every Y, without changing it if $\bigcap \mathcal{U}_n^Y$ was already a Martin-Löf test in the first place. However, we shall see in Section 7.3 that we cannot do that anymore in the higher setting.

From the definition of continuous randomness relativization, we have from Proposition 4.2.1 the interesting following fact:

Fact 4.3.1

The set of $\mathcal{O}$-continuously Π^1_1-Martin-Löf randoms coincides with the set of Martin-Löf randoms relatively to $\mathcal{O}$.

4.3.2 Continuous relativization for semi-measures

We now define continuous relativization for higher discrete semi-measures, and first for higher prefix-free machines.

Definition 4.3.3. *An* ***A-continuous $\mathbf{\Pi^1_1}$-prefix-free machine*** *M is a Π^1_1 set $M \subseteq 2^{<\mathbb{N}} \times 2^{<\mathbb{N}} \times 2^{<\mathbb{N}}$ such that $M^A = \{(\sigma_1, \sigma_2) : \exists \tau \prec A\ (\tau, \sigma_1, \sigma_2) \in M\}$ is a prefix-free machine. If M is also an X-continuous Π^1_1-prefix-free machine for every X then it is an* ***oracle-continuous-$\mathbf{\Pi^1_1}$-prefix-free machine****.* ∎

Definition 4.3.4. *An* ***A-continuous $\mathbf{\Pi^1_1}$-discrete semi-measure*** *M is a Π^1_1 set $M \subseteq 2^{<\mathbb{N}} \times 2^{<\mathbb{N}} \times \mathbb{Q}$ such that $M^A = \{(\sigma_1, q) : \exists \tau \prec A\ (\tau, \sigma_1, q) \in M\}$ is a discrete semi-measure. If M is also an X-continuous Π^1_1 discrete semi-measure for every X then it is an* ***oracle-continuous $\mathbf{\Pi^1_1}$-discrete semi-measure****.* ∎

Definition 4.3.5. *An A-**continuous $\mathbf{\Pi^1_1}$-continuous semi-measure** μ is a Π^1_1 set $\mu \subseteq 2^{<\mathbb{N}} \times 2^{<\mathbb{N}} \times \mathbb{Q}$ such that $\mu^A = \{(\sigma_1, q) : \exists \tau \prec A\ (\tau, \sigma_1, q) \in \mu\}$ is a continuous semi-measure. If μ is also an X-continuous Π^1_1 continuous semi-measure for every X then it is an* ***oracle-continuous $\mathbf{\Pi^1_1}$-continuous semi-measure****.* ∎

With continuous relativization of Π^1_1-prefix-free machines, naturally comes a continuous relativization of Π^1_1-Kolmogorov complexity:

Definition 4.3.6. *Given an A-continuous Π^1_1-prefix-free machine M, the* ***A-continuous Π^1_1-Kolmogorov complexity*** *with respect to the machine M and the oracle A is given by* $\mathrm{hK}^A_M(\sigma) = \min\{|\tau| \ : \ M^A(\tau) = \sigma\}$. ■

Unfortunately, it is not clear anymore that the randomness notions defined from continuously relativized measures, coincide with the continuous relativization of Π^1_1-Martin-Löf randomness. We give here the obvious implications.

Theorem 4.3.1:
Given sequences X, A, consider the four following propositions.

1. *The sequence X is A-continuously Π^1_1-Martin-Löf random.*
2. *For any A-continuous Π^1_1-continuous semi-measure μ, there exists a c such that $\mu(X\upharpoonright_n) \leq 2^{-n}2^c$ for every n.*
3. *For any A-continuous Π^1_1-discrete semi-measure M, there exists a c such that $M(X\upharpoonright_n) \leq 2^{-n}2^c$ for every n.*
4. *For any A-continuous Π^1_1-prefix-free machine M, there exists a c such that $\mathrm{hK}^A_M(X\upharpoonright_n) \geq n - c$ for every n.*

We have $(1) \to (2) \to (3) \to (4)$.

Proof: Each of the implications is a continuous relativization of the proof of Theorem 3.7.13.

(1) $\Longrightarrow$ (2): It is enough to see that given an A-continuous Π^1_1-continuous semi-measure μ, the set $\mathcal{U}_c = \{(\tau, \sigma) \mid \mu^\tau(\sigma) \geq 2^{-|\sigma|}2^c\}$ is a oracle continuous Π^1_1-open set such that $\lambda(\mathcal{U}^A_c) \leq 2^{-c}$.

(2) $\Longrightarrow$ (3): It is enough to see that given an A-continuous Π^1_1-discrete semi-measure M, the function $\mu^\tau(\sigma) = \sum_{\rho \geq \sigma} M^\tau(\rho)$ is a an A-continuous Π^1_1-continuous semi-measure such that μ^A dominates M^A.

(3) $\Longrightarrow$ (4): It is enough to see that given an A-continuous Π^1_1-prefix-free machine M, the function $2^{-\mathrm{hK}^A_M}$ is an A-continuous Π^1_1-discrete semi-measure. ■

The non-relativized implication (4) $\Longrightarrow$ (1), proved with Theorem 3.7.13, uses the fact that we can always assume that a Π^1_1-open set $\mathcal{U}$ can be described with an ε-prefix-free set of string, uniformly in ε. It is not clear anymore that this can be done with the continuous relativization. It also uses the higher KC-theorem, but here again, it is not clear that the higher KC theorem remains true with the continuous relativization. For this reason it is also not clear that we have (4) $\Longrightarrow$ (3). Basically, each of the other possible implication remains open. This will be discussed in Section 7.3.6.

4.3.3 The van Lambalgen theorem

The van Lambalgen theorem can be seen as an effective version of Fubini's theorem. For classical randomness we have:

Theorem 4.3.2 (van Lambalgen):
The sequence $X \oplus Y$ is Martin-Löf random iff X is Martin-Löf random and Y is Martin-Löf random relatively to X.

Note that this implies that Y is Martin-Löf random relatively to X iff X is Martin-Löf random relatively to Y. The theorem holds in the higher setting, where relativization is understood as continuous relativization. However, the proof needs to be twisted a little bit to work in the higher setting. The proof of the direction "X is Martin-Löf random and Y is Martin-Löf random relatively to X implies $X \oplus Y$ is Martin-Löf random" is the same in both the higher and the lower setting. But the other direction uses the existence of a universal oracle-continuous Martin-Löf test. We will prove in Section 7.3 that there is no universal oracle-continuous Π^1_1-Martin-Löf test. We will even prove that for some oracle X, there is no universal X-continuous Π^1_1-Martin-Löf test. Fortunately we can get rid of this hypothesis to prove a higher version of van Lambalgen, and this will be achieved with the help of the following lemma:

Lemma 4.3.1 *Given an oracle-continuous Π^1_1-open set $\mathcal{U} \subseteq 2^{<\mathbb{N}} \times 2^{<\mathbb{N}}$ one can define uniformly in $n \in \mathbb{N}$ and in $\varepsilon \in \mathbb{Q}^+$ an oracle-continuous Π^1_1-open set $\mathcal{V} \subseteq 2^{<\mathbb{N}} \times 2^{<\mathbb{N}}$ such that:*

- *If $\lambda(\mathcal{U}^X) \leq 2^{-n}$ then $\mathcal{U}^X = \mathcal{V}^X$.*
- *$\lambda(\{X \ : \ \lambda(\mathcal{V}^X) > 2^{-n}\}) \leq \varepsilon$.*

Proof: Let n be fixed. Recall that $p : \omega_1^{ck} \to \omega$ is the projectum function. At stage 0 we set $\mathcal{V}_0 = \emptyset$. At successor stage s, suppose that (σ, τ) is enumerated in $\mathcal{U}$. Let us consider the Δ^1_1-open set $\mathcal{W} = \{X \ : \ \lambda(\mathcal{V}^X_{s-1} \cup [\tau]) > 2^{-n}\}$. Let us find a finite set of strings B such that $[B]^{\prec} \cup \mathcal{W} = [\sigma]$ and such that $\lambda([B]^{\prec} \cap \mathcal{W}) \leq \varepsilon \times 2^{-p(s)}$. For any string ρ in B we then add (ρ, τ) in $\mathcal{V}$ at stage s. At limit stage s we define $\mathcal{V}_s$ to be the union of $\mathcal{V}_t$ for $t < s$.

It is obvious that if $\lambda(\mathcal{U}^X) \leq 2^{-n}$, then $\mathcal{U}^X = \mathcal{V}^X$. Also by construction, at successor stage s, we add in $\{X \ : \ \lambda(\mathcal{V}^X) > 2^{-n}\}$ something of measure at most $\varepsilon \times 2^{-p(s)}$. It follows that $\lambda(\{X \ : \ \lambda(\mathcal{V}^X) > 2^{-n}\}) \leq 2^{-n}$. ∎

We can now prove the higher van Lambalgen theorem:

Theorem 4.3.3 (Higher van Lambalgen):
The sequence $X \oplus Y$ is Π^1_1-Martin-Löf random iff X is Π^1_1-Martin-Löf random and Y is X-continuously Π^1_1-Martin-Löf random.

Proof: Suppose first that some sequence $X \oplus Y$ is captured by some Π^1_1-Martin-Löf test $\bigcap_n \mathcal{U}_n$. For $\mathcal{U}_n = \bigcup [\sigma_1 \oplus \sigma_2]$, note that we clearly have $\lambda(\bigcup [\sigma_1 \oplus \sigma_2]) = \lambda(\bigcup [\sigma_1] \times [\sigma_2])$. Also we can consider that the pair (X, Y) is not Π^1_1-Martin-Löf random in the product space $2^{\mathbb{N}} \times 2^{\mathbb{N}}$.

Let $\bigcap_n \mathcal{U}_n$ be a uniform intersection of Π^1_1-open sets of $2^{\mathbb{N}} \times 2^{\mathbb{N}}$ with $\lambda(\mathcal{U}_n) \le 2^{-n}$ and $(X,Y) \in \bigcap_n \mathcal{U}_n$. We are going to use Theorem 1.8.3 saying that for any Borel set $\mathcal{B}$ of $2^{\mathbb{N}} \times 2^{\mathbb{N}}$ we have $\lambda(\{X : \lambda(\mathcal{B}_X) > \sqrt{\lambda(\mathcal{B})}\}) \le \sqrt{\lambda(\mathcal{B})}$. For a string σ and an integer n, let us denote by $\mathcal{U}_n^\sigma$ the Π^1_1-open set $\{Y : \forall X \succ \sigma\ (X,Y) \in \mathcal{U}_n\}$.

Let $\mathcal{V}_n$ be the X-continuously Π^1_1-open set containing Y and equal to $\bigcup_{\sigma \prec X} \mathcal{U}_{2n}^\sigma$. Suppose that for all but finitely many n we have $\lambda(\mathcal{V}_n) \le 2^{-n}$. Then Y is not X-continuously Π^1_1-Martin-Löf random. Otherwise there are infinitely many n such that $\lambda(\mathcal{V}_n) > 2^{-n}$. Also consider now for each n the Π^1_1-open set $\mathcal{S}_n = \{Z : \lambda(\mathcal{U}_{2n}^Z) > 2^{-n}\}$. We have for infinitely many n that $X \in \mathcal{S}_n$ and as $\mathcal{S}_n \subseteq \{Z : \lambda(\mathcal{U}_{2n}^Z) > \sqrt{\lambda(\mathcal{U}_{2n})}\}$ we have by Theorem 1.8.3 that $\lambda(\mathcal{S}_n) \le \sqrt{\lambda(\mathcal{U}_{2n})} \le 2^{-n}$. Also $\{\mathcal{S}_n\}_{n \in \mathbb{N}}$ is a Π^1_1-Solovay test capturing X, which is then not Π^1_1-Martin-Löf random.

Conversely, suppose that X is not Π^1_1-Martin-Löf random or that Y is not X-continuously Π^1_1-Martin-Löf random. It is enough to deal with the last case, as if X is not Π^1_1-Martin-Löf random it is certainly not Y-continuously Π^1_1-Martin-Löf random either.

So suppose that Y is in some X-continuous Π^1_1-Martin-Löf test $\bigcap_n \mathcal{U}_n^X$ where each $\mathcal{U}_n$ can be seen as a Π^1_1 subset of $2^{<\mathbb{N}} \times 2^{<\mathbb{N}}$. From Lemma 4.3.1 we can consider that each $\mathcal{U}_n$ is such that $\lambda(\{Z : \lambda(\mathcal{U}_n^Z) > 2^{-n}\} \le 2^{-n})$ still with $Y \in \bigcap_n \mathcal{U}_n^X$. It is clear that the set $\bigcup_{\tau \in 2^{<\mathbb{N}}} [\tau] \times \mathcal{U}_n^\tau$ is a Π^1_1-open subset of $2^{\mathbb{N}} \times 2^{\mathbb{N}}$, defined uniformly in n and which contains (X,Y). Let us prove that it has measure smaller than 2^{-n+1}.

Since for $\tau \le \tau'$ we have $\mathcal{U}_n^\tau \subseteq \mathcal{U}_n^{\tau'}$, we then have $\lambda(\bigcup_{\tau \in 2^{<\mathbb{N}}} [\tau] \times \mathcal{U}_n^\tau) = \sup_m \sum_{|\tau|=m} \lambda([\tau] \times \mathcal{U}_n^\tau)$. Also for each m, the measure of the set of strings τ of length m such that $\lambda(\mathcal{U}_n^\tau) > 2^{-n}$ is of $\varepsilon_m \le 2^{-n}$, whereas on other strings τ of length m we have $\lambda(\mathcal{U}_n^\tau) \le 2^{-n}$. We then have:

$$\sum_{|\tau|=m} \lambda([\tau] \times \mathcal{U}_n^\tau) \le (1 - \varepsilon_m) 2^{-n} + \varepsilon_m \le 2^{-n+1}$$

It follows that $\lambda(\bigcup_{\tau \in 2^{<\mathbb{N}}} [\tau] \times \mathcal{U}_n^\tau) \le 2^{-n+1}$ and we then have a Π^1_1-Martin-Löf test capturing (X,Y). ■

4.3.4 The XYZ theorem

In classical randomness, we have the following theorem from Miller and Yu [62]:

Theorem 4.3.4 (XYZ theorem):
Suppose that X, Y are Martin-Löf random and that $X \ge_T Y$. Suppose also that X is Martin-Löf random relatively to Z. Then Y is also Martin-Löf random relatively to Z.

Proof: Let Φ be the functional such that $\Phi(X) = Y$. We first use the fact that $\mu([\sigma]) = \lambda(\Phi^{-1}([\sigma]))$ is a left-c.e. continuous semi-measure. It follows from a lower equivalent to Theorem 3.7.13 that for prefixes σ of Y, we have some constant c such that $\mu([\sigma]) \le 2^{-|\sigma|} 2^c$.

Now, given some uniform intersection of oracle Σ^0_1-open sets $\mathcal{U}_n^Z$ of measure less than 2^{-n} and with $Y \in \bigcap_n \mathcal{U}_n^Z$, for any $(\sigma, \tau) \in \mathcal{U}_n$, for any string ρ enumerated in a set of strings describing $\Phi^{-1}([\tau])$, we enumerate (σ, ρ) in some set $\mathcal{V}_n$, but stopping the enumeration when $\lambda(\Phi^{-1}([\tau]))[s]$ becomes bigger than $2^{-|\tau|}2^c$.

Because Y is random, we are sure that $X \in \bigcap_n \mathcal{V}_n^Z$: For $\tau \prec Y$ and $\sigma \prec Z$ such that $(\sigma, \tau) \in \mathcal{U}_n$, the open set $\Phi^{-1}([\tau])$ never has to be trimmed and then $\Phi^{-1}([\tau]) \subseteq \mathcal{V}_n^\sigma$. Also we have $\lambda(\mathcal{V}_n^Z) \leq 2^{-n}2^c$ for every n. Therefore, the function $n \to \lambda(\mathcal{V}_n^Z)$ goes to 0 with a computable bound and X is not Martin-Löf random relatively to Z. ∎

The previous theorem says that we have a randomness inheritance by Turing reduction. Of course a random sequence X can always compute something which is not random. But if it computes a random sequence Y, then Y needs to be 'as random as' X. In particular, a direct consequence of the previous theorem is that if an α-random sequence X computes a Martin-Löf random sequence Y, then Y is also α-random. It is not very hard to check that the proof also works with weak-α-randomness, for any computable α.

We shall now prove that the same theorem hold with Π^1_1-Martin-Löf randomness. The difficulty is however that if X higher Turing computes Y via some higher Turing functional Φ, the function $\lambda(\Phi^{-1}[\sigma])$ is not necessarily a Π^1_1 continuous semi-measure anymore, because Φ might be inconsistent on some oracles. According to later Corollary 7.3.2, the inconsistency cannot be completely removed. However, it can be 'reduced' as much as we want, from a measure-theoretical point of view.

Lemma 4.3.2 *From any higher functional Φ one can obtain effectively in ε a higher functional Ψ so that:*

1. *The correct computations are unchanged in Ψ: For all X, Y such that $\Phi(X) = Y$, we also have $\Psi(X) = Y$*
2. *The measure of the Π^1_1-open set on which Ψ is inconsistent is smaller than ε:*

$$\lambda(\{X \mid \exists n_1, n_2\ \exists \tau_1 \perp \tau_2\ (X\upharpoonright_{n_1}, \tau_1) \in \Psi \wedge (X\upharpoonright_{n_2}, \tau_2) \in \Psi\}) \leq \varepsilon$$

Proof: Let us build Ψ uniformly in Φ and ε. Recall that $p : \omega_1^{ck} \to \omega$ is the projectum function. We can assume that at most one pair enters Φ at each stage. At stage s, if (σ_1, τ_1) enters $\Phi[s]$, we compute the Δ^1_1 set of strings:

$$\mathcal{U}_s = \{\sigma_2\ :\ \sigma_2 \text{ is compatible with } \sigma_1 \text{ and } (\sigma_2, \tau_2) \in \Psi[< s] \text{ for some } \tau_2 \perp \tau_1)\}$$

We then find uniformly in $\mathcal{U}_s$ and s a finite set of strings C with $[C]^\prec \subseteq [\sigma_1]$, such that $[C]^\prec \cup \mathcal{U}_s$ covers $[\sigma_1]$ and such that $\lambda([C]^\prec \cap \mathcal{U}_s) \leq 2^{-p(s)}\varepsilon$. Then we put in $\Psi[s]$ all the pairs (σ, τ_1) for $\sigma \in C$.

We shall prove that (1) and (2) are satisfied. Suppose $\Phi(X) = Y$ and that $(X\upharpoonright_{n_1}, Y\upharpoonright_{n_2})$ enters $\Phi[s]$ at stage s. By definition of $\Phi(X) = Y$, we have no m and no $\tau \perp Y\upharpoonright_{n_2}$ such that $(X\upharpoonright_m, \tau)$ is in $\Phi[< s]$. Then also we have no m and no $\tau \perp Y\upharpoonright_{n_2}$ such that $(X\upharpoonright_m, \tau)$ is in $\Psi[< s]$, because $(X\upharpoonright_m, \tau) \in \Psi$ implies $(X\upharpoonright_n, \tau) \in \Phi$ for $n \leq m$. Therefore $X \notin \mathcal{U}_s$ and as $\mathcal{U}_s \cup C$ covers $X\upharpoonright_{n_1}$, we then have a prefix of X that is mapped to $Y\upharpoonright_{n_2}$ in $\Psi[s]$. Then we have (1). Also by construction, at stage s, we add a measure of at most $2^{-p(s)}\varepsilon$ of inconsistency. Then the total inconsistency is at most of ε, which gives us (2). ∎

We can now prove:

Theorem 4.3.5 (higher XYZ theorem):
Suppose that $X \geq_{\mathrm{hT}} Y$ for two Π^1_1-Martin-Löf randoms X, Y. Suppose also that X is Z-continuously Π^1_1-Martin-Löf random. Then also Y is Z-continuously Π^1_1-Martin-Löf random.

Proof: Suppose we have a higher Turing functional Φ such that $\Phi(X) = Y$. Using Lemma 4.3.2, uniformly in ε we can build a functional Φ_ε with $\Phi_\varepsilon(X) = Y$ and such that the open set of oracles on which Φ_ε is not consistent is smaller than ε.

Let $\{\mathcal{U}_n\}_{n\in\mathbb{N}}$ be a uniform sequence of oracle-continuous Π^1_1-open sets such that $\lambda(\mathcal{U}^Z_n) \leq 2^{-n}$ and such that $Y \in \bigcap_n \mathcal{U}_n$. Let us fix a function $q : 2^{<\mathbb{N}} \to \mathbb{Q}^+$ such that $\sum_{\sigma\in 2^{<\mathbb{N}}} q(\sigma) \leq 1$.

Let us define the left-Π^1_1 function ν to be $\nu([\sigma]) = \lambda(\Phi^{-1}_{q(\sigma)}([\sigma]))$. We have that $\nu(\sigma) + q(\sigma) \geq \nu(\sigma\hat{\ }0) + \nu(\sigma\hat{\ }1)$. It follows that:

$$\nu(\sigma) + \sum_{\tau\geq\sigma} q(\tau) \geq \left(\nu(\sigma\hat{\ }0) + \sum_{\tau\geq\sigma\hat{\ }0} q(\tau)\right) + \left(\nu(\sigma\hat{\ }1) + \sum_{\tau\geq\sigma\hat{\ }1} q(\tau)\right)$$

We define the left-Π^1_1 function μ by $\mu([\sigma]) = \nu([\sigma]) + \sum_{\tau\geq\sigma} q(\tau)$. As we have $\nu(\epsilon) + \sum_{\tau\geq\epsilon} q(\tau) \leq 2$, the function $\mu/2$ is then a Π^1_1-continuous semi-measure.

It follows that since Y is Π^1_1-Martin-Löf random, we have by Theorem 3.7.13 some constant c such that $\mu(Y \upharpoonright_n) \leq 2^{-n}2^c$ for every n. We can now do as in the proof of Theorem 4.3.5: For (σ, τ) enumerated in $\mathcal{U}_n$, we enumerate (σ, ρ) in $\mathcal{V}_n$, for any string ρ enumerated in a set of string describing $\Phi^{-1}_{q(s)}([\sigma])$, as long as $\lambda(\Phi^{-1}_{q(s)}([\sigma]))[s]$ is smaller than $2^{-|\sigma|}2^c$. The verification is then similar. ∎

We can adapt the proof of the previous theorem for many different notions of tests. In particular, we have for example:

Porism 4.3.1 (Variant of the higher XYZ theorem):
Suppose that $X \geq_{\mathrm{hT}} Y$ for two Π^1_1-Martin-Löf randoms X, Y. Suppose also that X is weakly-Π^1_1-random. Then also Y is weakly-Π^1_1-random.

4.4 Refinement of the notion of higher Δ^0_2

In this section we discuss the higher analogue of the notion of being Δ^0_2. After discussing the difference between higher Δ^0_2 and merely Δ^0_2, we will give different various restrictions of the former notion, each of them arising naturally from the study of higher randomness. This study will then allow us to separate in Section 5.3.2 the notion of weak-Π^1_1-randomness and Π^1_1-randomness. We first give a higher version of Shoenfield's limit lemma:

4.4.1 The higher limit lemma

Proposition 4.4.1:
Let $A \in 2^{\mathbb{N}}$. The following are equivalent for $f \in \mathbb{N}^{\mathbb{N}}$.

1. $f \leq_{\mathrm{hT}} \mathrm{hJ}^A$.
2. *There is a A-continuously higher Turing computable sequence $\{f_s\}_{s<\omega_1^{ck}}$ of functions from $\mathbb{N}$ to $\mathbb{N}$ with $\lim_{s\to\omega_1^{ck}} f_s = f$.*

Proof: Recall that $p : \omega_1^{ck} \to \omega$ is the projectum function.

(2) $\Longrightarrow$ (1). Let $m : \omega \to \omega_1^{ck}$ be the modulus of the sequence $\{f_s\}_{s<\omega_1^{ck}}$: The value $m(n)$ is the least s such that for all $t \geq s$ we have $f_t(n) = f_s(n)$. Let $W = \{(n, p(s)) : s < m(n)\}$; the set W is A-continuously Π^1_1 : to enumerate $(n, p(s))$ into W, what we need from A is the value $f_s(n)$ and a different value $f_t(n)$ for some $t > s$; both can be found with using a finite prefix of A. So $W \leq_{\mathrm{hT}} \mathrm{hJ}^A$. Now, from one pair $(n, p(s)) \notin W$, we output $f(n) = f_s(n)$. So $f \leq_{\mathrm{hT}} W \leq_{\mathrm{hT}} \mathrm{hJ}^A$.

(1) $\Longrightarrow$ (2). Let Ψ be a higher Turing functional such that $\Psi(\mathrm{hJ}^A) = f$. Note that the sequence $\{\mathrm{hJ}^A_s\}_{s<\omega_1^{ck}}$ is *not* necessarily A-continuously Turing computable. To help us, we have to use the projectum function $p : \omega_1^{ck} \mapsto \omega$. For $s < \omega_1^{ck}$ let $f_s = \Psi(\mathrm{hJ}_s^{A\upharpoonright p(s)})[s]$. For all n there is some $t < \omega_1^{ck}$ such that $p(s) \geq n$ for all $s \geq t$, which ensures that $\{f_s\}_{s<\omega_1^{ck}}$ converges to f. ■

It follows, using the fact that $\mathrm{hJ}^{\varnothing}$ is many-one equivalent to $\mathcal{O}$ and using Proposition 4.2.1 that a function f is Turing computable by Kleene's $\mathcal{O}$, iff it is the limit of a higher Turing computable sequence $\{f_s\}_{s<\omega_1^{ck}}$:

Corollary 4.4.1:
Let $f \in \mathbb{N}^{\mathbb{N}}$. Then the following are equivalent:

1. $f \leq_T \mathcal{O}$
2. $f \leq_{\mathrm{hT}} \mathcal{O}$
3. *There is a higher Turing computable sequence $\{f_s\}_{s<\omega_1^{ck}}$ of functions from $\mathbb{N}$ to $\mathbb{N}$ with $\lim_{s\to\omega_1^{ck}} f_s = f$.*

Such a function is said to be **higher Δ^0_2 function**. There is a topological difference between a Δ^0_2 approximation $\{f_s\}_{s<\omega}$ and a higher Δ^0_2 approximation $\{g_s\}_{s<\omega_1^{ck}}$. In the first case the set $\{f\} \cup \{f_s : s < \omega\}$ is a closed set, whereas in the second case, the set $\{g\} \cup \{g_s : s < \omega_1^{ck}\}$ needs not to be closed. Also we study in this section various restrictions of the notion of higher Δ^0_2 functions, that are built around this crucial point. We will explain why each of them makes sense (in particular in relation with higher randomness), why there are used and how they relate to each other. We will pursue a full study of higher Δ^0_2 functions in Section 5.4 and a picture on how all those notions interact with each other is given in Section 5.4.6.

4.4.2 Higher left-c.e. approximations

We start by the strongest restriction of higher Δ^0_2, which can be seen as a higher analogue of left-c.e.

Definition 4.4.1. *A **higher left-c.e. approximation** is a higher Turing computable sequence $\{f_s\}_{s<\omega_1^{ck}}$ such that for any stage $s_1 < s_2$ we have f_{s_1} smaller than f_{s_2} for the lexicographic order. Note that this implies that for any n, the sequence $\{f_s(n)\}_{s<\omega_1^{ck}}$ changes at most 2^n times and then that $\{f_s\}_{s<\omega_1^{ck}}$ converges. A function f is **higher left-c.e.** if there is a higher left-c.e. approximation converging to f.* ■

Just like left-c.e. binary sequences are exactly the leftmost path of Π^0_1 sets, it is not hard to see that higher left-c.e. binary sequences are the leftmost path of Σ^1_1-closed sets.

4.4.3 Higher ω-computable approximations

We continue with the second strongest restriction of higher Δ^0_2, which can be seen as a higher analogue of ω-computably approximable.

Definition 4.4.2. *A **higher ω-computable approximation** is a higher Turing computable sequence $\{f_s\}_{s<\omega_1^{ck}}$ such that for any n, the sequence $\{f_s(n)\}_{s<\omega_1^{ck}}$ changes at most $h(n)$ times, where h is a hyperarithmetic function of n. Note that this implies the convergence of $\{f_s\}_{s<\omega_1^{ck}}$. A function f is **higher ω-computably approximable** if there is a higher ω-computable approximation converging to f.* ■

In the classical case we have that a function f is ω-computably approximable iff it is weakly-truth-table reducible to $\emptyset^{(1)}$. If furthermore f is $\{0,1\}$-valued, then also f is weak-truth-table reducible to $\emptyset^{(1)}$ iff it is truth-table reducible to $\emptyset^{(1)}$. We will now see that the same holds here, with a higher analogue of weakly-truth-table reducibility, where the use of $\mathcal{O}$ is bounded by a hyperarithmetic function; and with a higher analogue of truth-table reducibility, where the functional is a higher Turing functional, total on every oracle.

Proposition 4.4.2:
A function f is higher ω-computably approximable iff it is higher weakly-truth-table reducible to $\mathcal{O}$. If f is $\{0,1\}$-valued, then f is higher ω-computably approximable iff it is higher truth-table reducible to $\mathcal{O}$.

Proof: The proof is similar to the one in the classical case. Let us first prove that a $\{0,1\}$-valued and higher ω-computably approximable function f is higher truth-table reducible to $\mathcal{O}$. Let $h : \mathbb{N} \to \mathbb{N}$ be the hyperarithmetic bound on the number of changes. To compute $f(n)$ using $\mathcal{O}$, we ask $\mathcal{O}$ the question: 'Will $f(n)$ change more than 1 time during the approximation?' If the answer is yes we ask : 'Will $f(n)$ change more than

2 times during the approximation?'. We can continue until the answer is no or until we have asked $h(n)$ times. At the end of the process we know the number of times $f(n)$ will change. Also we can always assume that the first value of $f(n)$ is 0. Thus if the number of times $f(n)$ will change is even then $f(n) = 0$, otherwise $f(n) = 1$. It is clear that this terminates for any oracle, therefore f is higher truth-table reducible to $\mathcal{O}$.

Let us now assume that f is higher ω-computably approximable, but not necessarily $\{0,1\}$-valued. Let us prove it is higher weakly-truth-table reducible to $\mathcal{O}$. The process is similar except that once we have the value m of the number of times $f(n)$ will change we then compute the actual approximation, until we reach m changes. The value of $f(n)$ is then the current one in the approximation. Also as the number of question we ask to $\mathcal{O}$ is bounded by $h(n)$, then also we have an hyperarithmetic bound on the use of $\mathcal{O}$.

Let us now suppose that f is higher weakly-truth-table reducible to $\mathcal{O}$ via the functional Φ, with bound $h(n)$ and let us show that f is higher ω-computably approximable. For every $n < \omega$ and every stage $s < \omega_1^{ck}$ we simply set $f_s(n) = \Phi(\mathcal{O}\upharpoonright_{h(n)})[s]$ (if Φ returns no value on $\mathcal{O}\upharpoonright_{h(n)}[s]$ at stage s we set $f_s(n) = 0$). As the approximation of $\mathcal{O}$ is higher left-c.e., the number of changes in the approximation $\{f_s(n)\}_{s<\omega_1^{ck}}$ of $f(n)$ is bounded by $2^{h(n)}$. ∎

4.4.4 Higher closed and compact approximations

Definition 4.4.3. *A* ***higher compact approximation*** *is a converging higher Turing computable sequence $\{f_s\}_{s<\omega_1^{ck}}$ such that $\{f\} \cup \{f_s : s < \omega_1^{ck}\}$ is a compact set. A function $f \in \mathbb{N}^{\mathbb{N}}$ is* ***higher compact approximable*** *if there is a higher compact approximation converging to f.* ∎

In particular, any approximation $\{f_s\}_{s<\omega_1^{ck}}$ where the number of changes of $\{f_s(n)\}_{s<\omega_1^{ck}}$ is finite for each n, is a compact approximation (see later Fact 5.4.1). This implies in particular that any ω-computably approximable function also has a compact approximation. We shall see that the converse does not hold in Section 5.4.

The notion of compact approximation is very important because it could be considered as the true counterpart of the lower notion of Δ^0_2 functions, because the lower Δ^0_2 functions are exactly those with a 'compact approximation'. However, we then don't have anymore the counterpart with 'being computable by $\emptyset^{(1)}$'. We shall see in Section 5.4 refinements of the notion of having a compact approximation that have an analogue in term of higher computability and $\mathcal{O}$.

Also, working in the Baire space rather than the Cantor space, we need to make a distinction between closed and compact approximations.

Definition 4.4.4. *A* ***higher closed approximation*** *is a converging higher Turing computable sequence $\{f_s\}_{s<\omega_1^{ck}}$ such that $\{f\} \cup \{f_s : s < \omega_1^{ck}\}$ is a closed set. A function $f \in \mathbb{N}^{\mathbb{N}}$ is* ***higher closed approximable*** *if there is a higher closed approximation converging to f.* ∎

The notion of compact approximation will mainly be used in Section 5.3 to study weak-Π^1_1-randomness.

4.4.5 Higher self-unclosed approximations

We shall now see an even weaker restriction of higher Δ^0_2, which will lead us to a separation of Π^1_1-randomness from weak-Π^1_1-randomness:

Definition 4.4.5. *A* ***higher self-unclosed approximation*** *of a function f is a higher Turing computable sequence $\{f_s\}_{s<\omega_1^{ck}}$ converging to f and such that for every stage s, the function f is not in the closure of $\{f_t \ : \ t < s\}$ unless it is already an element of $\{f_t \ : \ t < s\}$. Such a function f is then said to be* ***higher self-unclosed approximable.***■

The purpose of self-unclosed approximations is to find a criterion as weak as possible, for higher Δ^0_2 functions to collapse ω_1^{ck} (to make ω_1^{ck} computable in X):

Theorem 4.4.1:
If $f \in \mathbb{N}^\mathbb{N}$ is not Δ^1_1 and has a self-unclosed approximation then $\omega_1^f > \omega_1^{ck}$. In particular if $X \in 2^\mathbb{N}$ has a self-unclosed approximation, then X is not Π^1_1-random.

Proof: Suppose f has a self-unclosed approximation $\{f_s\}_{s<\omega_1^{ck}}$. We can define the $\Pi^1_1(f)$ total function $g : \omega \to \omega_1^{ck}$ which to n associates the smallest ordinal s_n so that $f_{s_n} \upharpoonright_n = f \upharpoonright_n$. Then we have that f is in the closure of $\{f_t\}_{t<s}$ for $s = \sup s_n$. Therefore we have $\sup s_n = \omega_1^{ck}$. Also as g is $\Pi^1_1(f)$ and total it is also $\Delta^1_1(f)$. Then we can define a $\Delta^1_1(f)$ sequence of computable ordinals, unbounded in ω_1^{ck} which implies $\omega_1^f > \omega_1^{ck}$, by the Σ^1_1-boundedness principle. ■

It is clear that a closed approximation is also a self-unclosed approximation. We shall see in Section 5.3.2 that the converse does not hold, by building an element with a self-unclosed approximation, which is weakly-Π^1_1-random. By later Theorem 5.3.1 such an element cannot have a closed approximation.

The self-unclosed approximable elements are also well-behaved with respect to the continuity issues that might occur in the higher setting. We shall prove later with Theorem 7.4.1 that higher Turing computations and fin-h computations coincide for elements having a self-unclosed approximation. Also the self-unclosed approximable elements are well behaved with respect to continuous relativization of randomness as we will see in theorem 7.4.4.

On the other hand, we will prove with Corollary 7.3.2 that there are some higher Δ^0_2 sequence Y, X such that $Y \geq_{hT} X$ but $Y \ngeq_{fin-h} X$. We will also prove with Theorem 7.3.2 that there are some higher Δ^0_2 sequence X such that there is no X-continuous universal Π^1_1-Martin-Löf test.

4.5 Continuously low for Π^1_1-Martin-Löf randomness

The sequences which are low for Martin-Löf randomness have been extensively studied. We shall transpose in this section the main results of the lower setting to the higher setting, using continuous relativization.

4.5.1 hK-trivial sequences

Definition 4.5.1. *A sequence A is* hK-*trivial if for some constant d,* $\mathrm{hK}(A\upharpoonright_n) \leq \mathrm{hK}(n) + d$. ■

It is obvious that any Δ^1_1 sequence is hK-trivial, because up to an index for such a sequence A, the information about the length of a prefix of A is enough to retrieve that prefix. We shall see that just like for the lower setting, there are non Δ^1_1 and hK-trivial sequences. Solovay was the first in [86] to build an incomputable K-trivial sequence. Later, Hjorth and Nies showed that similarly, there are incomputable hK-trivial sequences. Both proofs are similar in the lower and in the higher setting.

Theorem 4.5.1 (Hjorth, Nies [30]):
There is a hK-*trivial which is not* Δ^1_1.

Proof: **The construction :**

We want to build a Π^1_1 hK-trivial sequence X which is co-infinite and which intersect any infinite Π^1_1 set. Let $\{P_e\}_{e\in\mathbb{N}}$ be an enumeration of the Π^1_1 sets and let U be a universal Π^1_1-prefix-free machine. We enumerate X and build at the same time a Π^1_1-bounded request set M such that $\inf\{m : (m, X\upharpoonright_n) \in M\} \leq \mathrm{hK}_U(n)+1$. We keep track of a set of Boolean values R_e, initialized to false and meaning that X does not intersect P_e yet.

At successor stage s, at substage e for which R_e is false, if there is $n \in P_{e,s}$ with $n \geq 2e$ and such that the weight of M at stage s and substage $e-1$, restricted to strings of length bigger than n, is smaller than 2^{-e-1}, then we enumerate n in X at stage s, we set R_e to true, and for every pair $(l, X_{s-1}\upharpoonright_m)$ in M at stage s and substage $e-1$, for $m \geq n$, we put $(l, X_s\upharpoonright_m)$ in M at stage s and substage e.

After all substages e, if (σ, n) is enumerated in U at stage s, we enumerate $(|\sigma|+1, X_s\upharpoonright_n)$ in M at stage s.

The verification :

We should prove that $\mathrm{wg}(M) \leq 1$. The weight of all the pairs we enumerate in M because of some (σ, n) in U, is bounded by $1/2$. Then for each e, the additional weight we put in is bounded by 2^{-e-1}. Therefore the weight of M is bounded by 1.

We should now prove that X is not Δ^1_1. It is clearly co-infinite, as for each e we add in X at most one integer bigger than $2e$. Suppose that P_e is infinite. Then at some stage s it is already infinite, by the Σ^1_1-boundedness principle. Also at any stage t we have

$\mathrm{wg}(M) \leq 1$. Therefore there is a smallest length n such that the weight of M at stage s, restricted to strings of length bigger than n, is smaller than 2^{-e-1}. At this point, the integer n is enumerated in X if R_e is still false. So X intersects every infinite Π^1_1 set.

Also by construction it is clear that $\inf\{m : (m, X\upharpoonright_n) \in M\} \leq \mathrm{hK}_U(n) + 1$. Therefore X is hK-trivial. ■

Chaitin proved in [6] that there are only countable many K-trivial sequences. With a similar proof, we also have that there are only countably many hK-trivial sequences.

Theorem 4.5.2 (Hjorth, Nies [30]):
There is a constant c, such that for each constant d and each n, there are at most $c \times 2^d$ many strings σ of length n such that $\mathrm{hK}(\sigma) \leq \mathrm{hK}(|\sigma|) + d$.

Proof: Let M be the machine which on a string τ outputs $|U(\tau)|$. If τ is a short description for any string of length n via U, then τ is a short description for n, via the machine M. Also by the coding theorem (Theorem 3.7.12) we have $P_M(n) < 2^{-\mathrm{hK}(n)} \times c_M$ for some constant c_M (recall P_M from Definition 3.7.15). We now claim that for any length n and any d, there are at most $c_M \times 2^d$ strings of length n such that $\mathrm{hK}(\sigma) \leq \mathrm{hK}(n) + d$. Suppose otherwise for a given length n. Then $P_M(n) \geq c_M \times 2^d \times 2^{-\mathrm{hK}(n)-d} = c_M \times 2^{-\mathrm{hK}(n)}$, which is a contradiction. ■

Corollary 4.5.1:
There is a constant c, such that for each constant d there are at most $c \times 2^d$ many sequences X such that $\mathrm{hK}(X\upharpoonright_n) \leq \mathrm{hK}(n) + d$ for every n. In particular there are at most countably many hK-trivial sequences.

Proof: With c the constant of the previous theorem, if there are more than $c \times 2^d$ many sequences X such that $\mathrm{hK}(X\upharpoonright_n) \leq \mathrm{hK}(n) + d$ for every n, then also for n large enough, there are more than $c \times 2^d$ many strings σ of length n such that $\mathrm{hK}(\sigma) \leq \mathrm{hK}(|\sigma|) + d$. ■

The previous theorem will allow us to determine that hK-trivial sets are actually fairly simple to describe: They are all higher Δ^0_2. Also we can even put them in a sharper class, which will be useful for the notion of higher randomness, continuously relativized to an hK-trivial sequence: The class of self-unclosed approximable sequences (see Definition 4.4.5):

Proposition 4.5.1:
Every hK-trivial sequence A has a self-unclosed approximation $\{A_s\}_{s<\omega_1^{ck}}$ such that every A_s looks hK-trivial at stage s, that is, for every stage s and every n we have $\mathrm{hK}_s(A_s\upharpoonright_n) \leq \mathrm{hK}_s(n) + d$ where d is the hK-triviality constant of A. In particular, every hK-trivial sequence is higher Δ^0_2.

PROOF: Suppose that A is hK-trivial with constant d. For each stage $s < \omega_1^{ck}$, let $T_s = \{\sigma : \forall \tau \leq \sigma\ \mathrm{hK}_s(\tau) \leq \mathrm{hK}_s(|\tau|) + d\}$, where hK_s is the approximation of hK at stage s. Each tree T_s is Δ^1_1 uniformly in s. Let us prove the following convergence claim:

Convergence claim: At any limit stage s and for any string σ, if $\sigma \in T_s$ then also there is a stage $t < s$ such that for all stages $t \leq r < s$ the string σ is in T_r. Similarly if $\sigma \notin T_s$ then also there is a stage $t < s$ such that for all stages $t \leq r < s$ the string σ is not in T_r.

We only argue the case where $\sigma \in T_s$, the other one being similar. Suppose $\sigma \in T_s$. Then also every prefix of σ is in T_s. By induction suppose the claim is true for every strict prefix of σ. In particular as there are finitely many of them, there is a stage $t_1 < s$ such that for every stage $t_1 \leq r < s$, every strict prefix of σ is in T_r. Also the approximation $\{\mathrm{hK}_s\}_{s<\omega_1^{ck}}$ is pointwise decreasing and always above 0, in particular it is a higher ω-computable approximation and it is harmless to suppose $\lim_{t<s} \mathrm{hK}_t = \mathrm{hK}_s$ for every limit stage s (see the notion of partial continuous approximation in Section 5.4). It implies that the truth value of the assertion '$\mathrm{hK}_r(\sigma) \leq \mathrm{hK}_r(|\sigma|) + d$' can change at most finitely often over stages $r < \omega_1^{ck}$. It follows that as $\mathrm{hK}_s(\sigma) \leq \mathrm{hK}_s(|\sigma|) + d$ there is a stage $t_2 < s$ such that for every stage $t_2 \leq r < s$, we have $\mathrm{hK}_r(\sigma) \leq \mathrm{hK}_r(|\sigma|) + d$. Then at every stage $\max(t_1, t_2) \leq r < s$ we have $\sigma \in T_r$ and the convergence claim is true for σ.

It follows that at limit stage s, the tree T_s is the 'pointwise limit' of the sequence $\{T_t\}_{t<s}$, each 'point' of the pointwise limit being the truth value of an assertion '$\sigma \in T_t$' for some σ. We now define T to be the pointwise limit tree of $\{T_s\}_{s<\omega_1^{ck}}$ and we have similarly that $T = \{\sigma : \forall \tau \leq \sigma\ \mathrm{hK}(\tau) \leq \mathrm{hK}(|\tau|) + d\}$. In particular we have $A \in [T]$ and as $[T]$ has only finitely many elements, there is a prefix σ of A such that A is the only elements of $[T] \cap [\sigma]$.

We now build a Π^1_1 function $g : \omega_1^{ck} \mapsto \omega_1^{ck}$ such that for any stage $g(s)$, there are at most $c \times 2^d$ sequences which looks hK-trivial at stage $g(s)$, where c is the constant of the previous theorem. Let $g(0) = 0$. At successor stage $s+1$ we define $g(s+1)$ the following way: Let $t_0 = s$ and for any n, let t_{n+1} be the smallest stage bigger than t_n such that for every $i \leq n+1$, there are at most $c \times 2^d$ strings of length i in $T_{t_{n+1}}$. We know that such a stage always exists by the convergence claim and because in T we actually have at most $c \times 2^d$ strings of length i for every $i \leq n+1$. By the Σ^1_1-boundedness principle we have $\sup_n t_n < \omega_1^{ck}$ and then we can let $g(s+1) = \sup_n t_n$. By the convergence claim, the sequence of trees $\{T_t\}_{t<g(s+1)}$ converges 'pointwise' to $T_{g(s+1)}$ which implies that at stage $g(s+1)$, for every n there are at most $c \times 2^d$ strings of length n in $T_{g(s+1)}$. Finally at limit stage s we define $g(s) = \sup_{t<s} g(t)$. Note that by the convergence claim, we then also have at most $c \times 2^d$ strings of length n in $T_{g(s)}$, for every n. We call g-stage a stage of the form $g(s)$ for some s. Note that the supremum of a set of g-stages is also a g-stage (or equal ω_1^{ck}).

Recall that σ is a prefix of A such that $[T] \cap [\sigma]$ contains only A. We now build a Π^1_1 function $h : \omega_1^{ck} \mapsto \omega_1^{ck}$ which looks for g-stages s for which $[T_s] \cap [\sigma]$ is not empty. Let $h(0) = g(0)$. At successor stage $s+1$ we define $h(s+1)$ the following way: Let $t_0 = h(s)$ and for every n, let t_{n+1} be the smallest g-stage bigger than t_n such that there is a string of length $n+1$ compatible with σ in $T_{t_{n+1}}$. Again we know that such a stage always exists by the convergence claim and because there exists such a string in T. Also by the Σ^1_1-boundedness principle we have $\sup_n t_n < \omega_1^{ck}$ and then we can let $h(s+1) = \sup_n t_n$. As $h(s+1)$ is a supremum of g-stages then also $h(s+1)$ is a g-stage. Also by the convergence claim, there are strings of longer and longer length extending σ in $T_{h(s+1)}$, which implies that $[T_{h(s+1)}] \cap [\sigma]$ is non-empty, by compactness. Finally at limit stage s we define $h(s) = \sup_{t<s} h(t)$. Here again by the convergence claim and by compactness, the set $[T_{h(s)}] \cap [\sigma]$ is non-empty. We call h-stages the stages of the form $h(s)$ for some s. It is clear that each h-stage is a g-stage and that the supremum of a set of h-stages is also a h-stage (or equal ω_1^{ck}).

We now build an approximation $\{A_s\}_{s<\omega_1^{ck}}$ of A only along h-stages: At stage s we simply let A_s be the leftmost path of $[T_{h(s)}]\cap[\sigma]$. As $[T]\cap[\sigma]$ contains only one path, and using the convergence claim, it is clear that $\lim_{s<\omega_1^{ck}} A_s$ converges to A. By design, it is also clear that for each s we have $\mathrm{hK}_s(A_s \upharpoonright_n) \leq \mathrm{hK}_s(n) + d$ for every n. We also claim that as long as A is not Δ^1_1, the approximation is self-unclosed. Suppose otherwise, that is, for some lengths $n_1 < n_2 < \dots$ and some stages $s_1 < s_2 < \dots$ such that $s = \sup_i h(s_i) < \omega_1^{ck}$, we have $A\upharpoonright_{n_i} \prec A_{h(s_i)}$ for each $i \in \mathbb{N}$. We have by the convergence claim that $A \in [T_s]$. Also as s is a g-stage, the Δ^1_1 tree T_s contains at most $c \times 2^d$ many paths and then A is Δ^1_1. ∎

Corollary 4.5.2 (Hjorth, Nies [30]):
If X is hK-trivial and X is not Δ^1_1, then $\omega_1^X > \omega_1^{ck}$.

Proof: By Theorem 4.4.1 a non Δ^1_1 sequence with a self-unclosed approximation does not preserve ω_1^{ck}. ∎

4.5.2 Low for hK and low for Π^1_1-Martin-Löf randomness

Definition 4.5.2. *A sequence X is continuously low for* hK *if for any X-continuous Π^1_1 prefix-free machine M we have a constant c_M such that* $\mathrm{hK}(\sigma) \leq \mathrm{hK}^X_M(\sigma) + c_M$. ∎

Definition 4.5.3. *A sequence X is continuously low for Π^1_1-Martin-Löf randomness if the A-continuous Π^1_1-Martin-Löf randoms coincide with the Π^1_1-Martin-Löf randoms.* ∎

Proposition 4.5.2:
If a sequence X is continuously low for hK*, then it is* hK*-trivial.*

Proof: Let U be a universal Π^1_1-prefix-free machine and let M be the Π^1_1 set of triples where we enumerate $\{\sigma, \tau, \sigma\}$ in M at stage s if $U(\tau) = |\sigma|$ at stage s. We have for every oracle X that $\{(\tau, \sigma) \; : \; \exists \rho \prec X \; (\rho, \tau, \sigma) \in M\}$ is a prefix-free set of strings such that for any $\sigma \prec X$ we have $\mathrm{hK}^X_M(\sigma) = \mathrm{hK}(n)$.

Now because X is low for hK we have $\mathrm{hK}(X\upharpoonright_n) \leq \mathrm{hK}^X_M(X\upharpoonright_n) + c_M = \mathrm{hK}(n) + c_M$ which makes X hK-trivial as well. ∎

We can deduce using the fact that hK-trivial sequences have a self-unclosed approximation, that also the continuously low for hK sequences have a self-unclosed approximation. We can then use this to prove that being continuously low for hK implies being continuously low for Π^1_1-Martin-Löf randomness. We shall prove later with Corollary 4.5.5 that if a sequence A is continuously low for Π^1_1-Martin-Löf randomness, then also it is continuously low for hK.

Corollary 4.5.3:
If a sequence A is continuously low for hK *then it is continuously low for Π^1_1-Martin-Löf randomness. Also if A is continuously low for* hK *there are:*

1. *A A-universal oracle-continuous Π^1_1-prefix-free machine.*
2. *A A-universal oracle-continuous Π^1_1-discrete semi-measure.*
3. *A A-universal oracle-continuous Π^1_1-continuous semi-measure.*
4. *A A-universal oracle-continuous Π^1_1-Martin-Löf test.*

Proof: If A is continuously low for hK it is hK-trivial and then by Proposition 4.5.1 it has a self-unclosed approximation. Also we shall prove in Section 7.4.1 that if a sequence A has a self-unclosed approximation, then any nullset $\mathcal{B}$ corresponding to a A-continuous Π^1_1-Martin-Löf test is included in a nullset corresponding to a A-continuous Π^1_1-prefix-free machine M, that is:

$$\mathcal{B} \subseteq \{X \ : \ \forall c \ \exists n \ \mathrm{hK}^A_M(X\upharpoonright_n) \leq n - c\}$$

But as A is low for hK we have

$$\{X \ : \ \forall c \ \exists n \ \mathrm{hK}^A_M(X\upharpoonright_n) \leq n - c\} = \{X \ : \ \forall c \ \exists n \ \mathrm{hK}(X\upharpoonright_n) \leq n - c\}$$

which is a Π^1_1-Martin-Löf test containing $\mathcal{B}$. Then A is continuously low for Π^1_1-Martin-Löf randomness.

We also prove in Section 7.4.1 that in case A has a self-unclosed approximation, there is a universal oracle continuous object for the four notions mentioned above. ∎

We also have the following corollary, proved by Nies and Hjorth in [30]. We say that a sequence X is $\Pi^1_1(A)$-Martin-Löf random if X is in no uniform intersection of $\Pi^1_1(A)$ open sets effectively of measure 0. We say that a sequence A is low for Π^1_1-Martin-Löf randomness with full relativization if $\Pi^1_1(A)$-Martin-Löf randomness coincides with Π^1_1-Martin-Löf randomness.

Corollary 4.5.4:
A sequence A is low for Π^1_1-Martin-Löf randomness, with full relativization, iff it is Δ^1_1.

Proof: Suppose A is not continuously low for Π^1_1-Martin-Löf randomness, then it is certainly not low for Π^1_1-Martin-Löf randomness using full relativization. Now if it is low for Π^1_1-Martin-Löf randomness it is then hK-trivial. If furthermore it is not Δ^1_1, by Corollary 4.5.2 we then have $\omega_1^A > \omega_1^{ck}$. Also then by Theorem 3.5.1 we have $A \geq_h \mathcal{O}$ and then Ω, the leftmost path of a Σ^1_1 closed set containing only Π^1_1-Martin-Löf randoms, is hyperarythmetically reducible to A, and then not $\Pi^1_1(A)$-Martin-Löf random. So A is not low for Π^1_1-Martin-Löf randomness, with full relativization. ∎

Our next theorem is the hard part of this section. Nies proved (with Hirschfeldt) in [68] that for classical randomness, the class of K-trivials coincides with the class of low-for-K. The theorem remains the same in the higher setting, however, as often, the proof needs some adaptation (in the higher setting). There are basically two differences : In the classical proof, we need several times to pick a number *bigger* than any number used so far in the algorithm. This cannot be done anymore here, and this should be replaced by picking a number *different* than any number picked so far in the algorithm. This number should however also be 'large enough'. This will be made precise in the proof. The second difference is that in the classical proof, we need to work at special stages, stages at which the current approximation of our sequence 'looks' K-trivial for longer and longer prefixes. Here we can actually have stages for which the sequence looks K-trivial everywhere.

Theorem 4.5.3:
If a sequence A is hK*-trivial, then it is continuously low for* hK.

The rest of this section is dedicated to the proof of Theorem 4.5.3. Let M_u be a universal Π^1_1-discrete semi-measure. Suppose that A is hK-trivial, namely there is some c such that $M_u(A \upharpoonright_n) > M_u(n) \times 2^{-c}$ for any n. We will prove that A is then low for hK. Recall that by Corollary 4.5.3, since A is hK-trivial, there exists a A-universal oracle-continuous Π^1_1-discrete semi-measure M^A. Then to prove that A is low for hK, we should prove that there exists a Π^1_1-discrete semi-measure $\widetilde{M}$ such that $\widetilde{M} >^* M^A$.

The general idea

The universal Π^1_1-discrete semi-measure M_u is given by a Π^1_1 subset of $2^{<\mathbb{N}} \times \mathbb{Q}$ and the A-universal oracle-continuous Π^1_1-discrete semi-measure M^A is given by a Π^1_1 set $M \subseteq 2^{<\mathbb{N}} \times 2^{<\mathbb{N}} \times \mathbb{Q}$ with:

$$M^A(x) = \sup\{q \mid \exists \sigma \leq A \ s.t.\ (\sigma, x, q) \in M\}$$

Note that we have $\sum_x M^X(x) \leq 1$ for any oracle X.

As A is hK-trivial, we know it has a self-unclosed approximation. Also the principle is to try to match the A-continuous Π^1_1-discrete semi measure with an oracle-free Π^1_1-discrete semi-measure, using the approximation of A. First we wait to see the oracle-continuous Π^1_1-discrete semi-measure put weight on some integer e using the first m bits of some approximation A_s of A. But first we don't trust that the m first bits of A_s are the final approximation and then we don't want our measure $\widetilde{M}$ to match it yet. First this approximation has to prove itself 'solid' over time. This is done with the help of another Π^1_1-discrete semi-measure denoted by M_d which will put some weight q on some $n > m$ and wait for the universal Π^1_1-discrete semi-measure M_u to match the weight q on some $A_t \upharpoonright_n$ for $t \geq s$. Because A is hK-trivial, as long as M_d is truly a Π^1_1-discrete semi-measure, we know that this will necessarily happen, at least when we have the correct approximation of A (there are some details to handle with the hK-triviality constant, this is done accurately later). If during this process the first m bits of A_s don't change, then those bits have proven themselves solid and we can let $\widetilde{M}$ match M on the integer e. Otherwise we wait for the next approximation of A to be used to put some weight on e. Of course, even when $A_s \upharpoonright_m$ has proven itself worthy it does not mean that these m bits won't ever change. But we will show that they cannot change too many times without making $\sum_n M_u(n)$ bigger than 1.

We borrow the word 'process' from computer engineering, which fits well to the present situation: A process is an instance of a computer program that is being executed. Here our process will be 'running' Π^1_1 predicates. In the construction we will have infinitely many of them, all trying to match M^A with their own Π^1_1-discrete semi-measure $\widetilde{M}$, however all those processes will be involved in the enumeration of the same shared Π^1_1-discrete semi-measure M_d, whose role has been discussed above. Also, they will all be linked in an infinitely branching tree (but of finite depth) of recursive calls of processes, with a unique father process. Let d denote a code for the Π^1_1 description of the set $M_d \subseteq 2^{<\mathbb{N}} \times \mathbb{Q}$. We have a hK-triviality constant c such that $M_u(A\upharpoonright_n) > M_d(n) \times 2^{-d} \times 2^{-c}$ for every n. Let k be an integer such that $k \times 2^{-d} \times 2^{-c} > 2$. By the fixed point theorem, we can suppose that the Π^1_1 description of the set $M_d \subseteq 2^{<\mathbb{N}} \times \mathbb{Q}$ can 'access' its own code and then can also 'access' an integer k such that $k \times 2^{-d} \times 2^{-c} > 2$. The integer k will be a parameter of the Π^1_1 predicate M_d.

The notion of i-sets

We introduce the notion of i-set. We say $E \subseteq \mathbb{N}\times\mathbb{Q}$ is an i-set at stage s if for all $(n,\eta) \in E$ we have that (n,η) has been enumerated in M_d at some stage $t \leq s$ and if we have i distinct approximations $A_r\upharpoonright_n$ of the n first bits of A with $t \leq r \leq s$ which are such that $M_{u,r}(A_r\upharpoonright_n) > \eta\times 2^{-d}\times 2^{-c}$. In addition of that, we require that $(n_1,\eta_1) \in E$ and $(n_2,\eta_2) \in E$ implies $n_1 \neq n_2$. We call **weight** of an i-set the value $\sum_{(n,\eta)\in E} \eta$, also denoted by $\mathrm{wg}(E)$.

Each of our processes will create some i-sets for some $i \leq k$ (where k is the integer described above, such that $k \times 2^{-d} \times 2^{-c} \geq 2$). The starting point of the future golden run argument lies in the following lemma, about the weight of a k-set:

Lemma 4.5.1 *If E is a k-set, then* $\mathrm{wg}(E) < \frac{1}{2}$.

PROOF: For any $(n,\eta) \in E$ there are then k distinct strings σ of length n such that $M_u(\sigma) > \eta \times 2^{-d} \times 2^{-c}$, but then $\sum_\sigma M_u(\sigma) \geq k \times 2^{-d} \times 2^{-c} \times \frac{1}{2} > 1$, which contradicts that M_u is a discrete semi-measure. ∎

To achieve the creation of i-sets, we now give the informal description of k distinct processes $P_1,\dots,P_k$. Each process P_i has three parameters. The first one p, a rational smaller than 1, is a **goal** to reach, that is, the process P_i will try to enumerate a i-set F_i of weight p. The second one, a rational δ smaller than p, corresponds to 'how fast' we try to make $\mathrm{wg}(F_i)$ reach its goal p, that is, we will enumerate in F_i some pairs (n,δ) for some integers n. For this reason also, p should be a multiple of δ. We briefly explain the idea of the parameter δ, and the reason it has to be chosen carefully. By definition of an i-set, in order to enumerate (n,δ) into F_i, we first need to enumerate (n,δ) into M_d. One problem is that for technical reasons explained later, the weight δ put on n in M_d might be 'lost'. In order not to lose too much weight, the parameter δ has to be small enough, so that the procedure P_i fills F_i step by step, with a potential 'loss' of at most δ. For this reason, the parameter δ is also called the **garbage** parameter. Finally the last parameter of each P_i corresponds to some length that will have to increase at each recursive call. This mechanism will clarified itself later.

The tree of processes

The algorithm will consist of a dynamic tree of processes, where each node corresponds to a process with the relation that P is a child of Q if the process P has been called by the process Q. Each node stays in the tree as long as its corresponding process is being executed, and disappears (with all its children) when the process ends, or is 'killed', as we will see in details later.

Recall that k is chosen so that $k \times 2^{-d} \times 2^{-c} \geq 2$. The algorithm starts by executing the process P_k, corresponding to the root of our tree of processes. Then P_k will call infinitely many 'child processes' P_{k-1}, and so on, inductively for any node of the tree of depth $i < k$ that corresponds to a process P_{k-i}. Also, nodes of the tree of depth k, corresponding to processes P_1 will remain leaves of the tree of processes.

Each process P_i will be in charge of enumerating an i-set with the help of every process it has called. So each process P_1 starts the enumeration of a 1-set F_1. When the weight of F_1 reaches the required goal, the process P_1 stops, and the 1-set enumerated by P_1 is then intended to be 'emptied' into the 2-set enumerated by the father of P_1. After P_1 has stopped, it is the father of P_1 which decides when it is time to transfer the 1-set enumerated by P_1, into its own 2-set, because it is the father of P_1 which knows at what stage elements of the 1-set of P_1 can now be 'upgraded' into elements of a 2-set (We will see that elements of the 1-set can be upgraded into a 2-set when the approximation of some prefix of A has changed). The same happens then for any process P_i and its father. This back and forth recursion is pursued, with as final target, the enumeration of the k-set F_k enumerated by the one process P_k.

The dispatcher process and the notation s^*

In the algorithm, we will need to use several times 'an integer that has not been used yet'. In classical computability we can always pick an integer bigger than any other integer used so far. In higher computability this can be achieved with the use of the projectum function $p : \omega_1^{ck} \to \omega$.

However, one difficulty is that we have infinitely many processes that will be run in a dynamic tree of processes, where some process will be killed and some other will be born along the 'ordinal times of computation'.

Also, each process P will be described independently as an algorithm running through all the computable ordinal stages, but it should be clear that each of them is meant to be ruled by a unique 'dispatcher process', which allocates some ordinal computational time to each process of our recursive tree, in such a way that all of them are 'executed until the end' (meaning we assign them some ordinal time of computation cofinally below ω_1^{ck}), and also such that one action of one process corresponds to one stage.

So inside a process P, the notation s^* refers to the current global stage, rather than the current stage s of P, and s^* will then be used with the projectum function, in order to get 'a number that has no been used so far'.

A Π^1_1 description of the unique 'dispatcher process' would be very tedious to give, and we hope that the paragraph of Section 3.6.2 on the formal way to handle ordinal substages will convince the reader that this Π^1_1 description can be derived from the description we will give of the recursive call of each process of the tree of processes.

The golden run

The heart of the proof consists in a clever idea to which Andre Nies gave the elegant James Bond-ish name:

The golden run[1].

Each procedure P_i will try to enumerate the description of a discrete semi-measure $\widetilde{M}$, such that $\widetilde{M} >^* M^A$. We saw that the purpose of building k-sets is that P_k cannot reach its goal of $\frac{3}{4}$. A first idea is to try to show that if P_k does not reach its goal, this implies $\widetilde{M} >^* M^A$. However it is possible that P_k does not reach its goal simply because some of its children P_{k-1} do not return, which prevent us from saying anything, because then a change in the approximation of A can never prove itself worthy. But as we will see, there must exist a node of tree of processes, called the golden run, which does not reach its goal but which is such that all the its children processes return at some point (or are canceled as we will see in the details).

The special stages of computation

Before giving a description of a standard process P, we need to introduce a last notion, which actually presents a simplification compared to the same proof in the lower setting. We need to identify some stages s such that A looks hK-trivial at stage s, that is, such that for all n we have $M_{u,s}(A_s \upharpoonright_n) > M_{u,s}(n) \times 2^{-c}$. Such a thing could not be achieved for every n in the lower setting, leading to some complications. However, in the higher setting, we can prove:

Lemma 4.5.2 *Cofinally below ω_1^{ck}, there are some stages $s < \omega_1^{ck}$ such that for every n we have $M_{u,s}(A_s \upharpoonright_n) > M_{u,s}(n) \times 2^{-c}$. Furthermore, we can effectively enumerate those stages in order, along the computable ordinals.*

Proof: It follows directly from Proposition 4.5.1, which says that if A is hK-trivial with constant d, then it has a self-unclosed approximation $\{A_s\}_{s<\omega_1^{ck}}$ such that at every stage s we have $\mathrm{hK}_s(A_s \upharpoonright_n) \leq \mathrm{hK}_s(n) + d$ for every n. We can always consider that our universal semi-measure M_u is equal to $2^{-\mathrm{hK}}$ (see Corollary 3.7.2). ∎

We will call *special stages*, the stages of this form. Also we should now consider that any stage of the whole algorithm is actually a *special stage*, that is at stages for which A does not look hK-trivial we do nothing, and we will do something only at stages for which A looks hK-trivial. As we will work only with those *special stages*, we will simply call them *stages*. Why do we do this? Once some (n, η) has been enumerated into M_d, instead of waiting for $M_u(A \upharpoonright_n)$ to be bigger than $\eta \times 2^{-c} \times 2^{-d}$, we will wait for $M_u(n)$ to be bigger than $\eta \times 2^{-d}$ on some stage s. Then, because s is a *stage*, we know that also we have $M_{u,s}(A_s \upharpoonright_n) > \eta \times 2^{-c} \times 2^{-d}$, but also for any *stage* $t > s$ we have $M_{u,t}(A_t \upharpoonright_n) > M_{u,t}(n) > M_{u,s}(n) > \eta \times 2^{-c} \times 2^{-d}$. Also once we have $M_{u,s}(A_s \upharpoonright_n) > \eta \times 2^{-c} \times 2^{-d}$, it then remains true even after $A_s \upharpoonright_n$ changes; this will be useful to upgrade a i-set into a $i+1$-set.

[1] Andre Nies assured the author of this thesis that this name has nothing to do with the famous fictional British secret agent; the underlying idea behind this appellation thus remains a mystery...

Description of the algorithm

Recall that we have an A-universal oracle-continuous Π^1_1-discrete semi-measure given by a Π^1_1-predicate $M \subseteq 2^{<\mathbb{N}} \times \mathbb{N} \times \mathbb{Q}$. Each process of the tree tries to match it with its own local Π^1_1-discrete semi-measure $\widetilde{M}$. Similarly, every set F_i is local to the current process. On the contrary, recall that the description of M_d is shared among all of them (the leaves of the tree of processes being the only nodes that enumerate things in M_d). We start by executing the process $P_k(\frac{3}{4}, \frac{1}{4} \times 2^{-p(0)}, 0)$ described below, and we let the tree of processes build itself inductively.

Process $P_{i,e,q}$ for $1 < i \leq k$

Input : The goal p, The garbage δ and some length w
for all *stage* s **do**
  for substage (e,q) with $e,q \in \mathbb{N} \times \mathbb{Q}$ **do**
    if (e,q) is marked available **then**
      if $\exists\ \sigma \leq A_s$ s.t. (σ, e, q) is enumerated at some stage below s in M **then**
        Set $\sigma_{e,q}$ to be the smallest prefix of A_s of length strictly bigger than w, such that this is so. If needed, we extend the length of the largest such prefix by taking as many bits as we need in A_s to make it strictly longer than w.

        Set $w_{e,q}$ to be the length of $\sigma_{e,q}$ (i.e. the use of A_s such that $M^{A_s}(e) \geq q$).

        Let δ' be the biggest rational smaller than $\frac{1}{4} \times 2^{-p(s^*)}$ such that $q \times \delta$ is a multiple of δ' and call the process $P_{i-1,e,q}(q \times \delta, \delta', w_{e,q})$.

        Mark (e,q) as unavailable.
      end
    end

    if (e,q) is marked unavailable after the call of some process $P_{i-1,e,q}$ **then**
      if the process $P_{i-1,e,q}$ has returned **then**
        enumerate $(e, q \times \frac{\delta}{p+\delta})$ into $\widetilde{M}$.
      end

      if $A_s \upharpoonright_{w_{e,q}}$ is different from $\sigma_{e,q}$ **then**
        Mark (e,q) as available.

        If the process $P_{i-1,e,q}$ has not returned, cancel it and recursively cancel all its sub-processes.

        Put $F_{i-1,e,q}$ into F_i.

        if $\text{wg}(F_i) \geq p$ **then**
          Stop the current process and recursively stop all its sub-processes.
        end
      end
    end
  end
end

Procedure $P_{1,e,q}$

Input : The goal p, The garbage δ and a length w

for all *stage* s **do**

  Let m be the smallest number such that $p(\omega \times s^* + m) > w$ and let $n = p(\omega \times s^* + m)$.

  Enumerate (n, δ) into M_d.

  Wait for some *stage* t such that $M_{u,t}(n) > \delta \times 2^{-d}$.

  Put (n, δ) into F_1.

  if $\text{wg}(F_1) \geq p$ **then**
    Stop the process.
  end
end

We now turn to the formal verification:

Let us prove that each F_i is a i-set:

Let us prove that each F_1 is a 1-set at each *stage*. We should first argue that we never have (n, η_1) and (n, η_2) into some F_1 for $\eta_1 \neq \eta_2$. If P_1 enumerates (n_1, η_1) in M_d and later enumerates (n_2, η_2) in M_d, it does it at two distinct *stages* $s_1^* \neq s_2^*$. Also then $n_1 = p(\omega \times s_1^* + m_1)$ and $n_2 = p(\omega \times s_2^* + m_2)$ for some m_1, m_2, which implies that $n_1 \neq n_2$. Note that this argument holds even for two distinct 1-sets.

Now suppose that (n, η) enters into F_1 at some *stage* s. Then by construction, at some *stage* $t < s$ we have put (n, η) into M_d, and that at stage s we have $M_{u,t}(n) > \eta \times 2^{-d}$. But because t, s are *stages* we have that $M_{u,s}(A_s \upharpoonright_n) > \eta \times 2^{-d} \times 2^{-c}$.

Suppose now that each F_i associated to a process P_i corresponding to a node of depth $k - (i)$ of the tree of processes, is a i-set at each *stage*. Suppose that at *stage* s we have (n, η) which enters into some F_{i+1}. First, by the above argument, we cannot already have (n, η') in F_{i+1} at stage s for $\eta \neq \eta'$. Also by construction we have:

- There is a *stage* $t < s$ and an integer $m < n$ such that $A_t \upharpoonright_m \neq A_s \upharpoonright_m$.
- At some *stage* r with $t < r < s$ the couple (n, η) goes in some 1-set F_1.
- At some *stage* between r and s the couple (n, η) goes in some i-set F_i.

We have that $M_{u,s}(A_s \upharpoonright_n) \geq \eta \times 2^{-d} \times 2^{-c}$ because s is a *stage* and because $s > r$. So $A_s \upharpoonright_n$ is a good candidate to make n the element of the i+1-set F_{i+1}, but we need to verify that $A_s \upharpoonright_n$ is different from all the other strings $\sigma_1, \dots, \sigma_{i-1}$ of length n that makes (n, η) the element of F_i. Indeed, it is possible that prefixes in the approximation of A come back to previous values. It is enough to prove that A_t shares the same first m bits with each σ_j for $1 \leq j \leq i$, because as we have $A_t \upharpoonright_m \neq A_s \upharpoonright_m$, then $A_s \upharpoonright_n$ is different from each σ_j for $1 \leq j \leq i$. So suppose this is not the case. By definition of a i-set, and by construction, we then have a *stage* r' with $t < r < r' < s$ such that $M_u(A_{r'} \upharpoonright_n) \geq \eta \times 2^{-d} \times 2^{-c}$ with also $A_{r'} \upharpoonright_m \neq A_t \upharpoonright_m$. Here $A_{r'} \upharpoonright_n$ corresponds to one of the strings σ_j. But then if $A_{r'} \upharpoonright_m \neq A_t \upharpoonright_m$, we have by construction that the process P_i is canceled by its father at *stage* r', before (n, η) could enter F_i, which is a contradiction. In order for this argument to be valid, we need to assume that every process is being taken care of, with a priority corresponding to its depth in the tree of processes (nodes of lower depth have higher priority), which is a harmless hypothesis.

Let us prove that M_d is a discrete semi-measure:

The total weight which is put in M_d by some process $P_1(p, \delta)$ creating a 1-set F_1 is bounded by $\mathrm{wg}(F_1) + \delta$. Indeed, by construction, for every (n, δ) that P_1 enumerates in M_d, also (n, δ) is enumerated into F_1, except maybe for one (n, δ), in case the procedure is canceled before it finds a stage t such that $M_{u,t}(n) > \delta \times 2^{-d}$.

In addition to that, note that all the 1-sets enumerated in the whole algorithm are pairwise disjoint along theirs first components, because if in one instance of P_1 we enumerate (n_1, δ_1) in M_d and in another instance of P_1 we enumerate (n_2, δ_2) in M_d (possibly with $\delta_1 = \delta_2$), we do so at stages $s_1^* \neq s_2^*$. Also then $n_1 = p(\omega \times s_1^* + m_1)$ and $n_2 = p(\omega \times s_2^* + m_2)$ which implies that $n_1 \neq n_2$.

Now let $P_i(p,\delta)$ be a process enumerating an i-set F_i. Let C_{i-1} be the (disjoint) union of all the i-1-sets F_{i-1} that are enumerated by a process P_{i-1} which is called by P_i at some point. We denote by $C_{i-1} - F_i$ the elements which are in C_{i-1} at some point, but never enter F_i. Let us prove that $\mathrm{wg}(C_{i-1} - F_i) \leq \delta$. Note first that we have $\mathrm{wg}(C_{i-1} - F_i) \leq \sup_{s<\omega_1^{ck}} \mathrm{wg}(C_{i-1,s} - F_{i,s})$. It would then be enough to prove that at any *stage* s we have $\mathrm{wg}(C_{i-1,s} - F_{i,s}) \leq \delta \times \mathrm{wg}(M^{A_s})$, and this is what we now prove.

First, by construction, for any *stage* s, any i-1 set $F_{i-1,s}$ enumerated by a child of P_i, corresponds to an enumeration of some $(A_t\upharpoonright_m, e, q)$ in M_t for some m, e, q and a *stage* $t \leq s$. Now suppose that $A_s\upharpoonright_m \neq A_t\upharpoonright_m$, then by construction we have that everything which is in the i-1-set corresponding to the enumeration of $(A_t\upharpoonright_m, e, q)$ in M_t will be put into F_i. Therefore the weight of $(C_{i-1,s} - F_{i,s})$ is bounded by the sum of the weight of the i-1-sets corresponding to enumerations $(A_s\upharpoonright_m, e, q)$ in M_s, for some m, e, q. Also for each of those enumerations, the corresponding i-1-set has its weight bounded by $\delta \times q$ because by construction we fill it with quantities of the form $(\delta \times q)/m$ for some integer m until it reaches the value $\delta \times q$. But then we have $\mathrm{wg}(C_{i-1,s} - F_{i,s}) \leq \delta\, \mathrm{wg}(M^{A_s})$, which implies that $\mathrm{wg}(C_{i-1} - F_i) \leq \delta$.

Now combining this with the fact that all 1-sets are pairwise disjoint, and therefore also all i-sets for any i, it follows that the total weight which is put in M_d is bounded by the weight of the unique k-set F_k plus the sum of all the garbage parameters $\frac{1}{4} \times 2^{-p(s^*)}$. Also as the goal of P_k is of $1/2$ and as the goal of a process is never exceeded (or by at most the garbage parameter), we have $\mathrm{wg}(F_k) < \frac{3}{4}$; and by definition of s^* we have that the sum of all garbage parameters $\frac{1}{4} \times 2^{-p(s^*)}$ is smaller than $\frac{1}{4}$. It follows that $\mathrm{wg}(M_d) \leq 1$ and M_d is a discrete semi-measure.

End of the proof : The Golden run

We should now prove that there is one process $P_i(p,\delta,w)$ for $1 < i \leq k$ such that P_i is never canceled, never reaches its goal p (and then never returns) and such that every process it calls returns, unless canceled. Such a process is called **the golden run**.

We know that P_k cannot be canceled by any other process, as it is the root of the tree, and we also know that it never reaches its goal p, by Lemma 4.5.1. Also if any process P_{k-1} that it calls returns unless canceled, then P_k is the golden run. Otherwise, at least one process P_{k-1} called by P_k is never canceled and never returns. In particular P_{k-1} never reaches its goal. We can then continue the induction starting from P_{k-1} : Either it is the golden run, or it calls a process P_{k-2} which is never canceled and never reaches its goal. Also either the induction will stop when we find a golden run P_i for $2 < i \leq k$, or it will reach a process P_2, which is never canceled and never reaches its goal. But then such a process P_2 is necessarily the golden run, because by construction, any process P_1 reaches its goal, unless canceled.

So let $P_i(p,\delta,w)$ be the golden run. We should now prove that the predicate $\widetilde{M}$ which is enumerated by the golden run is a discrete semi-measure such that $\widetilde{M} > M^A \times \frac{\delta}{p+\delta}$.

Let us first prove that $\mathrm{wg}(\widetilde{M}) \leq 1$. When $(e, q \times \frac{\delta}{p+\delta})$ enters $\widetilde{M}$, it is because some (σ, e, q) is enumerated in M. Also let C_1 be the set containing all pairs (e,q) such that $(e, q \times \frac{\delta}{p+\delta})$ is enumerated in $\widetilde{M}$ because of some (σ, e, q) is enumerated in M for $\sigma \nprec A$, and let C_2 be the set containing all pairs (e,q) such that such that $(e, q \times \frac{\delta}{p+1})$ is enumerated in $\widetilde{M}$ because of some (σ, e, q) enumerated in M for $\sigma \prec A$. We have that $\mathrm{wg}(\widetilde{M}) \leq$

$\frac{\delta}{p+\delta}(\mathrm{wg}(C_1)+\mathrm{wg}(C_2))$.

We have $\mathrm{wg}(C_1) \leq \frac{p}{\delta}$ because if (e,q) is in C_1, the corresponding called process has stopped and reached its goal of $q \times \delta$, and the corresponding i-1-set will then be put in the i-set of the golden run. Also, as we are in the golden run, this i-set never reaches its goal of p, implying $\mathrm{wg}(C_1) \leq \frac{p}{\delta}$. Also we clearly have $\mathrm{wg}(C_2) \leq \mathrm{wg}(M^A) \leq 1$. Then $\mathrm{wg}(C_1)+\mathrm{wg}(C_2) \leq \frac{p+\delta}{\delta}$ and then $\mathrm{wg}(\widetilde{M}) \leq 1$ and $\widetilde{M}$ is a discrete semi-measure.

Let us now prove that $\widetilde{M} \geq M^A \times \frac{\delta}{p+\delta}$. If (σ,e,q) is enumerated in M for $\sigma \prec A$, then the corresponding called process will never be canceled, as $A\upharpoonright_{|\sigma|} = \sigma$. Also because we are in the golden run it will return, and therefore we will enumerate $(e, q \times \frac{\delta}{p+\delta})$ in $\widetilde{M}$. Then $\widetilde{M} \geq M^A \times \frac{\delta}{p+\delta}$, which concludes the proof.

4.5.3 Base for randomness

We now deal with another notion, that is equivalent to K-triviality in the lower setting, and whose higher analogue turns out be equivalent to hK-triviality in the higher setting.

Definition 4.5.4. *The sequence A is a* ***base for continuous Π^1_1-Martin-Löf randomness*** *if there is some A-continuous Π^1_1-Martin-Löf random sequence Z such that $Z \geq_{\mathrm{hT}} A$.* ■

We can first observe that any sequence which is continuously low for hK is also a base for continuous Π^1_1-Martin-Löf randomness.

Proposition 4.5.3:
If A is continuously low for hK*, then A is a base for continuous Π^1_1-Martin-Löf randomness.*

Proof: If is pretty clear, as being continuously low for hK is the same as being continuously low for Π^1_1-Martin-Löf randomness, and as by Theorem 4.1.2, for any sequence A, there is a Π^1_1-Martin-Löf random Z such that $Z \geq_{\mathrm{hT}} A$. Also as A is continuously low for Π^1_1-Martin-Löf randomness, then also Z is A-continuously Π^1_1-Martin-Löf random. ■

Hirschfeldt, Nies and Stephan proved in [29] that the two notions actually coincide in the lower setting. The result can be transfered in the higher setting, but again, the proof needs to be modified due to the usual topological issues of higher computability.

Theorem 4.5.4:
If A is a base for continuous Π^1_1-Martin-Löf randomness, then A is continuously low for hK*.*

Before proving the theorem, we deduce the following corollary:

Corollary 4.5.5:
If a sequence A is continuously low for Π^1_1-Martin-Löf randomness, then also it is continuously low for hK.

Proof: Suppose A is continuously low for Π^1_1-Martin-Löf randomness. By the higher Kučera-Gács theorem (Theorem 4.1.2), there is a Π^1_1-Martin-Löf random sequence Z which higher Turing computes A. But Z is also A-continuously Π^1_1-Martin-Löf random, making A a continuous base for Π^1_1-Martin-Löf randomness. Therefore A is low for hK. ■

The rest of the section is dedicated to the proof of Theorem 4.5.4. Suppose that Z is A-continuously Π^1_1-Martin-Löf random and suppose that $\Phi(Z) = A$ for some higher Turing functional Φ. We can assume that if (τ, σ) is in Φ then Φ also contains (τ, σ') for each $\sigma' \leq \sigma$. Let M be any higher A-continuous Π^1_1-discrete semi-measure. We have that $\sum_x \sup\{q \mid \exists \sigma \leq A \quad M(\sigma, x, q)\} \leq 1$, but M needs not describe a semi-measure on other oracles. We can assume without loss of generality that the q's in M are only powers of 2. We also can assume that each triple (τ, σ, q) is enumerated ω_1^{ck}-cofinally many times in M. For each integer d we will describe an algorithm having d as a parameter. Each instance of the algorithm will enumerate some Π^1_1 set of strings $C_{\sigma,x,q}$ for each triple $(\sigma, x, q) \in 2^{<\mathbb{N}} \times \mathbb{N} \times \mathbb{Q}$ (so called 'hungry sets' by Hirschfeldt, Nies and Stephan) and will enumerate a Π^1_1-discrete semi-measure described by a predicate $N \subseteq \mathbb{N} \times \mathbb{Q}$.

The algorithm for a parameter d

Before giving the algorithm, let us first fix for each triple (σ, x, q) a rational $\delta_{\sigma,x,q}$ such that $\sum_{\sigma,x,q} \delta_{\sigma,x,q} \leq 1$. Recall also that $p : \omega_1^{ck} \to \omega$ is the projectum function.

At the beginning of the algorithm, for each triple (σ, x, q) we set $C^0_{\sigma,x,q} = \varnothing$. Then at successor stage s of the algorithm, let (σ, x, q) be the new triple enumerated in M_s. Look at all pairs (τ, σ) enumerated in Φ at stage $t < s$ until two conditions are met: First the string τ should not be marked as used (as defined below). Then we must have $\lambda([C^s_{\sigma,x,q}]^{\prec}) + 2^{-|\tau|} \leq 2^{-d} q$. If no such pair (τ, σ) is found then we go to the next stage.

Otherwise we want to add τ to $C^s_{\sigma,x,q}$. But we also want to keep all the open sets described by each $C^s_{\sigma,x,q}$ pairwise disjoint. Since it is not be always possible, we keep them 'mostly disjoint'. Let U^s be the set of all the strings in any of the $C^s_{\sigma,x,q}$ which are compatible with τ. It is possible that $[\tau] - [U^s]^{\prec}$ is not an open set. To remedy this, just like in the proof of Lemma 3.7.1, let B^s be a finite set of strings such that $[B^s]^{\prec} \cup [U^s]^{\prec} = [\tau]$ and such that $\lambda([B^s]^{\prec} \cap [U^s]^{\prec}) \leq 2^{-p(s)} \delta_{\sigma,x,q}$. Note that it is Δ^1_1 uniformly in s to find such a set B^s. Then we mark τ and all strings extending τ as 'used' and we set $C^{s+1}_{\sigma,x,q} = C^s_{\sigma,x,q} \cup B^s$. Then (and this is important), if $\lambda([C^{s+1}_{\sigma,x,q}]^{\prec}) > 2^{-d-1} q$ we enumerate the pair $(x, q \times 2^{-d-1})$ into N.

Finally, at limit stage s we set each $C^s_{\sigma,x,q}$ to be $\bigcup_{t<s} C^t_{\sigma,x,q}$.

Verification : Semi-measure

We have to prove that for each d, the predicate N created by the instance of the algorithm with parameter d, describes a discrete semi-measure. In other words we have to prove that $\mathrm{wg}(N) = \sum_x \sup\{q \mid N(x,q)\} \leq 1$. Also it is clear that we have $\mathrm{wg}(N) \leq \frac{1}{2}\sum_{\sigma,x,q}\lambda([C_{\sigma,x,q}]^{\prec})$ because each $[C_{\sigma,x,q}]^{\prec}$ has measure at most $2^{-d} \times q$, and for each of them we enumerate at most once some $(e, 2^{-d-1} \times q)$ into N. So it is enough to prove that $\sum_{\sigma,x,q}\lambda([C_{\sigma,x,q}]^{\prec}) \leq 2$. Let

$$E = \bigcup_{(\sigma',x',q')\neq(\sigma,x,q)} ([C_{\sigma,x,q}]^{\prec} \cap [C_{\sigma',x',q'}]^{\prec})$$

and let $E_{\sigma,x,q}$ be the set of strings which enter $C_{\sigma,x,q}$ *after* it has enter some $C_{\sigma',x',q'}$ for $(\sigma',x',q') \neq (\sigma,x,q)$. Let $E'_{\sigma,x,q}$ be the set of strings in E which enter $C_{\sigma,x,q}$ before it enters any other $C_{\sigma',x',q'}$ for $(\sigma',x',q') \neq (\sigma,x,q)$. We have:

$$\sum_{\sigma,x,q}\lambda([C_{\sigma,x,q}]^{\prec}) \leq \sum_{\sigma,x,q}\lambda([C_{\sigma,x,q}]^{\prec} - E) + \sum_{(\sigma,x,q)}\lambda(E'_{\sigma,x,q}) + \sum_{(\sigma,x,q)}\lambda(E_{\sigma,x,q})$$

Clearly $\sum_{\sigma,x,q}\lambda([C_{\sigma,x,q}]^{\prec} - E) + \sum_{(\sigma,x,q)}\lambda(E'_{\sigma,x,q}) \leq 1$ because all the sets involved are pairwise disjoint, by the definition of E and $E'_{\sigma,x,q}$. Let us prove that $\sum_{(\sigma,x,q)}\lambda(E_{\sigma,x,q}) \leq 1$. We have:

$$\begin{aligned}
\sum_{(\sigma,x,q)}\lambda(E_{\sigma,x,q}) &\leq \sum_{(\sigma,x,q)}\sum_{s<\omega_1^{ck}}\lambda([B^s]^{\prec} \cap [U^s]^{\prec}) \\
&\leq \sum_{(\sigma,x,q)}\sum_{s<\omega_1^{ck}} 2^{-p(s)} \times \delta_{\sigma,x,q} \\
&\leq 1
\end{aligned}$$

Therefore N describes a discrete semi-measure.

Verification : Martin-Löf test

Let $C^d_{\sigma,x,q}$ be the set of strings $C_{\sigma,x,q}$ created by an instance of the algorithm with d as parameter. Let $C^A_d = \bigcup C^d_{\sigma\leq A,x,q}$. By construction we have that $\lambda([C^A_d]^{\prec}) \leq \sum_{\sigma\leq A,x,q}\lambda([C_{\sigma,x,q}]^{\prec}) \leq \sum_{x,q}\{q \mid M^A(x) > q\} \times 2^{-d}$. As M^A is an A-continuous Π^1_1-discrete semi-measure and as the q's are only powers of 2 we have that $\sum_{x,q}\{q \mid M^A(x) > q\} \leq 2$ and then $\lambda([C^A_d]^{\prec}) \leq 2^{-d+1}$. Then $\bigcap_d [C^A_d]^{\prec}$ is a A-continuous Π^1_1-Martin-Löf test. This implies by hypothesis that there is some d such that $Z \notin [C^A_d]^{\prec}$.

Verification : Low for hK

We now only consider the algorithm with d as a parameter where $Z \notin [C^A_d]^{\prec}$. We pretend that if (σ,x,q) is enumerated in M for $\sigma \leq A$ then $(x, q\times 2^{-d-1})$ will be enumerated in N. Suppose not, then it means that $\lambda([C_{\sigma,x,q}]^{\prec}) \leq 2^{-d-1} \times q$. Let $\tau \leq Z$ be large enough so that $\lambda([C_{\sigma,x,q}]^{\prec}) + 2^{-\tau} < 2^{-d} \times q$. There exists s such that (σ,x,q) is enumerated in M at stage s and such that for some $t \leq s$ we have (τ,σ) which is enumerated in Φ at stage t (or (τ',σ) for τ' extending τ). At this stage, if τ was marked as used it means that some prefix of τ was already enumerated in another $C^s_{\sigma',x,q}$ for $\sigma' \prec A$, and so that Z is in $[C^A_d]^{\prec}$ which is a contradiction. If τ was not marked as used then some B^s has been created such that $\tau = [B^s]^{\prec} \cup [U^s]^{\prec}$. If a prefix of Z is in B^s then Z is in $[C^{s+1}_{\sigma,x,q}]^{\prec}$ otherwise Z was already in some $[C^s_{\sigma,x,q}]^{\prec}$. In either case it is a contradiction. Therefore $(x, q\times 2^{-d-1})$ will be enumerated in N, and we have $N \geq M^A \times 2^{-d-1}$. As this can be achieved for any M we have that A is continuously low for hK.

Chapter 5

Further studies on higher randomness

Est-il possible de raisonner sur des objets qui ne peuvent pas être définis en un nombre fini de mots ? Est-il possible même d'en parler en sachant de quoi l'on parle, et en prononçant autre chose que des paroles vides ? Ou au contraire doit-on les regarder comme impensables ? Quant à moi, je n'hésite pas à répondre que ce sont de purs néants.

Dernières Pensées, Henri Poincaré

5.1 Higher difference randomness

Franklin and Ng defined in [22] a notion of test in classical randomness, which exactly captures the sequences which are either not Martin-Löf random, or Turing compute the halting problem. They called **difference randomness** this notion of randomness, which has been very useful to prove various theorems.

Something analogous can be done in higher randomness, to capture exactly the Π^1_1-Martin-Löf random sequences which higher Turing compute $\mathcal{O}$.

Definition 5.1.1. *A sequence X is not higher difference random if there is a Σ^1_1-closed set $\mathcal{F}$ and a uniform sequence of Π^1_1-open sets $\{\mathcal{U}_n\}_{n\in\mathbb{N}}$ such that $\lambda(\mathcal{U}_n \cap \mathcal{F}) \leq 2^{-n}$ and such that $X \in \bigcap_n(\mathcal{U}_n \cap \mathcal{F})$.* ■

Theorem 5.1.1 (Yu):
Given X a Π^1_1-Martin-Löf random sequence, we have that X is not higher difference random iff X higher Turing computes $\mathcal{O}$.

Proof: Suppose that X higher Turing compute $\mathcal{O}$. Then also X can higher Turing computes Ω, the leftmost path of a Σ^1_1-closed set containing only Π^1_1-martin-Löf random sequences. Let Φ be such that $\Phi(X) = \Omega$. From Lemma 4.3.2, uniformly in ε, we can find a functional Φ_ε such that the open set of sequences on which Φ is not consistent is smaller than ε. Just like in the proof of Theorem 4.3.5, we fix a function $q : 2^{<\mathbb{N}} \to \mathbb{Q}^+$ such that $\sum_{\sigma\in 2^{<\mathbb{N}}} q(\sigma) \leq 1$.

Then, similarly we have can define $\mu([\sigma]) = \lambda(\Phi^{-1}_{q(\sigma)}([\sigma])) + \sum_{\tau\geq\sigma} q(\tau)$ and we have that $\mu/2$ is a Π^1_1-continuous semi-measure, and then that there is a constant c such that $\mu(\Omega\upharpoonright_n) \leq 2^{-n}2^c$ for every n. Therefore also $\lambda(\Phi^{-1}_{q(\Omega\upharpoonright_n)}([\Omega\upharpoonright_n])) \leq 2^{-n}2^c$. In what follows, the notation $\Phi^{-1}([\sigma])$ implicitly means $\Phi^{-1}_{q(\sigma)}([\sigma])$.

For every n, we define the Π^1_1 open set $\mathcal{U}_n$ to be $\bigcup_{s<\omega_1^{ck}} \Phi^{-1}([\Omega_s\upharpoonright_n])$. Then we define the Π^1_1-open set $\mathcal{V}$ to be $\bigcup_{n<\mathbb{N}}\bigcup_{s<\omega_1^{ck}}\{\Phi^{-1}([\Omega_s\upharpoonright_n])\ :\ \Omega_s\upharpoonright_n\neq\Omega_{s+1}\upharpoonright_n\}$.

Because Ω is higher left-c.e. we clearly have $X\in\bigcap_n(\mathcal{U}_n\cap\mathcal{V}^c)$. Also $\mathcal{U}_n\cap\mathcal{V}^c$ is actually equal to $\Phi^{-1}([\Omega\upharpoonright_n])$ and therefore its measure is smaller than $2^{-n}2^c$ for every n. Thus X is not difference random.

For the converse, suppose that a Π^1_1-Martin-Löf random X belongs to $\bigcap_n(\mathcal{U}_n\cap\mathcal{F})$ with $\lambda(\mathcal{U}_n\cap\mathcal{F})\leq 2^{-n}$. We build a Π^1_1-Solovay test $\{\mathcal{V}_m\}_{m\in\mathbb{N}}$. If m enter $\mathcal{O}$ at stage s, we search for the smallest ordinal $t>s$ such that $\lambda(\mathcal{U}_{m,t}\cap\mathcal{F}_t)\leq 2^{-m}$ and we set $\mathcal{V}_m=\mathcal{U}_{m,t}\cap\mathcal{B}_t$ with $\mathcal{B}_t\supseteq\mathcal{F}_t$ a clopen set such that $\lambda(\mathcal{U}_{m,t}\cap\mathcal{B}_t)<2^{-m+1}$. Note that we can find $\mathcal{B}_t$ uniformly in $\mathcal{U}_{m,t}$, $\mathcal{F}_t$ and m.

As X is Π^1_1-Martin-Löf random, there is some n such that for all $m\geq n$, the sequence X is not in $\mathcal{V}_m$. Also to know if $m\geq n$ is in $\mathcal{O}$, with the help of X, we search for the smallest stage s such that $X\in\mathcal{U}_{m,s}$. We claim that $m\in\mathcal{O}$ iff $m\in\mathcal{O}_s$. Suppose otherwise, that is, $m\in\mathcal{O}$ but $m\notin\mathcal{O}_s$. Note that for every stage $t\geq s$ we have $X\in\mathcal{U}_{m,t}\cap\mathcal{F}_t$, because otherwise X could not be in $\mathcal{U}_m\cap\mathcal{F}$. Now for t the smallest stage bigger than s such that $m\in\mathcal{O}_t$ and such that $\lambda(\mathcal{U}_{m,t}\cap\mathcal{F}_t)\leq 2^{-m}$, we then have that $\mathcal{U}_{m,t}\cap\mathcal{B}_t$ is enumerated in $\mathcal{V}_m$. But then $X\in\mathcal{V}_m$ which is a contradiction. ■

Corollary 5.1.1:
Higher difference randomness is strictly stronger than Π^1_1-Martin-Löf randomness.

Proof: It is clear that a Π^1_1-Martin-Löf test is also a higher difference test. So the set of higher difference randoms is included in the set of Π^1_1-Martin-Löf randoms.

Also using the higher Kučera-Gács theorem (see Theorem 4.1.2), there is some Π^1_1-Martin-Löf random sequence which higher Turing computes $\mathcal{O}$ and which is then not higher difference random, so the inclusion is strict. ■

5.2 Π^1_1-Martin-Löf[$\mathcal{O}$]-randomness

Recall Theorem 2.1.4 saying that the two following statements are equivalent :

1. X is weakly-2-random.

2. X is in no set $\bigcap_n \mathcal{U}_{f(n)}$ with $f : \mathbb{N} \to \mathbb{N}$ a $\emptyset^{(1)}$-computable function and with $\lambda(\mathcal{U}_{f(n)}) \leq 2^{-n}$.

Also recall the proof (1) $\implies$ (2), in which we transform the set $\bigcap_n \mathcal{U}_{f(n)}$ into the set $\bigcap_{n,t} \bigcup_{s\geq t} \mathcal{U}_{f_s(n)}$. This proof uses a 'time trick'. Indeed, the fact that the time of computation is in ω implies that the set $\bigcap_{n,t} \bigcup_{s\geq t} \mathcal{U}_{f_s(n)}$ is a Π^0_2 set, but if now the time of computation is in ω_1^{ck} (assuming that f is now higher Δ^0_2), each open set of the intersection is now indexed by a computable ordinal, and we do not have anymore an intersection of Π^1_1-open sets uniformly in $n \in \omega$.

We shall indeed prove that the notion of Π^1_1-Martin-Löf randomness, where Kleene's $\mathcal{O}$ can be used for the index of each component is much stronger than weak-Π^1_1-randomness, and even stronger than Π^1_1-randomness. We call this notion Π^1_1-Martin-Löf[$\mathcal{O}$]-randomness (to be pronounced, for a mysterious reason: Π^1_1-Martin-Löf 'plop O' randomness).

Definition 5.2.1. *A sequence X is Π^1_1-Martin-Löf[$\mathcal{O}$]-random if X is in no set $\bigcap_n \mathcal{U}_{f(n)}$ with f higher Turing computable by $\mathcal{O}$ and with $\lambda(\mathcal{U}_{f(n})) \leq 2^{-n}$ for each n.* ∎

So as we will see, we don't have the equivalence between Π^1_1-Martin-Löf[$\mathcal{O}$]-randomness and weak-Π^1_1-randomness. Nevertheless there is a way to remove $\mathcal{O}$ from this definition, in order to get a better understanding of it:

Proposition 5.2.1:
The following are equivalent for a sequence $X \in 2^{\mathbb{N}}$:

1. *X is Π^1_1-Martin-Löf[$\mathcal{O}$]-random.*
2. *X does not belong to any test $(\mathcal{U}_s)_{s<\omega_1^{ck}}$ not necessarily nested where each $\mathcal{U}_s$ is a Π^1_1-open set uniformly in s, and such that $\lambda(\bigcap_s \mathcal{U}_s) = 0$.*

Proof: Let us show that (2) implies (1). Let $\bigcap_n \mathcal{U}_{f(n)}$ be an Π^1_1-Martin-Löf[$\mathcal{O}$] test. Recall that $p : \omega_1^{ck} \to \omega$ is the projectum function and let us define $\mathcal{V}_s = \bigcap_{n<p(s)} \bigcup_{t>s} \mathcal{U}_{f_t(n)}$. It is clear that $\bigcap_n \mathcal{U}_{f(n)} \subseteq \bigcap_s \mathcal{V}_s$. To prove that $\lambda(\bigcap_s \mathcal{V}_s) = 0$, let us prove that $\bigcap_s \mathcal{V}_s \subseteq \bigcap_n \mathcal{U}_{f(n)}$. For each n there exists s large enough such that $n \leq p(s)$ and $\forall m \leq n \quad \bigcup_{t>s} \mathcal{U}_{f_t(m)} = \mathcal{U}_{f(m)}$. Then we have for that n and s that $\mathcal{V}_s \subseteq \bigcap_{m\leq n} \mathcal{U}_{f(m)}$ and then $\bigcap_s \mathcal{V}_s \subseteq \bigcap_n \mathcal{U}_{f(n)}$.

Let us show that (1) implies (2). Suppose now that we have a test $(\mathcal{U}_s)_{s<\omega_1^{ck}}$ with $\lambda(\bigcap_s \mathcal{U}_s) = 0$. Then using $\mathcal{O}$ we can higher Turing compute the measure of each $\mathcal{U}_s$ uniformly in s. Then for each n, $\mathcal{O}$ can higher Turing compute s_n such that $\lambda(\mathcal{U}_{s_n}) \leq 2^{-n}$ and then we can find an equivalent Π^1_1-Martin-Löf[$\mathcal{O}$] test, by setting $\mathcal{V}_n = \mathcal{U}_{s_n}$. ∎

We shall now see that Π^1_1-Martin-Löf[$\mathcal{O}$]-randomness is strictly stronger than Π^1_1-randomness. For this we first prove:

Proposition 5.2.2:
If $X \in 2^{\mathbb{N}}$ higher Turing computes a non Δ^1_1 higher Δ^0_2 sequence Y, then X is not Π^1_1-Martin-Löf[$\mathcal{O}$]-random.

PROOF: The set $\mathcal{A} = \bigcap_{n,s} \bigcup_{t \geq s} \Phi^{-1}(Y_t \upharpoonright_n)$ is also equal to the set $\bigcap_n \Phi^{-1}(Y_t \upharpoonright_n)$. Also by Sack's theorem (Theorem 3.4.2), as Y is not Δ^1_1, the set of sequences which higher Turing compute Y is a nullset. However the function Φ can also be inconsistent. Therefore the measure of the set $\mathcal{A}$ is bounded by the measure of the Π^1_1-open set on which Φ is inconsistent. Also by Lemma 4.3.2 we can transform Φ uniformly in any ε so that the measure of this open set is smaller than ε, without damaging the right computations of Φ. But then uniformly in n we can define the set $\mathcal{A}_n$ like above, but with the measure of $\mathcal{A}_n$ bounded by 2^{-n}. Also by Proposition 5.2.1, we then have that $\bigcap_n \mathcal{A}_n$ is a Π^1_1-Martin-Löf[$\mathcal{O}$] test, and by design, it contains X. ∎

Theorem 5.2.1:
Π^1_1-Martin-Löf[$\mathcal{O}$]-randomness is strictly stronger than Π^1_1-randomness.

PROOF: By the proposition above we have that Π^1_1-Martin-Löf[$\mathcal{O}$]-randomness is either incomparable with Π^1_1-randomness, or strictly stronger than Π^1_1-randomness: Indeed, by the Gandy basis theorem, there is a higher Δ^0_2 sequence which is Π^1_1-random. All that remains to be proved is that Π^1_1-Martin-Löf[$\mathcal{O}$]-randomness is stronger than Π^1_1-randomness.

By Theorem 6.1.2, proved later, if X is Δ^1_1-random but not Π^1_1-random, then there exists a uniform intersection of Π^1_1-open sets $\bigcap_n \mathcal{U}_n$ such that $X \in \bigcap_n \mathcal{U}_n$ but X is in no Σ^1_1-closed set $\mathcal{F}$ with $\mathcal{F} \subseteq \bigcap_n \mathcal{U}_n$. Let us argue that there is an effective enumeration $\{\mathcal{F}_s\}_{s<\omega_1^{ck}}$ of the Σ^1_1-closed sets included in $\bigcap_n \mathcal{U}_n$. For a given Σ^1_1-closed set $\mathcal{F}$, we can build the Π^1_1 function $f : \omega \to \omega_1^{ck}$ which to n associates the least t such that $\mathcal{F}_t \subseteq \bigcap_{m \leq n} \mathcal{U}_{m,t}$. If we really have $\mathcal{F} \subseteq \bigcap_n \mathcal{U}_n$ then f is total and then its range is bounded by some computable ordinal t, for which we already have $\mathcal{F}_t \subseteq \bigcap_n \mathcal{U}_{n,t} \subseteq \bigcap_n \mathcal{U}_n$.

So if a Σ^1_1-closed set is included in $\bigcap_n \mathcal{U}_n$ we will know at some computable ordinal stage. Then we can easily get an effective enumeration $\{\mathcal{F}_s\}_{s<\omega_1^{ck}}$ of the Σ^1_1-closed sets included in $\bigcap_n \mathcal{U}_n$ by checking at each stage t and for each index of a Σ^1_1-closed set $\mathcal{F}$ if we have $\mathcal{F}_t \subseteq \bigcap_n \mathcal{U}_{n,t}$. Also we have that X is in $\bigcap_n \mathcal{U}_n \cap \bigcap_{s<\omega_1^{ck}} \mathcal{F}_s^c$ which is a set of measure 0 and therefore, by Proposition 5.2.1 a Π^1_1-Martin-Löf[$\mathcal{O}$] test. ∎

This theorem yields a natural question, which is still open at the moment. We have that no sequence computing a higher Δ^0_2 sequence is Π^1_1-Martin-Löf[$\mathcal{O}$]-random. Does the converse hold on Π^1_1-Martin-Löf random sequences? Using Theorem 6.3.1 proved later, we already know that the Π^1_1-Martin-Löf randoms which are not Π^1_1-random can higher Turing computes higher Δ^0_2 sequences (even Π^1_1 sequences). But what about the sequences which are Π^1_1-random but not Π^1_1-Martin-Löf[$\mathcal{O}$]-random?

Question 5.2.1 *Is there some X which is Π^1_1-random, not Π^1_1-Martin-Löf[$\mathcal{O}$]-random, and which does not higher Turing compute any higher Δ^0_2 sequence?*

5.3 weak-Π^1_1-randomness

5.3.1 An equivalent test notion

We develop here a new type of test whose corresponding notion of randomness turns out to be weak-Π^1_1-randomness, thus giving a better understanding of this notion. We start by generalization of a result from Liang Yu and C.T. Chong (see [8]) which says that every higher left-c.e. sequence can be captured by a weak-Π^1_1-randomness test. Recall Definition 4.4.3 and Definition 4.4.4 of compact and closed approximations. Note also that as we now work in the Cantor space, those two notions coincide.

Theorem 5.3.1:
No sequence $X \in 2^{\mathbb{N}}$ with a higher closed approximation is weakly-Π^1_1-random.

Proof: Let $\{X_s\}_{s \leq t}$ be a closed approximation of X. Let us denote the closed set $\{X_s\}_{s \leq \omega_1^{ck}}$ by $\mathcal{C}$. Let $\mathcal{U}_n = \bigcup_{s<\omega_1^{ck}} [X_s \upharpoonright_n]$ and let us prove that $\bigcap_n \mathcal{U}_n \subseteq \mathcal{C}$. If an element is in $\mathcal{U}_n$ then its distance to the closed set $\mathcal{C}$ is smaller than 2^{-n} (it shares the same first n bits with an element of $\mathcal{C}$). Thus if it is in all the $\mathcal{U}_n$, its distance to the closed set $\mathcal{C}$ is null and thus it is an element of $\mathcal{C}$. Therefore we have $\bigcap_n \mathcal{U}_n \subseteq \mathcal{C}$ and as $\mathcal{C}$ is countable it has measure 0. Therefore we have that $\bigcap_n \mathcal{U}_n$ is a weak-Π^1_1-randomness test containing X. ∎

Corollary 5.3.1:
weak-Π^1_1-randomness is strictly stronger than higher difference randomness.

Proof: Let us first argue that the set of weakly-Π^1_1-randoms is included in the set of higher difference randoms. Consider the leftmost path Ω of a Σ^1_1 closed set containing only Π^1_1-Martin-Löf randoms. In particular Ω is higher left-c.e. and then it is Turing computable by $\mathcal{O}$. Also if Z higher Turing computes $\mathcal{O}$ it also higher Turing computes Ω. From the previous theorem, Ω can be captured by a weak-Π^1_1-randomness test, and using a variant of the higher XYZ theorem (see Porism 4.3.1), we then also have that Z is captured by a weak-Π^1_1-randomness test.

Now to prove that the inclusion is strict. Let Ω_1, Ω_2 be the two halves of Ω, that is, $\Omega = \Omega_1 \oplus \Omega_2$. By the higher van Lambalgen theorem, we have that Ω_1 is Ω_2-continuously Π^1_1-Martin-Löf random. In particular Ω_2 does not higher Turing computes Ω_1 (despite possible inconsistency, we still have that if X higher Turing computes Y, then Y is not X-continuously Π^1_1-Martin-Löf random, see Fact 7.3.1). A fortiori Ω_2 does not higher Turing compute $\mathcal{O}$. It follows that Ω_2 is higher difference random. However Ω_2 still has a higher closed approximation (actually even a higher finite-change approximation, see Definition 5.4.2). Therefore it is not weakly-Π^1_1-random. ∎

We now bring the technique of Theorem 5.3.1 to its full generalization, by giving an equivalent notion of weak-Π^1_1-randomness test, that uses compact approximations of elements of the Baire space. This is done by giving another notion of weak-Π^1_1-randomness test, in the style of Π^1_1-Martin-Löf[$\mathcal{O}$]-randomness. The idea is simple: instead of using a Δ^0_2 function f to find the indices of open components of a test, we now allow only functions with a compact approximation.

Theorem 5.3.2:
Let $\{\mathcal{U}_e\}_{e\in\omega}$ be a standard enumeration of the Π^1_1-open sets. For a sequence X we have that the following is equivalent :

1. *X is in no uniform intersection of Π^1_1-open sets $\bigcap_n \mathcal{U}_{f(n)}$ where f has a compact approximation and with $\lambda(\mathcal{U}_{f(n)}) \leq 2^{-n}$.*
2. *X is weakly-Π^1_1-random.*
3. *X is in no uniform intersection of Π^1_1-open sets $\bigcap_n \mathcal{U}_{f(n)}$ where f has a finite change approximation and with $\lambda(\mathcal{U}_{f(n)}) \leq 2^{-n}$.*

Proof: (1) $\implies$ (3) : Trivial

(2) $\implies$ (1) : Consider a set $\bigcap_n \mathcal{U}_{f(n)}$ with $\{f_s\}_{s<\omega_1^{ck}}$ a compact approximation of f, with $\lambda(\mathcal{U}_{f(n)}) \leq 2^{-n}$ and with $X \in \bigcap_n \mathcal{U}_{f(n)}$. Let us prove that X is not weakly-Π^1_1-random. To do so consider the set $\mathcal{A} = \bigcup_{s\leq\omega_1^{ck}} \bigcap_{n<\omega} \mathcal{U}_{f_s(n)}$ and the set $\mathcal{B} = \bigcap_{n<\omega} \bigcup_{s<\omega_1^{ck}} \bigcap_{m\leq n} \mathcal{U}_{f_s(m)}$.

Let us prove that $\mathcal{B} \subseteq \mathcal{A}$. Suppose that $Y \in \mathcal{B}$. Then for all n there is a smallest stage s_n so that $Y \in \bigcap_{m\leq n} \mathcal{U}_{f_{s_n}(m)}$. As f has a closed approximation we have that every limit point of $\{f_{s_n}\}$ is equal to f_t for some $t \leq \omega_1^{ck}$. And as the approximation is also compact, at least one limit point exists. Fix f_s such a limit point. For any k there is $i \geq k$ be such that $f_{s_i} \upharpoonright_k = f_s \upharpoonright_k$ and then such that $\bigcap_{m\leq k} \mathcal{U}_{f_{s_i}(m)} = \bigcap_{m\leq k} \mathcal{U}_{f_s(m)}$. Now we have by definition of the $\{s_n\}_{n\in\omega}$ that $Y \in \bigcap_{m\leq i} \mathcal{U}_{f_{s_i}(m)}$ and therefore we have that $Y \in \bigcap_{m\leq k} \mathcal{U}_{f_s(m)}$. Since this holds for any k, this shows that Y belongs to $\bigcap_k \mathcal{U}_{f_s(k)}$ and thus we have $Y \in \mathcal{A}$.

Let us prove that $\lambda(\mathcal{B}) = 0$. By measure countable subadditivity we have

$$\lambda(\mathcal{A}) \leq \sum_{s\leq\omega_1^{ck}} \lambda\left(\bigcap_n \mathcal{U}_{f_s(n)}\right)$$

And for each $s \leq \omega_1^{ck}$ we have $\lambda(\bigcap_n \mathcal{U}_{f_s(n)}) = 0$ and then that $\lambda(\mathcal{A}) = 0$ (for this, note that we can suppose $\lambda(\mathcal{U}_{f_s(n)}) \leq 2^{-n}$ for every s, as we can always trim the open set otherwise). But then as $\mathcal{B} \subseteq \mathcal{A}$ we have $\lambda(\mathcal{B}) = 0$.

Let us prove that $X \in \mathcal{B}$. For all n, there is some stage s_n such that $f_{s_n} \upharpoonright_n = f \upharpoonright_n$. Then at stage s_n we have $X \in \bigcap_{m\leq n} \mathcal{U}_{f_{s_n}(m)}$. As this is true for all n, we have $X \in \mathcal{B}$. We can then conclude that $\mathcal{B}$ is in a weak-Π^1_1-randomness test containing X.

(3) $\implies$ (2) : Suppose now that X is not weakly-Π^1_1-random in order to prove that it is in some set $\bigcap_n \mathcal{U}_{f(n)}$ where f has a finite change approximation. Suppose that $X \in \bigcap_n \mathcal{V}_n$ with $\lambda(\bigcap_n \mathcal{V}_n) = 0$. We define $f(n)$ to be the smallest m such that $\lambda(\mathcal{V}_m) \leq 2^{-n}$. We have for every n that $\lambda(\mathcal{V}_{f(n)}) \leq 2^{-n}$ and $X \in \mathcal{V}_{f(n)}$. All we need to prove is that f has a finite change approximation $\{f_s\}_{s\in\omega_1^{ck}}$. We simply let $f_s(n)$ be the smallest m such that $\lambda(\mathcal{V}_m[s]) \leq 2^{-n}$. Then we clearly have for each n that the set $\{s : f_s(n) \neq f_{s+1}(n)\}$ is finite. ■

The previous theorem can also be used to provide another proof of Yu and Chong's result [8], saying that for any hyperdegree above $\mathcal{O}$, there is a Π^1_1-Martin-Löf random in that degree, which is not weakly-Π^1_1-random.

Now, recall Theorem 2.1.5 of classical randomness: For a sequence X Martin-Löf random we have that the three following properties are equivalent:

1. X is weakly-2-random.
2. X does not compute any non-computable Δ^0_2 sequence.
3. X does not compute any non-computable c.e. set.

The higher counterpart of (1) $\leftrightarrow$ (2) cannot work, as by the Gandy basis theorem, there is a sequence which is both higher Δ^0_2 and Π^1_1-random. We shall see that a higher counterpart of (1) $\leftrightarrow$ (3) also fails. It will be a consequence of Theorem 6.3.1 which says that Π^1_1-randomness is actually the right randomness notion for the equivalence (1) $\leftrightarrow$ (3). Then the separation of weak-Π^1_1-randomness and Π^1_1-randomness will allow us to conclude.

5.3.2 Separation of weak-Π^1_1-randomness and Π^1_1-randomness

We now separate the notion of weak-Π^1_1-randomness and the notion of Π^1_1-randomness. This is actually done by building a self-unclosed approximable sequence X which is weakly-Π^1_1-random (recall Section 4.4 about refinements of the notion of higher Δ^0_2 approximation). In practice, we will use a refinement for the notion of self-unclosed approximation. We say that a sequence Y has a ω-self-unclosed approximation if the number of changes in the approximation above a correct prefix of Y is finite. Formally:

Definition 5.3.1. *A higher Δ^0_2 approximation $\{f_s\}_{s<\omega_1^{ck}}$ of a function f is said to be ω-**self-unclosed** if for any n, there is no infinite sequence of ordinals $s_0 < s_1 < \dots$ such that $f\upharpoonright_n = f_{s_i}\upharpoonright_n$ and such that $f_{s_i}(n) \neq f_{s_{i+1}}(n)$.* ■

It is clear that an ω-self-unclosed approximation is also a self-unclosed approximation: Suppose that Y has a Δ^0_2 approximation such that Y is in the closure of $\{Y_t : t < s\}$ for some smallest stage s. We can suppose that Y is not the only limit point of $\{Y_t : t < s\}$ as otherwise Y would be Δ^1_1. But then there are several limit points and this implies infinitely many changes above some prefix of Y. We now show that there is a weakly-Π^1_1-random with a ω-self-unclosed approximation.

Theorem 5.3.3:
There is a weakly-Π^1_1-random X with a ω-self-unclosed approximation. In particular, there is a weakly-Π^1_1-random X that is not Π^1_1-random.

The rest of the section is dedicated to the proof of theorem 5.3.3. Let $\{\mathcal{S}_i\}_{i\in\omega}$ be an enumeration of all the higher Σ^0_2 sets. For each $\mathcal{S}_i$ and each j let us define the Σ^1_1 closed set $\mathcal{F}_{i,j}$ so that $\mathcal{S}_i = \bigcup_j \mathcal{F}_{i,j}$.

Sketch of the proof:

We will build X as a limit point of some $\{X_s\}_{s<\omega_1^{ck}}$. Each X_s is built as the unique limit point of a sequence $\{[\sigma^n_s]\}_{n<\omega}$, where $\sigma^1_s \prec \sigma^2_s \prec \dots$.

At each stage we will ensure that X_s is in some sense *weakly-Π^1_1-random at stage s*. What do we mean by this? For some s and some n, as long as $\lambda(\mathcal{S}_n[s]) = 1$, we believe that X_s should belong to $\mathcal{S}_n[s]$. If at some point we have $\lambda(\mathcal{S}_n[s]) < 1$ (which is by the Σ^1_1-boundedness principle equivalent to $\lambda(\mathcal{S}_n) < 1$) then n is removed from the set of indices that we use to make X_s weakly-Π^1_1-random at stage s.

Concretely we have at each stage s a set of indices $\{e_n\}_{n\in\omega}$ which are initialized at stage 0 with $e_n = n$. Suppose that at stage s we have for each n that $\lambda(\mathcal{S}_{e_n}[s]) = 1$. Then it is easy to build a Δ^1_1 sequence X_s in $\bigcap_n \mathcal{S}_{e_n}[s]$:

We can suppose that e_0 is such that $\mathcal{F}_{e_0,i} = 2^{\mathbb{N}}$ for all i. So for $d_0 = 0$ and $\sigma_0 = \epsilon$ we have $\lambda(\mathcal{F}_{e_0,d_0} \mid [\sigma_0]) \geq 1$. Then, inductively, assuming that for some n we have $\lambda(\bigcap_{k\leq n} \mathcal{F}_{e_k,d_k} \mid [\sigma_n]) \geq 2^{-n}$, we then continue recursively the construction as follows:

Step 1: We find one strict extension σ_{n+1} of σ_n so that $\lambda(\bigcap_{k\leq n} \mathcal{F}_{e_k,d_k} \mid [\sigma_{n+1}])[s] \geq 2^{-n}$.

Step 2: We find some index d_{n+1} such that $\lambda(\bigcap_{k\leq n+1} \mathcal{F}_{e_k,d_k} \mid [\sigma_{n+1}])[s] \geq 2^{-n-1}$.

This way we have an intersection of closed sets containing at most one point X_s. Also by the measure requirement, this intersection is not empty at each step and then we really have $X_s \in \bigcap_n \mathcal{S}_{e_n}[s]$. Note that for the actual construction we will need different lower bounds for the measure requirements, due to some technicality, explained in the next paragraphs.

We only try here to give the general idea. To have that the X_s converge to some X, we have to keep the chosen strings and closed sets at stage $s+1$ equal if possible to those of stage s. When do we have to change them? Three things can happens :

1. We might have $\lambda(\mathcal{S}_{e_n})[s] = 1$ for all $s < t$ but $\lambda(\mathcal{S}_{e_n})[t] < 1$.
2. We might have a smallest n such that (3) does not happen up to $n-1$ and such that the measure of $\bigcap_{k\leq n} \mathcal{F}_{e_k,d_k}$ inside $[\sigma_{n+1}]$ drops below 2^{-n} at stage t.
3. We might have a smallest n such that (2) does not happen up to n and such that the measure of $\bigcap_{k\leq n+1} \mathcal{F}_{e_k,d_k}$ inside $[\sigma_{n+1}]$ drops below 2^{-n-1} at stage t.

If (1) happens then the index e_n is set to some fixed index a so that $\lambda(\mathcal{S}_a) = 1$, therefore each index e_n can change at most once. If (2) happens, it is the responsibility of the string σ_{n+1} to change, and if (3) happens it is the responsibility of the index d_{n+1} to change.

For (2), we are sure that there exists one extension σ_{n+1} of σ_n of length $|\sigma_n| + 1$ such that the measure inside $[\sigma_{n+1}]$ does not drop below 2^{-n}. So as long as the construction is stable 'below the choice of σ_{n+1}', the string σ_{n+1} can change at most once. We will see that in practice we will need extensions of length $|\sigma_n| + 2n$, but for the same reason, the string σ_{n+1} can then change at most finitely often.

For (3), as long as $\lambda(\mathcal{S}_{e_{n+1}}) = 1$, we are sure that we will change only finitely often of index d_{n+1}. However if $\lambda(\mathcal{S}_{e_{n+1}}) < 1$ it can happen that d_{n+1} will change infinitely often at stages $s_1 < s_2 < \dots$, and that $t = \sup_n s_n$ is the first stage for which we witness $\lambda(\mathcal{S}_{e_{n+1}})[t] < 1$ (then at stage t the integer e_{n+1} is set to a the fixed index such that $\lambda(\mathcal{S}_a) = 1$).

There is nothing we can do to prevent those infinitely many changes, which could lead as well to infinitely many changes of the string σ_{n+2}. However we can still ensure that if this happens, the string σ_{n+1} will then change, and its previous value will be banished forever, so that the approximation of the sequence X is still ω-self-unclosed.

To do so, we need to take extensions sufficiently long, so that the current closed set still has positive measure inside at least two of them. That way we can afford to banish one of them. So before the formal proof, we recall here Lemma 4.1.1 that helps us to achieve this:

Lemma 5.3.1 *let σ be a string and $\mathcal{F}$ a closed set so that $\lambda(\mathcal{F} \mid [\sigma]) \geq 2^{-n}$. Then there is at least two extensions τ_1, τ_2 of σ of length $|\sigma| + n + 1$ so that for $i \in \{1,2\}$ we have $\lambda(\mathcal{F} \mid [\tau_i]) \geq 2^{-n-1}$.*

Before the construction:

Let $\{\mathcal{S}_i\}_{i \in \mathbb{N}}$ be an enumeration of all the higher Σ^0_2 sets, with $\mathcal{S}_i = \bigcup_{j \in \mathbb{N}} \mathcal{F}_{i,j}$ where each $\mathcal{F}_{i,j}$ is a Σ^1_1 closed set. We can assume that each union is increasing.

We start by deciding in advance the length m_n of each extension. We set $m_0 = 0$ and then recursively we set $m_{n+1} = m_n + (2n+1)$. Finally, let a be an integer so that $\mathcal{F}_{a,i} = 2^{\mathbb{N}}$ for every i.

For each stage s and each n we will define indices e^n_s and d^n_s for the closed set $\mathcal{F}_{e^n_s, d^n_s}$, as well as strings σ^n_s. Also to simplify the reading, we define three predicates:

$$\begin{array}{rll} A(n,s) & \text{means} & \lambda(\bigcap_{k \leq n} \mathcal{F}_{e^k_s, d^k_s} \mid [\sigma^n_s])[s] \geq 2^{-2n} \\ A(n,s,\sigma) & \text{means} & \lambda(\bigcap_{k \leq n} \mathcal{F}_{e^k_s, d^k_s} \mid [\sigma])[s] \geq 2^{-2n-1} \\ A(n,s,\sigma,d) & \text{means} & \lambda(\bigcap_{k \leq n} \mathcal{F}_{e^k_s, d^k_s} \cap \mathcal{F}_{e^{n+1}_s, d} \mid [\sigma])[s] \geq 2^{-2n-2} \end{array}$$

The construction:

At stage 0 we define for each n the set P^n_0 to be the set of strings of length m_n, ordered lexicographically. We initialize each string σ^n_0 to be the first string of P^n_0 (so they are all a range of 0), we initialize e^0_0 to a and e^{n+1}_0 to n. Then we initialize to 0 each index d^n_0 of the sets $\mathcal{F}_{e^n_0, d^n_0}$.

At successor stage $s+1$ and substage 0, we set $e^0_{s+1} = e^0_s = a$, $\sigma^0_{s+1} = \sigma^0_s = \epsilon$ and $d^0_{s+1} = d^0_s = 0$. Now assume that at substage n we have defined e^k_{s+1}, d^k_{s+1} and σ^k_{s+1} for $k \leq n$ and that we have $A(n, s+1)$. Let us now define e^{n+1}_{s+1}, d^{n+1}_{s+1} and σ^{n+1}_{s+1} at substage $n+1$.

Def. of e^{n+1}_{s+1} : If $\lambda(\mathcal{S}_{e^{n+1}_s})[s+1] = 1$, set $e^{n+1}_{s+1} = e^{n+1}_s$ and $P^{n+1}_{s+1} = P^{n+1}_s$, otherwise set $e^{n+1}_{s+1} = a$ and $P^{n+1}_{s+1} = P^{n+1}_s - \{\sigma^{n+1}_s\}$.

Def. of σ^{n+1}_{s+1} : If $A(n, s+1, \sigma^{n+1}_s)$ and σ^{n+1}_s extends σ^n_{s+1}, set $\sigma^{n+1}_{s+1} = \sigma^{n+1}_s$. Otherwise set σ^{n+1}_{s+1} to be the first string of P^{n+1}_{s+1} extending σ^n_{s+1} such that $A(n, s+1, \sigma^{n+1}_{s+1})$.

Def. of d^{n+1}_{s+1} : If $A(n, s+1, \sigma^{n+1}_{s+1}, d^{n+1}_s)$ set $d^{n+1}_{s+1} = d^{n+1}_s$. Otherwise set d^{n+1}_{s+1} to be the smallest integer such that $A(n, s+1, \sigma^{n+1}_{s+1}, d^{n+1}_{s+1})$.

Finally after every substage, define X_{s+1} to be the unique element in $\bigcap_n [\sigma^n_{s+1}]$.

At limit stage s, for each $n \geq 0$ set e^n_s to be the convergence value of $\{e^n_t\}_{t<s}$ and set P^n_s to be the convergence value of $\{P^n_t\}_{t<s}$. (among other things we will have to prove that we always have convergence).

At substage n, if $\{\sigma_t^n\}_{t<s}$ does not converge, set σ_s^n to be the first string of P_s^n extending σ_s^{n-1}, otherwise set σ_s^n to be the convergence value. If $\{d_t^n\}_{t<s}$ does not converge, set d_s^n to 0, otherwise set it to its convergence value.

Finally after every substage, define X_s to be the unique element in $\bigcap_n[\sigma_s^n]$.

The verification:

Claim 1: For every n the sequence $\{e_s^n\}_{s<\omega_1^{ck}}$ can change at most once. In particular, for every s and every n we have that $\{e_t^n\}_{t<s}$ converges:

It is clear because $e_{s+1}^n \neq e_s^n$ only if $\lambda(\mathcal{S}_{e_s^n}[s+1]) < 1$. Also when this happens we have $e_{s+1}^n = a$ and then it can happen only once.

Claim 2: For every stage s, any string τ of size m_n and any closed set $\mathcal{F}$ such that $\lambda(\mathcal{F} \mid [\tau]) \geq 2^{-2n}$, there is a string $\sigma \in P_s^{n+1}$ which extends τ so that $\lambda(\mathcal{F} \mid [\sigma]) \geq 2^{-2n-1}$.

Suppose that $\lambda(\mathcal{F} \mid [\tau]) \geq 2^{-2n}$ for $|\tau| = m_n$. Using Lemma 4.1.1 we have two strings τ_1 and τ_2 of length $m_n + 2n + 1$ so that for $i \in \{1,2\}$ we have $\lambda(\mathcal{F} \mid [\tau_i]) \geq 2^{-2n-1}$. Also $m_{n+1} = m_n + 2n + 1$ and then $\tau_1, \tau_2 \in P_0^{n+1}$. By construction and by Claim 1, at any stage s we have that P_0^{n+1} contains at most one more string than P_s^{n+1}. Then at any stage s we have at least one string $\sigma \in P_s^{n+1}$ which extends τ and so that $\lambda(\mathcal{F} \mid [\sigma]) \geq 2^{-2n-1}$.

Claim 3: The construction converges, in particular the sequence $\{X_s\}_{s<\omega_1^{ck}}$ converges to X:

There is no difficulty here.

Claim 4: The sequence X_s is ω-self-unclosed:

Let $D(s,n)$ be the sentence : "There is an infinite sequence of ordinal $s_0 < s_1 < \dots$ with $\sup_i s_i = s$, such that $X_{s_i} \upharpoonright_n = X_{s_{i+1}} \upharpoonright_n$, and such that $X_{s_i}(n) \neq X_{s_{i+1}}(n)$".

For $\{X_s\}_{s<\omega_1^{ck}}$ to be self-unclosed, it is enough to prove that for any s and any n, if $D(s,n)$ is true, then $X \upharpoonright_n \neq X_s \upharpoonright_n$.

Let s be the smallest stage such that $D(s,n)$ is true for some n. Let n be the smallest integer such that $D(s,n)$ is true, and let $s_0 < s_1 < \dots$ be a sequence of ordinals making $D(s,n)$ true.

Let us prove that there is some i such that $\{X_t \upharpoonright_n\}_{s_i \leq t < s}$ is stable. If $n = 1$ it is clear because $X_t \upharpoonright_1 = 0$ for every $t < \omega_1^{ck}$. If $n > 1$, then by minimality of n, we necessarily have that $\{X_t \upharpoonright_2\}_{t<s}$ converges, otherwise $D(s,1)$ would be true. So for some i we have that $\{X_t \upharpoonright_2\}_{s_i \leq t < s}$ is stable. We continue inductively to prove that there is some i such that $\{X_t \upharpoonright_n\}_{s_i \leq t < s}$ is stable.

Let us now identify m such that $\{\sigma_t^m\}_{s_i \leq t < s}$ is stable, and such that $\sigma_{s_j}^{m+1} \neq \sigma_{s_{j+1}}^{m+1}$ for $j \in \mathbb{N}$. We shall now prove that for at least one $k \leq m$ (presumably for $k = m$), the sequence $\{d_t^k\}_{s_i \leq t < s}$ does not converge. Suppose otherwise, that is, the sequence $\{d_t^k \mid k \leq m\}_{s_i \leq t < s}$ converges, then there is some $j \geq i$ such that $\{d_t^k \mid k \leq m\}_{s_j \leq t < s}$ is stable. But then for all t with $s_j \leq t < s$ we have $A(m,t)$ and then we also have $A(m,s)$. Then using Claim 2 with $\bigcap_{k \leq m} \mathcal{F}_{e_s^k, d_s^k}[s]$ as the closed set $\mathcal{F}$, we have at least one string σ in P_s^{m+1} extending σ_s^m

such that $A(m,s,\sigma)$ is true and then such that $A(m,t,\sigma)$ is true for every t with $s_j \leq t < s$. Also this contradicts that $\{\sigma^{m+1}_t\}_{s_i \leq t < s}$ does not converge.

So let $k \leq m$ be the smallest integer such that $\{d^k_t\}_{s_i \leq t < s}$ does not converge, equivalently $\lim_{t<s} d^k_t = \infty$. In particular we have $A(k-1,s,\sigma^k_s)$, but there is no d large enough such that $A(k-1,s,\sigma^k_s,d)$. This is only possible if $\lambda(\mathcal{S}_{e^k_s})[s] < 1$. Then at stage $s+1$ we have that $\sigma^k_s \leq \sigma^m_s \prec X_s \upharpoonright_n$ is banished, that is, removed from P^k_s.

It follows that we have $X \upharpoonright_n \neq X_s \upharpoonright_n$, but also that for any other $n' > n$ such that $D(s,n')$ is true, we have $X \upharpoonright_{n'} \neq X_s \upharpoonright_{n'}$. Indeed, if $D(s,n')$ is true for $n' > n$, with stages $s'_0 < s'_1 < \dots$, by minimality of s we have $\sup_i s'_i = s$ and then, as $\{\sigma^m_t\}_{s_i \leq t < s}$ is stable, also $\{\sigma^m_t\}_{s'_j \leq t < s}$ is stable for some j, which implies $\sigma^m_s \prec X_{s'_i} \upharpoonright_{n'}$ for every i and then that $X \upharpoonright_{n'} \neq X_s \upharpoonright_{n'}$.

We can then continue inductively with the smallest stage $s' > s$ such that $D(s',n)$ is true for some n, and then with the smallest n such that $D(s',n)$ is true.

Claim 6: The sequence X is weakly-Π^1_1-random:

It is clear that if $\lambda(\mathcal{S}_n) = 1$, then $e^{n+1} = \lim_{s<\omega_1^{ck}} e^{n+1}_s$ is equal to n. Therefore any sequence in $\bigcap_n \mathcal{S}_{e^n}$ is weakly-Π^1_1-random. We shall then simply prove that we have $X \in \bigcap_n \mathcal{S}_{e^n}$.

Let s_n be the smallest ordinal such that $\{(e^k_t, d^k_t) \mid k \leq n\}_{s_n \leq t < \omega_1^{ck}}$ is stable and equal to $\{(e^k, d^k) \mid k \leq n\}$. In particular we have that $\mathcal{A} = \{X_{s_n}\}_{n \in \mathbb{N}} \cup \{X\}$ is a closed set and that $\bigcap_{k \leq n} \mathcal{F}_{e^k, d^k} \cap \mathcal{A}$ is not empty because it contains X_{s_n}. Then also $\bigcap_{k \leq \omega} \mathcal{F}_{e^k, d^k} \cap \mathcal{A}$ is not empty and it then contains X, as it is the only non Δ^1_1 point of $\mathcal{A}$.

5.4 Further studies on higher Δ^0_2 approximations

We started in Section 4.4 a study of different restrictions of the notion of higher Δ^0_2, and we introduced later Definition 5.3.1 that appeared naturally in the separation of weak-Π^1_1-randomness and Π^1_1-randomness. We shall pursue here the study of all the defined notions, as well as the study of new notions. In particular, we will establish the separation between all of them. For this study, we introduce the notion of partial continuity of a Δ^0_2 approximation:

Definition 5.4.1. *We say that a higher Δ^0_2 approximation $\{f_s\}_{s<\omega_1^{ck}}$ is **partially continuous** if for any limit stage s, whenever $\{f_t\}_{t<s}$ converges to some function f', we have $f_s = f'$.* ∎

It is easy to check that without loss of generality, we can consider that any higher Δ^0_2 approximation is partially continuous:

Lemma 5.4.1 *For any higher Δ^0_2 approximation $\{f_s\}_{s<\omega_1^{ck}}$ that converges to f, there is a partially continuous higher Δ^0_2 approximation $\{f'_s\}_{s<\omega_1^{ck}}$ that converges to f.*

Proof: For each successor stage s we define $f'_s = f_s$. For each limit stage s, in case $\{f_t\}_{t<s}$ converges to some function f', we define $f_s = f'$, and otherwise we define $f'_s = f_s$. It is easy to check that if $\{f_s\}_{s<\omega_1^{ck}}$ converges to f, then also $\{f'_s\}_{s<\omega_1^{ck}}$ converges to f. ∎

Quite often in this section we will also use uniform enumerations of (possibly non-converging) Δ_2^0 approximations, which contains all the converging Δ_2^0 approximations. Let us argue that such an object exists: given the code for a sequence of partial function $\{f_s\}_{s<\omega_1^{ck}}$, we can always assume that each f_s is total without damaging the possible convergence of $\{f_s\}_{s<\omega_1^{ck}}$: If f_s is not total we can replace it by f'_s where each bit of $f'_s(n)$ is equal to the convergence value of $\{f'_t(n)\}_{t<s}$ if it exists, and 0 otherwise. If by keeping only stages at which f_s was total we have convergence of $\{f_s\}_{s<\omega_1^{ck}}$ to f, then also we have convergence of $\{f'_s\}_{s<\omega_1^{ck}}$ to f.

5.4.1 Higher finite change approximations

Just like the ω-self-unclosed approximations are the self-unclosed approximation such that the number of change of any value above a correct prefix of the function is finite, we define the higher finite-change approximations to be the closed approximations such that the number of change of any value is finite:

Definition 5.4.2. *A function f has a **higher finite-change approximation** if it has a Δ_2^0 approximation $\{f_s\}_{s<\omega_1^{ck}}$ such that for any n, the value $f_s(n)$ can change at most finitely often, that is, the set $\{s : f_s(n) \neq f_{s+1}(n)\}$ is finite.* ∎

Fact 5.4.1

A function f has a **finite-change approximation** iff f has an approximation $\{f_s\}_{s<\omega_1^{ck}}$ so that for any limit ordinal s we have $f_s = \lim_{t<s} f_t$.

Suppose that f has a finite-change approximation. Without loss of generality, we can consider that the approximation is partially continuous. Now because the approximation is a finite-change approximation, for every n and s we have that $\{f_t(n)\}_{t<s}$ converges, we then have $f_s = \lim_{t<s} f_t$. The other direction is similar.

We already know that a function is higher Δ_2^0 iff it is higher Turing computable by $\mathcal{O}$. We also have that a function if higher ω-c.a. iff it is higher truth-table-computable by $\mathcal{O}$. We should establish here an equivalence in terms of higher Turing computability, of the notion of finite-change approximable functions:

Proposition 5.4.1:
Let f be function. Then the following are equivalent:

1. *The function f has a finite change approximation.*
2. *The function f is higher Turing computable by $\mathcal{O}$ with a higher functional Φ which is defined on any subset of $\mathcal{O}$.*

PROOF: Suppose that f is higher Turing reducible to $\mathcal{O}$ with a higher functional Φ which is defined on any subset of $\mathcal{O}$, then as $\mathcal{O}_s \subseteq \mathcal{O}$ we can define $f_s = \Phi(\mathcal{O}_s)$. Also because Φ is continuous we have $f_s = \Phi(\lim_{t<s} \mathcal{O}_t) = \lim_{t<s} \Phi(\mathcal{O}_t) = \lim_{t<s} f_t$. Thus we have that each value $f_s(n)$ changes at most finitely often.

Let us now suppose that f has a Δ_2^0 approximation $\{f_s\}_{s<\omega_1^{ck}}$ such that for each n, the number of change of $f(n)$ is finite. In order to compute f we can ask to $\mathcal{O}$, for any (n, m) if the value of $f_s(n)$ will change at least m times. This is a Π_1^1 question and then the answer is yes iff some bit of $\mathcal{O}$ is 1. As in the proof of Proposition 4.4.2 we continue to ask the question until the answer is no. It is clear that on every subset of $\mathcal{O}$, the process will halt, and that the final answer we get on the number of change of $f(n)$, denoted by m, is always smaller than or equal to its actual number of changes. Therefore the process of approximating f until we get m changes, always stops, on any subset of $\mathcal{O}$. ∎

By definition, every higher ω-computable approximation of a function f is also a higher finite-change approximation of f. It is easy to prove, by transposing a well-known technique of classical computability to the higher setting, that the converse does not hold.

Proposition 5.4.2:
There is a function f with a higher finite-change approximation, but no higher ω-c.a. approximation.

Proof: There is a classical trick in computability theory which can be used to enumerate a uniform list of the approximations of every ω-c.a. function. That is, we can compute a c.e. sequence $\{f_{n,t}\}_{n,t<\omega}$ such that for every n, the sequence $\{f_{n,t}\}_{t<\omega}$ converges to some f_n with the number of changes of each $f_n(m)$ bounded by a computable function of m; and such that every ω-c.a. function is listed. This is done by listing all pairs $(h, \{g_t\}_{t<\omega})$ where h is a partial computable function and $\{g_t\}_{t<\omega}$ a sequence of computable function, that we can without loss of generality suppose total. Now given a pair $(h, \{g_t\}_{t<\omega})$, we can uniformly define the ω-c.a. function f, whose approximation is equal on the input m, to 0 as long as $h(m)$ does not halt, and is equal to the successive versions of $g_t(m)$, with up to $h(m)$ changes otherwise.

We can apply the same technique with all the pairs $(h, \{g_t\}_{t<\omega_1^{ck}})$ where h is a partial Π_1^1 function and $\{g_t\}_{t<\omega_1^{ck}}$ a sequence of total Δ_1^1 functions. Then a mere diagonalization can be used to build f with a higher finite-change approximation, but with no higher ω-computable approximation. ∎

5.4.2 Higher closed unbounded approximations

We now investigate a new notion, which on elements of the Baire space is incomparable with higher Δ_2^0, but which lies between ω-self-unclosed and self-unclosed approximations once restricted to $\{0, 1\}$-valued functions. We use a well-known tool of set theory: the notion of **closed unbounded** set of ordinals, also called **club sets**. Here by unbounded, we mean unbounded below ω_1^{ck}, and closed means topologically closed for the order topology on ordinals, that is, if a sequence of ordinals $s_1 < s_2 < s_3 < \dots$ is in our set, then the ordinal $\sup_n s_n$ is also in our set.

Definition 5.4.3. *A sequence $\{f_s\}_{s<\omega_1^{ck}}$ is said to be a* ***closed unbounded approximation*** *of f if for any n, the set $\{s : f_s\restriction_n = f\restriction_n\}$ is closed unbounded.* ∎

With this definition, $\{f_s\}_{s<\omega_1^{ck}}$ does not need to converge. It is however straightforward to verify that the defined function f is unique because for $\sigma_1 \neq \sigma_2$ we cannot have that both sets $\{s \ : \ f_s \upharpoonright_n = \sigma_1\}$ and $\{s \ : \ f_s \upharpoonright_n = \sigma_2\}$ are closed unbounded, as otherwise their intersection would be non-empty (actually closed unbounded) which is a contradiction. Also we will see that when f is $\{0,1\}$-valued, it is possible to transform closed unbounded approximations of f into higher Δ^0_2 approximations of f. On the other hand, if f can take its values in ω, it needs not even be Δ^0_2. However, we still have that any function with a closed unbounded approximation collapses ω_1^{ck}.

Proposition 5.4.3:
If f is not Δ^1_1 and has a closed unbounded approximation, then $\omega_1^f > \omega_1^{ck}$.

Proof: The proof is almost exactly the same as the one of Theorem 4.4.1. We can define the $\Pi^1_1(f)$ total function $g : \omega \to \omega_1^{ck}$ which to n associates the smallest ordinal s_n so that $f_{s_n} \upharpoonright_n = f \upharpoonright_n$. If $s = \sup_n s_n < \omega_1^{ck}$ then $f_s = f$ and f is then Δ^1_1. Therefore we have $\sup_n s_n = \omega_1^{ck}$. Also as g is $\Pi^1_1(f)$ and total, it is also $\Delta^1_1(f)$. Then we have defined a $\Delta^1_1(f)$ sequence of computable ordinals, unbounded in ω_1^{ck} which implies that $\omega_1^f > \omega_1^{ck}$. ∎

In particular, as the set of Π^1_1-randoms is a Σ^1_1 set whose every element is not Δ^1_1 and preserves ω_1^{ck}, by the Gandy basis theorem we can find a higher Δ^0_2 sequence which is not Δ^1_1 and preserves ω_1^{ck}, which proves the existence of higher Δ^0_2 functions with no closed unbounded approximation. We shall now prove that the two notions are incomparable, by building a function with a closed unbounded approximation, but no higher Δ^0_2 approximation:

Proposition 5.4.4:
There is some f with a closed unbounded approximation, but no higher Δ^0_2 approximation.

Proof: Recall that $p : \omega_1^{ck} \to \omega$ is the projectum function. Let $\{f_{n,s}\}_{n<\omega, s<\omega_1^{ck}}$ be a list of all (possibly non-converging) Δ^0_2 approximations. We will build an 'approximation' $\{g_s\}_{s<\omega_1^{ck}}$ of a function g such that for any n, if $\{f_{n,s}(n)\}_{s<\omega_1^{ck}}$ converges to $f_n(n)$ then $g(n) \neq f_n(n)$; and for any n, the set $\{s \ : \ g_s(n) = g(n)\}$ is closed unbounded. It will follow that for any n the set $\{s \ : \ g_s \upharpoonright_n = g \upharpoonright_n\}$ is closed unbounded.

The construction:

At stage 0 we set $g_0(n)$ to be 0. At successor stage s, for any n, if $g_{s-1}(n) = 0$ and $f_{n,s}(n) = 0$ we set $g_s(n)$ to be equal to $p(s)$. If $g_{s-1}(n) = m \neq 0$ and $f_{n,s}(n) = m$ we set $g_s(n)$ to be equal to 0. Otherwise we set $g_s(n) = g_{s-1}(n)$.

At limit stage s, for any n, if $\{g_t(n)\}_{t<s}$ converges then we set $g_s(n)$ to be the convergence value. Otherwise we set it to 0.

The verification:

Let us prove that for every n, there is an m such that $\{s : g_s(n) = m\}$ is closed unbounded. It is clear because oscillations of $g_s(n)$ are only between 0 and other values, and because at limit stage s, if we do not have convergence of $\{g_s(n)\}_{t<s}$, then $g_s(n)$ is always 0. So either we have oscillations between 0 and something else, unboundedly below ω_1^{ck} (or we have $g_s(n) = 0$ for every stages $s \geq t$ for some t) in which case we take $m = 0$, and both the unbounded and the closed requirement for the set $\{s : g_s(n) = m\}$ are satisfied; or there is a smallest stage s such that $g_s(n) = p(s)$ and such that $g_t(n) = p(s)$ for every $t \geq s$, in which case $m = p(s)$. This implies for this case that the unbounded requirement is satisfied. Also as in this case s is the first ordinal such that $g_s(n) = m$, the closed requirement is satisfied, as the set $\{t < s : g_t(n) = m\}$ is empty and the set $\{t \geq s : g_t(n) = m\}$ is equal to everything. The function g is then defined by taking for each n this corresponding value m.

It is clear by construction that as long as $f_{n,s}(n)$ converges, we have $g(n) \neq f_n(n)$. ■

5.4.3 $(\omega + 1)$-self-unclosed approximations

If a function with a closed unbounded approximation is $\{0,1\}$-valued, it is possible to transform it into higher Δ_2^0 approximations of f, where the number of changes in the approximation above any prefix (including the correct one) can be infinite, but just once. We call such approximations $(\omega + 1)$-self-unclosed approximations:

Definition 5.4.4. *A function f has a $(\omega+1)$-**self-unclosed approximation** $\{f_s\}_{s<\omega_1^{ck}}$ if for any n, if there is an infinite sequence of ordinals $s_0 < s_1 < \dots$ such that $f_{s_i} \restriction_n = \sigma$ and such that $f_{s_i}(n) \neq f_{s_{i+1}}(n)$ for all i, then for any stage $t \geq \sup_i s_i = s$ such that $f_t \restriction_n = \sigma$, we have $f_t(n) = f_s(n)$.* ■

Proposition 5.4.5:
A $\{0,1\}$-valued function f that has a closed unbounded approximation, also has a $(\omega+1)$-self-unclosed approximation.

Proof: Let $\{f_s\}_{s<\omega_1^{ck}}$ be a closed unbounded approximation of a $\{0,1\}$-valued function f. We can suppose without loss of generality that each f_s is $\{0,1\}$-valued. We transform the sequence $\{f_s\}_{s<\omega_1^{ck}}$ into a $(\omega+1)$-self-unclosed approximation $\{g_s\}_{s<\omega_1^{ck}}$ of the same function f. We keep track of a set of 'banished' strings, which is empty at first. In what follows, for a set of ordinals A, we say that s is limit in A if there is no largest ordinal strictly smaller than s in A. Otherwise s is successor in A.

The construction:

At stage s, inductively for every n, let A_s be the set of stages t smaller than s such that $g_t \restriction_n = g_s \restriction_n$.

If A_s is empty we set $g_s(n) = f_s(n)$. If s is successor in A_s and if $g_s \restriction_n \hat{}\, i$ is banished for $i = 0$ or $i = 1$, we then set $g_s(n)$ to be the unbanished value. Otherwise, if $g_s \restriction_n = f_s \restriction_n$, we set $g_s(n) = f_s(n)$, and if $g_s \upharpoonright_n \neq f_s \upharpoonright_n$ we set $g_s(n)$ to be $g_t(n)$ where s is the successor of t in A_s. If s is the first such stage, then we set $g_s(n) = 0$.

If s is limit in A, if $\{g_t(n)\}_{t\in A}$ converges, we set $g_s(n)$ to be the convergence value. Otherwise we set $g_s(n)$ to be $f_s(n)$ and we banish the string $g_s \upharpoonright_n \hat{}\, i$ for $i \neq f_s(n)$.

The verification:

The use of the set of banihsed strings in the construction clearly ensures that $\{g_s\}_{s<\omega_1^{ck}}$ is $(\omega+1)$-self-unclosed. We should now prove that $\{g_s\}_{s<\omega_1^{ck}}$ converges to f.

Let us first prove the following: If a string σ is banished at stage s then $\sigma \nprec f$. So suppose that $\sigma \hat{}\, i$ is banished at stage s where σ is possibly the empty word. In particular s is the smallest stage such that $\{g_t(n)\}_{t\in A_s}$ does not converge (recall that A_s be the set of stages smaller than s such that $g_t \upharpoonright_n = g_s \upharpoonright_n = \sigma$). Let t_1 be the smallest stage of A_s, then t_2 be the smallest stage of A_s bigger than t_1 such that $g_{t_1}(n) \neq g_{t_2}(n)$, and let t_i be defined analogously for any $i \in \mathbb{N}$. We have by minimality of s that $\sup_i t_i = s$, and by construction and minimality of t_i, each t_i is necessarily successor in A_s. Also by minimality of each t_i we have $g_{t_i}(n) \neq g_{t'_i}(n)$ where t'_i is the predecessor of t_i in A_s. This implies by construction that $f_{t_i}(n) = g_{t_i}(n)$ and $f_{t_i} \upharpoonright_n = g_{t_i} \upharpoonright_n = \sigma$ for any i. But then also we have a set of stages $B \subseteq A_s$, unbounded in A_s such that $f_t \upharpoonright_n = \sigma$ for $t \in B$ and such that $\{f_t(n)\}_{t\in B}$ does not converges. Therefore $f \upharpoonright_{n+1} \neq \sigma \hat{}\, i$ where $i \neq f_s(n)$.

We can then prove by induction on n that for every n there is a stage r such that that $g_t \upharpoonright_n = f \upharpoonright_n$ for any stage $t \geq r$. Suppose $g_t \upharpoonright_n = f \upharpoonright_n$ for any t bigger than some stage r and let us prove that $g_t \upharpoonright_{n+1} = f \upharpoonright_{n+1}$ for t bigger than some stage $r' \geq r$. Either at some stage r', some string $g_r \upharpoonright_n \hat{}\, i$ for $i \in \{0,1\}$ is banished in which case $g_t \upharpoonright_{n+1} = f \upharpoonright_{n+1}$ for any $t \geq \sup(r, r')$, or it is not, which means that $g_t(n)$ changes only finitely often for $t \geq r$ and then that $g_t(n)$ is stable for any $t \geq r' \geq r$. Also this necessarily imply that $g_t \upharpoonright_{n+1} = f \upharpoonright_{n+1}$.■

It is clear that every ω-self-unclosed approximation is also $(\omega+1)$-self-unclosed. We should see later that the converse does not hold. For now, we prove that $(\omega+1)$-self-unclosed approximations are also self-unclosed approximations:

Proposition 5.4.6:
An $(\omega+1)$-self-unclosed approximation of a function f, is also a self-unclosed approximation of the function f.

Proof: Let us suppose that a function f, not Δ^1_1, has a $(\omega+1)$-self-unclosed approximation $\{f_t\}_{t<s}$. Suppose for contradiction that this approximation $\{f_t\}_{t<s}$ of f is not self-unclosed. Then there is a smallest stage s such that f is in the closure of $\{f_t\}_{t<s}$. As f is not Δ^1_1, there is a largest n such that $f_s \upharpoonright_n = f \upharpoonright_n$, with an infinite set of stages smaller than s, denoted by A, such that for any $t \in A$ we have $f_t \upharpoonright_n = f \upharpoonright_n$, but $\{f_t(n)\}_{t\in A}$ switches between $f(n)$ and another value unboundedly often below s. But then as $f_s(n) \neq f(n)$, and as the sequence $\{f_s\}_{s<\omega_1^{ck}}$ is $(\omega+1)$-self-unclosed, it cannot converge to f, which is a contradiction. ■

5.4.4 Separation of $(\omega+1)$-self-unclosed and ω-self-unclosed approximations

We prove here that the converse of Proposition 5.4.6 does not hold, that is, $(\omega+1)$-self-unclosed approximations give us strictly more power than just ω-self-unclosed approximations. But first let us argue that we have a uniform enumeration of (possibly non converging) Δ^0_2 approximations, that contains all the ω-self-unclosed approximations.

As we did after Lemma 5.4.1, given the code for a sequence of partial function $\{f_s\}_{s<\omega_1^{ck}}$, if f_s is not total we can replace it by f'_s where each bit of $f'_s(n)$ is equal to the convergence value of $\{f'_t(n)\}_{t<s}$ is it exists, and 0 otherwise. We easily verify that if by keeping only stages at which f_s is total, $\{f_s\}_{s<\omega_1^{ck}}$ is an ω-self-unclosed approximation, then also $\{f'_s\}_{s<\omega_1^{ck}}$ is an ω-self-unclosed approximation of the same function. We can now prove:

Proposition 5.4.7:
There is some $\{0,1\}$-valued function f with a $(\omega+1)$-self-unclosed approximation, but no ω-self-unclosed approximation.

Proof: Let $\{f_{n,s}\}_{n<\omega,s<\omega_1^{ck}}$ be a list of all (possibly non-converging) $\{0,1\}$-valued Δ^0_2 approximations and containnig all the ω-self-unclosed approximations. We build an $(\omega+1)$-self-unclosed $\{0,1\}$-valued approximation of a function g such that for every n, as long as $\{f_{n,s}\}_{n<\omega,s<\omega_1^{ck}}$ is an ω-self-unclosed approximation of a function f_n, we have $g\upharpoonright_{n+1}\neq f_n\upharpoonright_{n+1}$. We keep track of a set of banished strings that is initialized to the empty set.

The construction:

We start by setting $g_0(n)\neq f_{n,0}(n)$ for every n. At stage s, inductively for every n, if $g_s\upharpoonright_n\hat{\ }1$ is banished then set $g_s(n)=0$. Otherwise let A_s be the set of stages $t<s$ such that $g_t\upharpoonright_n = g_s\upharpoonright_n$. We say that any stage r is successor in A_s if there is a largest stage $t<r$ in A_s. Otherwise we say that r is limit in A_s.

If A_s is empty let $g_s(n)\neq f_{n,s}(n)$. If s is successor in A_s let t be the predecessor of s in A_s. If $g_s\upharpoonright_n\hat{\ }g_t(n)\neq f_{n,s}\upharpoonright_{n+1}$ let $g_s(n)=g_t(n)$. Otherwise let $g_s(n)\neq f_{n,s}(n)$. If s is limit in A_s and if $\{g_t(n)\}_{t\in A_s}$ does not converge we set $g_s(n)=0$ and we banish $g_s\upharpoonright_n\hat{\ }1$. Otherwise let i be the convergence value of $\{g_t(n)\}_{t\in A_s}$. If $g_s\upharpoonright_n\hat{\ }i\neq f_{n,s}\upharpoonright_{n+1}$ let $g_s(n)=i$, otherwise let $g_s(n)\neq f_{n,s}(n)$.

The verification:

It is clear by construction (with the system of banished strings) that $\{g_s\}_{s<\omega_1^{ck}}$ is an $(\omega+1)$-self-unclosed approximation of a function g. We should now prove that for any n, as long as $\{f_{n,s}\}_{s<\omega_1^{ck}}$ is an ω-self-unclosed approximation of a function f_n, we have $g\upharpoonright_{n+1}\neq f_n\upharpoonright_{n+1}$.

Let us suppose that $\{f_{n,s}\}_{s<\omega_1^{ck}}$ is an ω-self-unclosed approximation of a function f_n. In the construction, at any stage s, if $g_s\upharpoonright_n\hat{\ }1$ is not banished at stage s of before stage s, it is clear that we always have $g_s\upharpoonright_{n+1}\neq f_{n,s}\upharpoonright_{n+1}$. Also it would be enough to prove that if a string $\sigma\hat{\ }1$ is banished for any σ of length n, then $f_n\upharpoonright_n\neq\sigma$. So suppose that a string σ is banished at stage s. In particular, s is the smallest stage such that $g_s\upharpoonright_n=\sigma$ and such that $\{g_t(n)\}_{t\in A_s}$ does not converge. But by construction, this can happen only if there are infinitely many stages $r_0 < r_1 < \dots$ in A_s with $f_{n,r_i} \upharpoonright_n = \sigma$ and with $f_{n,r_{i+1}}(n) \neq f_{n,r_i}(n)$, which implies $f_n \upharpoonright_n \neq \sigma$. ■

5.4.5 Separation of $(\omega+1)$-self-unclosed and closed approximations

We will now show that the notion of $(\omega+1)$-self-unclosed approximation and the notion of closed approximation are incomparable. Theorem 5.3.3, that constructs a weakly-Π_1^1-random sequence with an ω-self-unclosed approximation, together with Theorem 5.3.1 saying that no sequence with a closed approximation is weakly-Π_1^1-random, implies that there is a $(\omega+1)$-self-unclosed function f that does not have a closed approximation. It remains to show that there is a function with a closed approximation that does not have a $(\omega+1)$-self-unclosed approximation. We will do so with a $\{0,1\}$-valued such function.

It does not seem possible to obtain a uniform list of the (possibly non-converging) $\{0,1\}$-valued Δ_2^0 approximations that contains every $(\omega+1)$-self-unclosed approximation, because at limit stage s, when a bit has changed infinitely often above some prefix, we need to decide for a value of that bit at stage s if it does not have one yet, but doing so we might pick the wrong one. However we easily see how to obtain (like we did after Lemma 5.4.1) a uniform list of the (possibly non-converging) $\{0,1\}$-valued Δ_2^0 approximations such that for any $(\omega+1)$-self-unclosed approximation, the list contains a $(\omega+2)$-self-unclosed approximation of the same function, where the notion of $(\omega+2)$-self-unclosed approximation is defined analogously to the one of $(\omega+1)$-self-unclosed approximation, but where a bit can change once more after it has changed infinitely often above some prefix. Also we shall now see that those approximations can be listed:

Proposition 5.4.8:
One can uniformly transform a (possibly non-converging) $\{0,1\}$-valued Δ_2^0 approximation $\{f_s\}_{s<\omega_1^{ck}}$ into a $\{0,1\}$-valued $(\omega+2)$-self-unclosed approximations $\{g_s\}_{s<\omega_1^{ck}}$ such that each $f_s = g_s$ if $\{f_s\}_{s<\omega_1^{ck}}$ was already an $(\omega+2)$-self-unclosed approximation.

Proof: We keep track of a set of banished strings, as well as a set of warned strings (strings which are about to be banished).

The construction:

At stage s, we define $g_s(n)$ inductively for every n the following way: Let A_s be the set of stages $t < s$ such that $g_t \upharpoonright_n = g_s \upharpoonright_n$. If $g_s \upharpoonright_n \hat{}\, i$ is banished for $i \in \{0,1\}$, then we set $g_s(n)$ to be $j \neq i$. Otherwise we set $g_s(n)$ to be $f_s(n)$. Furthermore if s is limit in A_s and if $\{g_t(n)\}_{t\in A_s}$ does not converge, we put a warning on the string $g_s \upharpoonright_{n+1}$. If s is successor in A_s (we denote its predecessor in A_s by $s-1$), if $g_s(n) \neq g_{s-1}(n)$ and if $g_{s-1} \upharpoonright_{n+1}$ has been warned, we banish $g_{s-1} \upharpoonright_{n+1}$.

The verification:

The system of banished strings ensures that $\{g_s\}_{s<\omega_1^{ck}}$ is $(\omega+2)$-self-unclosed. Also it is clear that if σ is the first string to be banished at some stage s, then it is because for infinitely many stages $r_1 < r_2 < \dots < r_\omega < s$ we have $f_{r_i} \upharpoonright_n = \sigma$ but $f_{r_i}(n) \neq f_{r_{i+1}}(n)$ and

$f_{r_\omega}(n) \neq f_s(n)$. Then if $\{f_s\}_{s<\omega_1^{ck}}$ is $(\omega+2)$-self-unclosed, for no stage $t \geq s$ we have $\sigma \prec f_t$. We can continue by induction on stages to show that for no string σ banished at stage s we have $\sigma \prec f_t$ for $t \geq s$ (as long as the approximation $\{f_s\}_{s<\omega_1^{ck}}$ is $(\omega+2)$-self-unclosed).■

Using the previous proposition we can obtain a uniform list of $(\omega+2)$-self-unclosed approximations such that for any $(\omega+1)$-self-unclosed approximation of a function f, the list contains a $(\omega+2)$-self-unclosed approximation of the same function. using this we now prove:

Proposition 5.4.9:
There is some $\{0,1\}$-valued function f with a closed approximation, but no $(\omega+1)$-self-unclosed approximation.

PROOF: Let $\{f_{n,s}\}_{n,\omega,s<\omega_1^{ck}}$ uniform list of $(\omega+2)$-self-unclosed approximations such that for any $(\omega+1)$-self-unclosed approximation of a function f, the list contains a $(\omega+2)$-self-unclosed approximation of the same function. The goal is to build a function g with a closed approximation, such that for each n there exists a k with $g(k) \neq f_n(k)$.

The construction:

At stage 0, for each n we set $k_0^n = n$. Then we define $g_0(k_0^n) \neq f_{n,0}(k_0^n)$. At successor stage s, we look for the smallest n so that $f_{n,s}(k_s^n)$ is equal to $g_{s-1}(k_s^n)$. If such a n does not exists, then for every $i \in \mathbb{N}$ we set $g_s(i) = g_{s-1}(i)$ and $k_s^i = k_{s-1}^i$. Otherwise for every $i \leq n$ we set $k_s^i = k_{s-1}^i$ and for every $i < k_s^n$ we set $g_s(i) = g_{s-1}(i)$. Then we set $g_s(k_s^n)$ to be another value than $f_{n,s}(k_s^n)$, among $\{0,1\}$. Then for every $i > n$ we set $k_s^i = k_{s-1}^i + 1$, we set $g_s(k_s^i)$ to be a value different from $f_{i,s}(k_s^i)$ among $\{0,1\}$ and for every $k_s^i < j < k_s^{i+1}$ we set $g_s(j) = g_{s-1}(j)$.

At a limit stage s, we look for the largest n so that $\{k_t^n\}_{t<s}$ converges to some k. Note that we always have convergence of $\{k_t^0\}_{t<s}$. If no largest such n exists, we set each $g_s(i)$ and k_s^i to be the convergence value of the sequences $\{g_t(i)\}_{t<s}$ and $\{k_t^i\}_{t<s}$. Among other things, we will verify later that in this case, those convergence values always exist. Otherwise, if a largest such n exists, then for every $i \leq n$ we set k_s^i to be the convergence value of $\{k_t^i\}_{t<s}$ and for every $i > n$ we set inductively $k_s^i = k_s^{i-1} + 1$. Then for every $i < k_s^n$ we set $g_s(i)$ to be the convergence value of $\{g_t(i)\}_{t<s}$, we set $g_s(k_s^n) = 0$ and for every $i > k_s^n$ we set $g_s(i)$ to be the convergence value of $\{g_t(i)\}_{t<s}$. Among other things, we will verify later that in this case, those convergence values always exist. Then at stage $s+1$ we keep the exact sames values, except for $g_{s+1}(k_s^n)$ which is set to 1. We then continue the algorithm directly at stage $s+2$.

The verification:

First, let us notice that the approximation of g is partially continuous. In particular if $\{g_t(m)\}_{t<s}$ does not converge for some limit stage s, it is because of what happens at previous successor stages. That is, we have stages $r < s$ unbounded in s, so that $g_r(m) \neq g_{r+1}(m)$.

Let us prove that at any limit stage s and for any n, if $\{k_t^n\}_{t<s}$ converges to k, then $\{g_t(m)\}_{t<s}$ converges for any $m < k$. Let us suppose that $\{k_t^n\}_{t<s}$ converges to k and that

$\{g_t(m)\}_{t<s}$ does not converges for $m < k$. By hypothesis, we have stages $r < s$ unbounded in s, so that $g_r(m) \neq g_{r+1}(m)$. Also by construction $g_r(m) \neq g_{r+1}(m)$ only for $m = k^i_{r+1}$ for some i. But we also have that $k^i_t < k^{i+1}_t$ for any i and any t. In particular for all those stages r we have $m = k^i_{r+1} < k^n_{r+1} = k$. But then, by construction we have that $k^n_{r+1} > k^n_r$ (Note that it is independent of whether or not $k^i_r = k^i_{r+1}$). In particular we have that $\{k^n_t\}_{t<s}$ does not converges, which is a contradiction.

Let us prove that at any limit stage s, if there exists a largest n so that $\{k^n_t\}_{t<s}$ converges to k, then $\{g_t(i)\}_{t<s}$ converges for every i different from k, and diverges for $i = k$. By the previous paragraph, we already have that $\{g_t(i)\}_{t<s}$ converges for any $i < k$. Now if we suppose for contradiction that $\{g_t(k)\}_{t<s}$ converges, then we also have by construction that $\{k^{n+1}_t\}_{t<s}$ converges, which contradicts the maximality of n. Therefore there are some stages r, unbounded in s such that $g_r(k) \neq g_{r+1}(k)$. Now let p be the smallest stage so that k^i_t and $g_t(j)$ do not change for $p \leq t < s$, $i \leq n$ and $j < k$. Let $r_0 \leq s$ be the smallest stage bigger than p so that $\{g_t(k)\}_{t<r_0}$ does not converge. If $r_0 < s$ we continue by defining similarly $r_1 \leq s$ to be the smallest stage bigger than r_0 such that$\{g_t(k)\}_{t<r_1}$ does not converge, and so on, defining r_{m+1} for $m \in \omega$, as long as $r_m < s$. Let us prove that for some $m < \omega$, the stage $r_m = s$. Suppose that $r_{2^k-1} < s$. By construction, at stage r_{2^k}, the value $f_n(k)$ have moved infinitely often above all the strings of length k. But then g can move at most once above all the strings of length k after stage r_{2^k} and then $r_{2^k} = s$. Therefore, there is some $m < \omega$ such that $r_m = s$. Then for every $j < m$ we have that $\{g_t(i)\}_{r_j \leq t < r_{j+1}}$ converges for $i > k$, because after the i-th time that $g_t(k)$ has changed, the values of $g_t(i)$ is fixed because k^{n+1}_t is then bigger than i. Also as this is true for every $j \leq m$ and as there are only finitely many of them, it follows that $\{g_t(i)\}_{t<r_m}$ converges for $i > k$.

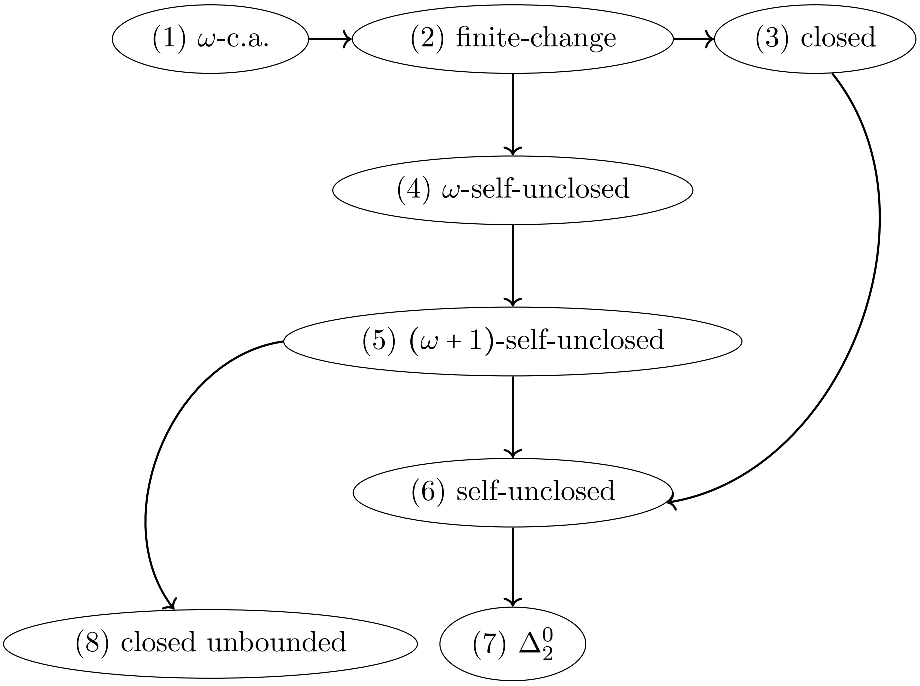
We should now prove that at each stage $s+1$ for s limit, we have that $\{g_t : t \leq s+1\}$ is a closed set. Suppose by induction that for any $r < s$ limit stage we have that $\{g_t : t \leq r+1\}$ is a closed set, by the previous paragraph, we have that the set $\{g_t : t < s\}$ has at most two limit points which are not limit points of any set $\{g_t : t \leq r+1\}$ for r limit below s. And also by construction, we have that g_s is the first of those limit points, and g_{s+1} the second of those limit points, if it exists. Therefore we have that $\{g_t : t \leq s+1\}$ is a closed set.

By a classic finite injury argument, we have that each sequence $\{k^n_t\}_{t<\omega_1^{ck}}$ and $\{g_t(n)\}_{t<\omega_1^{ck}}$ converges respectively to numbers k^n and $g(n)$. Let $\{s_n\}_{n<\omega}$ be a sequence of computable ordinals with $\sup_n s_n = \omega_1^{ck}$. By the previous paragraph we have that each $\{g_t : t \leq s_n + 1\}$ is also a closed set. As $\lim_{n<\omega} g_{s_n}$ converges to g, we also have that $\{g_t : t \leq \omega_1^{ck}\}$ is a closed set.

Also by construction, we have that $g(k^n)$ is different from $\lim_{t<\omega_1^{ck}} f_{n,t}(k^n)$ and the proof is complete. ■

5.4.6 A summary

We now sum up the different notions of being higher Δ^0_2 for elements of the Baire space:



All the implications are straightforward, except maybe for (5) → (6) which is proved in Proposition 5.4.6. All the implications are also strict.

- (2) $\not\to$ (1) is proved in Proposition 5.4.2.
- (3) $\not\to$ (5) is prove in Proposition 5.4.9 and implies that (3) $\not\to$ (2), (3) $\not\to$ (4) and (6) $\not\to$ (5).
- (5) $\not\to$ (4) is proved in Proposition 5.4.7.
- (4) $\not\to$ (3) is a consequence of Theorem 5.3.3, together with Theorem 5.3.1. It implies that (5) $\not\to$ (3) and (6) $\not\to$ (3).
- (7) $\not\to$ (6) is a consequence of Gandy Basis theorem, used to find Π_1^1-random which is higher Δ_2^0, combined with Theorem 4.4.1. It implies that (7) implies nothing smaller than (6).
- (8) $\not\to$ (7) is proved in Proposition 5.4.4. It implies that (8) implies nothing else.
- (7) $\not\to$ (8) is a consequence of Gandy Basis theorem, used to find Π_1^1-random which is higher Δ_2^0, combined with Proposition 5.4.3.

Chapter 6

Π^1_1-randomness and Σ^1_1-genericity

> Prenons par exemple la tâche de démontrer un théorème qui reste hypothétique (à quoi, pour certains, semblerait se réduire le travail mathématique). Je vois deux approches extrêmes pour s'y prendre. L'une est celle du marteau et du burin, quand le problème posé est vu comme une grosse noix, dure et lisse, dont il s'agit d'atteindre l'intérieur, la chair nourricière protégée par la coque. Le principe est simple : on pose le tranchant du burin contre la coque, et on tape fort. Au besoin, on recommence en plusieurs endroits différents, jusqu'à ce que la coque se casse - et on est content. Cette approche est surtout tentante quand la coque présente des aspérités ou protubérances, par où "la prendre". Dans certains cas, de tels "bouts" par où prendre la noix sautent aux yeux, dans d'autres cas, il faut la retourner attentivement dans tous les sens, la prospecter avec soin, avant de trouver un point d'attaque. Le cas le plus difficile est celui où la coque est d'une rotondité et d'une dureté parfaite et uniforme. On a beau taper fort, le tranchant du burin patine et égratigne à peine la surface - on finit par se lasser à la tâche. Parfois quand même on finit par y arriver, à force de muscle et d'endurance.
> Je pourrais illustrer la deuxième approche, en gardant l'image de la noix qu'il s'agit d'ouvrir. La première parabole qui m'est venue à l'esprit tantôt, c'est qu'on plonge la noix dans un liquide émollient, de l'eau simplement pourquoi pas, de temps en temps on frotte pour qu'elle pénètre mieux, pour le reste on laisse faire le temps. La coque s'assouplit au fil des semaines et des mois - quand le temps est mûr, une pression de la main suffit, la coque s'ouvre comme celle d'un avocat mûr à point ! Ou encore, on laisse mûrir la noix sous le soleil et sous la pluie et peut-être aussi sous les gelées de l'hiver. Quand le temps est mûr c'est une pousse délicate sortie de la substantifique chair qui aura percé la coque, comme en se jouant - ou pour mieux dire, la coque se sera ouverte d'elle-même, pour lui laisser passage.
>
> Récoltes et Semailles, Alexandre Grothendieck

6.1 The Borel complexity of the set of Π^1_1-randoms

We saw in Section 2.2.2 that classical randomness notions can be seen as genericity notions for a different topology. Similarly we will give here two equivalent genericity notions for respectively weak-Π^1_1-randomness and Π^1_1-randomness. This will allow us to conclude that the Borel complexity of the set of Π^1_1-randoms is $\mathbf{\Pi^0_3}$, and this will help us to answer later (see Section 6.5) the longstanding open question of what is lowness for Π^1_1-randomness.

Also our proof, together with a result of Liang Yu (see [71] and [96]), will imply that $\mathbf{\Pi^0_3}$ is the exact complexity of the set of Π^1_1-randoms.

Definition 6.1.1. *We say that X is **weakly-Σ^1_1-Solovay-generic** if it belongs to all sets of the form $\bigcup_n \mathcal{F}_n$ which intersect with positive measure all the Σ^1_1-closed sets of positive measure, where each $\mathcal{F}_n$ is a Σ^1_1-closed set uniformly in n.* ∎

Definition 6.1.2. *We say that X is **Σ^1_1-Solovay-generic** if for any set of the form $\bigcup_n \mathcal{F}_n$ where each $\mathcal{F}_n$ is a Σ^1_1-closed set uniformly in n, either X is in $\bigcup_n \mathcal{F}_n$ or X is in some Σ^1_1-closed set of positive measure $\mathcal{F}$, disjoint from $\bigcup_n \mathcal{F}_n$.* ∎

Proposition 6.1.1:
A sequence X is weakly-Σ^1_1-Solovay-generic iff it is weakly-Π^1_1-random.

Proof: Note first that X is weakly-Π^1_1-random iff it is in every uniform union of Σ^1_1-closed sets of measure 1. We shall prove that a uniform union of Σ^1_1-closed sets is of measure 1 iff it intersects with positive measure every Σ^1_1-closed set of positive measure.

Let us prove that a uniform union of Σ^1_1 closed sets of measure less than 1 cannot intersect all Σ^1_1-closed sets of positive measure. Let $\bigcup_n \mathcal{F}_n$ be a uniform union of Σ^1_1-closed sets of measure strictly smaller than 1. Let $\bigcap_n \mathcal{U}_n$ be its complement. We shall prove that already for some computable s we have that $\bigcap_n \mathcal{U}_{n,s}$ is of positive measure. We actually have that $\mathcal{A} = \bigcap_n \mathcal{U}_n - \bigcup_{s<\omega_1^{ck}} \bigcap_n \mathcal{U}_{n,s} \subseteq \{X \ : \ \omega_1^X > \omega_1^{ck}\}$. Indeed, if $X \in \mathcal{A}$ then the $\Pi^1_1(X)$ total function which to n associates the smallest s such that $X \in \bigcap_{m\leq n} \mathcal{U}_{m,s}$ has its range unbounded in ω_1^{ck}, implying that $\omega_1^X > \omega_1^{ck}$. Also using Theorem 3.7.3 saying that $\lambda(\{X \ : \ \omega_1^X > \omega_1^{ck}\}) = 0$ we then have $\lambda(\bigcap_n \mathcal{U}_n) = \lambda(\bigcup_{s<\omega_1^{ck}} \bigcap_n \mathcal{U}_{n,s})$, and as $\lambda(\bigcap_n \mathcal{U}_n) > 0$, there exists then some s such that $\lambda(\bigcap_n \mathcal{U}_{n,s}) > 0$. Also $\bigcap_n \mathcal{U}_{n,s}$ is a Δ^1_1 set of positive measure, and then by Theorem 1.8.1 there exists a Δ^1_1-closed set of positive measure $\mathcal{F} \subseteq \bigcap_n \mathcal{U}_{n,s} \subseteq \bigcap_n \mathcal{U}_n$. Thus $\bigcup_n \mathcal{F}_n$ does not intersect all Σ^1_1-closed sets of positive measure.

Conversely a uniform union of Σ^1_1-closed sets of measure 1 obviously intersects with positive measure any Σ^1_1-closed set of positive measure. Then the weakly-Σ^1_1-Solovay-generics are exactly the weakly-Π^1_1-randoms. ∎

We shall now prove that the notion of Σ^1_1-Solovay-genericity coincides with the notion of Π^1_1-randomness. We already know from Theorem 3.7.4 that if X is weakly-Π^1_1-random but not Π^1_1-random, then $\omega_1^X > \omega_1^{ck}$. We first should prove that if X is Σ^1_1-Solovay-generic then $\omega_1^X = \omega_1^{ck}$ (this is the difficult part of the equivalence).

Note first that $\omega_1^X > \omega_1^{ck}$ iff there is $a \in \mathcal{O}^X$ such that $|a|_o^X = \omega_1^{ck}$. In particular, $\omega_1^X > \omega_1^{ck}$ iff there is a Turing functional $\Phi : 2^{<\mathbb{N}} \times \mathbb{N} \to \mathbb{N}$ such that for any n we have $\Phi(X,n) \in \mathcal{O}^X_{<\omega_1^{ck}}$ and with $\sup_n |\Phi(X,n)|_o^X = \omega_1^{ck}$. We should show that if X is Σ^1_1-Solovay-generic and if we have some Φ such that $\Phi(X,n) \in \mathcal{O}^X_{<\omega_1^{ck}}$ for all n, then $\sup_n |\Phi(X,n)|_o^X < \omega_1^{ck}$. To show this we need an approximation lemma, which can be seen as an extension of Theorem 1.8.1, saying that any Δ^1_1 set can be approximated from below by a uniform union of Δ^1_1-closed sets of the same measure. We cannot extend this to all Σ^1_1 sets, but we can for a restricted type of Σ^1_1 set:

Lemma 6.1.1 *For a Σ^1_1 set $\mathcal{S} = \bigcap_{\alpha<\omega_1^{ck}} \mathcal{S}_\alpha$ where each $\mathcal{S}_\alpha$ is Δ^1_1 uniformly in α, one can find uniformly in an index for $\mathcal{S}$ and in any n, a Σ^1_1 closed set $\mathcal{F} \subseteq \mathcal{S}$ with $\lambda(\mathcal{S} - \mathcal{F}) \leq 2^{-n}$.*

PROOF: Recall that $p : \omega_1^{ck} \to \omega$ is the projectum function. Using Theorem 1.8.1, one can find uniformly in $\alpha < \omega_1^{ck}$ a Δ^1_1-closed set $\mathcal{F} \subseteq \mathcal{S}_\alpha$ such that $\lambda(\mathcal{S}_\alpha - \mathcal{F}_\alpha) \leq 2^{-p(\alpha)}2^{-n}$. We now define the Σ^1_1-closed set $\mathcal{F}$ to be $\bigcap_\alpha \mathcal{F}_\alpha$. We clearly have $\mathcal{F} \subseteq \mathcal{S}$ and we have:

$$\begin{array}{rcl} \lambda(\mathcal{S} - \mathcal{F}) & = & \lambda(\mathcal{S} - \bigcap_{\alpha<\omega_1^{ck}} \mathcal{F}_\alpha) \\ & = & \lambda(\bigcup_{\alpha<\omega_1^{ck}} (\mathcal{S} - \mathcal{F}_\alpha)) \\ & \leq & \lambda(\bigcup_{\alpha<\omega_1^{ck}} (\mathcal{S}_\alpha - \mathcal{F}_\alpha)) \\ & \leq & \sum_{\alpha<\omega_1^{ck}} \lambda(\mathcal{S}_\alpha - \mathcal{F}_\alpha) \leq 2^{-n}. \end{array}$$

We can now prove the desired theorem:

Theorem 6.1.1:
If Y is Σ^1_1-Solovay-generic then $\omega_1^Y = \omega_1^{ck}$.

PROOF: Suppose that Y is Σ^1_1-Solovay-generic. For any functional Φ, consider the set

$$\mathcal{P} = \{X \mid \forall n \;\; \exists \alpha < \omega_1^{ck} \;\; \Phi(X, n) \in \mathcal{O}^X_\alpha\}$$

Let $\mathcal{P}_n = \{X \mid \exists \alpha < \omega_1^{ck} \;\; \Phi(X, n) \in \mathcal{O}^X_\alpha\}$ and $\mathcal{P}_{n,\alpha} = \{X \mid \Phi(X, n) \in \mathcal{O}^X_\alpha\}$, so $\mathcal{P} = \bigcap_n \mathcal{P}_n$ and $\mathcal{P}_n = \bigcup_{\alpha<\omega_1^{ck}} \mathcal{P}_{n,\alpha}$.

Note that the complement of each $\mathcal{P}_n$ is a restricted type of Σ^1_1 set, on which we can then apply Lemma 6.1.1. So we can find uniformly in n a uniform union of Σ^1_1-closed sets included in $\mathcal{P}^c_n$ with the same measure as $\mathcal{P}^c_n$. From this we can find a uniform union of Σ^1_1-closed sets included in $\mathcal{P}^c$ with the same measure as $\mathcal{P}^c$. Suppose that Y is in $\mathcal{P}$. As it is Σ^1_1-Solovay-generic we have a Σ^1_1-closed set $\mathcal{F}$ of positive measure containing Y which is disjoint from $\mathcal{P}^c$ up to a set of measure 0, formally $\lambda(\mathcal{F} \cap \mathcal{P}^c) = 0$. In particular for each n we have $\lambda(\mathcal{F} \cap \mathcal{P}^c_n) = 0$ and then $\lambda(\mathcal{F}^c \cup \mathcal{P}_n) = 1$. Then let f be the Π^1_1 total function which to each pair $\langle n, m\rangle$ associates the smallest computable ordinal $\alpha < \omega_1^{ck}$ such that:

$$\lambda(\mathcal{F}^c_\alpha \cup \mathcal{P}_{n,\alpha}) > 1 - 2^{-m}$$

where $\{\mathcal{F}^c_\alpha\}_{\alpha<\omega_1^{ck}}$ is the co-enumeration of $\mathcal{F}^c$. Let $\alpha^* = \sup_{n,m} |f(n, m)|$. As f is total and Π^1_1, we have by Spector boundedness principle that $\alpha^* < \omega_1^{ck}$. Also

$$\begin{array}{rlcl} & \forall n \;\; \lambda(\mathcal{F}^c_{\alpha^*} \cup \bigcup_{\alpha<\alpha^*} \mathcal{P}_{\alpha,n}) & = & 1 \\ \to & \forall n \;\; \lambda(\mathcal{F}_{\alpha^*} \cap \bigcap_{\alpha<\alpha^*} \mathcal{P}^c_{\alpha,n}) & = & 0 \\ \to & \forall n \;\; \lambda(\mathcal{F} - \bigcup_{\alpha<\alpha^*} \mathcal{P}_{\alpha,n}) & = & 0 \\ \to & \lambda(\mathcal{F} - \bigcap_n \bigcup_{\alpha<\alpha^*} \mathcal{P}_{\alpha,n}) & = & 0 \end{array}$$

As Y is Σ^1_1-Solovay-generic it is in particular weakly-Σ^1_1-Solovay-generic and then weakly-Π^1_1-random. Thus by Theorem 3.7.6 it belongs to no Σ^1_1 set of measure 0. Then as $\mathcal{F} - \bigcap_n \bigcup_{\alpha<\alpha^*} \mathcal{P}_{\alpha,n}$ is a Σ^1_1 set of measure 0 we have that Y belongs to $\bigcap_n \bigcup_{\alpha<\alpha^*} \mathcal{P}_{\alpha,n}$ and then $\sup_n |\Phi(Y, n)|^Y_o \leq \alpha^* < \omega_1^{ck}$. ∎

We can now prove the equivalence:

Theorem 6.1.2:
The set of Σ^1_1-Solovay-generics coincides with the set of Π^1_1-randoms.

PROOF: Using Theorem 3.7.4 combined with the previous theorem, we have that the Σ^1_1-Solovay-generics are included in the Π^1_1-randoms. We just have to prove the reverse inclusion.

Suppose Y is not Σ^1_1-Solovay-generic. If $\omega_1^Y > \omega_1^{ck}$ then Y is not Π^1_1-random. Otherwise $\omega_1^Y = \omega_1^{ck}$ and also there is a sequence of Σ^1_1-closed sets $\bigcup_n \mathcal{F}_n$ of positive measure such that X is not in $\bigcup_n \mathcal{F}_n$ and such that any Σ^1_1-closed set of positive measure which is disjoint from $\bigcup_n \mathcal{F}_n$ does not contain Y. Let $\bigcap_n \mathcal{U}_n$ be the complement of $\bigcup_n \mathcal{F}_n$. As $\omega_1^Y = \omega_1^{ck}$ we have that $Y \in \bigcap_n \mathcal{U}_{n,s}$ for some computable ordinal s (the proof of this is like in the proof of Proposition 6.1.1). Also as $\bigcap_n \mathcal{U}_{n,s}$ is a Δ^1_1 set, either it is of measure 0 and then Y is not Δ^1_1-random, or it is of positive measure and can then be approximated from below, using Theorem 1.8.1 by a uniform union of Δ^1_1-closed sets, of the same measure. Also as Y is in none of them it is in their complement in $\bigcap_n \mathcal{U}_{n,s}$, which is a Δ^1_1-set of measure 0. Then Y is not Δ^1_1-random. ∎

The previous theorem gives a higher bound on the Borel complexity of the Π^1_1-randoms, and then on the Borel complexity of the biggest Π^1_1 nullset.

Corollary 6.1.1:
The set of Π^1_1-randoms is $\boldsymbol{\Pi^0_3}$.

The following result of Liang Yu (see [71]) can be used to prove that the set of Π^1_1-randoms is not $\boldsymbol{\Sigma^0_3}$.

Theorem 6.1.3 (Yu):
Let $\bigcap_n \mathcal{U}_n$ be a $\boldsymbol{\Pi^0_2}$ set containing only weakly-Π^1_1-randoms. Then the set

$$\{\mathcal{F} \mid \mathcal{F} \text{ is a } \Sigma^1_1\text{-closed set and } \bigcap_n \mathcal{U}_n \cap \mathcal{F} = \varnothing\}$$

intersects with positive measure any Σ^1_1-closed set of positive measure.

It follows that the set of weakly-Π^1_1-randoms cannot be $\boldsymbol{\Sigma^0_3}$ but also that the set of Π^1_1-randoms cannot be $\boldsymbol{\Sigma^0_3}$, and more generally:

Corollary 6.1.2:
No set $\mathcal{A}$ containing the set of Π^1_1-random sequences and contained in the set of weakly-Π^1_1-random sequences is $\mathbf{\Sigma^0_3}$.

Proof: Suppose that such a set $\mathcal{A}$ is equal to $\bigcup_n \bigcap_m \mathcal{U}_{n,m}$ each $\mathcal{U}_{n,m}$ being open. For each n let $\mathcal{B}_n = \bigcup\{\mathcal{F} \mid \mathcal{F}$ is a Σ^1_1-closed set and $\bigcap_m \mathcal{U}_{n,m} \cap \mathcal{F} = \varnothing\}$. We have $\bigcap_n \mathcal{B}_n \cap \bigcup_n \bigcap_m \mathcal{U}_{n,m} = \varnothing$. Also each set $\bigcap_m \mathcal{U}_{n,m}$ is a $\mathbf{\Pi^0_2}$ set containing only weakly-Π^1_1-randoms. Therefore by Theorem 6.1.3 we have that $\bigcap_n \mathcal{B}_n$ contains some Solovay-Σ^1_1-generic elements (some Π^1_1-random element), which contradicts that $\mathcal{A} = \bigcup_n \bigcap_m \mathcal{U}_{n,m}$ contains all of them.■

6.2 Randoms with respect to (plain) Π^1_1-Kolmogorov complexity

We can deduce from Corollary 6.1.2 another interesting corollary. Before stating it, we need to introduce a few notions. In classical randomness, we can define a non prefix-free Kolmogorov complexity $C : 2^{<\mathbb{N}} \to \mathbb{N}$, also called **plain complexity**. Also Miller [61] together with Nies, Stephan, and Terwijn [72] proved that a sequence X is 2-random iff infinitely many prefixes of X have maximal plain Kolmogorov complexity. We can make a similar definition in the higher setting:

Definition 6.2.1. *A Π^1_1-**machine** M is a Π^1_1 partial function $M : 2^{<\mathbb{N}} \to 2^{<\mathbb{N}}$. We denote by $\mathrm{hC}_M(\sigma)$ the Π^1_1-Kolmorogov complexity of a string σ with respect to the Π^1_1-machine M, defined to be the length of the smallest string τ such that $M(\tau) = \sigma$, if such a string exists, and by convention, ∞ otherwise.* ■

Just like we proved that there exists a universal Π^1_1-prefix-free machine (see Theorem 3.7.10) we can prove that there is a universal Π^1_1-machine (we leave the proof to the reader, as it is very similar to the proof of Theorem 3.7.10):

Theorem 6.2.1 (Universal Π^1_1-machine theorem):
There is a universal Π^1_1-machine U, that is, for each Π^1_1-machine M, there exists a constant c_M such that $\mathrm{hC}_U(\sigma) \leq \mathrm{hC}_M(\sigma) + c_m$ for any string σ.

We can then give a meaning to *the* Π^1_1-Kolmorogov complexity of a string:

Definition 6.2.2. *For a string σ, we define $\mathrm{hC}(\sigma)$ to be $\mathrm{hC}_U(\sigma)$ for a universal Π^1_1-machine U, fixed in advance.* ■

Let us now define the set $\mathcal{A}$ of sequences which have infinitely many prefixes of maximal Π^1_1-Kolmogorov complexity:

$$\mathcal{A} = \{X \mid \exists c\ \forall n\ \exists m \geq n\ \mathrm{hC}(X \upharpoonright_m) \geq m - c\}$$

It is clear that $\mathcal{A}$ is a Σ^1_1 set. It is also easy to prove that it is of measure 1. In particular it contains the set of Π^1_1-randoms, and we can also prove that it is contained in the set of Π^1_1-Martin-Löf random. However it follows directly from Corollary 6.1.2 that it does not coincide with the set of Π^1_1-randoms or with the set of weakly-Π^1_1-randoms:

Corollary 6.2.1:
The set $\mathcal{A}$ strictly contains the set of Π^1_1-randoms. The set $\mathcal{A}$ is not contained in the set of weakly-Π^1_1-randoms.

Proof: The set $\mathcal{A}$ is easily seen to be $\mathbf{\Sigma^0_3}$. The results follows then from Corollary 6.1.2.■

The following question remains open:

Question 6.2.1 *Does the set $\mathcal{A}$ contain the weakly-Π^1_1-randoms?*

6.3 Equivalent test notions for Π^1_1-randomness

We will now use Theorem 6.1.2 to give several equivalent definitions of Π^1_1-randomness.

6.3.1 First equivalence

Recall Theorem 2.1.5 of classical randomness: For a sequence Z Martin-Löf random the following are equivalent:

1. Z is weakly-2-random.
2. Z forms a minimal pair with $\emptyset^{(1)}$.
3. Z does not compute any non-computable c.e. set.

A first higher counterpart of (1) $\leftrightarrow$ (2) of Theorem 2.1.5 would be: 'For Z Π^1_1-Martin-Löf random, Z is weakly-Π^1_1-random iff Z forms a higher Turing minimal pair with Kleene's $\mathcal{O}$'. But this cannot be true, as by the Gandy Basis theorem, there is a Π^1_1-random, and therefore a weakly-Π^1_1-random, which is Turing computable by Kleene's $\mathcal{O}$.

A higher counterpart of (1) $\leftrightarrow$ (3) of Theorem 2.1.5 would be: 'For Z Π^1_1-Martin-Löf random, Z is weakly-Π^1_1-random iff Z does not higher Turing compute a Π^1_1 set which is non Δ^1_1'. Here again, we will see that this does not hold. One can easily see that the proof of direction (1) $\Longrightarrow$ (3) of theorem 2.1.5 does not work in the higher setting, as it uses a 'time trick'. We will indeed prove that this cannot be fixed, by proving that the correct higher counterpart of Theorem 2.1.5 is obtained by replacing weak-Π^1_1-randomness by Π^1_1-randomness. Thus the separation of the two notions, achieved in Section 5.3.2 implies in particular that there are some weakly-Π^1_1-random sequences which higher Turing computes Π^1_1 sets which are non Δ^1_1.

Theorem 6.3.1:
For a set Z Π^1_1-Martin-Löf random, the following are equivalent:

1. *Z is Π^1_1-random.*
2. *Z does not higher Turing compute a Π^1_1 sequence which is not Δ^1_1.*

PROOF: $(1) \Longrightarrow (2)$: This is the easy direction. Suppose that Z higher Turing computes a Π^1_1 sequence A which is not Δ^1_1. As A is Π^1_1, we have an approximation $\{A_s\}_{s<\omega_1^{ck}}$ of A such that for any limit ordinal s we have $\lim_{t<s} A_t = A_s$. As A is not Δ^1_1 it cannot be equal to A_s for some computable s. We can now define the $\Pi^1_1(A)$ total function $f : \omega \to \omega_1^{ck}$ by sending $f(n)$ to the smallest ordinal s such that $A_s \upharpoonright_n = A \upharpoonright_n$. Therefore we have $\sup_n f(n) = \omega_1^{ck}$. Also as A is higher Turing below Z we also have that f is $\Pi^1_1(Z)$, and as f is total it is also $\Delta^1_1(Z)$ and therefore the range of f is a $\Delta^1_1(Z)$ set of ordinals, cofinal in ω_1^{ck}, which implies that $\omega_1^Z > \omega_1^{ck}$.

$(2) \Longrightarrow (1)$: Suppose that Z is Π^1_1-Martin-Löf random but not Π^1_1-random. Then from Theorem 6.1.2 there is a uniform intersection of Π^1_1-open sets $\bigcap_n \mathcal{U}_n$ so that $Z \in \bigcap_n \mathcal{U}_n$ and so that no Δ^1_1-closed set $\mathcal{F} \subseteq \bigcap_n \mathcal{U}_n$ of positive measure contains Z. Then as Z is Δ^1_1-random we actually have that no Δ^1_1 closed set $\mathcal{F} \subseteq \bigcap_n \mathcal{U}_n$ contains Z. Let $\{W_e\}_{e<\omega}$ be an enumeration of the Π^1_1 subsets of ω. We will construct a Π^1_1 sequence A which is not Δ^1_1 and such that $Z \geq_{\mathrm{hT}} A$. The usual way to make A not Δ^1_1, is by meeting each requirement:

$$R_e : W_e \text{ infinite } \to A \cap W_e \neq \varnothing$$

making sure in the meantime that A is co-infinite.

Construction of A:

At stage s, at substage $\langle e, m, k\rangle$, if R_e is actively satisfied, go to the next substage, otherwise if $m \in W_e[s]$ with $m > 2e$, then consider the Δ^1_1 set $\bigcap_n \mathcal{U}_n[s]$ and compute an increasing union of Δ^1_1-closed sets $\bigcup_n \mathcal{F}_n$ with $\bigcup_n \mathcal{F}_n \subseteq \bigcap_n \mathcal{U}_n[s]$ and $\lambda(\bigcup_n \mathcal{F}_n) = \lambda(\bigcap_n \mathcal{U}_n[s])$.

If $\lambda(\mathcal{U}_m[s] - \mathcal{F}_k) \leq 2^{-e}$ then enumerate m into A at stage s, mark R_e as 'actively satisfied' and let $\mathcal{V}_{\langle m,e\rangle} = \mathcal{U}_m[s] - \mathcal{F}_k$.

This ends the algorithm. The sets $\mathcal{V}_{\langle m,e\rangle}$ are intended to form a higher Solovay test.

Verification that A is not Δ^1_1:

A is co-infinite because for each e at most one m is enumerated into A and this m is bigger than $2e$. Now suppose that W_e is infinite. By the Σ^1_1-boundedness principle there exists $s < \omega_1^{ck}$ so that $W_e[s]$ is infinite. Then there exists $t \geq s$ so that $\lambda(\bigcap_n \mathcal{U}_n - \bigcap_n \mathcal{U}_n[t]) < 2^{-e}$. Then there is a Δ^1_1-closed set $\mathcal{F}_k \subseteq \bigcap_n \mathcal{U}_n[t]$ so that $\lambda(\bigcap_n \mathcal{U}_n - \mathcal{F}_k) < 2^{-e}$. Then there exists an integer a such that for all $b \geq a$ we have $\lambda(\mathcal{U}_b - \mathcal{F}_k) < 2^{-e}$ and in particular $\lambda(\mathcal{U}_b[r] - \mathcal{F}_k) < 2^{-e}$ for any stage r. But as $W_e[t]$ is infinite we have some $m \in W_e[t]$ with $m > 2e$ such that $\lambda(\mathcal{U}_m[t] - \mathcal{F}_k) < 2^{-e}$. Then at stage t and substage $\langle e, m, k\rangle$, the integer m is enumerated into A good, if R_e is not met yet.

Verification that $\{\mathcal{V}_{\langle m,e\rangle}\}_{m,e\in\omega}$ is a higher Solovay test:

Note that each $\mathcal{V}_{\langle m,e\rangle}$ is well-defined uniformly in m and e. We implicitly have that $\mathcal{V}_{\langle m,e\rangle}$ enumerates nothing until the algorithm decides otherwise, which can happen at most once for a given pair (m, e), and even at most once for a given e, as when it happens, R_e is actively satisfied. Also as each $\mathcal{V}_{m,e}$ has measure smaller than 2^{-e}, we have a higher Solovay test.

Computation of A from Z:

We now just describe the algorithm to compute A from Z. The verification that the algorithm works as expected is given in the next paragraph. Let p be the smallest integer so that for any $m \geq p$, the set Z is in no $\mathcal{V}_{\langle m,e\rangle}$ for any e, which exists because Z passes the Solovay test $\mathcal{V}_{\langle m,e\rangle}$. To decide whether $m \geq p$ is in A, we look for the smallest s such that $Z \in \mathcal{U}_m[s]$. Then decide that m is in A iff m is in $A[s]$.

Verification that Z computes A:

Let p be the smallest integer so that for any $m \geq p$ the set Z is in no $\mathcal{V}_{\langle m,e\rangle}$ for any e. Suppose for contradiction that we have $m \geq p$ and $s < \omega_1^{ck}$ such that $Z \in \mathcal{U}_m[s]$ and $m \notin A[s]$, but $m \in A[t]$ for $t > s$. By construction, it means that we have some e and some Δ^1_1-closed set $\mathcal{F}_k \subseteq \bigcap_n \mathcal{U}_n$ with $\lambda(\mathcal{U}_m[t] - \mathcal{F}_k) < 2^{-e}$ and $\mathcal{V}_{\langle m,e\rangle} = \mathcal{U}_m[t] - \mathcal{F}_k$.

As Z does not belong to $\mathcal{V}_{\langle m,e\rangle}$ and does not belong to $\mathcal{F}_k$, it does not belong to $\mathcal{U}_m[t]$ which contradicts the fact that it belongs to $\mathcal{U}_m[s] \subseteq \mathcal{U}_m[t]$. ∎

Corollary 6.3.1:
Some weakly-Π^1_1-random computes a Π^1_1 set which is not Δ^1_1.

Proof: This follows from the previous theorem and from Theorem 5.3.3 saying that the set of Π^1_1-randoms is strictly included in the set of weakly-Π^1_1-randoms. ∎

6.3.2 Second equivalence

Theorem 6.3.1 can now be used to give another equivalent notion of test for Π^1_1-randomness, in the same spirit as the definition of higher difference randomness.

Theorem 6.3.2:
For a sequence X, the following are equivalent:

1. *X is captured by a set $\mathcal{F} \cap \bigcap_n \mathcal{U}_n$ with $\lambda(\mathcal{F} \cap \bigcap_n \mathcal{U}_n) = 0$ where $\mathcal{F}$ is a Σ^1_1 set and each $\mathcal{U}_n$ is a Π^1_1-open set uniformly in n.*

2. *X is not Π^1_1-random.*

3. *X is captured by a set $\mathcal{F} \cap \bigcap_n \mathcal{U}_n$ with $\lambda(\mathcal{F} \cap \bigcap_n \mathcal{U}_n) = 0$ where $\mathcal{F}$ is a Σ^1_1-closed set and each $\mathcal{U}_n$ is a Π^1_1-open set uniformly in n.*

PROOF: (1) $\Longrightarrow$ (2): Suppose first that X is captured by a set $\mathcal{F} \cap \bigcap_n \mathcal{U}_n$ of measure 0. Then either $\omega_1^X > \omega_1^{ck}$, in which case X is not Π^1_1-random, or there exists some stage s for which $X \in \bigcap_n \mathcal{U}_n[s]$. As also $X \in \mathcal{F}$ we then have $X \in \mathcal{U}_n[s] \cap \mathcal{F}$, which is a Σ^1_1 set of measure 0. Therefore X is not Δ^1_1-random and thus not Π^1_1-random.

(2) $\Longrightarrow$ (3): Suppose now that X is not Π^1_1-random. Then by Theorem 6.3.1, either it is not Π^1_1-Martin-Löf random, in which case we have (3) with $\mathcal{F} = 2^{\mathbb{N}}$ and $\{\mathcal{U}_n\}_{n<\omega}$ a Π^1_1-Martin-Löf test, or it higher Turing computes a Π^1_1 set Y which is not Δ^1_1, via a higher functional Φ. We define $\mathcal{U}_n = \bigcup_s \Phi^{-1}(Y_s \upharpoonright_n)$. We now define a Σ^1_1-closed set by defining its complement $\mathcal{F}^c$: We put in $\mathcal{F}^c$ at successor stage $s+1$, the open set $\Phi^{-1}(Y_s \upharpoonright_n)$ for every n as soon as we witness $Y_s \upharpoonright_n \neq Y_{s+1} \upharpoonright_n$. It follows that $\bigcap_n \mathcal{U}_n \cap \mathcal{F}$ contains only the sequences which higher Turing computes Y with the functional Φ, or some sequences on which Φ is not consistent. In particular, by Theorem 3.4.2, the set of sequences which higher Turing compute Y has measure 0. Therefore the measure of $\bigcap_n \mathcal{U}_n \cap \mathcal{F}$ is bounded by the measure of the inconsistency set of Φ.

Also recall Lemma 4.3.2 saying that uniformly in ε, we can obtain a version of Φ for which the inconsistency set of Φ has measure smaller than ε. We can then uniformly in ε define a uniform intersection of Π^1_1-open sets $\bigcap_n \mathcal{U}_n^\varepsilon$ such that $\lambda(\bigcap_n \mathcal{U}_n^\varepsilon \cap \mathcal{F}) \leq \varepsilon$. Note that we can keep the same set $\mathcal{F}$ for any ε. Then we have $\lambda(\bigcap_{\varepsilon,n} \mathcal{U}_n^\varepsilon \cap \mathcal{F}) = 0$ and $X \in \bigcap_{\varepsilon,n} \mathcal{U}_n^\varepsilon \cap \mathcal{F}$.

(3) $\Longrightarrow$ (1) is immediate. ■

6.3.3 Third equivalence

We now give a notion of test for Π^1_1-randomness, which has the same flavour as the notion of test defined in Theorem 5.3.2, proved to characterize weak-Π^1_1-randomness. Just like Theorem 5.3.2 generalizes the fact that no sequence with a closed approximation is weakly-Π^1_1-random, the following test notion generalizes the fact that no sequence with a self-unclosed approximation is Π^1_1-random.

Theorem 6.3.3:
For a sequence X, the following is equivalent:

1. *X is not Π^1_1-random.*
2. *X is captured by a set $\bigcap_n \mathcal{U}_{f(n)}$ with $\lambda(\mathcal{U}_{f(n)}) \leq 2^{-n}$, where f has a higher Δ^0_2 approximation $\{f_s\}_{s<\omega_1^{ck}}$ such that for every n, the set X is in at most finitely many versions $\mathcal{U}_{f_s(n)}$.*

PROOF: (2) $\Longrightarrow$ (1): This is the easy direction. Let $\bigcap_n \mathcal{U}_{f(n)}$ be a test which captures some X following the hypothesis of (2). Note that we can always suppose that the approximation of f is partially continuous, that is for s limit, if the limit of $\{f_t\}_{t<s}$ exists, then it is also equal to f_s. We can also always suppose that $\lambda(\mathcal{U}_{f_s(n)}) \leq 2^{-n}$ for any s and n, as it is harmless to trim $\mathcal{U}_{f_s(n)}$ if its measure becomes too big. Let us define the total $\Pi^1_1(X)$ function $g : \omega \to \omega_1^{ck}$ by:

$$\begin{aligned} g(0) &= 0 \\ g(n+1) &= \min\left\{ s > g(n) \;\middle|\; X \in \bigcap_{m \leq n} \mathcal{U}_{f_s(m)}[s] \right\} \end{aligned}$$

As the function g is total it is $\Delta^1_1(X)$. Suppose that we have $\sup_n g(n) = \omega_1^{ck}$ then the range of g is a $\Delta^1_1(X)$ set of ordinals cofinal in ω_1^{ck}, and therefore we have $\omega_1^X > \omega_1^{ck}$, implying (1). Suppose now that $\sup_n g(n) = s < \omega_1^{ck}$. Also for each m, there exists some n such that $f_{g(n)}(m) = f_{g(k)}(m)$ for any $k \geq n$, as otherwise X would be in infinitely many versions of $\mathcal{U}_{f_s(m)}$. Therefore $\lim_n f_{g(n)}$ exists and as the approximation is partially continuous, this limit is equal to f_s. But then $X \in \bigcap_m \mathcal{U}_{f_s(m)}$ and therefore it is not Π^1_1-Martin-Löf random.

(1) $\Longrightarrow$ (2): Suppose that X is not Π^1_1-random. If also X is not Π^1_1-Martin-Löf random then we have (2). Otherwise, by Theorem 6.3.1, the sequence X higher Turing computes some strictly Π^1_1 sequence A, via some functional Φ. Also the set $\bigcap_n \Phi^{-1}(A\upharpoonright_n)$ contains only sequences which higher Turing computes A or sequences on which Φ is not consistent. By Theorem 3.7.3 the measure of the set $\bigcap_n \Phi^{-1}(A\upharpoonright_n)$ is then bounded by the measure of the inconsistency set of Φ. Also recall that we showed in Lemma 4.3.2 that the measure of this set can be made as small as we want. For the rest of the proof we denote this set by $\mathcal{B}$.

Construction of f:

Let us define for each $s < \omega_1^{ck}$, each n and each m the Π^1_1-open set:

$$\mathcal{U}_{n,m,s} = \{\Phi^{-1}(A_s\upharpoonright_m) \mid \text{ with a version of } \Phi \text{ used so that } \lambda(\mathcal{B}) \leq 2^{-n}\}$$

Then let us define for each s, each n and each m the Π^1_1 open set $\mathcal{V}_{n,m,s}$ to be equal to $\mathcal{U}_{2n,m,s}$ truncated if necessary so that $\lambda(\mathcal{V}_{n,m,s}) \leq 2^{-n}$. We then define uniformly in $s < \omega_1^{ck}$ a Δ^1_1 function $g_s : \omega \to (\omega \times \omega_1^{ck})$. In what follows, if $g_s(n) = (m,t)$ then $g^1_s(n)$ refers to m and $g^2_s(n)$ refers to t.

$$\begin{array}{lcll}
g_0(n) & = & (0,0) & \\
g^1_{s+1}(n) & = & g^1_s(n) & \text{if} \quad \mathcal{V}_{n,g^1_s(n),g^2_s(n)}[s+1] \\
 & & & \quad\text{is not truncated so far.} \\
 & = & g^1_s(n)+1 & \text{otherwise} \\
g^2_{s+1}(n) & = & g^2_s(n) & \text{if} \quad A_{g^2_s(n)}\upharpoonright_{g^1_s(n)} = A_{s+1}\upharpoonright_{g^1_s(n)} \\
 & = & s+1 & \text{otherwise}
\end{array}$$

The computation of $g_s(n)$ at limit stage s needs a more complex definition. The reason for this more complex definition is that we do not want g to be able to 'come back to a previous value' (see further Claim 2). At limit stage s we check whether $\sup_{t<s} g^1_t(n) < \omega$. If so we first compute $A_s = \lim_{t<s} A_t$ (the limit is well defined as A has a Π^1_1 approximation), then we look for the smallest stage $t > s$ so that $\exists m \quad A_t\upharpoonright_m \neq A_s\upharpoonright_m$ and we look for the smallest such m. Such a stage t necessarily exists as A is not Δ^1_1. Then we set $g^1_s(n) = m$ and $g^2_s(n) = t$. Otherwise, if $\sup_{t<s} g^1_t(n) < \omega$ we set $g^1_s(n) = \sup_{t<s} g^1_t(n)$ and $g^2_s(n) = \sup_{t<s} g^2_t(n)$.

We finally define uniformly in $s < \omega_1^{ck}$ a function $f_s : \omega \to \omega$ by mapping $f_s(n)$ to the index of the open set $\mathcal{V}_{n,g^1_s(n),g^2_s(n)}$. The sequence $\{f_s\}_{s<\omega_1^{ck}}$ is intended to be a Δ^0_2 approximation that maches the hypothesis of (2). But first we have to prove the convergence of f:

Verification of the convergence of $\{f_s\}_{s<\omega_1^{ck}}$:

Claim 1: The sequence of pairs $\{g_s(n)\}_{s<\omega_1^{ck}}$ converges to a value $g(n)$ so that $\mathcal{V}_{n,g^1(n),g^2(n)}$ does not need to be truncated to have its measure smaller than 2^{-n} and so that $A_{g^2(n)}\upharpoonright_{g^1(n)}= A\upharpoonright_{g^1(n)}$.

Fix some n. There exists some smallest integer m so that we have $\lambda(\Phi^{-1}(A\upharpoonright_m)) \leq 2^{-n}$ as long as the version of Φ used is such that $\lambda(\mathcal{B}) \leq 2^{-2n}$. Let u_m be the smallest ordinal so that $A\upharpoonright_m= A_{u_m}\upharpoonright_m$. Note that u_m is necessarily a successor ordinal. Suppose first that we have $g^1_{u_m-1}(n) < m$. Then by definition of g, and by minimality of u_m and m, for $s \geq u_m$, the value $g^2_s(n)$ will never move anymore and the value $g^1_s(n)$ will move up to m. Suppose now that $g^1_{u_m-1}(n) \geq m$. Then by definition of g, and by minimality of u_m and m, for $s \geq u_m$, the value $g^1_s(n)$ will never move anymore (because by minimality of u_m we then have $g^2_{u_m}(n) = u_m$) and the value $g^2_s(n)$ will move until $A\upharpoonright_{g^1_s(n)}= A_{g^2_s(n)}\upharpoonright_{g^1_s(n)}$.

We can deduce that the sequence of functions $\{f_s\}_{s<\omega_1^{ck}}$ converges to some function f.

Verification that $X \in \bigcap_n \mathcal{U}_{f(n)}$:

Immediate from Claim 1.

Verification that for every m the set X is in finitely many versions of $\mathcal{U}_{f_s(m)}$:

Claim 2: The sequence $\{g_s(n)\}_{s<\omega_1^{ck}}$ 'never comes back to a previous value'. Formally if for a smallest stage $t > s$ we have $g_s(n) \neq g_t(n)$ then for any $u \geq t$ we have $g_s(n) \neq g_u(n)$.

If $g^2_s(n) \neq g^2_t(n)$ the Claim is immediate because the value $g^2(n)$ only increases. Otherwise, if $g^1_s(n) \neq g^1_t(n)$ but $g^2_s(n) = g^2_t(n)$, then by minimality of t, definition of $g(n)$ and the fact that $g^2_s(n) = g^2_t(n)$, we necessarily have that t is successor and $g^1_t(n) = g^1_s(n) + 1$. We can then prove by induction on stages u bigger than t that at least $g^1_s(n) < g^1_u(n)$ or $g^2_s(n) < g^2_u(n)$ (Note that the definition of g at limit stages is here important).

Claim 3: For any n and any sequence $s_1 < s_2 < \dots$ such that $\sup_m s_m = s < \omega_1^{ck}$ and such that for all i we have $g_{s_i}(n) \neq g_{s_{i+1}}(n)$, we have that X is in only finitely many $\mathcal{V}_{n,g^1_{s_i}(n),g^2_{s_i}(n)}$.

Suppose that Claim 3 is false for some n and that there exists a sequence $s_1 < s_2 < \dots$ such that $\sup_m s_m = s < \omega_1^{ck}$ and such that for all i we have $g_{s_m}(n) \neq g_{s_{m+1}}(n)$, with X in infinitely many $\mathcal{V}_{n,g^1_{s_m}(n),g^2_{s_m}(n)}$. Then using Claim 2, we can suppose without loss of generality that $X \in \mathcal{V}_{n,g^1_{s_m}(n),g^2_{s_m}(n)}$ for every s_m (still having $g_{s_m}(n) \neq g_{s_{m+1}}(n)$ for each m).

Let us first suppose for contradiction that the sequence $\{g^1_{s_m}(n)\}_{i\in\omega}$ is bounded. Then we must have infinitely many $g^1_{s_m}(n)$ which are equal to some integer k. But then using Claim 2, their corresponding values $g^2_{s_m}(n)$ must be all pairwise distinct. However by construction we have that $g^2_{s_i}(n) \neq g^2_{s_j}(n)$ while $g^1_{s_i}(n) = g^1_{s_j}(n) = k$ implies that $A_{g^2_{s_i}(n)}\upharpoonright_k \neq A_{g^2_{s_j}}\upharpoonright_k$. But the biggest set of pairwise distinct strings of length k is finite, which gives a contradiction.

Then for all k there is a i with $g^1_{s_i}(n)$ bigger than k. In particular there is an infinite subsequence $\{t_i\}_{i\in\omega}$ of the $\{s_i\}_{i\in\omega}$ so that $g^1_{t_i}(n) < g^1_{t_{i+1}}(n)$. Let $t = \sup_i g^2_{t_i}(n)$. We have that $A_t = \sup_i A_{g^2_{t_i}(n)}$ because A has a Π^1_1 approximation. Also as X belongs to all the

$\mathcal{V}_{n,g^1_{t_i}(n),g^2_{t_i}(n)}$ we have for each i that $X \in \Phi^{-1}(A_{g^2_{t_i}(n)} \upharpoonright_{g^1_{t_i}(n)})$. But for any k there exists i so that for all $j \geq i$ we have both $g^1_{t_j}(n) \geq k$ and $A_t \upharpoonright_k = A_{g^2_{t_j}(n)} \upharpoonright_k$. But then for any k we have $X \in \Phi^{-1}(A_t \upharpoonright_k)$ and then $\Phi(X) = A_t$. Also as A is not Δ^1_1 we have that $A \neq A_t$ and then $\Phi(X) \neq A$ which is a contradiction.

From claim 3 we can deduce that for each n the sequence X is in only finitely many versions of the open set indexed by $\{f_s(n)\}_{s<\omega_1^{ck}}$, which concludes the proof. ∎

6.4 A higher hierarchy of complexity of sets

The notion of weak-Π^1_1-randomness deals with uniform intersection of Π^1_1-open sets, the uniformity being along the natural numbers. Also one could think of iterating this notion. We could consider for example uniform union of uniform intersections of Π^1_1 open sets. Recall that we proved in Section 5.3.2 that weak-Π^1_1-randomness is strictly weaker than Π^1_1-randomness, that is, uniform intersections of Π^1_1-open sets, of measure 0, are not enough to cover the largest Π^1_1 nullset.

We shall see in this section that if we just allow a little bit more descriptional power to define our nullsets, that is allowing more successive intersection and union operations over Π^1_1-open sets, we can then define nullsets that capture every non Π^1_1-random sequence. We start by defining formally the new hierarchy on the complexity of sets, that we will use.

Definition 6.4.1. *A set is $\Sigma^{\omega_1^{ck}}_1$ if it is a Π^1_1-open set. It is $\Pi^{\omega_1^{ck}}_1$ if it is a Σ^1_1-closed set. It is $\Sigma^{\omega_1^{ck}}_{n+1}$ if it is an effective union over ω of a sequence of $\Pi^{\omega_1^{ck}}_n$ sets and it is $\Pi^{\omega_1^{ck}}_{n+1}$ if it is an effective intersection over ω of a sequence of $\Sigma^{\omega_1^{ck}}_n$ sets.* ∎

We did not iterate the definition through the computable ordinal, first because we will not use it, and then because it is not clear what should be the meaning of $\Sigma^{\omega_1^{ck}}_\omega$. Indeed, this new hierarchy has the unusual property that a $\Pi^{\omega_1^{ck}}_1$ set is not necessarily a $\Pi^{\omega_1^{ck}}_2$ set; more generally, a $\Pi^{\omega_1^{ck}}_n$ set is not necessarily $\Pi^{\omega_1^{ck}}_{n+p}$ for p odd, and a $\Sigma^{\omega_1^{ck}}_n$ set is not necessarily $\Sigma^{\omega_1^{ck}}_{n+p}$ for p odd. Indeed, $\Pi^{\omega_1^{ck}}_n$ sets for n odd and $\Sigma^{\omega_1^{ck}}_n$ for n even are all Σ^1_1 sets, but $\Pi^{\omega_1^{ck}}_n$ sets for n even and $\Sigma^{\omega_1^{ck}}_n$ for n odd are all Π^1_1 sets. We give here an illustration of this new hierarchy:

$$\begin{tikzcd}
\Sigma^{\omega_1^{ck}}_1 \arrow[dr] & \Sigma^{\omega_1^{ck}}_2 \arrow[dr] & \Sigma^{\omega_1^{ck}}_3 \arrow[dr] & \Sigma^{\omega_1^{ck}}_4 \arrow[dr] & \Sigma^{\omega_1^{ck}}_5 \arrow[dr] & \cdots \\
\Pi^{\omega_1^{ck}}_1 \arrow[ur] & \Pi^{\omega_1^{ck}}_2 \arrow[ur] & \Pi^{\omega_1^{ck}}_3 \arrow[ur] & \Pi^{\omega_1^{ck}}_4 \arrow[ur] & \Pi^{\omega_1^{ck}}_5 \arrow[ur] & \cdots
\end{tikzcd}$$

Figure 6.1: The higher hierarchy of complexity of sets.
The blue complexities correspond to Π^1_1 sets.
The green complexities correspond to Σ^1_1 sets.

Example 6.4.1:
Consider a Σ^1_1-closed set containing only higher Martin-Löf randoms. Such a set can be neither $\Pi_2^{\omega_1^{ck}}$, nor $\Sigma_3^{\omega_1^{ck}}$, etc..., because all those complexities characterizes Π^1_1 sets, and by Corollary 3.7.1, hyperarithmetic sequences form a basis for Π^1_1 sets of positive measure. ◇

Fact 6.4.1

The same phenomenon happens classically if one consider the Borel sets defined on some non Polish topological space. For example, let $\mathbb{T}(2^{\mathbb{N}})$ be the set of open sets of $2^{\mathbb{N}}$ and consider the topology on $\mathbb{T}(2^{\mathbb{N}})$ generated by the subbasis $[\![\sigma]\!] = \{\mathcal{U} \in \mathbb{T}(2^{\mathbb{N}}) : [\sigma] \subseteq \mathcal{U}\}$ for any string σ. Consider the closed set $\mathcal{F} = \{\mathcal{U} \in \mathbb{T}(2^{\mathbb{N}}) : [\sigma] - \mathcal{U} \neq \varnothing\}$ for a given string σ. Now as any open set in this topology contains the element $[\epsilon] = 2^{\mathbb{N}}$, also any intersection of open set contains $[\epsilon]$, which is not an element of $\mathcal{F}$.

With this higher complexity notion, we have by definition that any sequence is weakly-Π^1_1-random iff it is in no null $\Pi_2^{\omega_1^{ck}}$ set. The question we study here is :

What randomness notions do we obtain by considering null $\Pi_n^{\omega_1^{ck}}$ sets or null $\Sigma_n^{\omega_1^{ck}}$ sets?

Definition 6.4.2. *We say that X is $\Sigma_n^{\omega_1^{ck}}$-random, respectively $\Pi_n^{\omega_1^{ck}}$-random, if X is in no $\Sigma_n^{\omega_1^{ck}}$ nullset, respectively in no $\Pi_n^{\omega_1^{ck}}$ nullset.* ■

6.4.1 On the Σ^1_1 randomness notions in the higher hierarchy

It is clear that complexities corresponding to Σ^1_1 sets will give us a notion at least weaker than Σ^1_1-randomness and then than Δ^1_1-randomness. Concretely, the notion of being in no null $\Sigma_2^{\omega_1^{ck}}$ sets, or no null $\Pi_3^{\omega_1^{ck}}$ sets, etc... gives us a notion of randomness at least weaker than Σ^1_1-randomness. The notion of $\Pi_1^{\omega_1^{ck}}$-randomness has been studied by Kjos-hanssen, Nies, Stephan, and Yu in [36], under the name of Δ^1_1-Kurtz randomness. In particular they studied lowness for various notions of randomness, defined similarly to Δ^1_1-Kurtz randomness.

The notion of Δ^1_1-randomness where the Borel complexity of the null sets is restrained has also been studied by Chong, Nies and Yu in [7]. In particular, they observed that uniform intersection of Δ^1_1 open sets, effectively of measure 0, are enough to capture any non Δ^1_1-random. What we consider here is different, as we start our successive unions and intersections with Σ^1_1 closed sets.

Theorem 6.4.1:
We have:

$$\Pi_1^{\omega_1^{ck}}\text{-randomness} \leftrightarrow \Sigma_2^{\omega_1^{ck}}\text{-randomness} \leftarrow \Pi_3^{\omega_1^{ck}}\text{-randomness} = \Delta^1_1\text{-randomness}.$$

The reverse implication is strict. Also it follows from $\Pi_3^{\omega_1^{ck}}$-randomness $= \Delta_1^1$-randomness that $\Pi_{3+p}^{\omega_1^{ck}}$-randomness and $\Sigma_{2+p}^{\omega_1^{ck}}$-randomness for p even are all equivalent to Δ_1^1-randomness.

PROOF: It is clear that $\Pi_1^{\omega_1^{ck}}$-randomness is the same as $\Sigma_2^{\omega_1^{ck}}$-randomness, because in both cases the non random sequences are those which are in the union of all Σ_1^1-closed null sets.

Let us prove that $\Pi_3^{\omega_1^{ck}}$ nullsets are enough to cover any Δ_1^1 nullsets. Using Theorem 1.8.1 we can approximate from above any Δ_1^1 set by a uniform intersection of Δ_1^1-open sets $\bigcap_n \mathcal{U}_n$. Also as each $\mathcal{U}_n$ is Δ_1^1 uniformly in n, the predicate $\sigma \subseteq \mathcal{U}_n$ and the predicate $\sigma \nsubseteq \mathcal{U}_n$ are both Δ_1^1 which implies that we can easily define uniformly in n a Δ_1^1 total function $h_n : \omega \to 2^{<\omega}$ such that $\bigcup_m [h_n(m)] = \mathcal{U}_n$. We then define uniformly in (n,m) the Δ_1^1-closed set $\mathcal{F}_m^n$ to be $[h_n(m)]$. We then have $\bigcap_n \bigcup_m \mathcal{F}_m^n = \bigcap_n \mathcal{U}_n$.

Let us prove that $\Pi_1^{\omega_1^{ck}}$-randomness is strictly weaker than Δ_1^1-randomness. The proof is similar to the one that Kurtz-randomness (being in no Π_1^0 sets of measure 0) is strictly weaker than Martin-Löf randomness. We use here some Baire category notions: The set of $\Pi_1^{\omega_1^{ck}}$-randoms is a countable intersection of open sets of measure 1. Also it is clear that an open set of measure 1 is necessarily dense. But then this intersection contains some Cohen generic sequences for any notion of genericity which is strong enough. Also any X which is generic for even the weakest notion of genericity generally studied, namely weakly-1-generic, is not Martin-Löf random (because each open set of a universal Martin-Löf test is dense), and therefore certainly not Δ_1^1-random.

Now, as $\Pi_{3+p}^{\omega_1^{ck}}$ nullsets and $\Sigma_{2+p}^{\omega_1^{ck}}$ nullsets are all Σ_1^1 nullsets for p even, the corresponding randomness notions are all equivalent to Σ_1^1-randomness $= \Delta_1^1$-randomness. ∎

6.4.2 On the Π_1^1 randomness notions in the higher hierarchy

We know that the weakly-Π_1^1-randoms are exactly the elements which are $\Pi_2^{\omega_1^{ck}}$-random. Also it is clear that this notion coincides with $\Sigma_3^{\omega_1^{ck}}$-randomness, as in both case the non random elements are the unions of all the $\Pi_2^{\omega_1^{ck}}$ null sets. We shall now prove that $\Pi_4^{\omega_1^{ck}}$-randomness coincide with Π_1^1-randomness.

To do so, we will use Π_1^1 functionals Φ from $2^{\mathbb{N}}$ into sequences of computable ordinals, that is, $(\omega_1^{ck})^{\mathbb{N}}$. Concretely such a functional Φ is given by a Π_1^1 subset of $2^{<\mathbb{N}} \times \mathbb{N} \times \omega_1^{ck}$. We then say that Φ is defined on X, if for every n, there exists a unique α such that for some m we have $(X\upharpoonright_m, n, \alpha) \in \Phi$.

Note that just like for usual higher Turing reductions, we cannot guarantee that such a functional is consistent everywhere. Also if along some oracle X, some n is mapped to at least two distinct ordinals, then the functional is said to be inconsistent on X. It follows from the work of Chapter 7 that the inconsistency set cannot be completely removed, however, as in Lemma 4.3.2, it can be made of measure as small as we want. We will prove this formally in Lemma 6.4.1, but first we give a few notations.

The set of elements on which Φ is defined (and consistent) will be denoted by $\mathrm{Cdom}(\Phi)$. If for some X and n there is some α (not necessarily unique) such that $(X\upharpoonright_m, n, \alpha) \in \Phi$ for some m, we write $\Phi(X,n) = \alpha$. One can consider Φ^X as a multivalued function. Note that the equality symbol '=' used in the expression $\Phi(X,n) = \alpha$ does not mean that $\Phi(X,n)$ is equal to α in the strict sense of equality, but more than $\Phi(X,n)$ is mapped to α. Then the set of elements X such that for any n we have $\Phi(X,n) = \alpha$ for at least one α will be denoted by $\mathrm{dom}(\Phi)$. Formally:

$$\mathrm{dom}(\Phi) = \bigcap_n \{X \ : \ \exists m, \alpha_n \ (X\upharpoonright_m, n, \alpha_n) \in \Phi\}$$

One nice thing about $\mathrm{dom}(\Phi)$ is that it is a $\Pi_2^{\omega_1^{ck}}$ set, whereas $\mathrm{Cdom}(\Phi)$ is more complicated. We now prove, as a consequence of Theorem 6.3.1 (a sequence Z is Π^1_1-Martin-Löf random but not Π^1_1-random iff it higher Turing computes a strictly Π^1_1 sequence) that the measure of the inconsistency set of a functional Φ can be made as small as we want:

Lemma 6.4.1 *If Z is Π^1_1-Martin-Löf random and not Π^1_1-random, one can define uniformly in $\varepsilon \in \mathbb{Q}$ a Π^1_1 functional $\Phi \subseteq 2^{<\mathbb{N}} \times \mathbb{N} \times \omega_1^{ck}$ such that:*

- *Φ is defined (and consistent) on Z, and $\sup_n \Phi(Z,n) = \omega_1^{ck}$.*
- *The measure of the Π^1_1 open set on which Φ is not consistent is smaller than ε. Formally:*

$$\lambda(\{X \ : \ \exists n, m_1, m_2 \ \exists \alpha_1 \neq \alpha_2 \ \Phi(X\upharpoonright_{m_1}, n) = \alpha_1 \ \textit{and} \ \Phi(X\upharpoonright_{m_2}, n) = \alpha_2\}) \leq \varepsilon$$

Proof: From 6.3.1 we have a higher Turing functional Ψ so that $\Psi(Z) = A$ for A a Π^1_1 set which is not Δ^1_1. From Lemma 4.3.2, the measure of the inconsistency set of Φ can be made smaller than ε, uniformly in ε.

To define Φ, we enumerate (σ, n, α) in Φ if there exists τ of length bigger than n and α such that $(\sigma, \tau) \in \Psi$ and α is the first ordinal for which we have $\tau\upharpoonright_n = A_\alpha\upharpoonright_n$. We verify easily that such a functional Φ has the desired properties. ∎

Using those Π^1_1 functionals, we now state the following theorem, which is the heart of the proof that $\Pi_4^{\omega_1^{ck}}$-randomness coincide with Π^1_1-randomness.

Theorem 6.4.2:
For any Π^1_1 functional $\Phi \subseteq 2^{<\mathbb{N}} \times \mathbb{N} \times \omega_1^{ck}$, One can define, uniformly in an index for Φ, a $\Pi_4^{\omega_1^{ck}}$ nullset $\mathcal{A}$ such that $\{X \in \mathrm{Cdom}(\Phi) \ : \ \sup_n \Phi(X,n) = \omega_1^{ck}\} \subseteq \mathcal{A}$.

Before proving Theorem 6.4.2 we see some of its consequences, in particular using Lemma 6.4.1, it implies that $\Pi_4^{\omega_1^{ck}}$-randomness coincides with Π^1_1-randomness:

Theorem 6.4.3:
We have:

$$\Pi_2^{\omega_1^{ck}}\text{-randomness} \leftrightarrow \Sigma_3^{\omega_1^{ck}}\text{-randomness} \leftarrow \Pi_4^{\omega_1^{ck}}\text{-randomness} = \Pi_1^1\text{-randomness}.$$

The reverse implication is strict. Also it follows from $\Pi_4^{\omega_1^{ck}}$-randomness $=$ Π_1^1-randomness that $\Pi_{4+p}^{\omega_1^{ck}}$-randomness and $\Sigma_{3+p}^{\omega_1^{ck}}$-randomness are all equivalent to Π_1^1-randomness for p even and all weaker than Π_1^1-randomness for p odd.

Proof: Let us first prove that Theorem 6.4.2 implies that $\Pi_4^{\omega_1^{ck}}$-randomness $=$ Π_1^1-randomness. One direction is obvious as the largest Π_1^1 nullset covers any $\Pi_4^{\omega_1^{ck}}$ nullset. For the other direction, suppose that Z is not Π_1^1-random. If Z is not Π_1^1-Martin-Löf random it is by definition covered by a $\Pi_2^{\omega_1^{ck}}$ nullset. Otherwise we can define using Lemma 6.4.1 a Π_1^1 functional $\Phi \subseteq 2^{<\mathbb{N}} \times \mathbb{N} \times \omega_1^{ck}$ defined on Z, with $\sup_n \Phi(Z,n) = \omega_1^{ck}$. It follows using Theorem 6.4.2 that Z can be captured by a $\Pi_4^{\omega_1^{ck}}$ nullset.

It follows that $\Pi_2^{\omega_1^{ck}}$-randomness, corresponding to weak-Π_1^1-randomness, is strictly weaker than $\Pi_4^{\omega_1^{ck}}$-randomness, using Theorem 5.3.3 that separates weak-Π_1^1-randomness from Π_1^1-randomness. The fact that $\Sigma_3^{\omega_1^{ck}}$-randomness coincide with $\Pi_2^{\omega_1^{ck}}$-randomness is clear.

The rest of the proposition follows: For any n the null $\Sigma_n^{\omega_1^{ck}}$ or $\Pi_n^{\omega_1^{ck}}$ sets are either also null Π_1^1 sets, or covered by some null Π_1^1 sets. ∎

Corollary 6.4.1:
The set of Π_1^1-randoms is $\Pi_5^{\omega_1^{ck}}$.

Proof: We actually have an effective listing $\{\Phi_e\}_{e\in\mathbb{N}}$ of the Π_1^1 functionals $\Phi_e \subseteq 2^{<\mathbb{N}} \times \mathbb{N} \times \omega_1^{ck}$, as it is simply the listing of all the Π_1^1 subsets of $2^{<\mathbb{N}} \times \mathbb{N} \times \omega_1^{ck}$ (recall that inconsistency is allowed). Then using Theorem 6.4.2, we can define uniformly in e a $\Pi_4^{\omega_1^{ck}}$ null set $\mathcal{A}_e$ which captures:

$$\{X \in \mathrm{Cdom}(\Phi) \ : \ \sup_n \Phi_e(X,n) = \omega_1^{ck}\}$$

Also using Lemma 6.4.1 we know that as long as Z is not Π_1^1-random and Π_1^1-Martin-Löf random, it will be captured by some of those set $\mathcal{A}_e$. Therefore, the uniform union of all the sets $\mathcal{A}_e$, itself joined with the universal Π_1^1-Martin-Löf test, is a $\Sigma_5^{\omega_1^{ck}}$ nullset containing the biggest Π_1^1 nullset. And as a $\Sigma_5^{\omega_1^{ck}}$ set is itself Π_1^1, it actually coincides with the biggest Π_1^1 nullset. ∎

The rest of this section is dedicated to the proof of Theorem 6.4.2. So consider a Π_1^1 functional $\Phi \subseteq 2^{<\mathbb{N}} \times \mathbb{N} \times \omega_1^{ck}$. Let us fix some ε and let us assume that the inconsistency set of Φ has measure smaller than ε. From now on, the construction will remain uniform in Φ and then in ε.

The strategy:

The strategy is to define uniformly in each version of Φ that have an inconsistency set of measure smaller ε, a $\Pi_4^{\omega_1^{ck}}$ set $\mathcal{C}$ such that:

- $\{X \in \text{Cdom}(\Phi)\ :\ \sup_n \Phi(X,n) = \omega_1^{ck}\} \subseteq \mathcal{C} \subseteq \text{dom}(\Phi)$.
- $\{X \in \text{Cdom}(\Phi)\ :\ \sup_n \Phi(X,n) < \omega_1^{ck}\} \subseteq 2^{\mathbb{N}} - \mathcal{C}$.

In particular, it will follow that $\mathcal{C}$ contains either some element X such that $\omega_1^X > \omega_1^{ck}$, or some element $X \in \text{dom}(\Phi)$ such that Φ is not consistent on X. As by Theorem 3.7.3 the measure of the set of X such that $\omega_1^X > \omega_1^{ck}$ is null, it follows that the measure of $\mathcal{C}$ is bounded by ε, the measure of the inconsistency set of Φ. Also uniformly in ε we can define the $\Pi_4^{\omega_1^{ck}}$ set $\mathcal{C}_\varepsilon$ containing $\{X \in \text{Cdom}(\Phi)\ :\ \sup_n \Phi(X,n) = \omega_1^{ck}\}$ and of measure smaller than ε. It follows that the intersection over ε of the sets C_ε is a $\Pi_4^{\omega_1^{ck}}$ nullset containing $\{X \in \text{Cdom}(\Phi)\ :\ \sup_n \Phi(X,n) = \omega_1^{ck}\}$.

Some notations:

In what follows, we denote by R_e the e-th c.e. subset of $\mathbb{N} \times \mathbb{N}$, that is, $(n,m) \in R_e \leftrightarrow \langle n,m\rangle \in W_e$, where W_e is the usual e-th c.e. subset of $\mathbb{N}$. We will consider such a set as a c.e. binary relation. Also for a computable ordinal α we denote by R_α the c.e. binary relation coded by the smallest integer $a \in \boldsymbol{\mathcal{W}}$ such that $|a|_o = \alpha$. Recall the notation $\boldsymbol{\mathcal{W}}$ from Definition 1.4.1, denoting the set of codes for computable ordinals.

We also denote by $R_e\restriction_k$, the binary relation R_e restricted to elements 'smaller' than k in the sense of R, that is, the pair (n,m) is in $R_e\restriction_k$ iff the pair (m,k) and (n,m) are both in R_e ($(n,m) \in R_e$ is intended to be understood as $n < m$ in the sense of R_e). Note that $R_e\restriction_k$ is well defined for any e, but the underlying idea really makes sense when R_e represents an order, and we actually intend to use it only when R_e represents a linear order.

Finally, we say that a function $f : \mathbb{N} \to \mathbb{N}$ is a morphism from a linear order coded by a binary relation R_{e_1} to another linear order coded by a binary relation R_{e_2}, if f is total on $\text{dom}\, R_{e_1}$, with $f(\text{dom}\, R_{e_1}) \subseteq \text{dom}\, R_{e_2}$ and if $(x,y) \in R_{e_1} \to (f(x), f(y)) \in R_{e_2}$. Here $\text{dom}\, R_e$ denotes the set of integer a such that $(a,b) \in R_e$ or $(b,a) \in R_e$ for some b.

Definition of the $\Pi_4^{\omega_1^{ck}}$ set $\mathcal{C}$

We now do the proof of Theorem 6.4.2. Let us define uniformly in each integer e the sets $\mathcal{A}_e$ and $\mathcal{B}_e$:

$$\mathcal{A}_e = \left\{ X \in 2^{\mathbb{N}}\ :\ \begin{array}{l} \exists n\ \exists \alpha_n\ \Phi(X,n) = \alpha_n \text{ and} \\ \forall f\ f \text{ is not a morphism from } R_{\alpha_n} \text{ to } R_e \end{array} \right\}$$

and

$$\mathcal{B}_e = \left\{ X \in 2^{\mathbb{N}}\ :\ \begin{array}{l} \exists m\ \forall n\ \exists \alpha_n\ \Phi(X,n) = \alpha_n \text{ and} \\ \forall f\ f \text{ is not a morphism from } R_e\restriction_m \text{ to } R_{\alpha_n} \end{array} \right\}$$

Let us now define the Π_2^0 set G of integers e such that R_e is a linear order of $\mathbb{N}$. We finally define:

$$\mathcal{C} = \bigcap_{e \in G} (\mathrm{dom}(\Phi) \cap (\mathcal{A}_e \cup \mathcal{B}_e))$$

Proof that $\mathcal{C}$ is $\Pi_4^{\omega_1^{ck}}$:

We have that $\mathrm{dom}(\Phi)$ is $\Pi_2^{\omega_1^{ck}}$, that $\mathcal{A}_e$ is $\Sigma_1^{\omega_1^{ck}}$ uniformly in e and that $\mathcal{B}_e$ is $\Sigma_3^{\omega_1^{ck}}$ uniformly in e. Then the set $\mathrm{dom}(\Phi) \cap (\mathcal{A}_e \cup \mathcal{B}_e)$ is $\Sigma_3^{\omega_1^{ck}}$ uniformly in e. As G has a Π_2^0 description, we then have that $\mathcal{C}$ is a $\Pi_4^{\omega_1^{ck}}$ set.

Proof that $\mathcal{C}$ captures enough:

We should prove that $\{X \in \mathrm{Cdom}(\Phi) \ : \ \sup_n \Phi(X,n) = \omega_1^{ck}\} \subseteq \mathcal{C}$. Fix some $Z \in \mathrm{Cdom}(\Phi)$ and suppose that $\sup_n \Phi(Z,n) = \omega_1^{ck}$. Let us prove for any $e \in G$ that $Z \in \mathcal{A}_e \cup \mathcal{B}_e$. It will follow that $Z \in \mathcal{C}$.

Suppose first that R_e is a well-founded relation. As e is already in G we have that R_e is a c.e. well-ordered relation with $|R_e| < \omega_1^{ck}$. But then there is some n so that $\Phi(Z,n) = \alpha_n$ with $|\alpha_n| > |R_e|$ and we cannot have a morphism from R_{α_n} to R_e. Then $Z \in A_e$.

Suppose now that R_e is an ill-founded relation. There is then some m so that $R_e \upharpoonright_m$ is already ill-founded. But as R_{α_n} is well-founded for every $\alpha_n = \Phi(Z,n)$, then for every n we cannot have a morphism from $R_e \upharpoonright_m$ to R_{α_n}, and then $Z \in \mathcal{B}_e$.

Proof that $\mathcal{C}$ does not capture too much:

Let us now prove that for any $X \in \mathrm{Cdom}(\Phi)$, if $\sup_n \Phi(X,n) < \omega_1^{ck}$ then $X \notin \mathcal{C}$. Consider such a sequence X with $\sup_n \Phi(X,n) = \alpha < \omega_1^{ck}$. In particular there exists some integer $e \in G$ so that R_e is a well-order of order-type α. For this e we certainly have for all $\alpha_n = \Phi(X,n)$ a morphism from R_{α_n} into R_e and then $X \notin \mathcal{A}_e$.

Let us now prove that $X \notin \mathcal{B}_e$. For any m we have $|R_e \upharpoonright_m | < \alpha$. But because $\alpha = \sup_n \Phi(X,n)$ there is necessarily some n so that $\Phi(X,n) = \alpha_n > |R_e \upharpoonright_m|$. Thus there is a morphism from $R_e \upharpoonright_m$ into R_{α_n}. Then $X \notin \mathcal{B}_e$, and therefore $X \notin \mathcal{C}$. This ends the proof.

6.4.3 A lower bound on the higher complexity of randomness notions

We saw with Corollary 6.4.1 that the set of Π_1^1-randoms is $\Pi_5^{\omega_1^{ck}}$. Can it be made simpler? We will prove here that the set of weakly-Π_1^1-randoms and the set of Π_1^1-randoms both cannot be $\Pi_3^{\omega_1^{ck}}$. The question of whether the set of Π_1^1-randoms is $\Sigma_4^{\omega_1^{ck}}$ remains open, and the question of whether the set of weakly-Π_1^1-randoms is somewhere in the higher hierarchy also remains open.

Theorem 6.4.4:
For any $\Pi_3^{\omega_1^{ck}}$ set $\mathcal{A} = \bigcap_n \bigcup_m \mathcal{F}_{n,m}$ of measure 1, there is a sequence $X \in \mathcal{A}$ that has a finite change approximation.

PROOF: The proof can be considered to be a simpler version of the proof of the separation of Π_1^1-randomness from weak-Π_1^1-randomness.

Overview:

Let $\mathcal{A} = \bigcap_n \bigcup_m \mathcal{F}_{n,m}$ be of measure 1. Without loss of generality, we can suppose that for each n, the union $\bigcup_m \mathcal{F}_{n,m}$ is increasing. The goal is to find for each n some integer d^n and a string σ^n with $\sigma^0 \prec \sigma^1 \prec \sigma^2 \prec \dots$ such that for any n we have $\lambda(\bigcap_{m\le n} \mathcal{F}_{m,d^m} \cap [\sigma^n]) > 0$. It will follow that the limit point of $\{[\sigma^n]\}_{n\in\mathbb{N}}$ will be in $\bigcap_{m\le n} \mathcal{F}_{m,d^m}$ and then in $\mathcal{A}$.

The construction:

We can consider without loss of generality that $\mathcal{F}_{0,m} = 2^{\mathbb{N}}$ for any m. For any stage s we let $\mathcal{S}^0_s = \mathcal{F}_{0,0}$ and $\sigma^0_s = \epsilon$, the empty word. Note that we have $\lambda(\mathcal{S}^0_0) = \lambda(\mathcal{S}^0_0 \mid [\sigma^0_0]) = 1$.

Then at stage 0, for any $n \ge 1$ we set each σ^n_0 to be a range of n zeros and d^n_0 to be 0. At successor stage s, at substage $n+1$, let us suppose that we have already defined the set $\mathcal{S}^n_s$ and the string σ^n_s such that $\lambda(\mathcal{S}^n_s \mid [\sigma^n_s])[s] \ge 2^{-n}$. Let us define $\mathcal{S}^{n+1}_s$ and σ^{n+1}_s.

$$\lambda\left(\mathcal{S}^n_s \cap \mathcal{F}_{n+1,d^{n+1}_{s-1}} \;\middle|\; [\sigma^n_s]\right)[s] \ge 2^{-n-1}$$

we set $d^{n+1}_s = d^{n+1}_{s-1}$. Otherwise we set d^{n+1}_s to be the samllest integer d such that:

$$\lambda\left(\mathcal{S}^n_s \cap \mathcal{F}_{n+1,d} \;\middle|\; [\sigma^n_s]\right)[s] \ge 2^{-n-1}$$

Note that as by induction we have $\lambda(\mathcal{S}^n_s \mid [\sigma^n_s])[s] \ge 2^{-n}$, such an integer d exists, because $\lambda(\bigcup_m \mathcal{F}_{n,m}) = 1$ for every n. Then we set $\mathcal{S}^{n+1}_s$ to be $\mathcal{S}^n_s \cap \mathcal{F}_{n+1,d^{n+1}_s}$. Then if

$$\lambda\left(\mathcal{S}^{n+1}_s \;\middle|\; [\sigma^{n+1}_{s-1}]\right)[s] \ge 2^{-n-1}$$

we set $\sigma^{n+1}_s = \sigma^{n+1}_{s-1}$, otherwise we set σ^{n+1}_s to be $\sigma^n_s \hat{}\, i$ where $i \in \{0,1\}$ is such that:

$$\lambda\left(\mathcal{S}^{n+1}_s \;\middle|\; [\sigma^n_s \hat{}\, i]\right)[s] \ge 2^{-n-1}$$

Note that as by induction we have $\lambda(\mathcal{S}^{n+1}_s \mid [\sigma^n_s])[s] \ge 2^{-n-1}$, we easily verify that such an i always exists.

At limit stage s we set d^n_s to be the limit of $\{d^n_t\}_{t<s}$ for every n and σ^n_s to be the limit of $\{\sigma^n_t\}_{t<s}$ for every n. Among other things we should prove that the limit always exists.

The verification:

We should prove that at limit stage s, each $\{\sigma^n_t\}_{t<s}$ and $\{d^n_t\}_{t<s}$ converges. It is true for $n = 0$ as $\sigma^n_s = \sigma^n_0$ and $d^n_s = d^n_0$. Suppose it is true up to n and let us show it is true for $n+1$.

Let us first show that $\{d^{n+1}_t\}_{t<s}$ converges. Let r be the smallest stage such that for any $r \le t < s$ we have $\sigma^n_t = \sigma^n_r$ and $d^k_t = d^k_r$ for any $k \le n$. It follows that we have $\mathcal{S}^n_t = \mathcal{S}^n_r$ for $r \le t < s$. Let us argue that $\lambda(\mathcal{S}^n_r \mid [\sigma^n_r])[s] \ge 2^{-n}$. Suppose otherwise, then by compactness, there is a smallest successor stage t with $r < t < s$ such that $\lambda(\mathcal{S}^n_r \mid [\sigma^n_r])[t] < 2^{-n}$. But then by construction we have that $d^k_t \ne d^k_r$ or $\sigma^k_t \ne \sigma^k_r$ for some $k \le n$, which contradicts the hypothesis. Therefore $\lambda(\mathcal{S}^n_r \mid [\sigma^n_r])[s] \ge 2^{-n}$ and because $\lambda(\bigcup_m \mathcal{F}_{n+1,m}) = 1$, there exists some d such that:

$$\lambda\left(\mathcal{S}^n_r \cap \mathcal{F}_{n+1,d} \;\middle|\; [\sigma^n_r]\right)[s] \ge 2^{-n-1}$$

This implies that d_t^{n+1} is bounded by d for any $r \leq t < s$ and as d_t^{n+1} only increases, we have that $\{d_t^{n+1}\}_{t<s}$ converges.

We now prove similarly that $\{\sigma_t^{n+1}\}_{t<s}$ converges. Let r be the smallest stage such that for any $r \leq t < s$ we have $\sigma_t^n = \sigma_r^n$ and $d_t^k = d_r^k$ for any $k \leq n+1$. Also we have $\mathcal{S}_t^{n+1} = \mathcal{S}_r^{n+1}$ for $r \leq t < s$. Then by compactness again and by construction we have $\lambda(\mathcal{S}_r^{n+1} \mid [\sigma_r^n])[s] \geq 2^{-n-1}$. Therefore, for at least $i = 0$ or $i = 1$ we have $\lambda(\mathcal{S}_r^{n+1} \mid [\sigma_r^n \hat{\ } i])[s] \geq 2^{-n-1}$. It follows that σ_t^{n+1} can change at most once after stage r, to be equal to $\sigma_r^n \hat{\ } i$. Therefore $\{\sigma_t^{n+1}\}_{t<s}$ converges.

It follows that for any n and any stage s, there is no infinite sequence $r_1 < r_2 < \dots$ with $\sup_i r_i = s < \omega_1^{ck}$ such that $\sigma_{r_i}^n \neq \sigma_{r_{i+1}}^n$. Also by the Σ_1^1-boundedness principle we cannot have an infinite sequence $r_1 < r_2 < \dots$ with $\sup_i r_i = \omega_1^{ck}$ such that $\sigma_{r_i}^n \neq \sigma_{r_{i+1}}^n$. It follows that each sequence X_s, defined as the unique limit point of $\{[\sigma_s^n] \mid n \in \mathbb{N}\}$, converges to a sequence X and also that $\{X_s\}_{s<\omega_1^{ck}}$ is a finite change approximation of X.

Similarly we prove that each sequence $\{d_s^n\}_{s<\omega_1^{ck}}$ converges to an integer d^n and then that each $\{\mathcal{S}_s^n\}_{s<\omega_1^{ck}}$ converges to a Σ_1^1-closed set $\mathcal{S}^n$ with $\bigcap_n \mathcal{S}_n \subseteq \mathcal{A}$ and with $\lambda(\mathcal{S}^n \cap [X \upharpoonright_n]) > 0$ for each n. As each $\mathcal{S}_n$ is a closed set we have $X \in \bigcap_n \mathcal{S}^n \subseteq \mathcal{A}$. ∎

Corollary 6.4.2:
The set of Π_1^1-randoms and the set of weakly-Π_1^1-randoms are not $\Pi_3^{\omega_1^{ck}}$.

Proof: It is clear, as no sequence with a finite change approximation is weakly-Π_1^1-random (or Π_1^1-random). ∎

6.4.4 Open questions on higher complexity

It is tempting to try relaxing in Theorem 6.4.4 the hypothesis that the $\Pi_3^{\omega_1^{ck}}$ set $\mathcal{A}$ is of measure 1, by just considering that $\mathcal{A}$ is of positive measure. This indeed would imply that the set of weakly-Π_1^1-randoms and the set of Π_1^1-randoms are both not $\Sigma_4^{\omega_1^{ck}}$.

However, the measure 1 hypothesis seems crucial to conduct the proof of Theorem 6.4.4, and it does not seems that this technique could be used to prove that the sequences with a higher finite-change approximation are a basis for the $\Pi_3^{\omega_1^{ck}}$ sets of positive measure. On the other hand it also seems difficult to define a $\Sigma_3^{\omega_1^{ck}}$ set of measure less than 1 capturing every sequence which has a finite-change approximation. So we pose here the two following questions:

Question 6.4.1 *Is the set of Π_1^1-random $\Sigma_4^{\omega_1^{ck}}$?*

Question 6.4.2 *Is the set of weakly-Π_1^1-randoms somewhere in the higher hierarchy?*

6.5 Lowness for Π^1_1-randomness

6.5.1 Characterization of lowness for Π^1_1-randomness

Theorem 6.1.2 helps us here to solve the question of lowness for Π^1_1-randomness, which as been asked in [70] (question 9.4.11). We do not use here continuous relativization, but full relativization. So the question is, is there some sequence A which is not Δ^1_1 and such that the largest $\Pi^1_1(A)$ set equals the largest Π^1_1 set? We answer the question by the negative, in a strong sense, as we prove that if A is not Δ^1_1, then some $\Pi^1_1(A)$-Martin-Löf test already captures some Π^1_1-random sequence Z.

In what follows, for some X and some $\Pi^1_1(X)$-open set $\mathcal{U}$ of measure less than δ, we define the $\Pi^1_1(X)$-open set $\mathcal{U}^2$ the following way: First, using Lemma 3.7.1 relativized to X, let W be a $\Pi^1_1(X)$ set of strings such that $[W]^{<} = \mathcal{U}$ and such that $\sum_{\sigma\in W}\lambda([\sigma]) \leq \lambda(\mathcal{U})+\varepsilon$ for some ε that we pick to satisfy $\varepsilon+\delta<1$. Then let W^2 denote the $\Pi^1_1(X)$ set of strings $\{\sigma_1\hat{}\sigma_2 \mid \sigma_1,\sigma_2\in W\}$, and let $\mathcal{U}^2$ denotes the open set described by W^2. In particular we have $\lambda(\mathcal{U}^2)\leq(\lambda(\mathcal{U})+\varepsilon)^2$:

$$\begin{aligned}
\lambda(\mathcal{U}^2) &\leq \textstyle\sum_{\sigma_1,\sigma_2\in W}\lambda([\sigma_1\hat{}\sigma_2])\\
&\leq \textstyle\sum_{\sigma_1\in W}\sum_{\sigma_2\in W}\lambda([\sigma_1])\lambda([\sigma_2])\\
&\leq (\textstyle\sum_{\sigma_1\in W}\lambda([\sigma_1]))(\sum_{\sigma_2\in W}\lambda([\sigma_2]))\\
&\leq (\lambda(\mathcal{U})+\varepsilon)^2
\end{aligned}$$

We then define inductively $\mathcal{U}^{n+1}$ to be the $\Pi^1_1(X)$-open set described by the $\Pi^1_1(X)$ set of strings $W^{n+1}=\{\sigma_1\hat{}\sigma_2 \mid \sigma_1\in W^n,\ \sigma_2\in W\}$. Similarly we have $\lambda(\mathcal{U}^{n+1})\leq(\lambda(\mathcal{U})+\varepsilon)^{n+1}$ and thus the value $\lambda(\mathcal{U}^n)$ goes to 0 with a computable bound, as n goes to infinity.

Lemma 6.5.1 *For any sequence A not Δ^1_1, we have some $\Pi^1_1(A)$-open set $\mathcal{U}$ of measure less than 1, such that for any Π^1_1-open set $\mathcal{V}$ of measure less than 1, the set $\mathcal{U}$ is not contained in $\mathcal{V}$, that is, $\mathcal{U}\cap\mathcal{V}^c\neq\varnothing$.*

PROOF: Suppose that for any $\Pi^1_1(A)$-open set $\mathcal{U}$ of measure less than 1, there exists a Π^1_1-open set $\mathcal{V}$ of measure less than 1 with $\mathcal{U}\subseteq\mathcal{V}$, and let us prove that A is Δ^1_1.

Consider the universal $\Pi^1_1(A)$-Martin-Löf test $\bigcap_n\mathcal{U}_n$, with full relativization, that is each $\mathcal{U}_n$ is $\Pi^1_1(A)$ uniformly in n. By hypothesis we have some Π^1_1-open set $\mathcal{V}$ of measure less than 1 so that $\bigcap_n\mathcal{U}_n\subseteq\mathcal{V}$. As explained above, let W be a Π^1_1 set of strings describing $\mathcal{V}$ such that $\lim_n\lambda(\mathcal{V}^n=[W^n]^{<})=0$. We claim that also $\bigcap_n\mathcal{U}_n\subseteq\bigcap_n\mathcal{V}^n$.

Suppose not, that is, some $X\in\bigcap_n\mathcal{U}_n$ but $X\notin\bigcap_n\mathcal{V}^n$. In particular, there is a $n\geq1$ so that $X\in\mathcal{V}^n$ but $X\notin\mathcal{V}^{n+1}$. As $X\in\mathcal{V}^n$ there is some $\sigma\in W^n$ and some Y so that $X=\sigma\hat{}Y$. Also $X\in\bigcap_n\mathcal{U}_n$ and as $\bigcap_n\mathcal{U}_n$ contains all the non $\Pi^1_1(A)$-Martin-Löf random, it is closed under the operation of removing finite initial segments: As $X=\sigma\hat{}Y$ is in $\bigcap_n\mathcal{U}_n$ then also $Y\in\bigcap_n\mathcal{U}_n\subseteq\mathcal{V}$. But then by definition of $\mathcal{V}^{n+1}$ we also have $\sigma\hat{}Y\in\mathcal{V}^{n+1}$ which is a contradiction.

So we have $\bigcap_n\mathcal{U}_n\subseteq\bigcap_n\mathcal{V}^n$ and also the function which to n associates $\lambda(\mathcal{V}^n)$ goes to 0 with a computable bound. Thus the universal $\Pi^1_1(A)$-Martin-Löf test is covered by a Π^1_1-Martin-Löf test which means that A is low for Π^1_1-Martin-Löf randomness, with full relativization. Also by Corollary 4.5.4 A is then Δ^1_1. ■

Theorem 6.5.1:
Suppose A is not Δ^1_1. There is a Π^1_1-random X which is captured by a $\Pi^1_1(A)$-Martin-Löf test.

PROOF: We have by Theorem 6.1.2 that the set of Π^1_1-randoms coincide with the set of Solovay-Σ^1_1-generic. In particular it can be described as an intersection of unions of Σ^1_1-closed set $\bigcap_m \bigcup_n \mathcal{F}_{m,n}$, such that for each n we have $\lambda(\bigcup_n \mathcal{F}_{m,n}) = 1$. Without loss of generality we can also suppose that each $\mathcal{F}_{m,n}$ contains only Π^1_1-Martin-Löf random sequences (we can just replace each $\mathcal{F}_{m,n}$ by the uniform union of $\mathcal{F}_{m,n}$ intersected with each Σ^1_1-closed component of the universal Π^1_1-Martin-Löf test). We can also suppose without loss of generality that each union $\bigcup_n \mathcal{F}_{m,n}$ is increasing.

By Lemma 6.5.1 we have that if A is not Δ^1_1, there is some $\Pi^1_1(A)$-open set $\mathcal{U}$ of measure less than 1, such that for any Π^1_1-open set $\mathcal{V}$ of measure less than 1 we have $\mathcal{U} \cap \mathcal{V}^c \neq \varnothing$. As explained above, let W be a $\Pi^1_1(A)$ set of strings describing $\mathcal{U}$ such that $\lim_n \lambda(\mathcal{U}^n = [W^n]^{\prec}) = 0$. The goal is to create an element Z into $\bigcap_k \mathcal{U}^k$, which is also in every $\bigcap_m \bigcup_n \mathcal{F}_{m,n}$. We do it by defining strings $\sigma_1 \prec \sigma_2 \prec \sigma_3 \prec \dots$.

The construction is a forcing which does not have to be effective. For a start, take the first integer n_1 so that $\mathcal{F}_{1,n_1}$ has positive measure. By Lemma 6.5.1 we have that $\mathcal{U} \cap \mathcal{F}_{1,n_1} \neq \varnothing$. Also as $\mathcal{F}_{1,n_1}$ contains only Π^1_1-Martin-Löf random sequences, we actually have $\lambda(\mathcal{U} \cap \mathcal{F}_{1,n_1}) > 0$. Indeed, suppose otherwise, then for some cylinder $[\sigma] \subseteq \mathcal{U}$ we have $\lambda([\sigma] \cap \mathcal{F}_{1,n_1}) = 0$ and $[\sigma] \cap \mathcal{F}_{1,n_1} \neq \varnothing$, which contradicts that $\mathcal{F}_{1,n_1}$ contains only Π^1_1-Martin-Löf random sequences. Therefore, for some $\sigma \in W$ we have $\lambda(\mathcal{F}_{1,n_1} \mid [\sigma]) > 0$. We set $\sigma_1 = \sigma$.

Suppose that at step $m \geq 1$ we have some string $\sigma_m \in W^m$ and some integers $n_1, \dots, n_m$ such that $\lambda(\bigcap_{1 \leq i \leq m} \mathcal{F}_{i,n_i} \mid [\sigma_m]) > 0$. To ease the reading we now denote $\bigcap_{1 \leq i \leq m} \mathcal{F}_{i,n_i}$ by $\mathcal{F}$. We should define an extension σ_{m+1} of σ_m such that $\sigma_{m+1} \in W^{m+1}$ with still $\lambda(\mathcal{F} \mid [\sigma_{m+1}]) > 0$. Then, as $\bigcup_n \mathcal{F}_{m+1,n} = 1$, there is some n_{m+1} such that $\lambda(\mathcal{F} \cap \mathcal{F}_{m+1,n_{m+1}} \mid [\sigma_{m+1}]) > 0$, and we can continue the construction inductively.

Let $\sigma_m \hat{\ } W$ denotes the set of string $\{\sigma_m \hat{\ } \tau \mid \tau \in W\}$. Suppose for contradiction that $[\sigma_m \hat{\ } W]^{\prec} \cap \mathcal{F} = \varnothing$, that is, $[\sigma_m \hat{\ } W]^{\prec}$ is covered by $\mathcal{F}^c$. Let V be a Π^1_1 set of strings describing $\mathcal{F}^c \cap [\sigma_m]$. Then also $\mathcal{U} = [W]^{\prec}$ is covered by the Π^1_1-open set $\mathcal{F}^c \upharpoonleft_{\sigma_m}$, described by removing each prefix σ_m from all the strings enumerated in V. Furthermore as $\lambda(\mathcal{F} \mid [\sigma_m]) > 0$, the set $\mathcal{F}^c$ does not have full measure inside $[\sigma_m]$ and therefore $\lambda(\mathcal{F}^c \upharpoonleft_{\sigma_m}) < 1$, which contradicts Lemma 6.5.1. Therefore we have $[\sigma_m \hat{\ } W]^{\prec} \cap \mathcal{F} \neq \varnothing$ and then by the same argument as above we have $\lambda([\sigma_m \hat{\ } W]^{\prec} \cap \mathcal{F}) > 0$. Then there is some string $\tau \in W$ so that $\lambda(\mathcal{F} \mid [\sigma_m \hat{\ } \tau]) > 0$. Set $\sigma_{m+1} = \sigma_m \hat{\ } \tau$. We have $\sigma_{m+1} \in W^{m+1}$.

We define Z to be the unique limit point of $\{[\sigma_m]\}_{m \in \mathbb{N}}$. We have by construction that $Z \in \bigcap_n \mathcal{U}^n$ which implies that it is not $\Pi^1_1(A)$-Martin-Löf random. We also have by construction that $Z \in \bigcap_m \bigcup_n \mathcal{F}_{m,n}$, which implies that it is Π^1_1-random. ∎

Corollary 6.5.1:
A sequence A is low for Π^1_1-randomness iff A is Δ^1_1.

6.5.2 Further discussion

Chong, Nies and Yu, together with Harrington and Slaman proved in [7] a theorem making an interesting connection between lowness for Π^1_1-randomness, lowness for Δ^1_1-randomness.

Definition 6.5.1. *A sequence A is Π^1_1-**random cuppable** if there is some Π^1_1-random sequence Z such that $\omega_1^{A\oplus Z} > \omega_1^{ck}$.* ∎

The equivalence between $\omega_1^X > \omega_1^{ck}$ and $X \geq_h \mathcal{O}$ (see Theorem 3.5.1) allows us to see this notion as a higher counterpart of the notion of Martin-Löf random cuppability:

Definition 6.5.2. *A sequence A is **Martin-Löf random cuppable** if there is some Martin-Löf random sequence Z which does not Turing computes $\emptyset^{'}$, and such that $A \oplus Z$ Turing compute $\emptyset^{'}$.* ∎

Kučera asked in 2004 during a talk (see [69]) if Martin-Löf random cuppability could coincide with non K-triviality. The question has recently been answered by the affirmative in [13] by Day and Miller. Going back now to Π^1_1-random cuppablility, here is the connection we announced above:

Theorem 6.5.2 (Chong, Nies and Yu ; Harrington and Slaman):
A sequence Z is low for Π^1_1-randomness iff it is low for Δ^1_1-randomness and non Π^1_1-random cuppable.

Also still in [7] it is proved that some non Δ^1_1 sequences are low for Δ^1_1-randomness. However, by Corollary 6.5.1, all of them should then be Π^1_1-random cuppable. Also a characterization of this class is still open:

Question 6.5.1 *Does Π^1_1-random cuppablility coincide with non Δ^1_1?*

6.6 Higher generic sequences

Joint work with Noam Greenberg.

We study in this section higher genericity notions, and we will compare them with some higher randomness notions.

6.6.1 Definitions

After defining α-genericity for any computable ordinal α in Section 2.2, the simplest higher genericity notion we can give is certainly the one of being α-generic for any α. So just like we defined Δ^1_1-randomness in Section 3.7, we define here Δ^1_1-genericity.

Definition 6.6.1. *A sequence G is **Δ^1_1-generic** if G is in every dense Δ^1_1-open set.* ∎

Fact 6.6.1

A sequence G is Δ^1_1-generic iff G is weakly-Δ^1_1-generic, that is, for any Δ^1_1-generic sequence and any Δ^1_1-open set $\mathcal{U}$, either G is in $\mathcal{U}$ or G is in some $[\sigma]$ disjoint from $\mathcal{U}$. This comes from the fact that for any Δ^1_1-open set $\mathcal{U}$, the set $\mathcal{U}$ together with the interior of $2^{\mathbb{N}} - \mathcal{U}$ is a dense open set, which is still Δ^1_1.

Just like we defined Π^1_1-Martin-Löf randomness as a higher counterpart of Martin-Löf randomness, we define here weak-Π^1_1-genericity and Π^1_1-genericity as higher counterparts of weak-1-genericity and 1-genericity. We shall see however later that the notion of weak-1-genericity is in some sense comparable with the one of Σ^1_1-randomness, as it will be seen to coincide with Δ^1_1-genericity, whereas the notion of Π^1_1-genericity does not seem to have a counterpart in algorithmic randomness.

Definition 6.6.2. *A sequence G is **weakly-Π^1_1-generic** if G is in every dense Π^1_1-open set. It is **Π^1_1-generic** if for any Π^1_1-open set $\mathcal{U}$, either G is in $\mathcal{U}$ or G is in some $[\sigma]$ disjoint from $\mathcal{U}$.* ∎

For an open set $\mathcal{U}$ such that G is not in the interior of $\mathcal{U}^c$, we also say that $\mathcal{U}$ is **dense along** G, because $\mathcal{U}$ can then be described by a set of string W such that for any $\sigma \prec G$, there is some τ extending σ in W.

Feferman has proved in [20] that if G is sufficiently Cohen generic, then $\omega_1^G = \omega_1^{ck}$. We give in this thesis the exact genericity notion that is required for G so that G preserves ω_1^{ck}: It is obtained by considering Σ^1_1-open sets instead of Π^1_1-open sets.

Definition 6.6.3. *An open set $\mathcal{U}$ is Σ^1_1 if $\mathcal{U}$ can be described as a Σ^1_1 set of strings W, that is, $\mathcal{U} = [W]^\prec$. A sequence X is **weakly-Σ^1_1-generic** if X is in every dense Σ^1_1-open set. It is **Σ^1_1-generic** if for any Σ^1_1-open set $\mathcal{U}$, either X is in $\mathcal{U}$ or X is in some $[\sigma]$ disjoint from $\mathcal{U}$.* ∎

The notion of (weakly-)Σ^1_1-genericity can be considered as a higher counterpart of the notion of (weakly-)Π^0_1-genericity, defined by considering open sets described by Π^0_1 set of strings. In particular Jockusch noticed (see [44] and [45]) that weak-Π^0_1-genericity is equivalent to 2-genericity. Such an equivalence cannot work in the higher setting, when replacing 2-genericity by $\Pi^1_1(\mathcal{O})$-genericity. However we will find another equivalence, and we will see in particular that weak-Σ^1_1-genericity coincides with Σ^1_1-genericity.

We give here an illustration of the connections between the different higher genericity notions. we will prove that each implication of the following picture is correct and strict.

$$
\begin{tikzcd}
\text{weakly-}\Sigma^1_1\text{-generic} \arrow[dr, leftrightarrow, bend left] \arrow[dd, leftrightarrow, bend right] & & & & \\
& \Sigma^1_1\text{-generic} \arrow[r] & \Pi^1_1\text{-generic} \arrow[r] & \text{weakly-}\Pi^1_1\text{-generic} \arrow[r, leftrightarrow] & \Delta^1_1\text{-generic} \\
\Delta^1_1\text{-generic} \wedge \omega_1^X = \omega_1^{ck} \arrow[ur, leftrightarrow, bend right] & & & &
\end{tikzcd}
$$

Figure 6.2: Higher genericity

6.6.2 Π^1_1-genericity

We start by showing the following: just like Σ^1_1-randomness coincides with Δ^1_1-randomness, weak-Π^1_1-genericity coincides with Π^1_1-genericity.

> **Proposition 6.6.1:**
> *A sequence G is Δ^1_1-generic iff it is weakly-Π^1_1-generic.*

Proof: If G is weakly-Π^1_1-generic then G is clearly Δ^1_1-generic. For the other direction, let us prove that for any Π^1_1 dense set of strings, there exists a Δ^1_1 dense set of strings contained it it. It will follow that if G is in all the dense Δ^1_1-open sets, it is already in all the dense Π^1_1-open sets.

Suppose we have a Π^1_1 dense set of strings W. Let us define the Π^1_1 function $f : 2^{<\omega} \to 2^{<\omega}$ as follow: The function associates to the string σ the smallest string τ extending σ which is in W. By hypothesis W is dense and then the function is total and therefore Δ^1_1. Then also its range is Δ^1_1. ∎

The next proposition is a mere higher analogue for the existence of a left-c.e. weakly-1-generic sequence.

> **Proposition 6.6.2:**
> *There is a higher left-c.e. weakly-Π^1_1-generic sequence G.*

Proof: We will buid G left-c.e. and weakly-Π^1_1-generic with a classic finite injury construction. Let $M_e \subseteq 2^{<\omega}$ be the e-th Π^1_1 set of strings. Recall that we can consider each M_e to be enumerated along computable ordinal stages, such that at most one string is enumerated at each successor stages, and none of them is enumerated at limit stages.

We want to ensure that for each e, the requirement R_e : "If M_e is dense then G extends a string in M_e" is satisfied. During the construction, at any stage s we define strings $\sigma_{e,s} \prec \sigma_{e+1,s} \prec \dots$ with G_s defined as the unique limit point of $\{[\sigma_{e,s}]\}_{e\in\mathbb{N}}$.

The construction:

We start at stage 0 by letting $\sigma_{0,0} = 0$ and $\sigma_{e+1,0} = \sigma_{e,0}\hat{\ }0$. So X_0 is juste an infinite range of 0. Then we define by induction the values of each $\{\sigma_{e,s}\}_{e\in\mathbb{N}}$ assuming the values $\{\sigma_{e,t}\}_{e\in\mathbb{N}}$ have been defined for all $t < s$:

At substage 0 of the stage s, we start by checking if any string τ extending 1 has been enumerated in M_0 up to stage s. If so we define $\sigma_{0,s}$ to be the first of those strings τ, in the order of their enumeration along stages, otherwise $\sigma_{0,s}$ is equal 0. Now assuming that all the strings $\sigma_{e,s}$ have been defined up to substage e, we define $\sigma_{e+1,s}$ at substage $e+1$. We check if a string τ extending $\sigma_{e,s}\hat{\ }1$ has been enumerated in M_{e+1} up to stage s. If so we set $\sigma_{e+1,s}$ to the first of those strings τ, in the order of their enumeration along stages. Otherwise we set $\sigma_{e+1,s}$ to be $\sigma_{e,s}\hat{\ }0$.

The verification:

The construction stabilizes by a classic finite-injury argument. It is clear that the approximation of G is left-c.e., because a string $\sigma_{n,s}$ changes only if one of its prefix $\sigma_{m,s}$ for $m \leq n$ goes from $\sigma\hat{\ }0\hat{\ }\tau$ to $\sigma\hat{\ }1\hat{\ }\tau'$. Also it is clear that each requirement is satisfied: The requirement R_e is satisfied, since if M_e is dense it contains some string extending $\sigma_{e-1}\hat{\ }1$, and after $\{\sigma_{e-1,s}\}_{s<\omega_1^{ck}}$ has stabilized to $\sigma_{e-1}\hat{\ }0$, the sequence $\{\sigma_{e,s}\}_{s<\omega_1^{ck}}$ will eventually stabilize to some string $\sigma_e \succeq \sigma_{e-1}\hat{\ }1$ in M_e. ∎

We shall now separate weak-Π^1_1-genericiy from Π^1_1-genericity, by proving that no Π^1_1-generic sequence is higher left-c.e., in a strong sense, as we actually prove that no Π^1_1-generic sequence can higher Turing compute a higher left-c.e., which is not Δ^1_1. Note the following fact:

Fact 6.6.2

No Δ^1_1-generic sequence is Δ^1_1. Indeed, for X a Δ^1_1 sequence, the open set $2^{\mathbb{N}} - \{X\}$ is dense and Δ^1_1.

The same technique can be used in the lower setting to separate 1-genericity from weak-1-genericity.

Proposition 6.6.3:
No Π^1_1-generic sequence G higher Turing computes a non Δ^1_1 higher left-c.e. sequence A.

Proof: Suppose that G higher Turing computes a higher left-c.e. sequence A which is not Δ^1_1, via the functionnal Φ. Let us show that for any $\sigma \prec G$ there exists $\tau \succ \sigma$ such that Φ maps τ to a string strictly at the left of A. We will then use this to prove that G is not Π^1_1-generic.

Suppose that there exists $\sigma \prec G$ such that any $\tau \succ \sigma$ is mapped either to a string strictly at the right of A, or to a prefix of A. Then we claim that A is also right-c.e. which would make it Δ^1_1. We define an approximation of A by a sequence A'_s from the right the following way:

The construction:

Let $\sigma \prec G$ having the property stated above. We start at stage 0 by setting each $A'_0(n) = 1$ for each n. Suppose that A'_t has been defined for all $t < s$. Then at stage s and substage n, we check if there exists a string extending σ which is mapped in Φ_s to some string extending $A'_s \upharpoonright_n \hat{} 0$. If so we set $A'_s(n) = 0$, otherwise we set $A'_s(n) = 1$.

The verification:

It is clear by construction that $\{A_s\}_{s<\omega_1^{ck}}$ is right-c.e., as A_s is different from A_{s+1} only if $A_s = \tau \hat{} 1 \hat{} B$ for some τ, B and $A_{s+1} = \tau \hat{} 0 \hat{} B'$ for some B'. Also as $\{A_s\}_{s<\omega_1^{ck}}$ is right-c.e. it necessarily converges to some sequence A'.

We claim that $A' = A$. Recall that by hypothesis, the strings extending σ can only be mapped to strings at the right of A or to a prefix of A. Also, as $\Phi(G) = A$, we know that for infinitely many prefixes τ of A there is at least one string extending σ which is mappped to τ by Φ. Then for each n, there exists some stage s such that we have $A'_s \upharpoonright_n = A \upharpoonright_n$. As the approximation cannot move on the n first bits after stage s, we then have $A' = A$.

The conclusion:

It follows that if A is higher left-c.e. and not Δ^1_1 and if for some sequence G we have $\Phi(G) = A$, then we can define the Π^1_1 set of all strings W which are mapped via Φ to some string which is strictly at the left of $A_s \upharpoonright_n$ for any s and any n. By definition no string in W is mapped to a prefix of A and therefore $G \notin [W]^{\prec}$. However by the above argument, for any prefix σ of G, there is an extention of σ in [W] and therefore $[W]^{\prec}$ is dense along G. Therefor G is not Π^1_1-generic. Note that we never need to require the functional Φ to be consistent somewhere, except on G. ■

Corollary 6.6.1:
If G is Π^1_1-generic, then G does not higher Turing compute $\mathcal{O}$, or any non Π^1_1 set which is not Δ^1_1.

We now see that there are still some Π^1_1-genreric sequences which are quite easy to define, and in particular, Π^1_1-genrericity is not enough to ensure preservation of ω_1^{ck}.

Proposition 6.6.4:
There is a higher ω-c.a. Π^1_1-generic sequence G.

Proof: The proof is essentially the same than the one of Proposition 6.6.2. We want to ensure that for each e, the requirement R_e : "If M_e is dense along G then G extends a string in M_e" is satisfied. Recall that we can consider each M_e to be enumerated along computable ordinal stages, such that at most one string is enumerated at each successor stages, and none of them is enumerated at limit stages.

During the construction, at any stage s we define strings $\sigma_{e,s} \prec \sigma_{e+1,s} \prec \dots$ with G_s defined as the unique limit point of $\{[\sigma_{e,s}]\}_{e\in\mathbb{N}}$. After the construction we will show that $\{G_s\}_{s<\omega_1^{ck}}$ is an ω-c.a. approximation of some sequence G satisfying all the requirements R_e.

The construction:

We start at stage 0 by letting $\sigma_{0,0} = 0$ and $\sigma_{e+1,0} = \sigma_{e+1,0}\hat{\ }0$. So X_0 is just a range of 0's. Then we define by transfinite induction the values of each $\sigma_{e,s}$ assuming the values $\sigma_{e,t}$ have been defined for all $t < s$:

At substage 0 of the stage s, we start by checking if any string τ different from ϵ (the empty word) has been enumerated in $M_{0,s}$. If so we define $\sigma_{0,s}$ to be the first of those strings, in the order of their enumeration along stages, otherwise $\sigma_{0,s}$ is equal 0. Now assuming that all the strings $\sigma_{e,s}$ have been defined up to substage n, we will define $\sigma_{e+1,s}$ at substage $e+1$. We check if a string τ strictly extending $\sigma_{e,s}$ has been enumerated in $M_{e+1,s}$. If so we set $\sigma_{e+1,s}$ to the first of those strings τ, in the order of their enumeration along stages. Otherwise we set $\sigma_{e+1,s}$ to be $\sigma_{e,s}\hat{\ }0$.

The verification:

We claim that each string σ_e can change at most 2^e times. As R_0 is injured by no previous requirement, the approximation $\{\sigma_{0,s}\}_{s<\omega_1^{ck}}$ can change at most once, going from 0 to some string different from ϵ. Suppose that $\{\sigma_{e,s}\}_{s<\omega_1^{ck}}$ can change at most 2^e time, by construction, for each $\sigma_{e,s}$ there is two possible values for $\sigma_{e+1,s}$. The first one being equal to $\sigma_{e,s}\hat{\ }0$ and the second one strictly extending $\sigma_{e,s}$. Then $\{\sigma_{e+1,s}\}_{s<\omega_1^{ck}}$ can change at most 2^{e+1} time. We can deduce that the sequence $\{G_s\}_{s<\omega_1^{ck}}$ is an ω-c.a. approximation of some sequence G.

We claim that each requirement is satisfied. The requirement R_0 is satisfied since if M_0 is dense along any path, it is not empty and then σ_0 will eventually stabilize to its first enumerated string. Also as the number of changes for each $\{\sigma_{e,s}\}_{s<\omega_1^{ck}}$ is finite, there is a stage s such that $\{\sigma_{e,t}\}_{t\geq s}$ is stable. Then if M_{e+1} is dense along G, there is a first enumerated string in it, strictly extending $\sigma_e = \sigma_{e,s}$ and then $\{\sigma_{e+1,s}\}_{s<\omega_1^{ck}}$ will eventually stabilize to this element of M_{e+1}. Then each requirement is satisfied. ∎

Corollary 6.6.2:
For some Π^1_1-generic sequence G we have $\omega_1^G > \omega_1^{ck}$.

We say that G higher Turing computes a total function $f : \omega \to \omega_1^{ck}$ if there is a Π^1_1 functional $\Phi : 2^{<\mathbb{N}} \times \mathbb{N} \times \mathbb{N}$ such that for any n, there is a unique value a such that $\Phi(G, n) = a$, and furthermore $a \in \mathcal{O}$. One can easily prove that if G higher Turing computes a Π^1_1 sequence which is not Δ^1_1, then also G higher Turing computes a function $f : \omega \to \omega_1^{ck}$ with $\sup_n f(n) = \omega_1^{ck}$. We can show here that the converse does not hold.

Corollary 6.6.3:
There is a sequence G which higher Turing computes a function $f : \omega \to \omega_1^{ck}$ with $\sup_n |f(n)| = \omega_1^{ck}$, but such that G does not higher Turing compute a Π^1_1 sequence which is not Δ^1_1.

PROOF: Let G be a higher ω-c.a. and Π^1_1-generic. We can define Φ by enumerating (σ, n, s) in Φ for $|\sigma| = n$ if s is the first stage such that $\sigma = G_s \upharpoonright_n$. We easily prove that this is a higher Turing computation from G, of a function $f : \omega \to \omega_1^{ck}$ with $\sup_n |f(n)| = \omega_1^{ck}$. Also as G is Π^1_1-generic, we can conclude with Corollary 6.6.1. ∎

Note that we have a difference here with randomness. Indeed, we saw with Theorem 6.3.1 that if a sequence is Δ^1_1-random but does not preserves ω_1^{ck} (equivalently is not Π^1_1-random), then it can higher Turing compute a Π^1_1 sequence which is not Δ^1_1. This is not the case for a sequence which is Δ^1_1-generic, and which does not preserve ω_1^{ck}.

6.6.3 Σ^1_1-genericity

We now study Σ^1_1-genericity and we shall see that it is the exact level of genericity we need in order to preserve ω_1^{ck}. We start with a lemma extending Theorem 1.9.1, which can be seen as a categorical version of its measure theoretical analogue, Lemma 6.1.1, which similarly extends Theorem 1.8.1.

Lemma 6.6.1 *Let $\mathcal{P}$ be a Π^1_1 set of the form $\mathcal{P} = \bigcup_{\alpha<\omega_1^{ck}} \mathcal{P}_\alpha$ where each $\mathcal{P}_\alpha$ is a Π^0_α set uniformly in α. Then we can find uniformly in an index for $\mathcal{P}$ a Π^1_1-open set $\mathcal{U}$ and a countable union of Σ^1_1-closed sets of empty interior $\bigcup_n \mathcal{F}_n$, such that $\mathcal{P} = \mathcal{U} \triangle \mathcal{B}$ for some set $\mathcal{B} \subseteq \bigcup_n \mathcal{F}_n$. Note that the union itself is not effective in n.*

PROOF: Using Theorem 1.9.1 one can define uniformly for each α a Π^0_α-open set $\mathcal{U}_\alpha$ and an effective union of $\Delta^0_{\alpha+1}$-closed set of empty interior $\bigcup_n \mathcal{F}_{n,\alpha}$ such that $\mathcal{P}_\alpha = \mathcal{U}_\alpha \triangle \mathcal{B}_\alpha$ for some set $\mathcal{B}$ included in $\bigcup_n \mathcal{F}_{n,\alpha}$.

Just like in the last part of the proof of Theorem 1.9.1, we verify that $\mathcal{P} = (\bigcup_\alpha \mathcal{U}_\alpha) \triangle \mathcal{B}$ where $\mathcal{B}$ is equal to $\bigcup_\alpha \mathcal{P}_\alpha \triangle \bigcup_\alpha \mathcal{U}_\alpha$, and we then verify that $\mathcal{B} \subseteq \bigcup_\alpha \bigcup_n \mathcal{F}_{n,\alpha}$. We have that $\mathcal{U}$ is a Π^1_1-open set and that each $\mathcal{F}_{n,\alpha}$ is a Σ^1_1-closed set of empty interior. ∎

Recall the proof of Theorem 6.1.1 that Σ^1_1-Solovay-genericity preserves ω_1^{ck}. The proof that Σ^1_1-genericity also preserves ω_1^{ck} is very similar. For any X, we have that $\omega_1^X > \omega_1^{ck}$ iff there is a Turing functional $\Phi : 2^{<\mathbb{N}} \times \mathbb{N} \times \mathbb{N}$ such that for every n we have $\Phi(X, n) \in \mathcal{O}^X_{<\omega_1^{ck}}$ and such that $\sup_n |\Phi(X, n)|^X_o = \omega_1^{ck}$. We use this to prove:

Theorem 6.6.1:
If G is Σ^1_1-generic then $\omega_1^G = \omega_1^{ck}$.

PROOF: We prove that for any Turing functional $\Phi : 2^{<\mathbb{N}} \times \mathbb{N} \times \mathbb{N}$, if G is Σ^1_1-generic and if for every n we have $\Phi(G,n) \in \mathcal{O}^G_{<\omega_1^{ck}}$, then $\sup_n |\Phi(G,n)|^G_o < \omega_1^{ck}$. So consider such a Turing functional Φ and the set:

$$\mathcal{P} = \{X \mid \forall n \;\; \exists \alpha < \omega_1^{ck} \;\; \Phi(X,n) \in \mathcal{O}^X_{\leq\alpha}\}$$

Let $\mathcal{P}_n$ be the Π^1_1 set $\{X \mid \exists \alpha < \omega_1^{ck} \;\; \Phi(X,n) \in \mathcal{O}^X_{\leq\alpha}\}$ and $\mathcal{P}_{n,\alpha}$ be the Δ^1_1 set $\{X \mid \Phi(X,n) \in \mathcal{O}^X_{\leq\alpha}\}$, uniform in α, so that $\mathcal{P}_n = \bigcup_{\alpha<\omega_1^{ck}} \mathcal{P}_{n,\alpha}$ and $\mathcal{P} = \bigcap_n \mathcal{P}_n$.

Suppose now that G is Σ^1_1-generic with $G \in \bigcap_n \mathcal{P}_n$. From Lemma 6.6.1, uniformly in n, we can define a Π^1_1-open set $\mathcal{U}_n$ such that $\mathcal{P}_n = \mathcal{U}_n \vartriangle \mathcal{B}$ for $\mathcal{B}$ included in a union of Σ^1_1-closed sets of empty interior. Also each complement of those Σ^1_1-closed sets contains a Δ^1_1 dense open set (see Proposition 6.6.1 for the details), and as G is Δ^1_1-generic, it follows that $G \in \mathcal{P}_n$ iff $G \in \mathcal{U}_n$ for every n.

Let us now prove that there is a prefix σ of G such that every $\mathcal{U}_n$ is dense in $[\sigma]$. Suppose otherwise. Then for any prefix σ of G, there is an n and an extension τ of σ such that $[\tau]$ is in the interior of the Σ^1_1-closed set $\mathcal{U}^c_n$, the complement of $\mathcal{U}_n$. Also the interior of any Σ^1_1-closed set $\mathcal{F}$ is a Σ^1_1-open set, uniformly in an index for $\mathcal{F}$. Indeed, it can be described by the set of strings $\{\sigma \;:\; \forall \tau \geq \sigma,\; \tau \notin \mathcal{F}^c\}$ which is a Σ^1_1 predicate uniformly in an index for $\mathcal{F}$.

It follows that the union of the interior of each $\mathcal{U}^c_n$ is also a Σ^1_1-open set. But then this Σ^1_1-open set is dense along G which contradicts that G is Σ^1_1-generic. It follows that for some prefix σ of G, each $\mathcal{U}_n$ is dense in $[\sigma]$. But then we can define the Π^1_1 total function $f : \omega \to \omega_1^{ck}$ which to n associates the smallest computable ordinal α such that $\mathcal{U}_\alpha$, the enumeration of $\mathcal{U}$ up to stage α, is already dense in $[\sigma]$ (see Proposition 6.6.1 for the proof that such an α exists). As f is total it is Δ^1_1 and then its range is a Δ^1_1 set of computable ordinals, which is then bounded by some $\alpha < \omega_1^{ck}$, by the Σ^1_1-boundedness principle. It follows that $\mathcal{U}_{n,\alpha}$ is dense in $[\sigma]$ for every n and also that G is in $\mathcal{U}_{n,\alpha}$ for every n, because otherwise, the set $\mathcal{U}_{n,\alpha}$ together with $2^{\mathbb{N}} - [\sigma]$ is a dense Δ^1_1-open set which would not contain G.

Also we used Lemma 6.6.1 to prove that G is in $\mathcal{P}_n$ iff G is in $\mathcal{U}_n$, and similarly, we can prove that G is in $\mathcal{P}_{n,\alpha}$ iff it is in $\mathcal{U}_{n,\alpha}$, as it is easily seen, by slightly modifying the proof of Lemma 6.6.1, that $\mathcal{P}_{n,\alpha} = \mathcal{U}_{n,\alpha} \vartriangle \mathcal{B}$ for some $\mathcal{B}$ included in a union of Σ^1_1 closed sets of empty interior. Then also $G \in \bigcap_n \mathcal{P}_{n,\alpha}$ and we then have $\sup_n |\Phi(G,n)|^G_o \leq \alpha < \omega_1^{ck}$. ∎

Corollary 6.6.4:
The set $\{X \;:\; \omega_1^X > \omega_1^{ck}\}$ *is meager.*

We can now deduce a result which can be considered to be an analogue of Theorem 3.7.4, which states that a sequence Z is Π^1_1-random (Σ^1_1-Solovay-generic) iff it is Δ^1_1-random (Δ^1_1-Solovay-generic) and $\omega_1^Z = \omega_1^{ck}$.

Theorem 6.6.2:
A sequence G is Σ^1_1-generic iff it is Δ^1_1-generic and $\omega_1^G = \omega_1^{ck}$.

PROOF: We already have that if G is Σ^1_1-generic, then G is Δ^1_1-generic and $\omega_1^G = \omega_1^{ck}$. Let us now suppose that G is not Σ^1_1-generic, but Δ^1_1-generic, in order to prove $\omega_1^G > \omega_1^{ck}$.

So we have a Σ^1_1 set of strings which is dense along G and which contains no prefix of G. Let W be the complement of this set of strings. We define the $\Pi^1_1(G)$ function $f : \omega \to \omega_1^{ck}$ which to n associates the smallest s such that $G\upharpoonright_n$ is enumerated in W_s. As W contains every prefix of G we have that f is total and then its range is a $\Delta^1_1(G)$ set of ordinals. Also it cannot be the case that the range of f is bounded by $s < \omega_1^{ck}$, because then $2^{<\mathbb{N}} - W_s$ would be a Δ^1_1 set of strings, dense along G and not containing G, which would make G not Δ^1_1-generic. It follows that $\omega_1^G > \omega_1^{ck}$. ■

In many regards, Σ^1_1-genericity can be seen as a categorical analogue of Σ^1_1-Solovay-genericity (Π^1_1-randomness). In some sense we could also consider that weak-Σ^1_1-genericity is a categorical analogue of weak-Σ^1_1-Solovay-genericity (weak-Π^1_1-randomness). However, weak-Σ^1_1-Solovay-genericity has been proved to be different from Σ^1_1-Solovay-genericity. We shall see that at the contrary, weak-Σ^1_1-genericity coincides with Σ^1_1-genericity.

We will actually give an equivalent definition of weak-Σ^1_1-genericity, and we will see that this definition implies Π^1_1-genericity. We will then prove that if weak-Σ^1_1-genericity together with Π^1_1-genericity implies Σ^1_1-genericity. We could give a more direct proof that weak-Σ^1_1-genericity implies Π^1_1-genericity. However we believe that this new notion of genericity we introduce might reveal itself useful for other purposes (for example, maybe the question of lowness for Σ^1_1-genericity).

In the lower setting, we have that weak-Π^0_1-genericity implies 2-genericity and then both Π^0_1-genericity and 1-genericity (see [44] and [45]). The proof however uses a time trick, and it is anyway clear that weak-Σ^1_1-genericity does not imply $\Pi^1_1(\mathcal{O})$-genericity, because one can easily prove that $\mathcal{O}$ higher Turing computes a Σ^1_1-generic sequence, just like $\mathcal{O}$ higher Turing computes a Π^1_1-random sequence.

So we shall consider a restricted way to use $\mathcal{O}$ in the enumerations of our open sets. Also here again, the notion of finite-change approximation appears to be useful.

Definition 6.6.4. *An open set $\mathcal{U}$ is* ***dense higher finite-change*** *if there is a finite-change approximable function $f : 2^{<\mathbb{N}} \to 2^{<\mathbb{N}}$ with $\sigma \leq f(\sigma)$ for any σ, and such that $\mathcal{U} = \bigcup_\sigma [f(\sigma)]$.* ■

Theorem 6.6.3:
A sequence X is weakly-Σ^1_1-generic iff X is in every dense higher finite-change open set.

PROOF: Consider a dense Σ^1_1-open set $\mathcal{U}$ and let us define a finite-change dense open set $\mathcal{V} = \mathcal{U}$. We build a finite-change approximation $\{f_s\}_{s<\omega_1^{ck}}$ of a function f. Let W be a Π^1_1 set of strings such that $\mathcal{U} = [2^{<\omega} - W]^{<}$. At stage 0 we let $f_0(\sigma) = \sigma$ for every σ. At successor stage s, for each σ, we let $f_s(\sigma) = f_{s-1}(\sigma)$ if $f_{s-1}(\sigma) \notin W_s$ and we let $f_s(\sigma)$ be the first string extending σ which is not in W_s otherwise. As limit stage s we let $f_s = \lim_{t<s} f_t$. As $\mathcal{U}$ is dense, it is clear that the approximation of f is higher finite-change. We now prove the converse.

Let $\mathcal{U}$ be a dense higher finite-change open set and let us prove that there is a dense Σ^1_1-open set $\mathcal{V} \subseteq \mathcal{U}$. We define the set $\mathcal{V}$ by enumerating a Π^1_1 set of strings W, and by letting the set $\mathcal{V}$ be equal to $[2^{<\omega} - W]^{<}$.

The construction:

At stage 0 we start with $W_0 = \varnothing$. At stage s, we set $A_{s,0} = \varnothing$. At substage n, let m be the length of the longest string in $A_{s,n}$. For each string σ of length m which extends (or coincides with) no string in $A_{s,n}$, we compute $\tau = f_s(\sigma)$. Then we set $A_{s,n+1}$ to be $A_{s,n}$ together with such strings τ and all of their prefixes. We define A_s to be $\bigcup_{n<\omega} A_{s,n}$. The set W is then given by $\bigcup_{s<\omega_1^{ck}} A_s$.

The verification:

We say that a set of strings A is **strongly dense** if for any string σ, there is an extension τ of σ such that τ and every extension of τ is in A.

Claim 1 : For each stage s, the set $2^{<\mathbb{N}} - A_s$ is strongly dense.

It is clear, because at each substage n, each set $A_{s,n}$ is finite, and if a string σ is in $A_{s,n}$, no extension of σ will be enumerated anymore in $A_{s,m}$ for $m > n$.

Claim 2 : If A_1, A_2 are two strongly dense sets of strings, then $A_1 \cap A_2$ is also strongly dense.

Suppose A_1, A_2 strongly dense. For any string σ, there is an extension τ_1 of σ such that every extension of τ_1 is in A_1. Then there is an extension τ_2 of τ_1 such that every extension of τ_2 is in A_2, and then in $A_1 \cap A_2$.

Claim 3 : For any limit ordinal $s \leq \omega_1^{ck}$ and any string σ, there is an extension τ of σ and a stage $t < s$ such that for any stage $t \leq r < s$, the string τ and all its extensions are in $2^{<\mathbb{N}} - A_r$.

Claim 3 can be easily proved by a finite injury argument, using the fact that the approximation of f is a finite-change approximation. In particular, this implies that for any n there is a stage $t < s$ such that for any stage $t \leq r < s$, each set A_r has the same strings of length smaller than n. Claim 3 then follows.

Claim 4 : For any ordinal $s \leq \omega_1^{ck}$, the set of strings $\bigcap_{t<s}(2^{<\mathbb{N}} - A_t)$ is a strongly dense set of strings.

We prove Claim 4 by induction on stages. Suppose that the claim is true for every stage $t < s$ and let us prove it is true at stage s. If s is successor then by induction hypothesis, by Claim 1 and Claim 2, Claim 3 is then true at stage s. Suppose that s is limit and consider any string σ. By Claim 3, there is a stage $t < s$ and an extension τ of σ such that every extension of τ is in $2^{<\mathbb{N}} - A_r$ for any stage r with $t \leq r < s$. Also by induction hypothesis, Claim 4 is true at stage t and in particular $\bigcap_{r<t}(2^{<\mathbb{N}} - A_r)$ is strongly dense. Therefore there is then an extension ρ of τ such that all its extensions are in $\bigcap_{t<s}(2^{<\mathbb{N}} - A_t)$. As this is true for any string σ, Claim 4 is true at stage s.

It follows that the set $\mathcal{V} = [2^{<\mathbb{N}} - W]^{\prec} = [\bigcap_{s<\omega_1^{ck}}(2^{<\mathbb{N}} - A_s)]^{\prec}$ is a dense open set. We should now prove that $\mathcal{V} \subseteq \mathcal{U}$. Also suppose $X \notin \mathcal{U}$ and let us show that every $\sigma \prec X$ is enumerated in W. Consider $\sigma \prec X$. There is a stage s such that $\{f_t\}_{s\leq t<\omega_1^{ck}}$ is stable on every string of length smaller than $|\sigma|$ and also stable on every string of length smaller than $m = \max\{|f(\tau)| \ : \ \tau \text{ of length smaller than } |\sigma|\}$. In particular, as X is not in $\mathcal{U}$, on every prefix of $X \upharpoonright_m$, the function f returns a string incomparable with X. Then by construction we will necessarily have σ in A_s. ∎

Corollary 6.6.5:
If a sequence G is weakly-Σ^1_1-generic then it is Π^1_1-generic.

Proof: Consider a Π^1_1-open set $\mathcal{U} = [W]^{\prec}$. We claim that $\mathcal{U}$ together with the interior of its complement is a finite-change dense open set. At stage 0 we define $f_0(\sigma)$ to be σ. At successor stage s, for any string σ, let $f_s(\sigma)$ be the smallest (in the lexicographic order) extension of σ which is in W_s, and $f_s(\sigma)$ be σ if no such string exists. At limit stage s let $f_s = \lim_{t<s} f_t$.

It is clear that $\{f_s\}_{s<\omega_1^{ck}}$ is a finite-change approximation and that the corresponding finite-change open set is equal to $\mathcal{U}$ together with the interior of its complement. ∎

Theorem 6.6.4:
A sequence G is weakly-Σ^1_1-generic iff it is Σ^1_1-generic.

Proof: Suppose that G is not Σ^1_1-generic and consider a Σ^1_1 set of strings S, dense along G. Suppose first that for some prefix σ of G, the set $[S]^{\prec}$ is dense in $[\sigma]$. Then $[S]^{\prec} \cap [\sigma]$ together with the complement of $[\sigma]$ is a dense Σ^1_1 set of strings not containing G, then G is not weakly-Σ^1_1-generic.

Suppose now that for all prefixes σ of G, the set $[S]^{\prec}$ is not dense in $[\sigma]$, that is, there is an extention τ of σ such that no extension of τ is in S. Let W be the Π^1_1 set of all the strings σ such that every extension of σ is enumerated in $2^{<\mathbb{N}} - S$ (by the Σ^1_1-boundedness principle, if this happens, it happens at some computable ordinal stage). This set contains no prefix σ of G because as S is dense along G, there is some string extending σ which is in S. Also this set is dense along G because for any prefix σ of G, there is a string τ extending σ such that every string extending τ is enumerated in $2^{<\mathbb{N}} - S$, implying that τ is in W.

It follows that G is not Π^1_1-generic, and then by the previous corollary it is not weakly-Σ^1_1-generic. ■

6.6.4 Further discussion about lowness for higher genericity notions

We shall mainly discuss here the lowness notions for various notions of genericity. We define the relatized notion of genericity using full relativization, that is, considering $\Delta^1_1(A)$, $\Pi^1_1(A)$ or $\Sigma^1_1(A)$ open sets, for a given sequence A.

Definition 6.6.5. *We say that A is* ***low for Δ^1_1-genericity*** *if any Δ^1_1-generic is also $\Delta^1_1(A)$-generic. We define similarly* ***lowness for Π^1_1-genericity*** *and* ***lowness for Σ^1_1-genericity****.* ■

Considering the lower analogues, Greenberg and Miller proved (unpublished), together with Yu [95] that only computable sequences are low for 1-genericity. A proof that low for Π^1_1-genericity is Δ^1_1 works analogously. It uses a higher analogue of an important theorem of Posner and Robinson [74], whose interesting consequences are discussed below.

Theorem 6.6.5 (Higher Posner-Robinson theorem):
For any two sequences A, X such that A is not Δ^1_1, there is a Π^1_1-generic sequence G such that $G \oplus A$ higher Turing computes X.

Proof: We can suppose that A is not Π^1_1 (otherwise consider the complement of A). In particular for any Π^1_1 set of integers, we have either some n which is in A but not in W, or some n which is in W but not in A.

Let us denote by W_n the n-th Π^1_1 set of strings. We will define a sequence of strings $\sigma_1 \prec \sigma_2 \prec \dots$ such that for any n, if there is an extension of σ_n in W_n, then $\sigma_{n+1} \in W_n$. It will follow that G, the unique limit point of $\{[\sigma_n]\}_{n\in\mathbb{N}}$, will be Π^1_1-generic. Also we will do it in such a way that $G \oplus A$ higher Turing computes X.

The construction:

Start with σ_0 to be $X(0)$. Suppose that σ_n is defined and let us define σ_{n+1}. Consider the Π^1_1 set:

$$V_n = \{m \ : \ \text{there exists } \tau \in W_n \text{ extending } \sigma_n\hat{\ }0^m\hat{\ }1\}$$

Let m be the smallest integer such that either m is in V_n but not in A or m is in A but not in V_n. If we are in the first case, then we define σ_{n+1} to be the first string τ of W_n (in order of the enumeration), extending $\sigma_n\hat{\ }0^m\hat{\ }1$, concatenated with the bit $X(n+1)$. If we are in the second case, define σ_{n+1} to be $\sigma_n\hat{\ }0^m\hat{\ }1$, concatenated with the bit $X(n+1)$.

Verification:

It is clear that G is Π^1_1-generic. We verify that $G \oplus A$ higher Turing computes X. To do so we retrieve the construction with the help of A. The description we give below can easily be converted into a Π^1_1 functional $\Phi \subseteq 2^{<\mathbb{N}} \times 2^{<\mathbb{N}}$ with $\Phi(G \oplus A) = X$. Note however that Φ might not be consistent on other oracles.

Let σ_0 be the first bit of G. We have $\sigma_0 = X(0)$. Now assuming the higher Turing computation has identified σ_n together with $X\upharpoonright_{n+1}$, let us identify σ_{n+1} together with $X\upharpoonright_{n+2}$. To do so, we first count the number of 0's of G which follows its prefix σ_n. Let m be that number. If $m \notin A$ we deduce that $m \in V_n$. In this case we look for the first string τ of V_n which extends $\sigma_n \hat{} 0^m \hat{} 1$, we set σ_{n+1} to be $\tau \hat{} G(|\tau|)$ and $X\upharpoonright_{n+2}$ to be $X\upharpoonright_{n+1} \hat{} G(|\tau|)$ (note that we necessarily have $\tau < G$). If $m \in A$ we deduce that $m \notin V_n$. Then we set σ_{n+1} to be $\sigma_n \hat{} 0^m \hat{} 1 \hat{} G(|\sigma_n| + m + 1)$, and let $X\upharpoonright_{n+2}$ be $X\upharpoonright_{n+1} \hat{} G(|\sigma_n| + m + 1)$. ∎

A lower version of the previous theorem can be used to prove the Posner-Robinson theorem, which states that for any non computable sequence A, there is a 1-generic sequence G such that $A \oplus G \geq_T G'$. This is done by finding G such that $A \oplus G \geq_T \emptyset'$, and as we have for any 1-generic G that $G \oplus \emptyset' \geq_T G'$, we then have $A \oplus G \geq_T G'$. This interesting consequence can be interpreted the following way: The 'Turing computational distance' between a sequence and its jump can take arbitrarily small 'values'.

It is noticeable that this technique does not work anymore to prove that for any non Δ^0_2 set A, there is a 2-generic sequence G such that $A \oplus G \geq_T G''$. It was a much harder work, performed by Shore and Slaman in [82], to identify a different forcing, the Kumabe-Slaman forcing, in order to make the Posner-Robinson theorem relativize to any computable α:

Theorem 6.6.6 (Shore, Slaman):
For any computable α and any sequence A, not computable in $\emptyset^{(\beta)}$ for $\beta < \alpha$, there is a sequence G such that $A \oplus G \geq_T G^{(\alpha)}$.

Later, Day and Dzhafarov showed [12] that the Kumabe-Slaman forcing is indeed necessary, as for some non Δ^0_2 sequence A, there exists no 2-generic sequence G such that $A \oplus G \geq_T G''$.

Coming back to lowness, we now easily derived from Theorem 6.6.5 that only Δ^1_1 sequences are low for Π^1_1-genericity.

Corollary 6.6.6:
For any non Δ^1_1 sequence A, there is a Π^1_1-generic sequence G which is not $\Pi^1_1(A)$-generic.

Proof: Suppose first that $\omega_1^A > \omega_1^{ck}$. Then in particular $A \geq_h \mathcal{O}$ and therefore, any ω-c.a. approximable sequence is $\Delta^1_1(A)$. It follows from Proposition 6.6.4 that some Π^1_1-generic is not $\Pi^1_1(A)$-generic.

Suppose now that $\omega_1^A = \omega_1^{ck}$. Then in particular we have that $\mathcal{O}$ is not $\Delta^1_1(A)$. Also Proposition 6.6.3 is easily seen to relativize the following way: if $G \oplus A$ higher Turing computes a left-c.e. sequence which is not $\Delta^1_1(A)$, then G is not $\Pi^1_1(A)$-generic. Also by Theorem 6.6.5 there is a Π^1_1-generic G such that $G \oplus A$ higher Turing compute $\mathcal{O}$, a higher left-c.e. sequence which is not $\Delta^1_1(A)$. It follows that G is not $\Pi^1_1(A)$-generic. ∎

The question of lowness for Δ^1_1-genericity has not been directly studied. However Kjos-Hanssen, Nies, Stephan and Yu have characterized in [36] lowness for Δ^1_1-Kurtz randomness, which turns out to be the exact higher analogue of lowness for Kurtz-randomness, and weak-1-genericity (see Stephan and Yu [90]). Also it is very likely that the notion of lowness for Δ^1_1-Kurtz randomness coincides with the one of lowness for Δ^1_1-genericity.

The question of lowness for Σ^1_1-genericity has not been studied. Could it be different from Δ^1_1? The technique used in Theorem 6.6.5 does not seem to work if one now tries to build a Σ^1_1-generic sequence.

Question 6.6.1 *Is lowness for Σ^1_1-genericity different from Δ^1_1?*

As every Σ^1_1-generic sequence preserves ω_1^{ck}, we can also ask the question of cuppability, defined analogously here, than it was defined for Π^1_1-randomness in Section 6.5.2:

Question 6.6.2 *Is Σ^1_1-generic cuppability different from non Δ^1_1?*

6.7 Steel forcing : The Borel complexity of the set of sequences which collapse ω_1^{ck}

6.7.1 Motivation

We proved in Corollary 6.1.1 that the Borel complexity of the Π^1_1-randoms, a subset of $\{X \in 2^{\mathbb{N}} : \omega_1^X = \omega_1^{ck}\}$, is $\boldsymbol{\Pi^0_3}$. We also proved in Proposition 3.3.2 that the set $\{X \in 2^{\mathbb{N}} : \omega_1^X = \omega_1^{ck}\}$ is Σ^1_1 and the Gandy basis theorem implies that it is contained in no Π^1_1 set, except $2^{\mathbb{N}}$. In particular, it is itself not a Π^1_1 set. It is however easy to see that this set is Borel:

Proposition 6.7.1:
The set $\{X \in 2^{\mathbb{N}} : \omega_1^X = \omega_1^{ck}\}$ is $\boldsymbol{\Pi^0_{\omega_1^{ck}+2}}$.

Proof: For a given e and a given n, the set:

$$\mathcal{A}_{e,n} = \{X : \forall \alpha < \omega_1^{ck}\ \Phi_e(X,n) \notin \mathcal{O}_{<\alpha}\}$$

is $\boldsymbol{\Pi^0_{\omega_1^{ck}}}$. Also for a given e the set:

$$\mathcal{B}_e = \{X : \exists \alpha < \omega_1^{ck}\ \forall n\ \Phi_e(X,n) \in \mathcal{O}_{<\alpha}\}$$

is $\boldsymbol{\Sigma^0_{\omega_1^{ck}}}$. Then the set $\{X : \omega_1^X = \omega_1^{ck}\}$ is equal to $\bigcap_e((\bigcup_n \mathcal{A}_{e,n}) \cup \mathcal{B}_e))$ which is clearly a $\boldsymbol{\Pi^0_{\omega_1^{ck}+2}}$ set. ∎

The goal of this section is to prove that the complexity of $\{X \in 2^{\mathbb{N}} : \omega_1^X = \omega_1^{ck}\}$ cannot be simplified. To do this we are going to use Steel forcing. In [88], Steel introduced his forcing notion for the purpose of studying countable Δ^1_1 subsets of $\mathbb{N}^{\mathbb{N}}$, as well as independence results for subsystems of analysis.

In his paper, Steel also noticed, but without giving an actual proof, that his forcing notion can also be used to prove that Borel complexity of the set $\{X \in 2^{\mathbb{N}} : \omega_1^X = \omega_1^{ck}\}$ is not $\mathbf{\Sigma^0_{\omega_1^{ck}+2}}$. Following a work of [3], we will give an exposition of Steel's proof in terms of Baire category rather than in terms of forcing. We will then give a proof that the set $\{X \in 2^{\mathbb{N}} : \omega_1^X = \omega_1^{ck}\}$ is not $\mathbf{\Sigma^0_{\omega_1^{ck}+2}}$.

6.7.2 The forcing notion

The trees

Let $\mathcal{T}$ be the set of trees of the Baire space, both finite and infinite. Let us fix a computable bijection $b : \mathbb{N} \to \mathbb{N}^{<\mathbb{N}}$. We say that an element $X \in 2^{\mathbb{N}}$ represents a tree T if $n \in X$ iff $b(n) \in T$. We easily observes that the set of sequences representing elements of $\mathcal{T}$ is a closed subset of $2^{\mathbb{N}}$. Indeed, the condition for X to represent a tree is Π^0_1: "For every n, if $X(n) = 1$ then for every prefix τ of $b(n)$ we should have $X(b^{-1}(\tau)) = 1$".

It is clear that any tree is uniquely represented by a sequence this way. Also sometimes we will blur the distinction between an element of $\mathcal{T}$ and its representation in the Cantor space. We use on $\mathcal{T}$ the topology of the Cantor space induced on the set of representations of elements of $\mathcal{T}$. We easily verify that the set of representations of elements in $\mathcal{T}$ has no isolated point, therefore its elements are the paths of a perfect subtree of $2^{<\mathbb{N}}$. It follows that topologically, we have that $\mathcal{T}$ is essentially the same space as the Cantor space[1].

Also we will denote by F the set of 'finite trees' that correspond to a cylinder in the set of representation for elements of $\mathcal{T}$, that is, an element $p \in F$ specifies a set of nodes that are in the tree, and also a set of nodes that are not in the tree. That way we ensure that a sequence $p_1 \prec p_2 \prec \dots$ defines a unique tree. Given an element $p \in F$, we denote by $[p]$ the set of all trees of $\mathcal{T}$ that extend p. Also if $T \in \mathcal{T}$ extends p we write $p \prec T$, and if another finite tree q extends p we write $p \leq q$. It is clear that for any cylinder $[p]$, there are two finite sets of strings $\{\sigma_1, \dots, \sigma_n, \tau_1 \dots, \tau_m\}$ such that any tree T is in $[p]$ iff for $i \leq n$ we have $\sigma_i \in T$ and for $i \leq m$ we have $\tau_i \notin T$.

Recall that for a well-founded tree T, we write $|T|_o$ to denote the ordinal coded by T (with $|\sigma|_o = \sup_i(|\sigma\hat{\ }n_i|_o + 1)$) where $\{\sigma\hat{\ }n_i\}_{i\in\mathbb{N}}$ are all the children of σ). Also for every countable ordinal α, we denote by $\mathcal{T}_\alpha$ the set of trees T in $\mathcal{T}$ so that for every node $\sigma \in T$ which is not the root of T, either $|\sigma|_o < \alpha$ or the subtree of the nodes compatible with σ is ill-founded, in which case we write $|\sigma|_o = \infty$. So for any tree $T \in \mathcal{T}_\alpha$ we have either $|T|_o = \infty$ or $|T|_o \leq \alpha$.

The tagging

We now define the set P to be the set of elements p in F, paired with a valid tagging function h which assigns to each node of p a countable ordinal, or the value ∞. A tagging is said to be *valid* if for any $\sigma_1 \prec \sigma_2 \in p$, we have $h(\sigma_1) > h(\sigma_2)$. By convention, ∞ is considered greater than any countable ordinal, and also greater than itself.

So an element of P is given by a pair (p,h) where $p \in F$ and where h is a valid tagging of p. Also for a given $(p,h) \in P$, we write $[(p,h)]$ to denote the set of trees in $[p]$ such that for every node $\sigma \in p$, we have $|\sigma|_o = h(\sigma)$, where $|\sigma|_o$ is taken in T. For $(p,h) \in P$ and

[1] One can easily prove it directly by constructing the homeomorphism, or use Brouwer's theorem, saying that any compact, metrisable, perfect, 0-dimensional space is homeomorphic to the Cantor space, see [34] for details.

$(q,g) \in P$, we say that $(p,h) \leq (q,g)$ if $p \leq q$ and $h \leq g$ (the taggings g and h coincide on elements of p).

For any countable α, we then let P_α denotes the set of elements $(p,h) \in P$ such that h assigns to nodes in p distinct from the root, only values strictly smaller than α, or the value ∞. Also for a given $(p,h) \in P_\alpha$, we write $[(p,h)]^\alpha$ to denote the set $[(p,h)]$ intersected with the set $\mathcal{T}_\alpha$ (so if $T \in [(p,h)]^\alpha$, for every node $\sigma \in T$ distinct from the root, we have either $|\sigma|_o < \alpha$ or $|\sigma|_o = \infty$).

The forcing relation

For any countable α, β, we now define the forcing relation between $\mathbf{\Sigma^0_\alpha}$ or $\mathbf{\Pi^0_\alpha}$ subsets of $\mathcal{T}$ and elements of P_β. For some β, some $(p,h) \in P_\beta$ and some Borel set $\mathcal{A}$, the relation $(p,h) \Vdash_\beta \mathcal{A}$ is intended to "more or less" means "$\mathcal{A}$ is co-meager in $[(p,h)]^\beta$" (for some topology, given later). Why "more or less"? We shall see that if $(p,h) \Vdash_\beta \mathcal{A}$ then $\mathcal{A}$ is co-meager in $[(p,h)]^\beta$, but the converse however is not necessarily true. Also one could easily add some complexity in the forcing relation to make the converse true, but as we don't need it, and as we would like to keep the relation as simple as possible, we don't do it.

For $(p,h) \in P_\beta$ and $(q,g) \in P$, we say that $(p,h) \leq_\beta (q,g)$ if $(p,h) \leq (q,g)$ and if in addition we have $(q,g) \in P_\beta$. Let $(p,h) \in P_\beta$ and let us define the relation $\Vdash_\beta$ by induction on the Borel complexity of sets.

- If $\mathcal{A}$ is $\mathbf{\Delta^0_1}$ (a finite union of cylinders) we say that $(p,h) \Vdash_\beta \mathcal{A}$ iff $[p] \subseteq \mathcal{A}$.
- If $\mathcal{A}$ is $\mathbf{\Sigma^0_\alpha}$ with $\mathcal{A} = \bigcup_n \mathcal{A}_n$, we say that $(p,h) \Vdash_\beta \mathcal{A}$ iff $\exists n\ (p,h) \Vdash_\beta \mathcal{A}_n$.
- If $\mathcal{A}$ is $\mathbf{\Pi^0_\alpha}$, we say that $(p,h) \Vdash_\beta \mathcal{A}$ iff $\forall (q,g) \geq_\beta (p,h)\ (q,g) \nVdash_\beta \mathcal{A}^c$.

Note that the forcing relation that we gave might depend on the presentation of a given Borel set. Also for two different ways to write $\mathcal{A} = \bigcup_n \mathcal{A}_n$ or $\mathcal{A} = \bigcup_n \mathcal{A}'_n$, we might have that some $(p_1,h_1) \Vdash_\beta \bigcup_n \mathcal{A}_n$ but $(p_1,h_1) \nVdash_\beta \bigcup_n \mathcal{A}'_n$. In practice this will have no consequence, because on the other hand, we necessarily have in this case some $(q,g) \geq_\beta (p_1,h_1)$ with $(q,g) \Vdash_\beta \bigcup_n \mathcal{A}'_n$, which will be sufficient. Also one can prove by induction that as long as our unions are increasing, the forcing relations then does not depend anymore on the presentation of a given Borel set.

To simplify the reading, instead of writing (p,h) for elements of P, we sometimes simply write p, the tagging function being implicit. When we do so, we will always precise it, so that there is no ambiguity. This slight abuse of notation starts with the next lemma, for which the tagging function is implicit:

Lemma 6.7.1 *For a $\mathbf{\Pi^0_\alpha}$ set $\mathcal{A} = \bigcap_n \mathcal{A}_n$, any countable β and any $(p,h) \in P_\beta$, we have*

$$p \Vdash_\beta \mathcal{A} \text{ iff } \forall n\ \forall q \geq_\beta p\ \exists r \geq_\beta q\ r \Vdash_\beta \mathcal{A}_n$$

PROOF: Suppose $p \Vdash_\beta \mathcal{A}$, then by definition, $\forall q \geq_\beta p\ q \nVdash_\beta \bigcup_n \mathcal{A}^c_n$. Still following the rules of forcing we then have $\forall q \geq_\beta p\ \forall n\ q \nVdash_\beta \mathcal{A}^c_n$ with $\mathcal{A}^c_n$ a $\mathbf{\Pi^0_\gamma}$ set for some $\gamma < \alpha$, and then $\forall n\ \forall q \geq_\beta p\ \exists r \geq_\beta q\ r \Vdash_\beta \mathcal{A}_n$.

Suppose $p \nVdash_\beta \mathcal{A}$, then by definition, $\exists q \geq_\beta p\ q \Vdash_\beta \bigcup_n \mathcal{A}^c_n$. Still following the rules of forcing we have $\exists q \geq_\beta p\ \exists n\ q \Vdash_\beta \mathcal{A}^c_n$ with $\mathcal{A}^c_n$ a $\mathbf{\Pi^0_\gamma}$ set for some $\gamma < \alpha$, and then $\exists n\ \exists q \geq_\beta p\ \forall r \geq_\beta q\ r \nVdash_\beta \mathcal{A}_n$. ■

The β-topology:

For any ordinal β, we call β-topology, the topology on $\mathcal{T}_\beta$ generated by the subbasis $[(p,h)]^\beta$ for any $(p,h) \in P_\beta$. We would like to study genericity with respect to the β-topology, that is, elements of $\mathcal{T}_\beta$ which are in 'sufficiently many' dense open sets of this topological space.

But this study can make sense only after we proved that generic elements actually exist, that is, we should make sure that $\mathcal{T}_\beta$ endowed with the β-topology is a Baire space:

Proposition 6.7.2:
For any β, the set $\mathcal{T}_\beta$, together with the β-topology is a Baire space.

Proof: Suppose that we have a sequence $\{\mathcal{U}_n\}_{n\in\mathbb{N}}$ of subsets of $\mathcal{T}_\beta$ which are open in the β-topology. Each of them is a union of cylinders, so that for any n and any $(p,h) \in P_\beta$, there is some cylinder $[(q,g)]^\beta \subseteq \mathcal{U}_n$ so that $[(q,g)]^\beta \subseteq [(p,h)]^\beta$.

Consider any condition $(p,h) \in P_\beta$. There must exist some $[(p_0,h_0)]^\beta \subseteq \mathcal{U}_0$ which is such that $[(p_0,h_0)]^\beta \subseteq [(p,h)]^\beta$. Then inductively for any n, assuming (p_n,q_n) is defined, we define (p_{n+1},q_{n+1}). We define a pair (q,g) extending (p_n,q_n) the following way: we start by putting in (q,g) all tagged nodes of (p_n,q_n). Then for any node σ in p_n with tagging $\alpha+1$, we add $(\sigma\hat{\ }k,\alpha)$ in (q,g) for some $\sigma\hat{\ }k$ so that no string $\tau \geq \sigma\hat{\ }k$ is mentioned in p_n or in q so far.

For any node σ in p_n with tagging α limit, if no sequence $\{\alpha_m\}_{m\in\mathbb{N}}$ is assigned to σ yet, we assign one so that $\alpha = \sup_m \alpha_m$. Then we put $(\sigma\hat{\ }k,\alpha_n)$ in (q,g) for some $\sigma\hat{\ }k$ so that no string $\tau \geq \sigma\hat{\ }k$ is mentioned in p_n or in q so far.

Finally for every node σ in p_n with tagging ∞, we add $(\sigma\hat{\ }k,\infty)$ in (q,g) for some $\sigma\hat{\ }k$ so that no string $\tau \geq \sigma\hat{\ }k$ is mentioned in p_n or in q so far. Then as q should correspond to a cylinder in the set of representations of trees, we might need to actively specify that some nodes are not in q (and then not in any extension of q). If needed we do so.

Then (q,g) is a valid extension of (p_n,h_n) and then there must exists a cylinder $[(p_{n+1},h_{n+1})]^\beta \subseteq \mathcal{U}_{n+1}$ such that $[(p_{n+1},h_{n+1})]^\beta \subseteq [(q,g)]^\beta$.

It is clear by construction that $\bigcap_n[(p_n,q_n)]^\beta \subseteq \bigcap_n \mathcal{U}_n$. We should now prove that $\bigcap_n[(p_n,q_n)]^\beta$ is not empty. Because $p_0 \leq p_1 \leq p_2 \leq \dots$ and $h_0 \leq h_1 \leq h_2 \leq \dots$ we have that $\bigcap_n[p_n]$ contains a unique element T and $\bigcap_n[h_n]$ contains a unique element H tagging every node in T (and saying nothing on nodes which are not in T).

It is clear by construction (and can be prove formally by induction) that for any node $\sigma \in T$ we have $H(\sigma) \leq |\sigma|_o$. Also suppose $H(\sigma) < |\sigma|_o$ for some node σ. Then we can recursively look for a child node σ such that $H(\sigma) < |\sigma|_o$ but such that $H(\tau) = |\tau|_o$ for all children τ of σ. Also necessarily finitely many of those children are enough to witness $H(\sigma) < |\sigma|_o$, implying that h_n is an invalid tagging of p_n already for some n, which is a contradiction. ∎

We shall now prove that if $(p,h) \Vdash_\beta \mathcal{A} \subseteq \mathcal{T}$, then $\mathcal{A} \cap \mathcal{T}_\beta$ is comeager in $[(p,h)]^\beta$ for the β-topology (we will simply say that $\mathcal{A}$ is comeager in $[(p,h)]^\beta$). In particular, if an element is generic enough for the β-topology (belongs to sufficiently many dense open sets of the β-topology), and if it belongs to $[p,h]^\beta$, then it belongs to $\mathcal{A}$.

Lemma 6.7.2 *Let $\mathcal{A}$ be any $\mathbf{\Sigma}^0_{\boldsymbol{\alpha}}$ or $\mathbf{\Pi}^0_{\boldsymbol{\alpha}}$ set and let $(p,h) \in P_\beta$. If $(p,h) \Vdash_\beta \mathcal{A}$ then $\mathcal{A} \cap \mathcal{T}_\beta$ is comeager in $[(p,h)]^\beta$ for the β-topology.*

Proof: Consider $\mathcal{A}$ a $\mathbf{\Delta}^0_{\mathbf{1}}$ set and suppose that for any β and $(p,h) \in P_\beta$ we have $(p,h) \Vdash_\beta \mathcal{A}$. Then $[p] \subseteq \mathcal{A}$ and then also $[(p,h)]^\beta \subseteq \mathcal{A}$, so clearly $\mathcal{A}$ is comeager in $[(p,h)]^\beta$.

The tagging function is now implicit. Consider $\mathcal{A} = \bigcap_n \mathcal{A}_n$ a $\mathbf{\Pi}^0_{\boldsymbol{\alpha}}$ set and suppose that for any β and $p \in P_\beta$ we have $p \Vdash_\beta \mathcal{A} = \bigcap_n \mathcal{A}_n$. Then $\forall n\ \forall q \geq_\beta p\ \exists r \geq_\beta q\ r \Vdash_\beta \mathcal{A}_n$. Therefore, for all n, by induction hypothesis, the set $\mathcal{A}_n$ is comeager in a dense open subset of $[p]^\beta$. Therefore it is also comeager in $[p]^\beta$. Also as every $\mathcal{A}_n$ is co-meager in $[p]^\beta$, then $\bigcap_n \mathcal{A}_n$ is co-meager in $[p]^\beta$.

Consider $\mathcal{A} = \bigcup_n \mathcal{A}_n$ a $\mathbf{\Sigma}^0_{\boldsymbol{\alpha}}$ set and suppose that for any β and $p \in P_\beta$ we have $p \Vdash_\beta \mathcal{A}$. Then $p \Vdash_\beta \mathcal{A}_n$ for some n. By induction hypothesis we have that $\mathcal{A}_n$ is comeager in $[p]^\beta$ and then that $\bigcup_n \mathcal{A}_n$ is comeager in $[p]^\beta$. ∎

Just a small step now remains to prove the Baire property of any Borel $\mathcal{A}$, for the β-topology, that is, any Borel set $\mathcal{A}$ is equal to an open set, up to a meager set. The tagging function is implicit in the following lemma.

Lemma 6.7.3 *For any $\mathbf{\Sigma}^0_{\boldsymbol{\alpha}}$ or $\mathbf{\Pi}^0_{\boldsymbol{\alpha}}$ set $\mathcal{A}$ and any β, the set $\{[p]^\beta : p \in P_\beta \wedge (p \Vdash_\beta \mathcal{A} \vee p \Vdash_\beta \mathcal{A}^c)\}$ is dense in $\mathcal{T}_\beta$, for the β-topology.*

Proof: Let $\mathcal{A}$ be $\mathbf{\Sigma}^0_{\boldsymbol{\alpha}}$. Consider any $p \in P_\beta$. Then either $p \Vdash_\beta \mathcal{A}^c$ or $p \nVdash_\beta \mathcal{A}^c$, in which case $\exists q \geq_\beta p\ q \Vdash_\beta \mathcal{A}$. ∎

For a fixed β, the more dense open sets (for the β-topology) T belongs to, the more generic it is. We argue that for any β and any countably many Borel sets $\{\mathcal{A}_n\}_{n\in\omega}$, if a tree $T \in \mathcal{T}_\beta$ is generic enough, we have for any n that $T \in \mathcal{A}_n$ iff there is a prefix p of T such that $(p, |T|_o \upharpoonright_p) \Vdash_\beta \mathcal{A}_n$. In what follows, the tagging function $|T|_o \upharpoonright_p$ is implicit.

Pick some n and suppose that for some prefix p of T we have $p \Vdash_\beta \mathcal{A}_n$. Then using Lemma 6.7.2 we have that $\mathcal{A}_n$ is co-meager in $[p]^\beta$ and then if T is generic enough it belongs to $\mathcal{A}_n$. Suppose now that $T \in \mathcal{A}_n$. In particular if T is generic enough, it is in the dense open set $\{[p]^\beta : p \in P_\beta \wedge p \Vdash_\beta \mathcal{A}_n \wedge p \Vdash_\beta \mathcal{A}_n^c\}$. Also we cannot have that $p \Vdash_\beta \mathcal{A}^c$ for some $p < T$, as we just proved that in this case $T \in \mathcal{A}^c$. Therefore, for some prefix p of T we have $p \Vdash_\beta \mathcal{A}_n$.

6.7.3 The retagging lemma

We now prove the main lemma of Steel forcing. For any ordinal α, any two ordinals $\beta_1, \beta_2 \geq \omega\alpha$, and $(p,h_1) \in P_{\beta_1}$, $(p,h_2) \in P_{\beta_2}$, we write $(p,h_1) \sim_{\omega\alpha} (p,h_2)$ if for every node σ in p we have $h_1(\sigma) < \omega\alpha$ iff $h_2(\sigma) < \omega\alpha$ iff $h_1(\sigma) = h_2(\sigma)$.

Lemma 6.7.4 (The retagging tool) *Let β, α be countable ordinals with $\beta < \alpha$. Let $\beta_1, \beta_2 \geq \omega\alpha$ and $p \in F$ with $(p,h_1) \in P_{\beta_1}$, $(p,h_2) \in P_{\beta_2}$ and suppose $(p,h_1) \sim_{\omega\alpha} (p,h_2)$. Then for any $(q,g_1) \geq_{\beta_1} (p,h_1)$, there exists a retagging g_2 of q such that $(q,g_2) \geq_{\beta_2} (p,h_2)$ and with $(q,g_1) \sim_{\omega\beta} (q,g_2)$.*

PROOF: We simply build g_2. On nodes σ of p we set $g_2(\sigma) = h_2(\sigma)$, so the tagging g_2 will extend the tagging h_2. Also as $(p, h_1) \sim_{\omega\alpha} (p, h_2)$ then also $(p, g_1 \upharpoonright_p) \sim_{\omega\alpha} (p, g_2 \upharpoonright_p)$.

Also because $(p, h_1) \sim_{\omega\alpha} (p, h_2)$ and because $\omega\beta + \omega \leq \omega\alpha$, for every other node σ of q that is not in p and such that $g_1(\sigma) < \omega\beta + \omega$, we can set $g_2(\sigma) = g_1(\sigma)$ and have that g_2 is still a valid tagging so far.

Let M be the largest integer such that every node σ in q and not in p is tagged by something smaller than $\omega\beta + M$ by g_2 so far. Now for every other node σ in q and not in p such that $g_1(\sigma) \geq \omega\beta + \omega$, we have infinitely many values between $\omega\beta + M$ and $\omega\beta + \omega$ that we can use to tag them in a valid way by g_2. It is then easy to check that $(q, g_1) \sim_{\omega\beta} (q, g_2)$ and that $(q, g_2) \geq_{\beta_2} (p, h_2)$. ∎

Lemma 6.7.5 (The retagging lemma) *For any $\mathbf{\Pi^0_\alpha}$ or $\mathbf{\Sigma^0_{\alpha+1}}$ set $\mathcal{A}$, any countable ordinal $\beta_1, \beta_2 \geq \omega\alpha$ and any $p \in F$ with $(p, h_1) \in P_{\beta_1}$ and $(p, h_2) \in P_{\beta_2}$, if $(p, h_1) \sim_{\omega\alpha} (p, h_2)$, then $(p, h_1) \Vdash_{\beta_1} \mathcal{A}$ iff $(p, h_2) \Vdash_{\beta_2} \mathcal{A}$.*

PROOF: Suppose that $\mathcal{A}$ is a $\mathbf{\Pi^0_1}$ set. Let us suppose that $(p, h_1) \nVdash_{\beta_1} \mathcal{A}$ in order to show that $(p, h_2) \nVdash_{\beta_2} \mathcal{A}$. The converse is then be similar. If $(p, h_1) \nVdash_{\beta_1} \mathcal{A}$ then $\exists (q, g_1) \geq_{\beta_1} (p, h_1)$ $(q, g_1) \Vdash_{\beta_1} \mathcal{A}^c$. Also $\mathcal{A}^c$ is given by a union of clopen set $\bigcup_n \mathcal{A}_n$ and we have by definition that $[q] \subseteq \mathcal{A}_n$ for some n. Also, for any other valid tagging g_2 of q, whose range lies in $\beta_2 \cup \{\infty\}$, we have $(q, g_2) \Vdash_{\beta_2} \mathcal{A}_n$. If such a valid tagging exists, with in addition that $(q, g_2) \geq_{\beta_2} (p, h_2)$, it would then follow that $(p, h_2) \nVdash_{\beta_2} \mathcal{A}$. Also by the retagging tool with $\alpha = 1$ and $\beta = 0$, such a tagging exists.

Suppose now that $\mathcal{A}$ is a $\mathbf{\Pi^0_\alpha}$ set. Let us suppose that $(p, h_1) \nVdash_{\beta_1} \mathcal{A}$, to prove that $(p, h_2) \nVdash_{\beta_2} \mathcal{A}$. The converse is then similar. We have an extension $(q, g_1) \geq_{\beta_1} (p, h_1)$ such that $(q, g_1) \Vdash_{\beta_1} \mathcal{A}^c$. Also let $\bigcup_n \mathcal{A}_n$ be the complement of $\mathcal{A}$. Then for some n we have $(q, g_1) \Vdash_{\beta_1} \mathcal{A}_n$. Also $\mathcal{A}_n$ is a $\mathbf{\Pi^0_\beta}$ set for some $\beta < \alpha$. Also by the retagging tool we have a tagging g_2 with $(q, g_2) \sim_{\omega\beta} (q, g_1)$ and such that $(q, g_2) \geq_{\beta_2} (p, h_2)$.

Now by induction hypothesis, as we have $(q, g_2) \sim_{\omega\beta} (q, g_1)$ we then have $(q, g_2) \Vdash_{\beta_2} \mathcal{A}_n$. Also as $(q, g_2) \geq_{\beta_2} (p, h_2)$, it follows that $(p, h_2) \nVdash_{\beta_2} \mathcal{A}$.

Suppose now that the lemma is true for any $\mathbf{\Pi^0_\alpha}$ set. For any $\mathbf{\Sigma^0_{\alpha+1}}$ set $\mathcal{A} = \bigcup_n \mathcal{A}_n$, with $(p, h_1), (p, h_2), \beta_1, \beta_2$ under the condition of the lemma, we have $(p, h_1) \Vdash_{\beta_1} \mathcal{A}$ iff $(p, h_1) \Vdash_{\beta_1} \mathcal{A}_n$ for some n iff $(p, h_2) \Vdash_{\beta_2} \mathcal{A}_n$ iff $(p, h_2) \Vdash_{\beta_2} \mathcal{A}$. ∎

6.7.4 Preservation of ω_1^{ck}

We should bring some effectivity in the forcing relation. To do so, for $\beta < \omega_1^{ck}$, we can represent the tagging of elements of P_β by elements of $\mathcal{O}_{<\beta}$. It is clear that the set of all representations for elements in P_β is Δ^1_1, uniformly in any element of $\mathcal{O}_{=\beta}$.

Lemma 6.7.6 *For any $\alpha < \omega_1^{ck}$, any Σ^0_α or Π^0_α set $\mathcal{A}$, and any $\beta < \omega_1^{ck}$, the set $\{p \in P_\beta : p \Vdash_\beta \mathcal{A}\}$ is Δ^1_1 uniformly in an index of $\mathcal{A}$, and a code for β.*

PROOF: Suppose $\mathcal{A} = \bigcup_n \mathcal{A}_n$ is a Σ^0_1 set, with each $\mathcal{A}_n$ a clopen set. Then for any $\beta < \omega_1^{ck}$ and any $(p, h) \in P_\beta$ we have $(p, h) \Vdash_\beta \mathcal{A}$ iff $[p] \subseteq \mathcal{A}_n$ for some n, which is a Σ^0_1 condition uniformly in (p,h) and in $\mathcal{A}$. Then the set $\{(p \in P_\beta \ : \ p \Vdash_\beta \mathcal{A}\}$ is Δ^1_1 uniformly in an index of $\mathcal{A}$ and a code for β.

Suppose the lemma is true for any Σ^0_α sets with $\alpha < \omega_1^{ck}$ and any $\beta < \omega_1^{ck}$, and let us argue that the lemma is true for any Π^0_α sets and any $\beta < \omega_1^{ck}$. Let $\mathcal{A}$ be a Π^0_α set and let $\beta < \omega_1^{ck}$. In what follows, the tagging function is implicit.

For any $p \in P_\beta$ we have that $p \Vdash_\beta \mathcal{A}$ iff $\forall q \geq_\beta p \ q \nVdash_\beta \mathcal{A}^c$. Also by induction hypothesis, the set $\{q \in P_\beta \ : \ q \nVdash_\beta \mathcal{A}^c\}$ is Δ^1_1 uniformly in an index for $\mathcal{A}_c$, and a code for β. Then it is also the case for a restriction of this set to elements that extends p.

Suppose now that the lemma is true for any $\Pi^0_{<\alpha}$ set with $\alpha < \omega_1^{ck}$ and any $\beta < \omega_1^{ck}$, and let us show that the lemma is true for any Σ^0_α set and any $\beta < \omega_1^{ck}$. Let $\mathcal{A} = \bigcup_n \mathcal{B}_n$ be a Σ^0_α sets with each $\mathcal{B}_n$ a $\Pi^0_{<\alpha}$ set. We have $p \Vdash_\beta \mathcal{A}$ iff $\exists n \ p \Vdash_\beta \mathcal{B}_n$. Also by induction hypothesis, the set $\{p \in P_\beta \ : \ p \Vdash_\beta \mathcal{B}_n\}$ is Δ^1_1 uniformly in an index for $\mathcal{B}_n$ and a code for β. Therefore also the set $\{p \in P_\beta \ : \ p \Vdash_\beta \mathcal{A}\}$ is Δ^1_1 uniformly in an index for $\mathcal{B}_n$ and a code for β. ∎

Theorem 6.7.1:
If $T \in \mathcal{T}_{\omega_1^{ck}}$ is generic enough, then $\omega_1^T = \omega_1^{ck}$.

Proof: Consider a functional $\Phi : \mathcal{T} \times \omega \to \omega$ and the set

$$\mathcal{A} = \{T \ : \ \forall n \ \exists \alpha < \omega_1^{ck} \ \Phi(T,n) \in \mathcal{O}^T_{<\alpha}\}$$

Let $\mathcal{A}_n = \{T \ : \ \exists \alpha < \omega_1^{ck} \ \Phi(T,n) \in \mathcal{O}^T_{<\alpha}\}$ and $\mathcal{A}_{n,\alpha} = \{T \ : \ \Phi(T,n) \in \mathcal{O}^T_{<\alpha}\}$. Note that from Porism 1.6.1, for each $\alpha < \omega_1^{ck}$ and each e the set $\{X \ : \ e \in \mathcal{O}^X_{<\alpha}\}$ is $\Sigma^0_{\alpha+1}$ uniformly in e and a code for α. It follows that the set $\mathcal{A}_{n,\alpha}$ is $\Sigma^0_{\alpha+1}$ uniformly in n and a code for α.

Suppose that for some $T \in \mathcal{T}_{\omega_1^{ck}}$ we have $T \in \mathcal{A}$. Suppose also that T is generic enough, so that T belongs to some $[(p,h)]^{\omega_1^{ck}}$ such that $(p,h) \Vdash_{\omega_1^{ck}} \mathcal{A}$. In particular there is a smallest $\alpha_0 < \omega_1^{ck}$ so that $(p,h) \in P_{\alpha_0}$. In what follows the tagging is implicit.

Let us now define the Π^1_1 function $f : \omega_1^{ck} \to \omega_1^{ck}$ which to each $\alpha < \omega_1^{ck}$ gives the smallest ordinal $\beta \geq \omega\alpha$ such that:

$$\forall n \ \forall q \geq_{\omega\alpha} p \ \exists r \geq_\beta q \ r \Vdash_\beta \bigcup_{\gamma<\beta} \mathcal{A}_{n,\gamma}$$

The fact that f is Π^1_1 is a direct consequence of Lemma 6.7.6. We should argue that f is defined on every ordinal $\alpha \geq \alpha_0$. As we have $p \Vdash_{\omega_1^{ck}} \bigcap_n \bigcup_{\gamma<\omega_1^{ck}} \mathcal{A}_{n,\gamma}$, then also we have:

$$\forall n \ \forall q \geq_{\omega_1^{ck}} p \ \exists r \geq_{\omega_1^{ck}} q \ r \Vdash_{\omega_1^{ck}} \bigcup_{\gamma<\omega_1^{ck}} \mathcal{A}_{n,\gamma}$$

So consider any n and any $q \geq_{\omega\alpha} p$. In particular there must exist some $r \geq_{\omega_1^{ck}} q$ such that $r \Vdash_{\omega_1^{ck}} \bigcup_{\gamma<\omega_1^{ck}} \mathcal{A}_{n,\gamma}$. Therefore, by the definition of the forcing relation we must have

$r \Vdash_{\omega_1^{ck}} \mathcal{A}_{n,\gamma}$ already for some $\gamma < \omega_1^{ck}$. Also let β be the smallest ordinal bigger than $\max(\omega\gamma, \omega\alpha)$ such that $r \in P_\beta$. Then by the retagging lemma, as $\mathcal{A}_{n,\gamma}$ is a $\Sigma^0_{\gamma+1}$ set, we must have $r \Vdash_\beta \mathcal{A}_{n,\gamma}$ and then $r \Vdash_\beta \bigcup_{\gamma<\beta} \mathcal{A}_{n,\gamma}$. As we can find such a β for any n and any $q \geq_{\omega\alpha} p$, then by the Σ^1_1-boundedness principle, the supremum of all those β is still a computable ordinal. So the function is f is defined everywhere.

It is straightforward to check that the function f is continuous, that is, $f(\sup_n \alpha_n) = \sup_n f(\alpha_n)$. Therefore if we define $\alpha_1 = f(\alpha_0)$ and $\alpha_{n+1} = f(\alpha_n)$ for each n, we then have that $\alpha_\omega = \sup_n \alpha_n$ is a fixed point of f. Note that also $\omega\alpha_\omega = \alpha_\omega$. It follows that we have:

$$\forall n\ \forall q \geq_{\alpha_\omega} p\ \exists r \geq_{\alpha_\omega} q\ r \Vdash_{\alpha_\omega} \bigcup_{\gamma<\alpha_\omega} \mathcal{A}_{n,\gamma}$$

But then by the forcing definition we have $p \Vdash_{\alpha_\omega} \bigcap_n \bigcup_{\gamma<\alpha_\omega} \mathcal{A}_{n,\gamma}$. We now have to prove that $p \Vdash_{\omega_1^{ck}} \bigcap_n \bigcup_{\gamma<\alpha_\omega} \mathcal{A}_{n,\gamma}$. Note that we cannot apply the tagging lemma directly because $\bigcap_n \bigcup_{\gamma<\alpha_\omega} \mathcal{A}_{n,\gamma}$ is only a $\Pi^0_{\alpha_\omega+1}$ set. This is here that we need to use the possibility for a tagging to be ∞. Using this, we shall argue that we actually already have:

$$\forall n\ \forall q \geq_{\omega_1^{ck}} p\ \exists r \geq_{\omega_1^{ck}} q\ r \Vdash_{\omega_1^{ck}} \bigcup_{\gamma<\alpha_\omega} \mathcal{A}_{n,\gamma} \tag{*}$$

Consider any n and any $q \geq_{\omega_1^{ck}} p$, and let q^* be a retagging of q, so that every node in q that is tagged by something bigger or equal to α_ω is retagged by ∞ in q^*. Then we have some $r^* \geq_{\alpha_\omega} q^*$ with $r^* \Vdash_{\alpha_\omega} \bigcup_{\gamma<\alpha_\omega} \mathcal{A}_{n,\gamma}$. In particular for some $\beta < \alpha_\omega$ we have $r^* \Vdash_{\alpha_\omega} \mathcal{A}_{n,\beta}$. Also by the retagging tool, as $q \sim_{\omega\alpha_\omega} q^*$, we have some $r \geq_{\omega_1^{ck}} q$ with $r \sim_{\omega\beta} r^*$ and then, by the retagging lemma, we have $r \Vdash_{\omega_1^{ck}} \mathcal{A}_{n,\beta}$, as $\mathcal{A}_{n,\beta}$ is a $\Sigma^0_{\beta+1}$ set. It follows that $r \Vdash_{\omega_1^{ck}} \bigcup_{\gamma<\alpha_\omega} \mathcal{A}_{n,\gamma}$ and then that (*) is actually true. Then we have $p \Vdash_{\omega_1^{ck}} \bigcap_n \bigcup_{\gamma<\alpha_\omega} \mathcal{A}_{n,\gamma}$.

It follows that $\sup_n |\Phi(T,n)|_o^T \leq \alpha_\omega < \omega_1^{ck}$. As we have this for every functional Φ for T generic enough, we then have $\omega_1^T = \omega_1^{ck}$. ∎

6.7.5 The Borel complexity of $\{X : \omega_1^X > \omega_1^{ck}\}$

Theorem 6.7.2:
The set $\{X \in 2^{\mathbb{N}} : \omega_1^X = \omega_1^{ck}\}$ is not $\boldsymbol{\Sigma}^0_{\omega_1^{ck}+2}$.

PROOF: We shall prove that the set of representations of elements of $\mathcal{T}$ which preserve ω_1^{ck} is not $\boldsymbol{\Sigma}^0_{\omega_1^{ck}+2}$. As this set is a closed subset of the Cantor space, it follows that also the set $\{X \in 2^{\mathbb{N}} : \omega_1^X = \omega_1^{ck}\}$ is not $\boldsymbol{\Sigma}^0_{\omega_1^{ck}+2}$. In what follows, the tagging functions are implicit.

Suppose that $\{T \in \mathcal{T} : \omega_1^T = \omega_1^{ck}\} = \bigcup_m \bigcap_n \mathcal{A}_{n,m}$ where each $\mathcal{A}_{n,m}$ is a $\boldsymbol{\Sigma}^0_{\omega_1^{ck}}$ set. Then using Theorem 6.7.1, there must be some m such that the set $\bigcap_n \mathcal{A}_{n,m}$ contains some tree T which is generic enough for Steel forcing over $P_{\omega_1^{ck}}$, so that $\omega_1^T = \omega_1^{ck}$. In particular we have $p \Vdash_{\omega_1^{ck}} \bigcap_n \mathcal{A}_{n,m}$ for some $p \prec T$ with $(p, |T|_o \upharpoonright_p) \in P_{\omega_1^{ck}}$.

So also we have $\forall n\ \forall q \geq_{\omega_1^{ck}} p\ \exists r \geq_{\omega_1^{ck}} q\ r \Vdash_{\omega_1^{ck}} \mathcal{A}_{n,m}$. Let ω_2^{ck} be the smallest ordinal bigger than ω_1^{ck}, and of the form ω_1^X for some oracle X (we can actually take $X = \mathcal{O}$). We should now prove that we actually have:

$$\forall n\ \forall q \geq_{\omega_2^{ck}} p\ \exists r \geq_{\omega_2^{ck}} q\ r \Vdash_{\omega_2^{ck}} \mathcal{A}_{n,m} \tag{*}$$

Consider now any n and any $q \geq_{\omega_2^{ck}} p$ and let q^* be a retagged version of q where each ordinal bigger or equal to ω_1^{ck} in q is retagged by ∞ in q^*. Then $q^* \geq_{\omega_1^{ck}} p$ and in particular we have some $r^* \geq_{\omega_1^{ck}} q^*$ such that $r^* \Vdash_{\omega_1^{ck}} \mathcal{A}_{n,m}$. Let $\mathcal{A}_{n,m} = \bigcup_{k \in \mathbb{N}} \mathcal{A}_{n,m,k}$ with each $\mathcal{A}_{n,m,k}$ is $\boldsymbol{\Pi}^0_{<\omega_1^{ck}}$ set. In particular for some $\beta < \omega_1^{ck}$ we have $r^* \Vdash_{\omega_1^{ck}} \mathcal{A}_{n,m,k}$ where $\mathcal{A}_{n,m,k}$ is a $\boldsymbol{\Pi}^0_{\boldsymbol{\beta}}$ set. Also by the retagging tool, as $q \sim_{\omega\omega_1^{ck}} q^*$, we have some $r \geq_{\omega_2^{ck}} q$ with $r \sim_{\omega(\beta)} r^*$ and then, by the retagging lemma, we have $r \Vdash_{\omega_2^{ck}} \mathcal{A}_{n,m,k}$, as $\mathcal{A}_{n,m,k}$ is a $\boldsymbol{\Pi}^0_{\boldsymbol{\beta}}$ set. It follows that $r \Vdash_{\omega_2^{ck}} \mathcal{A}_{n,m}$ and then that (*) is actually true. Then we have $p \Vdash_{\omega_2^{ck}} \bigcap_n \mathcal{A}_{n,m}$.

It follows that any $q \geq_{\omega_2^{ck}} p$ also forces $\bigcap_n \mathcal{A}_{n,m}$. Take such an extension q with a node tagged by the ordinal ω_1^{ck}. For any $T \in [q]^{\omega_2^{ck}}$ we have $\omega_1^T > \omega_1^{ck}$. Also as $q \Vdash_{\omega_2^{ck}} \bigcap_n \mathcal{A}_{n,m}$, the set $\bigcap_n \mathcal{A}_{n,m}$ contains some generic tree that is in $[q]^{\omega_2^{ck}}$. Then $\bigcup_m \bigcap_n \mathcal{A}_{n,m}$ contains an element which collapses ω_1^{ck}, which is a contradiction. ∎

Chapter 7

The badly-behaved oracles

On peut dire des divers procédés que nous avons décrits et de ceux qui pourront être imaginés par les futurs mathématiciens ce que nous avons dit des entiers : le nombre de ces procédés est, en fait, fini et nous pouvons tout au plus le considérer comme dénombrable, si nous ne fixons aucune limite supérieure à la durée de l'espèce humaine et au nombre total des hommes à venir. Comme chacun de ces procédés ne peut utiliser effectivement qu'un nombre limité d'entiers (ou de nombres accessibles précédemment définis), le nombre total des nombres ainsi accessibles sera dénombrable, c'est-à-dire ne représentera qu'une partie infime de l'ensemble des nombres incommensurables qui resteront inaccessibles.

Les nombres inaccessibles theory, Émile Borel

Joint work with Noam Greenberg and Laurent Bienvenu.

7.1 Time tricks : example with Π^1_1-open sets

In classical computability, we sometimes use the fact that the time of computation lies in the same space as the lengths of the sequences we use: ω. We call any use of this equality a **time trick**. Also sometimes, the use of a time trick is just done because it is convenient, but can actually be avoided. An example can be found in the proof that no left-c.e. sequence is weakly-2-random. This is done in the proof of Proposition 2.1.1, using a time trick, and a higher version of it is done in the proof of Theorem 5.3.1, without a time trick. Also it is clear that the proof in the higher setting, also works in the lower setting.

We shall now see an example where the use of a time trick cannot be removed: Any open set with a Σ^0_1 description also has a Δ^0_1 description. Indeed, for any Σ^0_1 set of strings W, we can define the Σ^0_1 set V by enumerating σ in V at stage $|\sigma|$ iff some prefix of σ is enumerated in $W_{|\sigma|}$. It is clear that $[W]^\prec = [V]^\prec$. Also $2^{<\omega} - V$ is Σ^0_1 because $\sigma \notin V$ iff $\sigma \notin V_{|\sigma|}$.

The proof of the previous paragraph clearly uses a time trick. We shall now see that there are some open sets with a Π^1_1 description that do not have a Δ^1_1 description. To do so, we start by proving that there are some Π^1_1 open sets $\mathcal{U}$ such that for any prefix-free Π^1_1 set of strings W, we have $\mathcal{U} \neq [W]^\prec$. This also justifies the necessity of Lemma 3.7.1 to prove several results of this thesis. Recall that the lemma says that for any Π^1_1-open set $\mathcal{U}$, uniformly in ε, one can find a Π^1_1 set of strings W with $[W]^\prec = \mathcal{U}$, which is 'almost disjoint' in the sense that $\sum_{\sigma\in W} 2^{-|\sigma|} \leq \lambda(\mathcal{U}) + \varepsilon$. We now show that in some cases, we cannot have $\sum_{\sigma\in W} 2^{-|\sigma|} = \lambda(\mathcal{U})$:

Theorem 7.1.1:
There is a Π^1_1-open set $\mathcal{U}$ such that for any prefix-free Π^1_1 set of strings W, we have $\mathcal{U} \neq [W]^\prec$.

Proof: Let W_e be a list of all Π^1_1 set of strings. Let $\{\sigma_e\}_{e\in\mathbb{N}}$ be a sequence of pairwise disjoint strings. We define the enumertion of a Π^1_1 set of strings V such that if W_e is prefix-free, then $[V]^\prec \neq [W_e]^\prec$.

For any string σ_e, we define a computable set of strings $A_e = \{\tau_{e,n} \ : \ n \in \mathbb{N}\}$ such that A_e is dense along $\sigma_e\hat{\ }0^\infty$, but such that no prefix of $\sigma_e\hat{\ }0^\infty$ is in A_e. For any e we put A_e in V at stage 0. Then for any stage s, and substage e, we check if both $[A_e]^\prec \subseteq [W_{e,s}]^\prec$ but $\sigma_e\hat{\ }0^\infty \notin [W_{e,s}]^\prec$. If so, then we enumerate σ_e in V at stage s.

We now claim that if W_e is prefix-free, then $[V]^\prec \neq [W_e]^\prec$. If $[V]^\prec = [W_e]^\prec$, in particular we have $[A_e]^\prec \subseteq [W_e]^\prec$. If so, then by compactness, for each string τ in A_e, there are only finitely many strings in W_e whose union of corresponding cylinders covers $[\tau]$. Also by the Σ^1_1-boundedness principle, as A_e is computable, there is a smallest stage $s < \omega_1^{ck}$ at which we already have $[A_e]^\prec \subseteq [W_{e,s}]^\prec$.

Also at stage s, if $\sigma_e\hat{\ }0^\infty \in [W_{e,s}]^\prec$, by construction, $\sigma_e\hat{\ }0^\infty \notin [V_t]^\prec$ for $t \geq s$, and then $[V]^\prec \neq [W_e]^\prec$.

On the other hand, if at stage s, we have $\sigma_e\hat{\ }0^\infty \notin [W_{e,s}]^\prec$, then σ_e is enumerated in V at stage s, and either $\sigma_e\hat{\ }0^\infty \notin [W_e]^\prec$, in which case, also we have $[V]^\prec \neq [W_e]^\prec$; or a prefix τ of $\sigma_e\hat{\ }0^\infty$ will be enumerated in W_e after stage s. But then, as already at stage s we have that $[W_e]^\prec$ covers $[A_e]^\prec$ without containing $\sigma_e\hat{\ }0^\infty$, and as A_e is dense along $\sigma_e\hat{\ }0^\infty$, there is necessarily an extension of τ which is already in W_e at stage s. Therefore W_e is not prefix-free. ∎

Corollary 7.1.1:
There is a Π^1_1-open set $\mathcal{U}$ such that for any Δ^1_1 set of strings W, we have $\mathcal{U} \neq [W]^\prec$.

Proof: It is clear, because for any Δ^1_1 set of strings W, there is a prefix-free Δ^1_1 set of strings V with $[W]^\prec = [V]^\prec$:

$$V = \{\sigma \in W \ : \ \forall \tau \prec \sigma, \ \tau \notin W\}$$

7.2 Higher Turing computation and fin-h computation

We defined in Section 4.1 the notion of higher Turing reduction, as well as the notion of fin-h reduction. Also we announced that given X, Y, among the following notions, the first two are different, and the last two coincide:

1. There is a c.e. partial map $\Phi : 2^{<\mathbb{N}} \to 2^{<\mathbb{N}}$, consistent on prefixes of X, such that $\Phi(X) = Y$.
2. There is a c.e. partial map $\Phi : 2^{<\mathbb{N}} \to 2^{<\mathbb{N}}$, consistent everywhere, such that $\Phi(X) = Y$.
3. There is a c.e. partial map $\Phi : 2^{<\mathbb{N}} \to 2^{<\mathbb{N}}$, consistent everywhere and closed under prefixes, such that $\Phi(X) = Y$.

Replacing c.e. by Π^1_1, we defined higher Turing reduction as in (1) and fin-h reductions as in (3). We shall prove here only that (2) is equivalent to (3), but in a non uniform way (using the fixed point theorem and the 'treeshbone' technique described in the next section, it is possible to prove that the equivalence has to be non-uniform). The proof that (1) is different from (2) will be done in Corollary 7.3.2.

Theorem 7.2.1:
If we have a functional Φ, consistent everywhere such that $\Phi(X) = Y$, then there exists a fin-h functional Ψ such that $\Psi(X) = Y$.

Proof: For any stage s, we define a fin-h functional Ψ_s. Unlike the way it is usually done, the functional Ψ_{s+1} is not an extension of the functional Ψ_s. They are all unrelated for different stages. So let us define Ψ_s; for this, consider the set of strings σ such that at stage s, some extensions of σ are mapped to strings of bigger and bigger length:

$$A_s = \{\sigma \ : \ \forall n \ \exists \tau \geq \sigma \ |\Phi_s(\tau)| \geq n\}$$

Now for every string σ in lexicographic order, if $\sigma \in A_s$ then consider the longest string τ which is compatible with the mapping of every extension of σ by Φ_s (note that τ can always be ϵ, the empty string). If there is no longest such string, that is, $\Phi_s(\sigma) = Z$ already for some Z, then let $\tau = Z\upharpoonright_{|\sigma|}$. Then put (σ, τ) in Ψ_s.

Each functional Ψ_s is clearly closed by prefixes because A_s is. Let us show that it is consistent. Suppose $\Psi_s(\sigma_1)$ is defined and let us suppose $\Psi_s(\sigma_2)$ is defined for $\sigma_2 > \sigma_1$. In particular we have $\sigma_1, \sigma_2 \in A_s$. For any σ, let W_σ be the set of strings extending σ, which are mapped to something via Φ_s. Also by definition it is not possible to have two incomparable strings which are both compatible with the mapping of each string in W_{σ_1} (or W_{σ_1}). Therefore as we have $W_{\sigma_2} \subseteq W_{\sigma_1}$, we also have $\Psi_s(\sigma_1) \preceq \Psi_s(\sigma_2)$. So Ψ_s is consistent.

Let us suppose that $\Phi(X) = Y$ and that every prefix of X is in A_s in order to prove that $\Psi_s(X) = Y$. For any $\sigma < X$ the set W_σ is dense along X. Also as $\Phi(X)$ is defined and as Φ is consistent, for longer and longer prefixes σ of X, there are longer and longer prefixes τ of Y, compatible with the mapping of every string in W_σ. Therefore $\Psi_s(X) = Y$.

There is one last case to handle, when $\Phi(X) = Y$ and there is no stage s such that every prefix of X is in A_s. We now define a last fin-h functional Ψ to handle this: At stage

s, if $\sigma \in A_s - A_{<s}$ we search for the longest string τ such that a prefix of σ is mapped to τ in Φ_s, and we then map σ to τ in Ψ_s.

It is clear that Ψ is consistent, as Φ is. Also as A_s is closed under prefixes, then Ψ is (we can prove by induction that if $\sigma \in A_s - A_{<s}$ and if we have $\tau \prec \sigma$ with $\tau \in A_{<s}$ then Ψ already maps τ to something before stage s). Now suppose $\Phi(X) = Y$ and that there is no stage s such that every prefix of X is in A_s. However, by the Σ^1_1-boundedness principle, if Φ is defined on X, then every prefix of X enters A_s at some computable stage s. Consider any n such that $\Phi(\sigma) = Y \upharpoonright_n$ for $\sigma \prec X$. Let s be the smallest stage such that $\Phi_s(\sigma) = Y \upharpoonright_n$. Then by hypothesis we have τ with $\sigma \preceq \tau \prec X$ and $t \geq s$ such that $\tau \in A_t - A_{<t}$. Then by construction we have $\Psi_t(\tau) \succeq Y \upharpoonright_n$. As this is true for any n, we then have $\Psi(X) = Y$. ■

7.3 Non-universality in continuous relativization

7.3.1 The perfect treesh-bone

For a tree $T \subseteq 2^{<\mathbb{N}}$, recall that $\sigma \in T$ is a branching node of T is $\sigma\hat{\ }0$ and $\sigma\hat{\ }1$ are both in T, and recall that the stem of T, denoted by $\text{stem}(T)$, is the smallest branching node of T. We describe here a construction that will be performed on various perfect trees, to conduct two proofs of this chapter. The idea is simple: Given a perfect tree T, we want to obtain a perfect subtree T' (that we will call $\texttt{Nar}(T)$), together with countably many nodes $\{\sigma_i\}_{i\in\mathbb{N}}$ of T which do not belong to T', but which are dense along any path of T'. We now formally describe how we achieve this.

For a perfect tree $T \subseteq 2^{<\mathbb{N}}$, we will now describe, uniformly in T, a dense $\Sigma^0_1(T)$ open subset of $[T]$ whose complement in $[T]$ is perfect. Let $\psi_T\colon 2^{<\mathbb{N}} \to T$ be the map which induces the natural isomorphism between $2^{\mathbb{N}}$ and $[T]$; $\psi_T(\epsilon) = \text{stem}(T)$ and for all $\sigma \in 2^{<\mathbb{N}}$ and $i \in \{0,1\}$, $\psi_T(\sigma\hat{\ }i)$ is the next splitting node in T above $\psi_T(\sigma)\hat{\ }i$. So the strings $\psi_T(\sigma)$ are exactly the splitting nodes of T.

Let $\sigma_0(T), \sigma_1(T), \ldots$ be an enumeration of all strings of the form $\psi_T(\sigma\hat{\ }1)$ for strings σ of odd length, such that the elements of the enumeration form an antichain, and are minimal under prefix ordering. We let $\texttt{Nar}(T)$ (the narrow subtree of T) be the result of removing the strings $\sigma_k(T)$ and their extensions from T. Also for any k we let $T[\![k]\!]$ denote $T\upharpoonright_{\psi_T(\sigma_k)}$, that is, the collection of strings of T comparable with $\sigma_k(T)$. We now give a picture to illustrate these definitions:

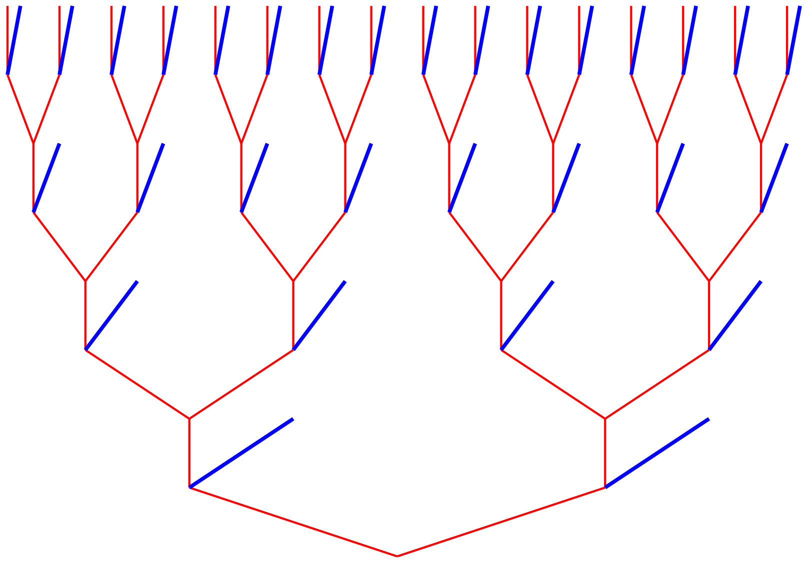


Figure 7.1: The treeshbone
The blue nodes correspond to nodes $\sigma_k(T)$.
The red subtree corresponds to $\mathtt{Nar}(T)$

Let us make a few remarks which will be widely used without explicit mention in what follows.

1. For any tree T and any k we have $\sigma_k(T) = \mathrm{stem}(T[\![k]\!])$
2. For any tree T and any k we have $\mathrm{stem}(T) \prec \mathrm{stem}(T[\![k]\!])$
3. For any tree T we have $\mathrm{stem}(T) = \mathrm{stem}(\mathtt{Nar}(T))$

7.3.2 The tree of trees

We now describe a general construction that will be used in two proofs of this chapter. What we build can be seen as a tree of trees. We define a subset $\mathcal{T}$ of the set of trees of the Cantor space.

Let $\mathcal{T}_0 = \{2^{<\omega}\}$. Let us define $\mathcal{T}_{n+1} = \{T[\![k]\!] \mid T \in \mathcal{T}_n, k \in \mathbb{N}\} \cup \{\mathtt{Nar}(T) \mid T \in \mathcal{T}_n\}$. Then $\mathcal{T}$ is defined to be the union of all the trees in $\mathcal{T}_n$ for some n. For two trees T_1, T_2 in $\mathcal{T}$, we say that T_2 extends T_1 or also $T_1 \preceq T_2$ if $T_2 \subseteq T_1$. In addition if $T_1 \neq T_2$ we write $T_1 \prec T_2$. We illustrate the tree of trees by the following picture:

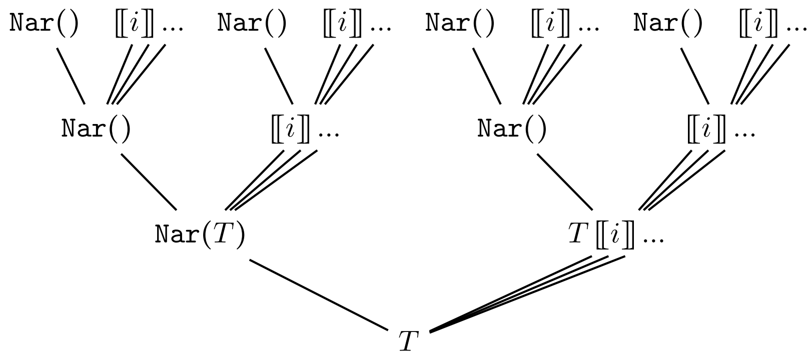


Figure 7.2: The tree of trees

We say that a sequence of trees $T_0 \prec T_1 \prec T_2 \prec \dots$, where each $T_i \in \mathcal{T}$, is a path of $\mathcal{T}$. If in addition, for infinitely many n we have $T_{n+1} \succ T_n [\![k]\!]$ for some k, the path is called **a shrinking path**, and in this case $\bigcap_n [T_n]$ contains exactly one sequence.

During the different constructions, we will pick at each stage s a shrinking path $T_0 \prec T_1 \prec \dots$ in the tree of trees. To do so, we will use what we will call a **strategy**. A strategy indicates which tree T_{n+1} we have to pick at some given stage to extend the current tree T_n. Also for each of the following construction, the number of time the tree T_{n+1} can change, assuming that the tree T_n does not change, will be bounded by $\omega + 2$. For this reason, we define a strategy to be an element of $(\omega + 2)^{<\mathbb{N}}$, that is, a finite sequence of ordinals all strictly smaller than $\omega + 2$.

Along with strategies, we use a function $F : (\omega + 2) \times \mathcal{T} \to \mathcal{T}$ such that for any $o \leq \omega + 2$ and any $T \in \mathcal{T}$ we have $T \prec F(o, T)$. The function F will depend on each construction. For the proof of Theorem 7.3.2, the function F will be partial Π^1_1 and for the proof of Theorem 7.3.1 it will be total computable. We then define by induction the function $G : (\omega+2)^{<\mathbb{N}} \times \mathcal{T}$ by $G(\epsilon) = 2^{<\omega}$ and $G(\alpha \hat{\ } o) = F(o, G(\alpha))$. Note that for any $\alpha, \beta \in (\omega+2)^{<\mathbb{N}}$ we have $\alpha \prec \beta$ iff $G(\alpha) \prec G(\beta)$. To emphasize that $G(\alpha)$ is a tree we will then write T_α instead of $G(\alpha)$.

7.3.3 No A-universal oracle continuous Π^1_1-Martin-Löf tests

Fact 7.3.1

If A higher Turing computes X, then X is not A-continuously Π^1_1-Martin-Löf random. The n-th oracle continuous Π^1_1-open set of the test is simply the enumeration of Φ, restricted to pairs with a second component of length longer than n.

Theorem 7.3.1:
Let $\mathcal{U}$ be an oracle continuous Π^1_1-open set such that for all sequences B, we have $\mathcal{U}^B \neq 2^\omega$. There is a higher Δ^0_2 sequence A such that $X \notin \mathcal{U}^A$ for some higher left-c.e. sequence X which is higher Turing computable in A.

PROOF: The strategies for this construction will be elements of $(\omega+1)^{<\mathbb{N}}$, together with the function $F : (\omega+1)^{<\mathbb{N}} \times \mathcal{T} \to \mathcal{T}$ defined by $F(k,T) = T[\![k]\!]$ for $k < \omega$ and $F(\omega,T) = \mathtt{Nar}(T)$. Let $\mathcal{U}$ be an oracle-continuous Π^1_1-open set and let us suppose that for any B we have $\mathcal{U}^B \neq 2^\omega$.

The construction:

At each stage $s < \omega_1^{ck}$ we define strategies $\alpha_{0,s} \prec \alpha_{1,s} \prec \dots$ and approximation X_s for X as follows:

At substage n, assuming $\alpha = \alpha_{n,s}$ and $X_s \upharpoonright_n$ have been defined, we define $\alpha_{n+1,s}$ and $X_s \upharpoonright_{n+1}$. We search for the least k such that for no $\sigma \in T_\alpha$ extending $\sigma_k(T_\alpha)$, we have $[X_s \upharpoonright_n \hat{}\, 0] \subseteq \mathcal{U}^\sigma_s$. If a least such k exists, we let $\alpha_{n+1,s} = \alpha_{n,s} \hat{}\, k$ (so $T_{\alpha_{n+1,s}} = T_\alpha[\![k]\!]$) and $X_s(n) = 0$. Otherwise $\alpha_{n+1,s} = \alpha_{n,s} \hat{}\, \omega$ (so $T_{\alpha_{n+1,s}} = \mathtt{Nar}(T_\alpha)$) and $X_s(n) = 1$.

The verification: Convergence

For a given stage s, let ξ_s be the unique limit point of $\{[\alpha_{n,s}]\}_{n\in\mathbb{N}}$. Note that by construction, the sequence $\{\xi_s\}_{s<\omega_1^{ck}}$ is higher left-c.e. Therefore, by Σ^1_1-boundedness principle, it reaches a limit sequence ξ and in particular, for any n, there is a stage $s < \omega_1^{ck}$ such that for every $s \leq t < \omega_1^{ck}$ we have $\xi_t \upharpoonright_n = \xi \upharpoonright_n$.

Also we then have for every n that the sequence $\{T_{\alpha_{n,s}}\}_{s<\omega_1^{ck}}$ converges to some tree T_{α_n}, and that $\{X_s\}_{s<\omega_1^{ck}}$ converges to some sequence X. Also as $\{\xi_s\}_{s<\omega_1^{ck}}$ is higher left-c.e., it implies that $\{X_s\}_{s<\omega_1^{ck}}$ is also higher left-c.e. Let $A \in \bigcap_n [T_n]$.

The verification: $X \notin \mathcal{U}^A$

We now prove that $X \notin \mathcal{U}^A$. By induction on n, we show that:

$$[X \upharpoonright_n] \nsubseteq \mathcal{U}^B \text{ for all } B \in [T_{\alpha_n}] \qquad (*)$$

The base case $n = 0$ holds by the assumption on $\mathcal{U}$. Suppose that this has been shown for n and let us show it for $n+1$. If $\xi(n) = k < \omega$ then by construction, for all $B \in [T_{\alpha_{n+1}}]$ we have $[X \upharpoonright_n \hat{}\, 0] \nsubseteq \mathcal{U}^B$, and also $X(n) = 0$ which proves (*) for $n+1$ in case $\xi(n) = k$. Suppose now that $\xi(n) = \omega$ and $X(n) = 1$. In particular, by construction, for every k there is σ extending $\sigma_k(T_{\alpha_n})$ such that $[X \upharpoonright_n \hat{}\, 0] \subseteq \mathcal{U}^\sigma$.

Also suppose that there is some $C \in [T_{\alpha_{n+1}}] = [\mathtt{Nar}(T_{\alpha_n})]$ such that $[X \upharpoonright_{n+1}] = [X \upharpoonright_n \hat{}\, 1] \subseteq \mathcal{U}^C$. By compactness, there is some $\sigma \in T_{\alpha_{n+1}}$ such that $[X \upharpoonright_n \hat{}\, 1] \subseteq \mathcal{U}^\sigma$. Also there is some $k < \omega$ such that $\sigma_k(T_{\alpha_n})$ extends σ, and there is some $B \in [T_{\alpha_n}[\![k]\!]] \subseteq [T_{\alpha_n}]$ such that $[X \upharpoonright_n \hat{}\, 0] \subseteq \mathcal{U}^B$. Therefore $[X \upharpoonright_n] \subseteq \mathcal{U}^B$, contradicting the induction hypothesis at level n.

The verification: A higher Turing computes X

We now prove that A higher Turing computes the sequence X. Let us define the Π^1_1 set of pairs of strings $\Phi = \left\{(\mathrm{stem}(T_{\alpha_{n,s}}), X_s \upharpoonright_n) : n \in \mathbb{N}, s < \omega_1^{ck}\right\}$.

Certainly for all n, we have $\mathrm{stem}(T_n) \prec A$. So to show that $\Phi(A) = X$, it remains to show that Φ is consistent on prefixes of A, that is, for all s and n, if $X_s \upharpoonright_n \nprec X$, then $\mathrm{stem}(T_{\alpha_{n,s}}) \nprec A$. This is done by induction on n. Suppose this is known for n and all stages s and let us suppose that $X_s \upharpoonright_{n+1} \nprec X$ in order to show $\mathrm{stem}(T_{\alpha_{n+1,s}}) \nprec A$.

Either $X_s \upharpoonright_n \nprec X$ in which case, by induction hypothesis we have that $\mathrm{stem}(T_{\alpha_{n+1,s}}) \nprec A$ because $\mathrm{stem}(T_{\alpha_{n,s}}) \preceq \mathrm{stem}(T_{\alpha_{n+1,s}})$, or $X_s \upharpoonright_n = X \upharpoonright_n$ and $X_s(n) \neq X(n)$. In particular $\xi \upharpoonright_{n+1} \neq \xi_s \upharpoonright_{n+1}$.

Let m be the smallest such that $\xi(m) \neq \xi_s(m)$, let $\alpha = \xi \upharpoonright_m$. If $m < n$, as $X_s \upharpoonright_n = X \upharpoonright_n$, the only possibility is that $X(m) = X_s(m) = 0$ and $\xi(m)$ and $\xi_s(m)$ are both smaller than ω. Also in this case we have $T_{\alpha\hat{}\xi_s(m)} = \sigma_i(T_\alpha) \neq \sigma_j(T_\alpha) = T_{\alpha\hat{}\xi(m)}$ which implies $\mathrm{stem}(T_{\alpha\hat{}\xi_s(m)}) \nprec A$.

If $m = n$, as X is higher left-c.e., the only possibility is that $X_s(n) = 0$ and $X(n) = 1$. But then we have $T_{\alpha\hat{}\xi_s(m)} = \sigma_i(T_\alpha)$ and $A \in \mathtt{Nar}(T_\alpha)$ for some i. Therefore we have $\mathrm{stem}(T_{\alpha_{n,s}}) \nprec A$. ∎

Corollary 7.3.1:
For some higher Δ^0_2 sequences A, there is no A-universal oracle-continuous Π^1_1-Martin-Löf test.

We can also use Theorem 7.3.1 to separate higher Turing computations from consistent higher Turing computations. This is done through the following proposition:

Proposition 7.3.1:
There is an oracle-continuous Π^1_1-Martin-Löf test $\bigcap_n \mathcal{U}_n$ such that for any oracle A and any X, if A higher Turing computes X with some functional Φ which is consistent everywhere, then $X \in \bigcap_n \mathcal{U}^A_n$.

Proof: First let us notice that we can enumerate the functionals which are consistent everywhere. For any Π^1_1 set $\Phi_e \subseteq 2^{<\mathbb{N}} \times 2^{<\mathbb{N}}$, we can build the Π^1_1 set $\Phi_{f(e)}$ which just copies Φ_e as long as everything is consistent, and stop the copy when some inconsistency if found, without copying the inconsistency itself.

Then let $\{\Phi_e\}_{e\in\omega}$ be an enumeration of the higher Turing functionals which are consistent everywhere. We define:

$$\mathcal{U}_n = \{(\sigma, \tau) \ : \ \exists e \ \Phi_e(\sigma) \succeq \tau \text{ with } |\tau| = n + e + 1\}$$

We then have $\mathcal{U}^A_n = \{\tau \ : \ \exists \sigma \prec A \ (\sigma, \tau) \in \mathcal{U}_n\}$. Also for each e there is at most one string τ of length $n + e + 1$ in $\mathcal{U}^A_n$, because each Φ_e is consistent everywhere. We then have $\lambda(\mathcal{U}^A_n) \leq 2^{-n}$. So $\bigcap_n \mathcal{U}_n$ is an oracle continuous Π^1_1-Martin-Löf test and by design, for any A, X, if $\Phi_e(A) = X$, we then have $X \in \bigcap_n \mathcal{U}^A_n$. ∎

Combining Theorem 7.3.1 with the previous proposition we can then deduce a separation between higher Turing computations and consistent higher Turing computations, that we mentioned in Section 7.2.

Corollary 7.3.2:
There is some higher Δ^0_2 sequence A and a higher left-c.e. sequence X, such that A higher Turing computes X, but cannot higher Turing compute X with a functional Φ which is consistent everywhere.

7.3.4 No A-universal A-continuous Π^1_1-Martin-Löf tests

We now strengthen Theorem 7.3.1 by proving that for some oracles A, there is no A-universal A-continuous Martin-Löf test.

Theorem 7.3.2:
There exists a higher Δ^0_2 oracle A such that for all oracle continuous Π^1_1-open set $\mathcal{U}_e$ we have either $\mathcal{U}^A_e = 2^{\mathbb{N}}$ or there exists some higher Δ^0_2 non A-continuously Π^1_1-Martin-Löf X_e such that $X_e \notin \mathcal{U}^A_e$.

Ordering the requirements:

In the proof of Theorem 7.3.1, each strategy α of length $e > 0$ is used to determine the value $X(e-1)$ such that X will defeat the oracle-continuous Π^1_1-open set $\mathcal{U}$. Here we have infinitely many sequences X_e to build, so for every pair $\langle e, d\rangle$, we need to decide the value of $X_e(d)$.

To do so, the obvious thing is to use a bijection between $\langle\rangle : \mathbb{N}\times\mathbb{N} \to \mathbb{N}$, so the strategies α of length $n = \langle e, d\rangle$ are used to determine the value $X_e(d)$ such that X_e will defeat the oracle open set $\mathcal{U}_e$. Also the bijection should be made in a way such that $\langle e, d\rangle < \langle e, d+1\rangle$, so that for every e we decide each bit of X_e in increasing order. This is what we are going to do, but for some reasons that will be clarified later, we also need to have a lots of consecutive strategies' length, which decide for consecutive bits of the same X_e.

We now precisely define the bijection. We split $\mathbb{N}$ into blocks of consecutive integers in the following way: We define $b_0 = 0$ and $b_{n+1} = 5b_n + n + 1$. The first values are $b_0 = 0$, $b_1 = 1$, $b_2 = 7$, $b_3 = 38$, ... We then define the block number n to be $[b_n, b_{n+1} - 1]$. So the first blocks are $[0, 0]$, $[1, 6]$, $[7, 37]$, ...

To each block $[b_i, b_{i+1} - 1]$ we assign a number e such that each number e is assigned to infinitely many blocks. A block to which the number e is assigned is called an e-**block**. The bijection $\langle,\rangle : \mathbb{N}\times\mathbb{N} \to \mathbb{N}$ is given by $\langle e, d\rangle = n$ if n is in some e-block and if n is the $(d-1)$-th value (using the order of integers) that appears in any e-block. It is easy to verify that the next two properties are met:

1. For any e and any d such that $\langle e, d\rangle$ and $\langle e, d\rangle + 1$ are in the same block, we have $\langle e, d+1\rangle = \langle e, d\rangle + 1$.
2. For any e and d_1, d_2 with $d_1 < d_2$ we have $\langle e, d_1\rangle < \langle e, d_2\rangle$.

The strategies:

In the proof of Theorem 7.3.1, the strategies were elements of $(\omega+1)^{<\mathbb{N}}$: if densely along any path in $\texttt{Nar}(T)$ we could find some strings $\tau \geq \sigma_k(T)$ for any k such that $\mathcal{U}_e^\tau$ covers $[0]$, we could then continue the construction in $\texttt{Nar}(T)$, and we knew that for no path $A \in \texttt{Nar}(T)$ we could have $\mathcal{U}_e^A$ covering $[1]$, as otherwise $\mathcal{U}_e^B$ would have covered the whole space for some other oracle B.

Here for every e the set $\mathcal{U}_e^B$ is allowed to cover the whole space on some oracles B. Also if densely along any path in $\texttt{Nar}(T)$ we can find some strings $\tau \geq \sigma_k(T)$ for any k such that $\mathcal{U}_e^\tau$ covers $[0]$, we still move the current tree to $\texttt{Nar}(T)$, however, it is then possible that for some $\tau \in \texttt{Nar}(T)$ to have that $\mathcal{U}_e^\tau$ covers $[1]$. In this case we cannot change the value of $X_e(0)$ anymore in order to defeat $\mathcal{U}_e$. However, we still can move back the current tree to some tree $T[\![k]\!]$ such that $\tau \leq \sigma_k(T)$, so we are sure that for any oracle A in this tree, the set $\mathcal{U}_e^A$ covers the whole space. It follows that $\mathcal{U}_e$ is still defeated as it cannot be the first component of an A-continuous Π^1_1-Martin-Löf test.

So the strategies for this construction will now be elements of $(\omega+2)^{<\mathbb{N}}$, where the value $\omega+1$ corresponds to having the approximation of the oracle coming back into some tree $T[\![k]\!]$ (actually some tree $T[\![k]\!][\![l]\!]$ as explained below). This simple idea requires a lot of technical complications that will be sketched in the paragraph "Removing noise". Also we now describe strategies as precisely as we can do it at this point.

The strategies come with the function $F : (\omega+2)^{<\mathbb{N}} \times \mathcal{T} \to \mathcal{T}$ defined by $F(k,T) = \texttt{Nar}(T[\![k]\!])$ for $k < \omega$ and $F(\omega,T) = \texttt{Nar}(T)$. The value of F cannot be given now on $\omega+1$, and depends on what will happens in the construction. Also for some tree T, it is possible that $F(\omega+1,T)$ will never be defined. However, if it is defined, it is always equal to $T[\![i]\!][\![j]\!]$ for some $i,j \in \mathbb{N}$. The choice of i and j will be determined at the first stage of computation at which the value $F(\omega+1,T)$ is required.

We say that $\alpha \in (\omega+2)^{<\mathbb{N}}$ is an **(e,d)-strategy** if the last bit of α is at a position $\langle e,d\rangle$, that is, $|\alpha| = n+1$ and $\langle e,d\rangle = n$. An (e,d)-strategy will help to determine the bit number d of the sequence X_e. Sometimes we do not need to know what d is, and in this case we simply write **e-strategy** to denote an (e,d)-strategy for some d. Also by extension, we call **strategy path** an element $\xi \in (\omega+2)^{\mathbb{N}}$.

Inputs and outputs

The algorithm takes as inputs all the oracle-continuous Π^1_1-open sets $\mathcal{U}_e$. At each stage $s < \omega_1^{ck}$ the algorithm outputs:

- A strategy path $\xi_s \in (\omega+2)^{<\mathbb{N}}$.
- For each e, one sequence $X_{e,s}$.

Also for each e such that $\mathcal{U}_e^A \neq 2^{\mathbb{N}}$, and uniformly in every integer j (but not in e) we will define some oracle-continuous Π^1_1-open set $\mathcal{V}_{e,j}$. We will prove that $\{\xi_s\}_{s<\omega_1^{ck}}$ and $\{X_{e,s}\}_{s<\omega_1^{ck}}$ converge respectively to a strategy path ξ and a sequence X_e. The strategy path ξ will always be such that $\{T_\alpha\}_{\alpha<\xi}$ is a shrinking path in $\mathcal{T}$, and then it defines a unique sequence $A \in \bigcap_{\alpha<\xi} T_\alpha$. For each e such that $\mathcal{U}_e^A \neq 2^{\mathbb{N}}$ we shall prove:

- For each $n = \langle e,d\rangle$ we have $[X_e \upharpoonright_{d+1}] \notin \mathcal{U}_e^A$.

- For each $n = \langle e, d\rangle$, if n is the last value of the block number j, then the string $X_e\upharpoonright_{d+1}$ is enumerated into the A-continuous Π^1_1-open set $\mathcal{V}^A_{e,j}$.
- For any j we have $\lambda(\mathcal{V}^A_{e,j}) \le 2^{-j}$.

Understanding the tree of trees:

It is obvious that for $T_1, T_2 \in \mathcal{T}$, if $T_1 \prec T_2$, then $\text{stem}(T_1) \preceq \text{stem}(T_2)$. The converse is not true as $\text{stem}(\texttt{Nar}(T)) \prec \text{stem}(T\,[\![k]\!])$ but $\texttt{Nar}(T) \nprec T\,[\![k]\!]$. It is however true in some special case, that is, if $\text{stem}(T_1) \prec \text{stem}(T_2)$ and if in addition we have $\text{stem}(T_2) \in T_1$ (then we have $T_1 \prec T_2$). Before we prove that, we give two facts that shall use several times in the proof:

Fact 7.3.2

For any $T, T' \in \mathcal{T}$ and any $\sigma \in \texttt{Nar}(T)$ such that $\text{stem}(T) \preceq \sigma$, if T' extends $T\,[\![k]\!]$ for some k then we have $\text{stem}(T') \npreceq \sigma$.

Proof: Suppose for contradiction that $\text{stem}(T') \preceq \sigma$. As a tree is closed by initial segment we have $\text{stem}(T') \in \texttt{Nar}(T)$ and then $\sigma_k(T) \in \texttt{Nar}(T)$. But by definition of $\texttt{Nar}(T)$ we cannot have $\sigma_k(T)$ in $\texttt{Nar}(T)$. So we have a contradiction. ■

Fact 7.3.3

For any $T, T' \in \mathcal{T}$, if $T' \succeq \texttt{Nar}(T)$ and if for some k, we have $\sigma_k(T)$ comparable with $\text{stem}(T')$, then $\text{stem}(T') \prec \sigma_k(T)$.

Proof: By Fact 7.3.2 we have $\sigma_k(T) \npreceq \text{stem}(T')$. Therefore $\text{stem}(T') \prec \sigma_k(T)$. ■

Lemma 7.3.1 *For $T_1, T_2 \in \mathcal{T}$, if $\text{stem}(T_2) \in T_1$ and $\text{stem}(T_1) \prec \text{stem}(T_2)$, then $T_1 \prec T_2$.*

Proof: Suppose that T_2 does not strictly extend T_1. We have four possibilities.

- Either T_1 extends T_2, then we have $\text{stem}(T_2) \preceq \text{stem}(T_1)$ which means $\text{stem}(T_1) \nprec \text{stem}(T_2)$.
- Or there is a tree $T \in \mathcal{T}$ and two integers $k_1 \neq k_2$ such that T_1 extends $T\,[\![k_1]\!]$ and T_2 extends $T\,[\![k_2]\!]$, but then as $k_1 \neq k_2$ we have $\sigma_{k_1}(T) \perp \sigma_{k_2}(T)$ which implies $\text{stem}(T_1) \perp \text{stem}(T_2)$ and then $\text{stem}(T_1) \nprec \text{stem}(T_2)$.
- Or there is a tree $T \in \mathcal{T}$ and a k such that T_1 extends $\texttt{Nar}(T)$ and T_2 extends $T\,[\![k]\!]$. In this case, we clearly have $\text{stem}(T_2) \notin T_1$.
- Or there is a tree $T \in \mathcal{T}$ and a k such that T_2 extends $\texttt{Nar}(T)$ and T_1 extends $T\,[\![k]\!]$. If $\text{stem}(T_2) \in T_1$ then $\sigma_k(T)$ is comparable with $\text{stem}(T_2)$ which implies using Fact 7.3.3 that $\text{stem}(T_2) \prec \sigma_k(T) \preceq \text{stem}(T_1)$ and then that $\text{stem}(T_1) \nprec \text{stem}(T_2)$. ■

Removing noise:

We now come to a construction lemma. One of the difficulty of the construction lies in the value $\omega + 1$ that can be taken by bits of a strategy and which introduce complications to build our A-continuous Martin-Löf tests, capturing sequences X_e. Also one difficulty is to make sure that our final oracle A does not extend too many strings of the type $\text{stem}(T_\alpha)$ for some strategy $\alpha \prec \xi_s$, where s is a stage of the construction.

The reason is that our A-continuous Martin-Löf tests will try to capture some sequences X_e with the knowledge of A. Also the knowledge of A is given to us in a higher Δ^0_2 way, as the unique elements of a shrinking path of $\mathcal{T}$, that evolves through time. Concretely, as for the proof of Theorem 7.3.1, at stage s, for the (e,d)-strategy α_s, we map $\text{stem}(T_{\alpha_s})$ to the current value of $X\upharpoonright_{d+1}$, but it might happen that strings comparable with $\text{stem}(T_{\alpha_s})$ are already mapped to some strings incomparable with $X\upharpoonright_{d+1}$:

As sketched in the paragraph "The strategies", if $\alpha\hat{\ }(\omega+1)$ is an $(e,0)$-strategy, then $\mathcal{U}_e$ is defeated. However $\mathcal{U}_e$ is defeated, by moving back $\text{stem}(T_{\alpha\hat{\ }(\omega+1)})$ so that it extends $\text{stem}(T_\alpha[\![k]\!])$ for some k. Doing so, it might happen that other (e',d)-strategies $\beta \succ \alpha\hat{\ }(\omega+1)$ are 'badly injured', because it is then possible that for two stage s,t we have (e',d)-strategies $\beta_s \neq \beta_t$ with both $\text{stem}(T_{\beta_s}) \prec A$ and $\text{stem}(T_{\beta_t}) \prec A$, together with $X_{e',s}(d) \neq X_{e',t}(d)$.

For this reason we do not try to higher Turing compute each X_e using A, but simply to put each X_e in an A-continuous Π^1_1-Martin-Löf test. To do so we need to ensure that for every (e,d)-strategy β_s, there is not too many distinct versions of β_s such that $\text{stem}(T_{\beta_s}) \prec A$. For this purpose, we introduce the following definition:

Definition 7.3.1. *Given a tree $T \in \mathcal{T}$ and a sequence $A \in [T]$, we say that a string τ* ***cancels noise around*** A ***above*** T *if for any $T' \geq T$ and any k we have $\sigma_k(T') \perp A$ implies $\sigma_k(T') \perp \tau$.* ∎

So when a (e,d)-strategy goes from $\alpha\hat{\ }(\omega)$ to $\alpha\hat{\ }(\omega+1)$, this definition will be used to choose both i and j, such that $T_{\alpha\hat{\ }(\omega+1)} = T_\alpha[\![i]\!][\![j]\!]$. We give here the lemma that we are going to use in order to pick those integers i and j.

Lemma 7.3.2 *Let us have a shrinking path $T_0, T_1, \dots$ in $\mathcal{T}$, with $A \in \bigcap_n [T_n]$. Suppose that for some m we have $T_{m+1} = \mathtt{Nar}(T_m)$. For any string σ with $\text{stem}(T_m) \preceq \sigma \prec A$ one can find (uniformly in $\{T_n\}_{n\in\mathbb{N}}$, in m and σ), an integer l such that $\sigma_l(T_m)$ extends σ and cancels noise around A above T_{m+1}.*

Proof: We first construct $\sigma_l(T_m)$. Find some integer $n > m$ such that $\sigma \preceq \text{stem}(T_n) \preceq A$. Let us effectively find a sequence B in $\bigcap_i \mathtt{Nar}^i(T_n)$. We know that the set of strings $\{\sigma_k(T_m)\}_{k<\omega}$ is dense along any path in $\mathtt{Nar}(T_m)$. In particular it is dense along B. But then there must exist some $\tau \in T_n$ such that $\text{stem}(T_m) \preceq \sigma \preceq \text{stem}(T_n) \preceq \tau \prec B$ and such that $\tau\hat{\ }1$ is equal to $\sigma_l(T_m)$ for some l. The string $\tau\hat{\ }1$ is the candidate.

Let us show that $\tau\hat{\ }1 = \sigma_l(T_m)$ cancels noise around A above $T_{m+1} = \mathtt{Nar}(T_m)$. Take any $T \in \mathcal{T}$ extending $\mathtt{Nar}(T_m)$ and any k such that $\sigma_k(T)$ is not a prefix of A, in order to show that $\sigma_l(T_m) \perp \sigma_k(T)$. As $\sigma_k(T) \in \mathtt{Nar}(T_m)$ and extends $\text{stem}(T_m)$, we can apply Fact 7.3.2 and deduce that $\tau\hat{\ }1 = \sigma_l(T_m) \npreceq \sigma_k(T)$. It follows that $\sigma_k(T) \perp \tau\hat{\ }1$, or that $\sigma_k(T) \preceq \tau$.

Suppose for contradiction that $\sigma_k(T) \preceq \tau$. Since $\sigma_k(T)$ is not a prefix of A it is not a prefix of $\mathrm{stem}(T_n)$. Also recall that $\mathrm{stem}(T_n) \preceq \tau$. We then have $\mathrm{stem}(T_n) \prec \sigma_k(T) \preceq \tau$. Also as τ is a prefix of $B \in \bigcap_i \mathtt{Nar}^i(T_n)$ we have $\tau \in T_n$ and in particular $\sigma_k(T) \in T_n$. We then have $\mathrm{stem}(T_n) \prec \mathrm{stem}(T[\![k]\!])$ with $\mathrm{stem}(T[\![k]\!]) \in T_n$, so we can apply Lemma 7.3.1 to deduce that $T[\![k]\!]$ strictly extends T_n.

Therefore $T[\![k]\!]$ extends a tree $(\mathtt{Nar}^j(T_n))[\![i]\!]$ for some $i, j \in \mathbb{N}$. Also as $B \in \bigcap_i \mathtt{Nar}^i(T_n)$ and as $\tau \prec B$ we have that $\tau \in \mathtt{Nar}(\mathtt{Nar}^j(T_n))$. In particular by Fact 7.3.2 we have that $\sigma_i(\mathtt{Nar}^j(T_n)) \npreceq \tau$ and as $\sigma_k(T) \succeq \sigma_i(\mathtt{Nar}^j(T_n))$ we also have that $\sigma_k(T) \npreceq \tau$ which is a contradiction. ■

Construction claims:

Before starting the construction we make a few claims which will be seen obvious from the construction and which are worth mentioning before, because they will needed for the construction:

Claim 1 : At any stage s, if $\alpha\hat{}o \prec \xi_s$ is a (e,d)-strategy, then if $o < \omega$ we have $X_{e,s}(d) = 0$, and otherwise we have $X_{e,s}(d) = 1$.

Claim 2 : If we have $\xi_s \upharpoonright_n = \alpha\hat{}\omega$, then for each $k < \omega$ there is a stage $t < s$ such that $\xi_t \upharpoonright_n = \alpha\hat{}k$.

Claim 3 : The approximation ξ_s is left-c.e. and partially continuous. In particular, if $\{\xi_t \upharpoonright_n\}_{t<s}$ converges, we have $\xi_s \upharpoonright_n = \lim_{t<s} \xi_t \upharpoonright_n$.

Claim 4 : At each stage s, the sequence $\{T_\alpha\}_{\alpha \prec \xi_s}$ is a shrinking path of $\mathcal{T}$.

The construction:

At stage $s = 0$ let us start $\xi_0 = 0$. We also set $X_e = 0^\infty$ for every e. Suppose that for all stages $t < s$ the strategy path ξ_t and the sequences $X_{e,t}$ have been defined. Let us define ξ_s and $X_{e,s}$ for each e.

Let us first suppose that s is successor. Using Construction claim 4 the sequence $A_{s-1} = \bigcap_{\alpha \prec \xi_{s-1}} T_\alpha$ is defined. We search for the smallest prefix $\alpha\hat{}o$ of ξ_{s-1}, such that if $\alpha\hat{}o$ is an (e,d)-strategy, either $o < \omega$ and $[X_{e,s}\upharpoonright_d \hat{}0] \subseteq \mathcal{U}_{e,s}^{A_{s-1}}$, or $o = \omega$ and $[X_{e,s}\upharpoonright_d \hat{}1] \subseteq \mathcal{U}_{e,s}^{A_{s-1}}$. Also, in case at stage $s-1$, some value $\xi_{s-1}(n)$ is equal to $\omega + 1$ whereas $\xi_{s-2}(n)$ is equal to ω (it can be the case for only one n at a time), the search for α is done in priority over e-strategies. If we find no such prefix, then we set $\xi_s = \xi_{s-1}$. Otherwise let $n = |\alpha|$ and do the following:

In case $o < \omega$ we set $\xi_s = \xi_{s-1}\upharpoonright_n \hat{}(o+1)\hat{}0^\infty$. In case $o = \omega$, let σ be the smallest prefix of A_{s-1} such that $[X_{e,s}\upharpoonright_d \hat{}1] \subseteq \mathcal{U}_{e,s}^{\sigma}$. Using Lemma 7.3.2 we find k such that $\sigma \preceq \sigma_k(T_\alpha)$ cancels noise around A_{s-1} above $\mathtt{Nar}(T_\alpha) = T_{\alpha\hat{}o}$. Using Construction claim 2 and claim 3, let t be the last stage smaller than s such that $\xi_t \upharpoonright_{n+1} = \alpha\hat{}k$. Note that we have $A_t \in [\mathtt{Nar}(T_\alpha[\![k]\!])]$ and by Construction claim 1, we have $X_{e,s}\upharpoonright_d = X_{e,t}\upharpoonright_d$ and $X_{e,t}(d) = 0$. Also because t is the last such stage, we necessarily have $[X_{e,s}\upharpoonright_d \hat{}0] \subseteq \mathcal{U}_{e,s}^{A_t}$. Then take the smallest string τ with $\sigma_k(T_\alpha) \preceq \tau \prec A_t$ and such that $[X_{e,s}\upharpoonright_d \hat{}0] \subseteq \mathcal{U}_{e,s}^{\tau}$. Then using

Lemma 7.3.2 a second time, find l such that $\tau \preceq \sigma_l(T_\alpha[\![k]\!])$ cancels noise around A_t above $\mathtt{Nar}(T_\alpha[\![k]\!])$. We then define $F(\omega+1, T_\alpha) = (T_\alpha[\![k]\!])[\![l]\!]$ and we set $\xi_s = \alpha\hat{}(\omega+1)\hat{}0^\infty$.

Let us now suppose that s is limit. We search for the smallest n such that $\{\xi_t\upharpoonright_n\}_{t<s}$ does not converge. If no such n exists then ξ_s is set to the convergence value. Otherwise let α be the convergence value of $\{\xi_t\upharpoonright_{n-1}\}_{t<s}$. We set ξ_s to be $\alpha\hat{}\omega\hat{}0^\infty$.

Now for each (e,d)-strategy $\alpha\hat{}o \prec \xi_s$ we set $X_{e,s}(d) = 0$ if $o < \omega$ and $X_{e,s}(d) = 1$ otherwise. This ends the construction.

Maybe only Claim 4 is not obvious. It follows from the fact that there are infinitely many e such that $\mathcal{U}_e$ enumerates nothing. Then at any stage s there are infinitely many n such that $\xi(n) = 0$, implying that $\{T_\alpha\}_{\alpha\prec\xi_s}$ is a shrinking path of $\mathcal{T}$.

The convergence:

By the Σ^1_1-boundedness principle, and because $\{\xi_s\}_{s<\omega_1^{ck}}$ is left-c.e. with the value of each of its bit bounded by $\omega+2$, it converges to some sequence ξ. It follows that each $\{X_{e,s}\}_{s<\omega_1^{ck}}$ converges to some sequence X_e and that $\{A_s\}_{s<\omega_1^{ck}}$, where A_s is defined to be the unique element of $\bigcap_{\alpha\prec\xi_s} T_\alpha$, converges to some sequence A. We shall now prove two lemmas about the possible use of the value $\omega+1$ inside bits of strategies. The first lemma basically says that the value $\omega+1$ is not 'stable', that is, if some (e,d) strategy α_s reaches the value $\omega+1$ at stage s for $d>0$, then at stage $s+1$ we will have that α_{s+1} is bigger than α_s in the lexicographic order. So only strategies of the form $(e,0)$ can keep the value $\omega+1$ (and when if this happens, after that strategies (e,d) may also keep this value).

Lemma 7.3.3 *Suppose $\xi_{s-1}(n) = \omega$ and $\xi_s(n) = \omega+1$, and suppose $\xi_s\upharpoonright_{n+1}$ is an (e,d)-strategy for $d>0$. Suppose also that for no e-strategy $\alpha \prec \xi_s\upharpoonright_{n+1}$ we have $\alpha(|\alpha|-1) = \omega+1$. Then there exists $m \le n$ such that $\xi_s\upharpoonright_{m+1}$ is an (e,d')-strategy with $d' < d$, and such that at stage $s+1$ we have $\xi_{s+1}(m) > \xi_s(m)$.*

Proof: By construction, if $\xi_s(n) = \omega+1$, we then have $[X_{e,s}\upharpoonright_d] \subseteq \mathcal{U}^\sigma_{e,s}$ with $\sigma = \mathrm{stem}(T_{\xi_s\upharpoonright_{n+1}})$. Also in this case, at stage $s+1$ the search in the construction is done in priority on e-strategies. So for some $m<n$ such that $\xi_s\upharpoonright_{m+1}$ is an (e,d')-strategy with $d'<d$ (presumably for $d' = d-1$), we will necessarily find out that $[X_{e,s}\upharpoonright_{d'+1}] \subseteq \mathcal{U}^\sigma_{e,s+1}$, forcing then $\xi_{s+1}(m) > \xi_s(m)$. ∎

Lemma 7.3.4 *Let $\alpha\hat{}(\omega+1) \prec \xi$ be an (e,d)-strategy. Then $\mathcal{U}_e^{\mathrm{stem}(T_{\alpha\hat{}(\omega+1)})} = 2^{\mathbb{N}}$.*

Proof: Without loss of generality, we can suppose that $\alpha\hat{}(\omega+1) \prec \xi$ is the smallest (e,d)-strategy for some d, in the prefix ordering. We then claim that we necessarily have $d=0$. Suppose otherwise for contradiction. Let $n = |\alpha|$ and let s be the smallest stage such that $\xi_s\upharpoonright_{n+1} = \alpha\hat{}(\omega+1)$. Using Lemma 7.3.3, it follows that $\xi_{s+1}\upharpoonright_{n+1}$ is bigger than $\alpha\hat{}(\omega+1)$ in the lexicographic order, and as $\{\xi_s\}_{s<\omega_1^{ck}}$ is left-c.e., it contradicts that $\xi\upharpoonright_{n+1} = \alpha\hat{}(\omega+1)$.

Therefore $\alpha\hat{}(\omega+1) \prec \xi$ is an $(e,0)$-strategy, and then we have $2^{\mathbb{N}} = [X_{e,s}\upharpoonright_d] \subseteq \mathcal{U}^\sigma_e$ where $\sigma = \mathrm{stem}(T_{\alpha\hat{}(\omega+1)})$. ∎

The sequences X_e

It is clear from the construction that for any e, the sequence X_e does not belong to the open set $\mathcal{U}_e^A$, as long as $\mathcal{U}_e^A$ does not cover the whole space. Indeed, using Lemma 7.3.4, if $\mathcal{U}_e^A$ does not cover the whole space, then for any (e,d)-strategy $\alpha \prec \xi$, the last bit of α is different from $\omega+1$. It follows that either it is equal to $k < \omega$ in which case, by construction, $\mathcal{U}_e^A$ does not cover $[X_e\upharpoonright_d \hat{}\, 0]$ and $X_e(d) = 0$, or that it is equal to ω in which case, by construction, $\mathcal{U}_e^A$ does not cover $[X_e\upharpoonright_d \hat{}\, 1]$ and $X_e(d) = 1$. Since for any $\sigma \prec X_e$ the set $\mathcal{U}_e^A$ does not cover $[\sigma]$, we have $X_e \notin \mathcal{U}_e^A$.

The noise canceling lemma

We now prove a lemma, that we will use to prove that if $\mathcal{U}_e^A$ does not cover the whole space, then X_e is not A-continuously Π_1^1-Martin-Löf random.

Lemma 7.3.5 (Noise canceling lemma) *Suppose that for some n, at some stage t we have $\xi_t(n) = o < \omega+1$. Suppose also that for every stage $s \geq t$ we have $\xi_s\upharpoonright_{n+1} \neq \xi_t\upharpoonright_n \hat{}\,(\omega+1)$. Then also for every stage $s \geq t$, and any $k < o$, we have $A_s \nsucc \mathrm{stem}(T_{\alpha\hat{}k})$ for $\alpha = \xi_t\upharpoonright_n$.*

Proof: First let us emphasize that the only "bad" case is that when for some $m < n$, the bit $\xi_s(m)$ takes the value $\omega+1$ at some stage s. Let t be such that $\xi_t(n) = o \neq \omega+1$, and suppose that for every $s \geq t$ we have $\xi_s(n) \neq \omega+1$. If we already have $\xi_t\upharpoonright_n = \xi\upharpoonright_n$, then only the value $\xi(n)$ can move, it can move only forward and is never equal to $\omega+1$. Therefore the lemma is obviously true.

Otherwise we can suppose without loss of generality (using Construction claim 3), that t is a stage such that $\xi_t\upharpoonright_n \neq \xi_{t+1}\upharpoonright_n$. We prove by induction on stages $s \geq t$, that:

$$A_s \nsucc \mathrm{stem}(T_{\alpha\hat{}k}) \text{ for any } k < o \text{ where } \alpha = \xi_t\upharpoonright_n \qquad (*)$$

The induction starts with t for which (*) is true. Suppose (*) is true for all stages smaller than s and let us prove (*) is true at stage s. We denote $\xi_t\upharpoonright_n$ by α. Also we let $m < n$ be the biggest length such that $\xi_t\upharpoonright_m = \xi_s\upharpoonright_m$, and let $\beta = \xi_t\upharpoonright_m = \xi_s\upharpoonright_m$.

If $\xi_s(m) = i < \omega$ and $\xi_t(m) = j < \omega$ with $i \neq j$. It is immediate that $\mathrm{stem}(T_\alpha) \npreceq A_s$. Also if $\xi_s(m) = \omega$ and $\xi_t(m) = o$ with $o < \omega$ or $o = \omega+1$, there exists some i such that T_α extends $T_\beta[\![i]\!]$. Then by Fact 7.3.2 we have for any $\sigma \in \mathtt{Nar}(T_\beta)$ that $\mathrm{stem}(T_\alpha) \npreceq \sigma$. In particular as $A_s \in [\mathtt{Nar}(T_\beta)]$ we have $\mathrm{stem}(T_\alpha) \npreceq A_s$ and then (*) is true at stage s. We decompose the rest of the induction into four cases:

- case 1 : We have $\xi_s(m) = \omega+1$ and $\xi_t(m) = \omega$, so $\beta\hat{}\,\omega \preceq \alpha$. In addition we suppose that s is not the first stage such that $\xi_s\upharpoonright_{m+1} = \beta\hat{}\,(\omega+1)$, so there is some stage s' with $t < s' < s$ such that $\xi_{s'}\upharpoonright_{m+1} = \xi_s\upharpoonright_{m+1} = \beta\hat{}\,(\omega+1)$. In particular $A_s, A_{s'} \in [T_{\beta\hat{}(\omega+1)}]$ and $T_{\beta\hat{}(\omega+1)} = (T_\beta[\![i]\!])[\![j]\!]$ for some i, j.

  We have that T_α extends $\mathtt{Nar}(T_\beta)$. Also if $\mathrm{stem}(T_\alpha) \prec A_s$, the string $\mathrm{stem}(T_\alpha)$ is comparable with $\mathrm{stem}(T_\beta[\![i]\!])$. Then by Fact 7.3.3 we have that $\mathrm{stem}(T_\alpha) \prec A_s$ iff $\mathrm{stem}(T_\alpha) \prec \sigma_i(T_\beta)$ iff $\mathrm{stem}(T_\alpha) \prec A_{s'}$. Therefore, by induction hypothesis, as (*) is true at stage s' it is true at stage s.

- case 2 : We have $\xi_s(m) = \omega+1$ and $\xi_t(m) = k < \omega$, so $\beta\hat{}\,k \preceq \alpha$. In addition we suppose that s is not the first stage such that $\xi_s\upharpoonright_{m+1} = \beta\hat{}\,(\omega+1)$, so there is some stage s' with $t < s' < s$ such that $\xi_{s'}\upharpoonright_{m+1} = \xi_s\upharpoonright_{m+1} = \beta\hat{}\,(\omega+1)$. In particular $A_s, A_{s'} \in [T_{\beta\hat{}(\omega+1)}]$ and $T_{\beta\hat{}(\omega+1)} = (T_\beta[\![i]\!])[\![j]\!]$ for some i, j.

We have that T_α extends $\mathtt{Nar}(T_\beta [\![k]\!])$. Then either $k \neq i$ in which case A_s cannot not extend $\mathrm{stem}(T_\alpha)$ and (*) is true at stage s, or $k = i$, in which case, if $\mathrm{stem}(T_\alpha) \prec A_s$, the string $\mathrm{stem}(T_\alpha)$ is comparable with $\mathrm{stem}((T_\beta [\![i]\!]) [\![j]\!])$. Then by Fact 7.3.3 we have that $\mathrm{stem}(T_\alpha) \prec A_s$ iff $\mathrm{stem}(T_\alpha) \prec \sigma_j(T_\beta [\![i]\!])$ iff $\mathrm{stem}(T_\alpha) \prec A_{s'}$. Therefore, by induction hypothesis, as (*) is true at stage s' it is true at stage s.

- case 3 : Like in case 1, we have $\xi_s(m) = \omega + 1$ and $\xi_t(m) = \omega$, but s is the first such stage. In particular $\xi_{s-1}(m) = \omega$. So at stage s, by construction, the oracle A_s will extend a string $\sigma_i(T_\beta)$ which cancels noise around A_{s-1} above $\mathtt{Nar}(T_\beta)$. By induction hypothesis (*) is true at stage $s-1$. Also since $\sigma_i(T_\beta) \prec A_s$ cancels noise around A_{s-1} above $\mathtt{Nar}(T_\beta)$, we have for any k that $A_{s-1} \perp \sigma_k(T_\alpha)$ implies $A_s \perp \sigma_k(T_\alpha)$ and then that (*) is true at stage s.

- case 4 : Like in case 2, we have $\xi_s(m) = \omega+1$ and $\xi_t(m) = k < \omega$, but s is the first such stage. In particular $\xi_{s-1}(m) = \omega$. We have that T_α extends $\mathtt{Nar}(T_\beta [\![k]\!])$ and that $T_{\beta\hat{\ }(\omega+1)} = (T_\beta [\![i]\!]) [\![j]\!]$ for some i, j. Also if $i \neq k$ then (*) is clearly true at stage s. Suppose then $i = k$. By construction there is a stage s' with $t \leq s' < s$ and $A_{s'}$ is in $[\mathtt{Nar}(T_\beta [\![i]\!])]$ so that $\sigma_j(T_\beta [\![i]\!])$ cancels noise around $A_{s'}$ above $\mathtt{Nar}(T_\beta [\![i]\!])$. Also by induction hypothesis we have that (*) is true at stage s'. And as for any l we have that $A_{s'} \perp \sigma_l(T_\alpha)$ implies $A_s \perp \sigma_l(T_\alpha)$, then (*) is true at stage s. ∎

The Martin-Löf test

We now fix some e. What follows can be done for any e, but not uniformly. We denote by $\alpha_0 \prec \xi$ the final $(e,0)$-strategy and s_0 the smallest stage such that $\xi_{s_0} \upharpoonright_{|\alpha_0|} = \alpha_0$. Also for any $n \geq |\alpha_0|$, we denote by S_n the set of all the strategies α of length n such that $\xi_s \upharpoonright_{|\alpha|} = \alpha$ for some stage $s \geq s_0$. We then denote by S the union of all S_n.

Lemma 7.3.6 *Suppose $\mathcal{U}_e^A \neq 2^\omega$. Then for any (e,d)-strategy $\alpha\hat{\ }o \geq \alpha_0$, with $\alpha\hat{\ }o \in S$, if* $\mathrm{stem}(T_\alpha) \prec A$*, then $o \neq \omega + 1$.*

Proof: Consider an e-strategy $\alpha\hat{\ }(\omega+1) \geq \alpha_0$ with $\alpha\hat{\ }(\omega+1) \in S$. Let $\beta \prec \alpha\hat{\ }(\omega+1)$ be the smallest such that $\beta\hat{\ }(\omega+1)$ is an (e,d)-strategy with $\beta\hat{\ }(\omega+1) \in S$. Note that because $\mathcal{U}_e^A \neq 2^\omega$ we have that $\alpha_0(|\alpha_0|-1) \neq \omega+1$, and then $\beta\hat{\ }(\omega+1) \succ \alpha_0$. In particular $\beta\hat{\ }(\omega+1)$ is an (e,d)-strategy for $d > 0$.

Let s be the smallest stage such that $\xi_s \upharpoonright_{|\beta|+1} = \beta\hat{\ }(\omega+1)$. We necessarily have $\xi_{s-1} \upharpoonright_{|\beta|} = \xi_s \upharpoonright_{|\beta|} = \beta$, $\xi_{s-1}(|\beta|) = \omega$ and $\xi_s(|\beta|) = \omega+1$. By Lemma 7.3.3, there is an integer m such that $\xi_s \upharpoonright_{m+1} \prec \beta\hat{\ }(\omega+1)$ is an (e,d')-strategy with $d' < d$ and $\xi_{s+1}(m) > \xi_s(m)$.

Also we cannot have $\xi_s(m) = \omega$ because otherwise we would have $\xi_s(m) = \omega+1$ which would contradicts the minimality of β. Then $\xi_s(m) = k < \omega$ and $\xi_{s+1}(m) = k+1$. Actually, by minimality of β the strategy $\xi_s \upharpoonright_m \hat{\ }(\omega+1)$ is not in S, therefore we can apply Lemma 7.3.5: For every stage $t \geq s+1$, we have $A_t \not\succ \mathrm{stem}(T_{\xi_s\upharpoonright_m\hat{\ }k})$. But as $\mathrm{stem}(T_{\xi_s\upharpoonright_m\hat{\ }k}) \preceq \mathrm{stem}(T_\alpha)$ we then have for any $t \geq s+1$ that $A_t \not\succ \mathrm{stem}(T_\alpha)$, and then that $A \not\succ \mathrm{stem}(T_\alpha)$ which is a contradiction. ∎

We can then deduce:

Lemma 7.3.7 *Suppose $\mathcal{U}_e^A \neq 2^\omega$. For any $\alpha \geq \alpha_0$, there is at most one value $o \leq \omega+1$ such that if $\alpha\hat{\ }o$ is an e-strategy with $\alpha\hat{\ }o \in S$, then* $\mathrm{stem}(T_{\alpha\hat{\ }o}) \prec A$.

PROOF: Fix α and let o be the greatest such that $\alpha\hat{}o$ is an e-strategy with $\alpha\hat{}o \in S$, and such that $\text{stem}(T_{\alpha\hat{}o}) \prec A$. By Lemma 7.3.6 we cannot have $o = \omega + 1$. It follows using Lemma 7.3.5 that $\text{stem}(T_{\alpha\hat{}k}) \nprec A$ for any $k < o$. ∎

We also trivially have:

Lemma 7.3.8 *For any n, there are at most 3^n many strategies $\alpha \in S_n$ such that $\text{stem}(T_\alpha) \prec A$.*

PROOF: It is clear. Given some α such that $\text{stem}(T_\alpha) \prec A$, we have at most $\text{stem}(T_{\alpha\hat{}\omega}) \prec A$, $\text{stem}(T_{\alpha\hat{}(\omega+1)}) \prec A$ and $\text{stem}(T_{\alpha\hat{}k}) \prec A$ for some k. ∎

Recall now the blocks from the beginning of the proof. A block will be used to build the component of an A-continuous Π^1_1-Martin-Löf test. So consider $[b_i, b_{i+1} - 1]$ to be the j-th block to which e has been assigned, so $b_{i+1} - 1 = \langle e, d\rangle$ for some d. Also let $\{\alpha_s\}_{s<\omega_1^{ck}}$ be the sequence $\{\xi_s \upharpoonright_{b_{i+1}-1}\}_{s_0 \le s < \omega_1^{ck}}$, recall that s_0 is the first stage such that $\xi_{s_0} \ge \alpha_0$. We define:

$$V_j = \{(\text{stem}(T_{\alpha_s}), X_{e,s}\upharpoonright_{d+1}) \ : \ s < \omega_1^{ck}\}$$

Using Lemma 7.3.8, there are at most 3^{b_i} many strategies $\alpha \in S_{b_i}$ such that $\text{stem}(T_\alpha) \prec A$. Then using Lemma 7.3.7, there are also at most 3^{b_i} many strategies $\alpha \in S_{b_{i+1}}$ such that $\text{stem}(T_\alpha) \prec A$. Therefore, inside V_j, some prefix of A is assigned to at most 3^{b_i} many strings of length at least $(b_{i+1} - 1) - b_i$ (one of them being $X_e\upharpoonright_{d+1}$). Also recall that $b_{i+1} - 1 = 5b_i + i$. A simple computation shows that $\lambda(V_j^A) \le 3^{b_i}2^{-4b_i - i} \le 2^{-i} \le 2^{-j}$.

It follows that $\bigcap_j V_j^A$ is a A-continuous Π^1_1-Martin-Löf test, which captures X_e. As this is true for any e such that $\mathcal{U}_e^A \ne 2^{\mathbb{N}}$, the theorem is then proved.

7.3.5 Higher A-continuously left-c.e. and Π^1_1-Martin-Löf randoms

Definition 7.3.2. *A sequence X is **higher A-continuously left-c.e.** if there is an oracle-continuous Π^1_1-open set $\mathcal{U}$ such that $\mathcal{U}^A$ does not cover the whole space and X is the leftmost path of $2^{\mathbb{N}} - \mathcal{U}^A$.* ∎

We prove here that despite the non existence of a A-universal A-continuous Π^1_1-Martin-Löf test for some oracle A, there is however always a A-continuously left-c.e. and Π^1_1-Martin-Löf random:

Theorem 7.3.3:
There is an oracle-continuous Π^1_1-open set $\mathcal{W}$ such that for any oracle A, the set $\mathcal{W}^A$ does not cover the whole space, and the leftmost path of the complement of $\mathcal{W}^A$ is A-continuously Π^1_1-Martin-Löf random.

PROOF: Let $\{\mathcal{U}_e\}_{e\in\mathbb{N}}$ be an enumeration of the oracle-continuous Π^1_1-open sets. We see $\mathcal{U}_e$ as a Π^1_1 subset of $2^{<\mathbb{N}} \times 2^{<\mathbb{N}}$. Uniformly in every strings σ, we define a Π^1_1 set of string W_σ the following way: At stage s, let us consider every possible finite lists $C_1, C_2, \ldots, C_n$ of every possible finite sets of strings such that:

1. Every string in C_k is also in $\mathcal{U}^{\sigma}_{k,s}$
2. $\lambda([C_k]^{\prec}) \leq 2^{-k-1}$
3. $\bigcup_{k\leq n}[C_k]^{\prec}$ is a clopen set equal to $[0,q]$ for some $q \in \mathbb{Q}$, when sequences are transposed into real numbers. More formally, there is a string τ such that $\bigcup_{k\leq n}[C_k]^{\prec}$ is exactly the set of sequences (not strictly) at the left of $\tau\hat{\ }1^{\infty}$, in the lexicographic order.

The set W_σ is the union of all such finite sets of strings, over all stages s. The conditions (2) and (3) ensure that for any σ we have $\lambda(W_\sigma) \leq 1/2$. Also we clearly have $\sigma_1 \leq \sigma_2$ implies $W_{\sigma_1} \subseteq W_{\sigma_2}$. It follows that the set $\mathcal{W} = \{(\sigma,\tau) \,:\, \tau \in W_\sigma\}$ is an oracle-continuous Π^1_1-open set such that for any A we have $\lambda(\mathcal{W}^A) \leq 1/2$. Note also that for any A, the set $\mathcal{W}^A$ is equal to $[0,X)$ for some real X which is the leftmost path of the complement of $\mathcal{W}^A$.

We should now prove that for every A, the leftmost path X of the complement of $\mathcal{W}^A$ is A-continuously Π^1_1-Martin-Löf random. Fix some A and suppose otherwise.

So we suppose that there is an A-continuous Π^1_1-Martin-Löf test $\bigcap_n \mathcal{V}_n$ which captures X. Also there is a computable function $f : \mathbb{N} \to \mathbb{N}$ which gives a code of $\mathcal{V}_n$ uniformly in n. Recall that K denotes prefix-free Kolmogorov complexity. We have $K(f(n)) \leq 2\log(n) + c$ for some c and any n, and therefore, for any c there is some n large enough such that $K(f(n)) < n + c$. It follows that for some n there is an index $a < n$ with $\mathcal{U}_a = \mathcal{U}_{f(n)} = \mathcal{V}_n$. We then have $\lambda(\mathcal{U}^A_a) \leq 2^{-n} \leq 2^{-a-1}$. Let $\tau \prec X$ be such that $\tau \in \mathcal{U}^A_a$.

We shall now find a finite sequence of finite sets of strings $C_1, C_2, \ldots, C_n$ with $n > a$ which covers an initial segment of the reals 'almost up to X'. The goal is then to replace C_a by $C_a \cup \{\tau\}$ to show that X is finally covered by $\mathcal{W}^A$ and get a contradiction. We have to distinguish the case where X has finitely many 1's from the case where X has infinitely many 1's.

Suppose first that X has finitely many 1's. Then there is a string $\rho \prec X$ such that $\mathcal{W}^A$ is the clopen set of sequences (not strictly) at the left of $\rho\hat{\ }0\hat{\ }1^{\infty}$, while $X = \rho\hat{\ }1\hat{\ }0^{\infty}$. But by compactness, there is some $\sigma \prec A$ large enough, so that $\tau \in \mathcal{U}^{\sigma}_a$, and W_σ 'contains' a finite sequence of finite set of strings $C_1, C_2, \ldots, C_n$ for $n > a$, verifying (1) and (2) above and such that $\bigcup_{k\leq n}[C_k]^{\prec}$ already equals $\mathcal{W}^A$.

Otherwise, if X has infinitely many 1's, there is a string ρ such that $\tau \leq \rho\hat{\ }1 \prec X$. In particular we have $[\rho\hat{\ }0] \subseteq [W_\sigma]^{\prec}$ for some $\sigma \prec A$ large enough, so that also $[\tau] \subseteq \mathcal{U}^{\sigma}_a$. Also by compactness, there is a finite sequence of finite sets of strings $C_1, C_2, \ldots, C_n$ with $n > a$ that verifies (1) and (2) above and such that $[\rho\hat{\ }0]$ is covered by $\bigcup_{k\leq n}[C_k]^{\prec}$.

In both cases we have identified a string $\tau \prec X$ and a finite sequence of finite set of strings $C_1, C_2, \ldots, C_n$ with $n > a$, such that $\bigcup_{k\leq n}[C_k]^{\prec} \cup [\tau]$ would verify (3) and cover X. Also we know that $\lambda(\mathcal{U}_a) \leq 2^{-a-1}$. Therefore whatever C_a is equal to, we can always replace it by $C_a \cup \{\tau\}$ without violating (1) or (2). But then X is covered by W_σ and then by $\mathcal{W}^A$ which contradicts the assumption that it is the leftmost path of the complement of $\mathcal{W}^A$. ∎

7.3.6 Further study on continuous relativization

We present here a few questions that haven't been investigated yet. For this section, we introduce short notations for the four notions of 'Π^1_1-Martin-Löf randomness' tests we have presented in this thesis:

1. We write A-pfm to denote "A-continuous Π^1_1-prefix-free machine"
2. We write A-dsm to denote "A-continuous Π^1_1-discrete semi-measure"
3. We write A-csm to denote "A-continuous Π^1_1-continuous semi-measure"
4. We write A-mlt to denote "A-continuous Π^1_1-Martin-Löf test"

We also denote by A-pfm-nullset, A-dsm-nullset, A-csm-nullset and A-mlt-nullset the corresponding nullsets notions. For example, given an A-pfm M, the corresponding A-pfm-nullset is the set $\{X \ : \ \forall c \ \exists n \ \mathrm{hK}^A_M(X\upharpoonright_n) < n-c\}$. We also call A-pfm-randomness, A-dsm-randomness, A-csm-randomness and A-mlt-randomness the corresponding randomness notions.

Recall that an A-pfm-nullset is always contained into an A-dsm-nullset, which is itself always contained into an A-csm-nullset, which is itself always contained into an A-mlt-nullset. However we could not prove that A-mlt-nullsets are contained into A-pfm-nullsets, because of two difficulties. The first one is the problem of finding, for a given A-continuous Π^1_1-open set, an A-continuous Π^1_1 'almost prefix-free' set of strings that describes it. The second one is the problem of obtaining a continuous relativization of the higher KC-theorem. So we address the following question:

Question 7.3.1 *Is any A-mlt-nullset included into an A-pfm-nullset for any oracle A? If not is any A-mlt-nullset included into an A-dsm-nullset for any oracle A? etc... the question extends straightforwardly to the remaining possible nullsets inclusions.*

The previous question addresses the problem of equivalence between randomness notions in a strong sense, by covering a nullset by another nullset. It might be the case for example that for some oracle A, an A-mlt-nullset cannot be cover by a A-csm-nullset, but could be cover by the union of all the A-csm-nullsets, which would show that A-csm-randomness implies A-mlt-randomness. So we address the following question:

Question 7.3.2 *Does A-pfm-randomness imply A-mlt-randomness for any oracle A? If not does A-dsm-randomness imply A-mlt-randomness or A-csm-randomness imply A-mlt-randomness for any oracle A? etc... the question extends straightforwardly to the remaining implications.*

The two previous questions might have different answers in particular because there are some oracles A with respect to which we do not have a A-universal A-mlt notion. It is almost certainly the case for any of the other three notions. Also we make the following conjecture:

Conjecture 7.3.1 *For some oracle A there is no A-universal A-pfm, for some oracle A there is no A-universal A-dsm for some oracle A there is no A-universal A-csm.*

We also think the following question is of interest:

Question 7.3.3 *Does the existence of an A-universal A-pfm, A-dsm, A-csm or A-mlt imply the existence of an A-universal object of any of the other type, for any oracle A?*

7.4 On well-behaved oracles

We study in this section the 'well-behaved' oracles, and in particular the self-unclosed approximable oracles, for which everything goes nicely.

7.4.1 Self-unclosed approximable oracles

If A is not Δ^1_1 and has a self-unclosed approximation $\{A_s\}_{s<\omega_1^{ck}}$, we can use the fact that for any stage t, there is a stage $s \geq t$ a string σ with both $\sigma \prec A_s$ and $\sigma \prec A$ such that no extension of σ has been a prefix of any A_r for $r < s$. Furthermore there are longer and longer such prefixes of A. We can first prove that on self-unclosed approximable oracles, the notion of higher Turing computation coincides with the notion of fin-h computation.

Theorem 7.4.1:
If Y has a self-unclosed approximation and if $X \leq_{\mathrm{hT}} Y$, then we have $X \leq_{fin-h} Y$.

PROOF: Suppose $\Phi(Y) = X$ with Φ a higher Turing functional, and let $\{Y_s\}_{s<\omega_1^{ck}}$ be a self-unclosed approximation of Y that we can suppose not Δ^1_1. We shall build a fin-h reduction Ψ for which the computation is unchanged on Y.

The construction:

We define Ψ_0 to be the emptyset. At successor stage $s > 0$, and at substage n, if $Y_s \upharpoonright_n$ is currently unmapped in Ψ_{s-1}, consider the longest string τ with $|\tau| \leq n$ such that some prefix of $Y_s \upharpoonright_n$ is mapped to τ in Φ_s, and such that adding $(Y_s \upharpoonright_n, \tau)$ in Ψ_{s-1} does not violate consistency. If no such string τ is found we add $(Y_s \upharpoonright_n, \epsilon)$ in Ψ at stage s. Otherwise we add $(Y_s \upharpoonright_n, \tau)$.

At limit stage s, we define Ψ_s as the union of Ψ_t for $t < s$.

The verification:

By construction, it is clear that Ψ defines a fin-h reduction: The pairing relation in Ψ is functional, as we add mapping in Ψ only for strings that are not already mapped to something. Also at each stage s, every prefix of Y_s, by order of their lengths, is mapped to something (unless already mapped). This implies that the mapping is closed by prefixes, and the consistency requirement is explicitly satisfied at each step.

We should now show that $\Psi(Y) = X$. Consider any m and let t is the first stage so that some prefix σ of Y is mapped to an extension of $X \upharpoonright_m$ in Φ_t. In particular we have a string τ with $|\tau| > n$ and $\sigma \leq \tau \prec Y$, and a smallest stage $s \geq t$ such that no extension of τ has been a prefix of Y_r for $r < s$. Therefore, as Φ is consistent on Y we necessarily have by construction that τ is mapped to an extension of $X \upharpoonright_m$ in Ψ at stage s.

Similarly we can prove that if A has a self-unclosed approximation, then we can find an 'almost prefix-free' A-continuously Π^1_1 set of strings describing any A-continuously Π^1_1-open set:

Theorem 7.4.2:
If A has a self-unclosed approximation, then for any oracle-continuous Π^1_1-open set $\mathcal{U}$ and any ε, one can uniformly define an oracle-continuous set of strings W such that $[W^A]^{\prec} = \mathcal{U}^A$ and $\sum_{\sigma \in W^A} 2^{-|\sigma|} \leq \lambda(\mathcal{U}^A) + \varepsilon$.

Proof: Let $\{A_s\}_{s<\omega_1^{ck}}$ be a self-unclosed approximation of A that we can suppose not Δ^1_1. Let $\mathcal{U} \subseteq 2^{<\omega} \times 2^{<\omega}$ describes an oracle-continuous Π^1_1-open set and fix ε.

The construction:

We start by $W_0 = \varnothing$. At successor stage s, we search for the smallest prefix σ of A_s such that σ is not a prefix of A_t for $t < s$. If such a string does not exists then we go to the next stage. Otherwise, we define the Δ^1_1 set of string $V \subseteq 2^{<\omega}$ to be the set $\mathcal{U}^\sigma_s$, rearranged, so that V is prefix-free.

Then for any $\tau \in V$, we define a finite set of strings B_τ with $[B_\tau]^{\prec} \subseteq [\tau]$ such that $\lambda([W^\sigma_{s-1}]^{\prec} \cap [B_\tau]^{\prec}) \leq \varepsilon \times 2^{-|\sigma|} 2^{-|\tau|}$ and such that $[\tau] \subseteq [W^\sigma_{s-1}]^{\prec} \cup [B_\tau]^{\prec}$. Then for each string ρ in B_τ we add (σ, ρ) in W at stage s.

Finally at limit stage s we define $W_s = \bigcup_{t<s} W_t$.

The verification:

We should first prove that $[W^A]^{\prec} = \mathcal{U}^A$. By construction and by the fact that $\{A_s\}_{s<\omega_1^{ck}}$ is a self-unclosed approximation of A, there are stages $s_1 < s_2 < \dots$ with $\sup_{n\in\mathbb{N}} s_n = \omega_1^{ck}$ and prefixes $\sigma_1 \prec \sigma_2 \prec \dots$ of A, such that $[W^A]^{\prec} = \bigcup_i [\mathcal{U}^{\sigma_i}_{s_i}]^{\prec}$. Also this is clearly equal to $\mathcal{U}^A$.

Let us now prove that the total overlap of what we add in W along prefixes of A, is smaller than ε. At stage s_i in the construction, we create a prefix-free set of string V describing $\mathcal{U}^{\sigma_i}_{s_i}$ and for each τ in V, the overlap between what we add and $W^A_{s_i-1}$, is smaller than $\varepsilon \times 2^{-|\sigma_i|} 2^{-|\tau|}$. Then the total overlap for strings in V is smaller than $\varepsilon \times 2^{-|\sigma_i|}$. It follows that the total overlap is bounded by ε. ∎

If A has a higher self-unclosed approximation, it is also possible to continuously relativize the higher KC theorem (Theorem 3.7.11). An A-continuous Π^1_1-bounded request set S is a Π^1_1 subset of $2^{<\mathbb{N}} \times \mathbb{N} \times 2^{<\mathbb{N}}$ such that $S^A = \{(l, \sigma) \ : \ \exists \tau \prec A \ (\tau, l, \sigma) \in S\}$ is a bounded request set (recall : $\sum_{(l,\sigma)\in S^A} 2^{-l} \leq 1$). We have:

Theorem 7.4.3:
If A has a self-unclosed approximation, then for any A-continuous Π^1_1-bounded request set S, there is a A-continuous Π^1_1-prefix-free machine M such that for any string σ, if $(l, \sigma) \in S^A$, then for a string τ of length l we have $M^A(\tau) = \sigma$.

PROOF: We only sketch the proof here, as the trick is similar than before. As for the previous proof, we can suppose that A is not Δ^1_1 and use the fact that it then has a self-unclosed approximation $\{A_s\}_{s<\omega_1^{ck}}$ such that for any stage t, there is a stage $s \geq t$ a string σ with both $\sigma \prec A_s$ and $\sigma \prec A$ such that no extension of σ has been a prefix of any A_r for $r < s$.

As for the previous proof, when at some stage s we find a smallest prefix σ of A_s such that σ is not a prefix of A_t for $t < s$. We consider every pair in S^σ_s that has not been dealt with yet to make our A-continuous Π^1_1-prefix-free machine and we deal with each of them in order of their enumeration, just like in the proof of the higher KC-theorem. ■

Corollary 7.4.1:
If A has a self-unclosed approximation, then the four notions of higher A-randomness with continuous relativization, that is, with Π^1_1-prefix-free machines, Π^1_1-discrete semi-measures, Π^1_1-continuous semi-measures and Π^1_1-Martin-Löf tests, coincide.

PROOF: With the two previous theorems, given an A-continuous Π^1_1-Martin-Löf test, we can define an A-continuous Π^1_1-prefix-free machine M such that for any X captured by the test and for any c, there exists some n such that $\mathrm{hK}_M(X\upharpoonright_n) < n - c$. This is done just like in the proof of Theorem 3.7.13. ■

We now prove similarly that if A has a self-unclosed approximation, then there is an oracle-continuous A-universal Π^1_1-prefix-free machine. The existence of universal objects for any of the other three notions then follows.

Theorem 7.4.4:
If A has a self-unclosed approximation, then there is an oracle-continuous A-universal Π^1_1-prefix-free machine.

PROOF: Let M be a Π^1_1 set of triples of strings. Let $\{A_s\}_{s<\omega_1^{ck}}$ be a self-unclosed approximation of A. We build a Π^1_1 set of triples of strings N such that for any oracle X, N^X is prefix-free machine, and if M^A is a prefix-free machine, then $N^A = M^A$. The construction is uniform in an approximation of A and a code for M. Then using this we can easily build an A-universal oracle continuous Π^1_1-prefix-free machine, by diagonalizing against every oracle-continuous Π^1_1-prefix-free machine, in a similar way to the proof of Theorem 3.7.10.

The construction:

At successor stage s, we search for the smallest prefix σ of A_s such that for any $t < s$, the string σ is not a prefix of A_t. If such a string does not exist, we go to the next stage. Otherwise if M^σ_s is a prefix-free machine, then for every pair (τ, ρ) in M we add the triple (σ, τ, ρ) in N at stage s. Otherwise we go to the next stage. At limit stage s we set N_s to be the union of N_t for $t < s$.

The verification:

We prove by induction the following for any stage $s \leq \omega_1^{ck}$:

$$\text{For any } \tau \text{ we have } N_s^\tau \subseteq M_s^\tau \text{ and } N_s^\tau \text{ is a prefix free machine.} \qquad (*)$$

It is clear that (*) is true at stage 0. Suppose it is true at any stage $t < s$. If s is limit then (*) is also true at stage s by induction hypothesis, as if N^τ is not prefix-free, it is because of finitely many triple $(\sigma, \tau, \rho) \in N_s$. If s is successor, we add something in N only if $\sigma \prec A_s$ and σ is not a prefix of A_t for $t < s$. Consider such a string σ. If M_s^σ is not prefix-free we do nothing and (*) is true at stage s. Otherwise for any $\tau \prec \sigma$ we have $N_s^\tau = N_{s-1}^\tau$ and then by induction hypothesis $(*)$ is true at stage s for each strict prefix of σ. Also by induction hypothesis we have $N_{s-1}^\sigma \subseteq M_{s-1}^\sigma$. It follows by construction that $N_s^\sigma = M_s^\sigma$. As M_s^σ is prefix-free, then also N_s^σ is prefix-free. Also by hypothesis on σ, we have $N_{s-1}^\sigma = N_{s-1}^\tau$ for any $\tau \succ \sigma$ and then we have $N_s^\tau \subseteq M_s^\tau$ and also that N_s^τ is a prefix-free machine for any string τ. So (*) is true at stage s.

Suppose now that M^A is prefix-free. As $\{A_s\}_{s<\omega_1^{ck}}$ is self-unclosed, there are stages $s_1 < s_2 < \dots$ with $\sup_n s_n = \omega_1^{ck}$ and prefixes $\sigma_1 \prec \sigma_2 \prec \dots$ of A with for each n that σ_n is the smallest prefix of A_{s_n} which is not a prefix of A_t for $t < s_n$. Also as M^A is prefix-free, we then have $N_{s_n}^{\sigma_n} = M_{s_n}^{\sigma_n}$ for every n, and then $N^A = M^A$. ∎

Corollary 7.4.2:
If A has a self-unclosed approximation, then there exist oracle-continuous A-universal objects, for the four notions of higher A-randomness with continuous relativization, that is, with Π_1^1-prefix-free machines, Π_1^1-discrete semi-measures, Π_1^1-continuous semi-measures and Π_1^1-Martin-Löf tests.

7.4.2 Random oracles

We end this section by showing that Z-continuous Z-universal objects also exist when Z is Π_1^1-Martin-Löf random. However this is a bit less powerful than for oracles with self-unclosed approximations. For example if A has a self-unclosed approximation we can prove the existence of a A-universal oracle-continuous Π_1^1-Martin-Löf test, in particular, the test is a Π_1^1-Martin-Löf test for every other oracle. If Z is random we have a Z-universal Z-continuous Π_1^1-Martin-Löf test, however, it might not be a test along other oracles:

Theorem 7.4.5:
If Z is Π_1^1-Martin-Löf random, there is a Z-universal Z-continuous Π_1^1-Martin-Löf test.

Proof: Using Lemma 4.3.1, given an oracle-continuous Π_1^1-open set $\mathcal{U}$ and a rational ε, uniformly in $n \in \mathbb{N}$, we define an oracle-continuous Π_1^1-open set $\mathcal{U}_n$ such that for any oracle X, if $\lambda(\mathcal{U}^X) \leq \varepsilon$, then $\mathcal{U}_n^X = \mathcal{U}^X$, and $\lambda(\mathcal{V}_n = \{X \ : \ \lambda(\mathcal{U}_n^X) > \varepsilon\}) \leq 2^{-n}$. Using this, if Z is Π_1^1-Martin-Löf random, it is in no test $\bigcap_n \mathcal{V}_n$. Furthermore, knowing the randomness deficiency of Z, one can find uniformly for each test $\bigcap_n \mathcal{V}_n$ an index m such that $X \notin \mathcal{V}_m$. Therefore, for a given oracle-continuous Π^1_1-open set $\mathcal{U}$ and any ε, we can use $\mathcal{U}_m$ (the integer m being found uniformly) instead of $\mathcal{U}$, being sure that $\lambda(\mathcal{U}^Z_m) \leq \varepsilon$, with $\mathcal{U}^Z_m = \mathcal{U}^Z$ if already $\lambda(\mathcal{U}^Z) \leq \varepsilon$. It follows that we can then diagonalize against every uniform intersection of oracle-continuous Π^1_1-open sets to get a Z-universal Z-continuous Π^1_1-Martin-Löf test, in a similar way to the proof of Theorem 2.1.1. ■

We can similarly obtain Z-continuous Z-universal objects for any of the three other notions of randomness, when Z is Π^1_1-Martin-Löf random. The question of the equivalence of the four notions of randomness when Z is Π^1_1-Martin-Löf random has not been investigated. We finish the section by an open question:

Question 7.4.1 *Is there a Π^1_1-Martin-Löf random sequence Z and a sequence A such that Z higher Turing computes A but Z does not fin-h compute A?*

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List of Symbols

Chapter 1

ϵ	the empty word, page 1
$2^{<\mathbb{N}}$	the space of strings, page 1
$2^{\mathbb{N}}$	the Cantor space, page 1
σ, τ, ρ	elements of $2^{<\mathbb{N}}$, page 1
X, Y, Z	elements of $2^{\mathbb{N}}$, page 1
$\sigma \leq \tau$	σ is a prefix of τ, page 1
$\sigma \prec \tau$	σ is a strict prefix of τ, page 1
$\sigma \prec X$	σ is a prefix of X, page 1
$\lvert\sigma\rvert$	the length of σ, page 1
$\sigma \perp \tau$	σ and τ are incomparable, page 1
$\sigma \parallel \tau$	σ and τ are comparable, page 1
$\sigma(n)$	the value of the n-th bit of σ, page 1
$\sigma \upharpoonright_n$	the restriction of σ to its n first bits, page 1
$X \upharpoonright_n$	the restriction of X to its n first bits, page 1
$\sigma \hat{\ } \tau$	the concatenation of τ to σ, page 1
$\langle , \rangle$	computable bijection from $\mathbb{N} \times \mathbb{N}$ to $\mathbb{N}$, page 2
$X \oplus Y$	the sequence Z with $Z(2i) = X(i)$ and $Z(2i+1) = Y(i)$, page 2
$\oplus_{i \in \mathbb{N}} X_i$	the sequence Z with $Z(\langle i, j \rangle) = X_i(j)$, page 2
$\mathbb{N}$	the set of natural numbers, page 2
$[\sigma]$	the cylinder given by σ: $\{X : \sigma \prec X\}$, page 2
$[W]^{\prec}$	$\bigcup_{\sigma \in W} [\sigma]$, page 2
$\mathbb{R}$	the set of real numbers, page 2
$\mathbb{N}^{<\mathbb{N}}$	the space of strings of the Baire space, page 3

$\mathbb{N}^{\mathbb{N}}$	the Baire space, page 3
σ, τ, ρ	elements of $\mathbb{N}^{<\mathbb{N}}$, page 3
f, g, h	elements of $\mathbb{N}^{\mathbb{N}}$, page 3
ϵ	the empty word, page 3
$[T]$	the set of infinite paths of the tree T, page 3
$\text{stem}(T)$	the first branching node of the tree T, page 3
$T\upharpoonright_{\sigma}$	the subtree of T obtained by keeping strings compatible with σ, page 3
$T\upharpoonleft_{\sigma}$	the tree obtained by 'shifting to the left' every string of $T\upharpoonright_{\sigma}$ by $\|\sigma\|$, page 3
$\sigma\hat{\ }T$	the tree obtained by 'shifting to the right' every string of T by σ, page 3
φ_e	computable function $\varphi_e : \mathbb{N} \to \mathbb{N}$ of code e, page 4
$a, b, c, d, e, i, j, k, l, m, n$	elements of $\mathbb{N}$, page 4
Φ_e	computable functional $\Phi_e : 2^{\mathbb{N}} \times \mathbb{N} \to \mathbb{N}$ of code e, page 4
Φ_e^X	currying of the computable functional of code e applied to X, page 4
$\Phi_e^X(n)\downarrow$	the computation $\Phi_e^X(n)$ halts, page 4
$\Phi_e^X(n)\uparrow$	the computation $\Phi_e^X(n)$ never halts, page 4
W_e	the domain of φ_e, page 4
W_e^X	the domain of Φ_e^X, page 4
$\Phi_e(X)$	the image of X by $\Phi_e : 2^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$, page 4
$\Phi_e(X, n)[t]$	the result of the computation up to time t, page 4
$\text{use}^X(n)$	the use of X on input n, page 4
$X \leq_m Y$	X is many-one reducible to Y, page 5
$X \equiv_m Y$	X is many-one equivalent to Y, page 5
$X \leq_T Y$	X is Turing reducible to Y, page 5
$X \equiv_T Y$	X is Turing equivalent to Y, page 5
$X \leq_{tt} Y$	X is truth-table reducible to Y, page 5
$X \equiv_{tt} Y$	X is truth-table equivalent to Y, page 5
$X \leq_{wtt} Y$	X is weakly truth-table reducible to Y, page 6
$X \equiv_{wtt} Y$	X is weakly truth-table equivalent to Y, page 6
α, β, γ	ordinals, page 8

$\lvert R\rvert$	the order-type of the well-founded relation R, page 8
α^+	successor of α, page 8
$\alpha+1$	successor of α, page 8
$\sup^+(A)$	least strict upper bound of A, page 8
$\lvert R\rvert_o$	the ordinal isomorphic to the well-order R, page 8
$\alpha+\beta$	sum of α and β, page 10
$\alpha\times\beta$	multiplication of α and β, page 10
$A\sqcup B$	disjoint union of A and B, page 10
$\mathcal{W}$	the set of codes for computable ordinals, page 11
$\lvert a\rvert_o$	the ordinal coded by $a\in\mathcal{W}$, page 11
ω_1^{ck}	the smallest non computable ordinal, page 11
ω_1^X	the smallest non X-computable ordinal, page 11
$\lvert\sigma\rvert_o$	the ordinal coded by the node σ of a well-founded tree T, page 12
$\lvert T\rvert_o$	the ordinal coded by a well-founded tree T, page 12
$\lvert\sigma\rvert_{KB}$	the ordinal coded by the node σ of a well-founded tree T, by the Kleene-Brouwer ordering, page 12
$\lvert T\rvert_{KB}$	the ordinal coded by a well-founded tree T, by the Kleene-Brouwer ordering, page 12
$\mathcal{T}$	the set of codes for c.e. well-founded trees, page 12
$\mathcal{T}_{<\alpha}$	the set of codes a for c.e. well-founded trees such that $\lvert a\rvert_o<\alpha$, page 12
$\mathcal{T}_{\leq\alpha}$	the set of codes a for c.e. well-founded trees such that $\lvert a\rvert_o\leq\alpha$, page 12
$\mathcal{T}_{=\alpha}$	the set of codes a for c.e. well-founded trees such that $\lvert a\rvert_o=\alpha$, page 12
$\mathcal{O}$	the set of codes for constructive ordinals, page 16
$\mathcal{O}_{<\alpha}$	the set of codes a for constructive ordinals such that $\lvert a\rvert_o<\alpha$, page 16
$\mathcal{O}_{\leq\alpha}$	the set of codes a for constructive ordinals such that $\lvert a\rvert_o\leq\alpha$, page 16
$\mathcal{O}_{=\alpha}$	the set of codes a for constructive ordinals such that $\lvert a\rvert_o=\alpha$, page 16
$T_1\simeq T_2$	T_1 is isomorphic to T_2, page 16
$a=\operatorname{succ} b$	a codes for the ordinal following the one encoded by b, page 16

$a = \sup_n b_n$	a codes for the ordinal being the supremum of the ordinals encoded by the b_n, page 16
$a +_o b$	the sum of a and b using $+_o : \mathcal{O} \times \mathcal{O} \to \mathcal{O}$, page 16
$\boldsymbol{\Sigma^0_\alpha}, \boldsymbol{\Pi^0_\alpha}, \boldsymbol{\Delta^0_\alpha}$	the level α of the Borel hierarchy, page 19
$\boldsymbol{\Sigma^0_{<\alpha}}$	$\boldsymbol{\Sigma^0_\beta}$ for $\beta < \alpha$, page 19
$\boldsymbol{\Pi^0_{<\alpha}}$	$\boldsymbol{\Pi^0_\beta}$ for $\beta < \alpha$, page 19
$\Sigma^0_\alpha, \Pi^0_\alpha, \Delta^0_\alpha$	the level α of the effective Borel hierarchy, page 21
$\Sigma^0_{<\alpha}$	Σ^0_β for $\beta < \alpha$, page 21
$\Pi^0_{<\alpha}$	Π^0_β for $\beta < \alpha$, page 21
$\Sigma^0_\alpha(X), \Pi^0_\alpha(X), \Delta^0_\alpha(X)$	the level α of the X-effective Borel hierarchy, page 21
$\Sigma^0_{<\alpha}(X)$	$\Sigma^0_\beta(X)$ for $\beta < \alpha$, page 21
$\Pi^0_{<\alpha}(X)$	$\Pi^0_\beta(X)$ for $\beta < \alpha$, page 21
Σ^0_α-open set	an open set described by a Σ^0_α set of strings, page 27
Π^0_α-open set	an open set described by a Π^0_α set of strings, page 27
Σ^0_α-closed set	complement of a Π^0_α-open set, page 27
Π^0_α-closed set	complement of a Σ^0_α-open set, page 27
$\emptyset^{(\alpha)}$	the Σ^0_α-complete set, page 30
$X^{(\alpha)}$	the $\Sigma^0_\alpha(X)$-complete set, page 30
$\emptyset^{(<\alpha)}$	the disjoint union of the Σ^0_β-complete sets for $\beta < \alpha$, page 30
$X^{(<\alpha)}$	the disjoint union of the $\Sigma^0_\beta(X)$-complete sets for $\beta < \alpha$, page 30
H_a	the H-set of code $a \in \mathcal{O}$, page 36
μ, ν, ξ	probability measures, page 41
$P(\mathcal{X})$	the set of subsets of $\mathcal{X}$, page 41
λ	Lebesgue measure, page 42
$\mu(\mathcal{A} \mid [\sigma])$	the relative measure of $\mathcal{A}$ inside $[\sigma]$, page 43
$\mu(\mathcal{A})[s]$	the measure of $\mathcal{A}$ at stage s, page 43
$A \vartriangle B$	the symmetric difference of A and B: $A - B \cup B - A$, page 47

Chapter 2

$\mathcal{U}[t]$	enumeration of $\mathcal{U}$ up to stage t, page 52

Chapter 3

$\boldsymbol{\Pi^1_1}$	analytic, page 65
$\boldsymbol{\Sigma^1_1}$	co-analytic, page 65
Π^1_1	effectively $\boldsymbol{\Pi^1_1}$, page 65
Σ^1_1	effectively $\boldsymbol{\Sigma^1_1}$, page 65
$X \geq_h Y$	Y is $\Delta^1_1(X)$, page 72
hK	higher Kolmogorov complexity, page 89

Chapter 4

hW_e	oracle-continuous e-th Π^1_1 set of integers, page 101
hW^σ_e	σ-continuous e-th Π^1_1 set of integers, page 101
hW^X_e	X-continuous e-th Π^1_1 set of integers, page 101
hJ^X	continuous higher jump of X, page 102
hK^A_M	higher Kolmogorov complexity continuously relativized to A and with respect to the machine M, page 109

Chapter 6

$\Sigma^{\omega_1^{ck}}_n$	the level Σ_n of the higher effective Borel hierarchy, page 168
$\Pi^{\omega_1^{ck}}_n$	the level Π_n of the higher effective Borel hierarchy, page 168

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