Teaching reductions : formal foundations

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Goal : develop a platform to help students learn complexity theory

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- ✓ Understand the classic reductions
- ☠ Design their own reductions, and get feedback
  - easy-to-grasp specification language for reductions
  - automatic tools to check the validity of such reductions
  - produce a counter-example if the reduction is incorrect

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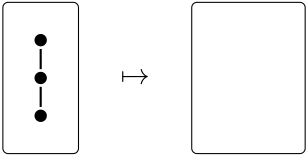
Best of both worlds :

- ✓ declarative
- ✓ left-to-right

Specification language : *cookbook reductions*

Cookbook reduction



k−Clique $\leq$ $(k+1)$−Clique

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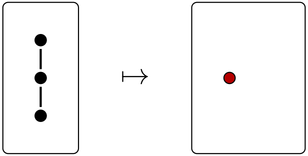

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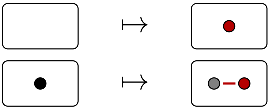

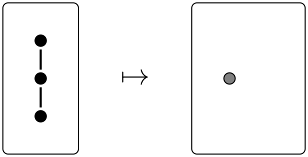

$$k\text{−}\textsc{Clique} \leq (k{+}1)\text{−}\textsc{Clique}$$

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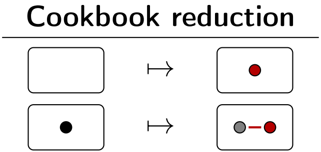

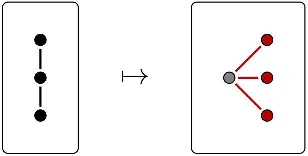

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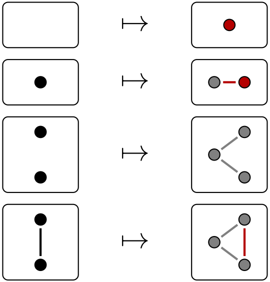

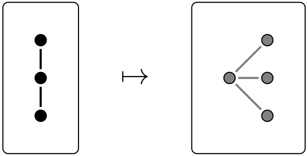

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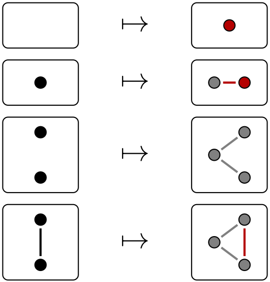

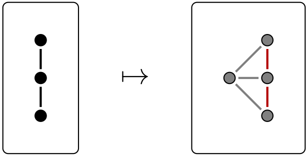

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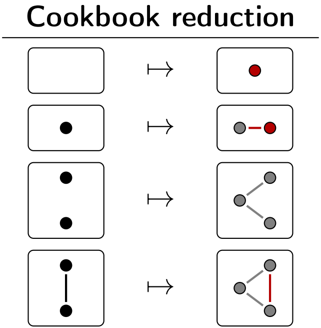

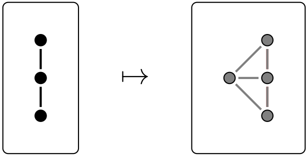

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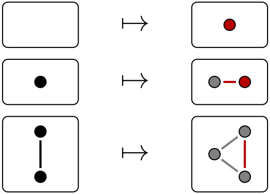

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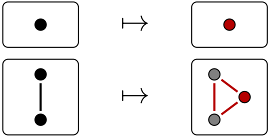
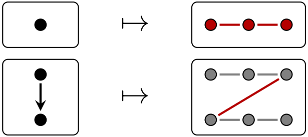
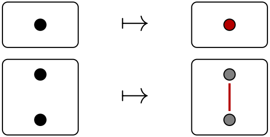


$k-\textsc{VertexCover} \leq k-\textsc{FVS}$



$\textsc{HamCycle}_d \leq \textsc{HamCycle}_u$



$k-\textsc{Clique} \leq k-\textsc{IndepSet}$

Specification language : *cookbook reductions*

Theorem

Every cookbook reduction is equivalent to a quantifier-free interpretation

...but not all QF-interpretations are cookbook reductions

Checking reduction correctness

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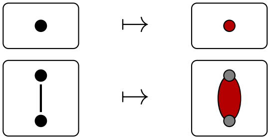

Edge-gadget reductions

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Fix any problem P and any $P^\star \in \mathsf{FO}$. One can decide whether a cookbook reduction of arity $\leq r$ is a valid reduction $P \leq P^\star$.

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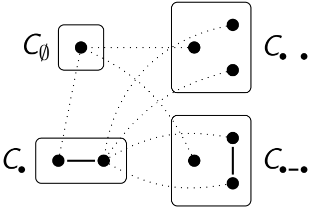

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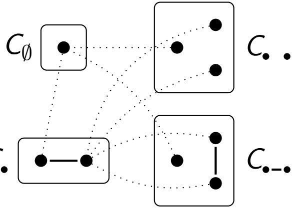

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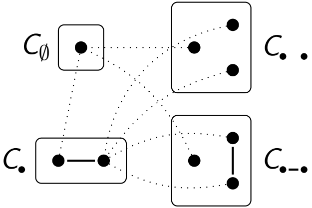

The recipe for the cookbook reduction ρ of arity 2 from $k-$Clique to $(k+1)-$Clique

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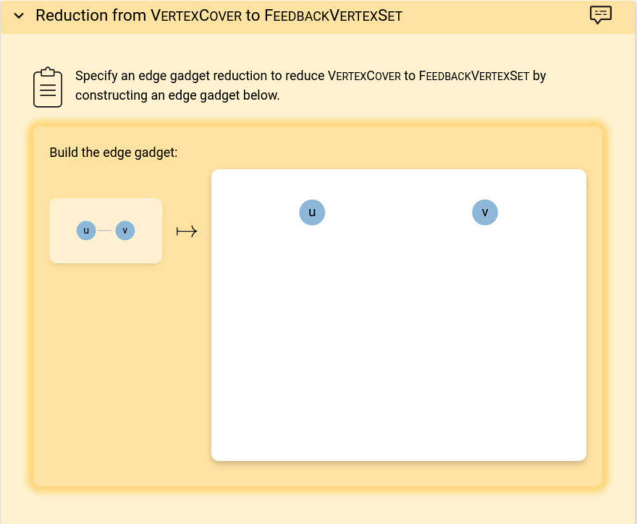
Hence, the correction of ρ only depends on the FO-type of its recipe at some depth.

Prototype on Iltis



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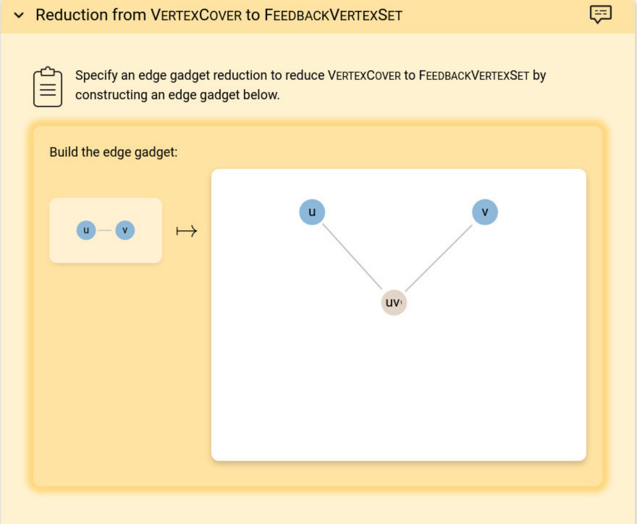

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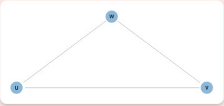
Prototype on Iltis

Your gadget does not induce a valid reduction.

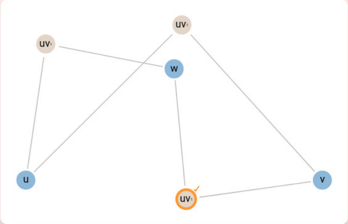
For instance, consider the following *negative* VERTEXCOVER instance:

$k = 1, G =$



Applying your gadget yields the following *positive* instance of FEEDBACKVERTEXSET:

$k^\star = 1, G^\star =$



A solution is given by the highlighted vertices.

Wrong reduction: feedback via counter-example

Conclusion and perspectives

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~~Theorem~~ Conjecture

Fix any problem P and any $P^\star \in$ ~~MSO~~ MSO^2. One can decide whether ~~an edge-gadget reduction~~ a cookbook reduction of arity $\leq r$ is a valid reduction $P \leq P^\star$.