Successor-Invariant First-Order Logic on Classes of Bounded Degree

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Introduction

Databases stored on disk come with an order

- Useful to scan the database
- Query results shouldn't depend upon it

Example of query using the successor relation

Example (Successor-dependent query)

$\mathcal{Q}_{\text{red}}$:= "The first node is red"

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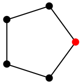

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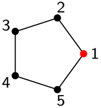

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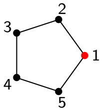

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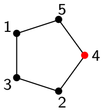

$\models \quad \neg\mathcal{Q}_{\mathsf{red}}$

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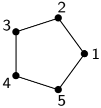

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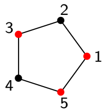

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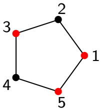

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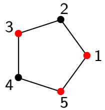

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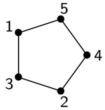

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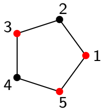

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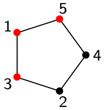

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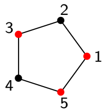

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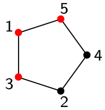

$\models \quad \mathcal{Q}_{\text{odd}}$

$\mathcal{Q}_{\text{odd}}$ defines odd structures.

Definition of Succ-inv FO

$\varphi \in \mathrm{FO}(\Sigma, S)$ is **successor-invariant** over a finite Σ-structure $\mathcal{A}$ if

$$\forall S_1, S_2, \qquad (\mathcal{A}, S_1) \models \varphi \quad \leftrightarrow \quad (\mathcal{A}, S_2) \models \varphi$$

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Definition (Successor-Invariant First-Order Logic)

Succ-inv $\mathrm{FO} := \{\varphi : \varphi$ is successor-invariant over every finite $\mathcal{A}\}$

Succ-inv FO doesn't have a recursive syntax.

Known results

Immerman-Vardi (1982):

- $\mathrm{PTIME}\,(=\!<\!\text{-inv LFP}\,) = \text{Succ-inv LFP}$

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Benedikt, Segoufin (2009):

- Succ-inv $\mathrm{FO} = \mathrm{FO}$ on trees
- Succ-inv $\mathrm{FO} \subseteq \mathrm{MSO}$ on
  - graphs of bounded degree
  - graphs of bounded treewidth

Succ-inv FO collapses to FO when the degree is bounded

Theorem

Let $\mathcal{C}_d$ be a class of degree at most d.

$$\text{Succ-inv FO} = \mathrm{FO} \textit{ on } \mathcal{C}_d$$

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Theorem

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$$\text{Succ-inv } \mathrm{FO} = \mathrm{FO} \textit{ on } \mathcal{C}_d$$

$$\varphi \in \text{ Succ-inv } \mathrm{FO}$$

$$\downarrow$$

$$\exists \psi \in \mathrm{FO}, \quad \psi \leftrightarrow \varphi \text{ on } \mathcal{C}_d$$

Proof of the collapse

$$\begin{array}{ccc} \mathcal{G}_1 & \equiv_{f(k)} & \mathcal{G}_2 \\ & \downarrow & \\ (\mathcal{G}_1, S_1) & \equiv_k & (\mathcal{G}_2, S_2) \end{array}$$

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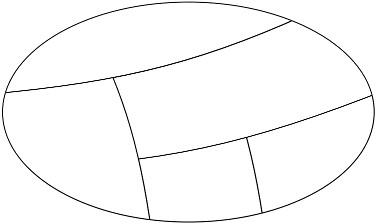

Equivalence classes for $\equiv_{f(k)}$
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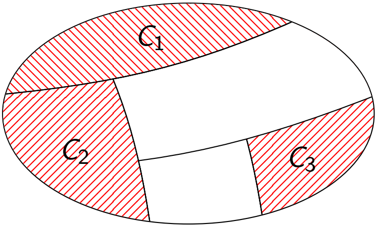

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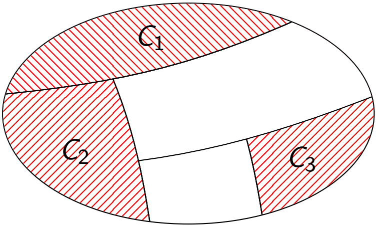
$$\varphi \quad \leftrightarrow \quad \underbrace{\psi_{C_1} \vee \psi_{C_2} \vee \psi_{C_3}}_{\in \text{ FO}} \text{ on } \mathcal{C}_d$$



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Hypothesis: $\mathcal{G}_1, \mathcal{G}_2$ have the same number of each neighborhood type (up to some threshold)

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Hypothesis: $\mathcal{G}_1, \mathcal{G}_2$ have the same number of each neighborhood type (up to some threshold)

Goal: Construct S_1, S_2 such that $(\mathcal{G}_1, S_1), (\mathcal{G}_2, S_2)$ have the same number of each neighborhood type (up to some threshold)

Neighborhood types with many occurrences in $\mathcal{G}_1$

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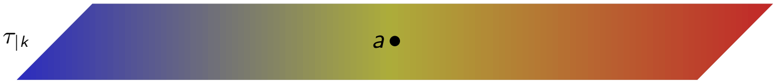
Neighborhood types with few occurrences in $\mathcal{G}_1$

- have the same number of occurrences in $\mathcal{G}_2$
- can be embedded among frequent neighborhood types in $(\mathcal{G}_1, S_1)$ and $(\mathcal{G}_2, S_2)$

Fractal types



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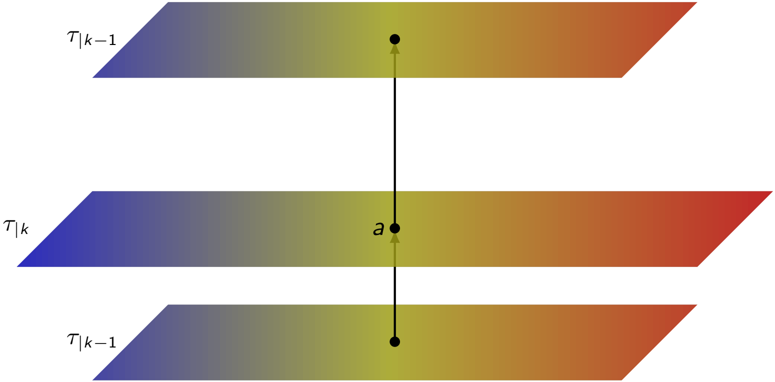

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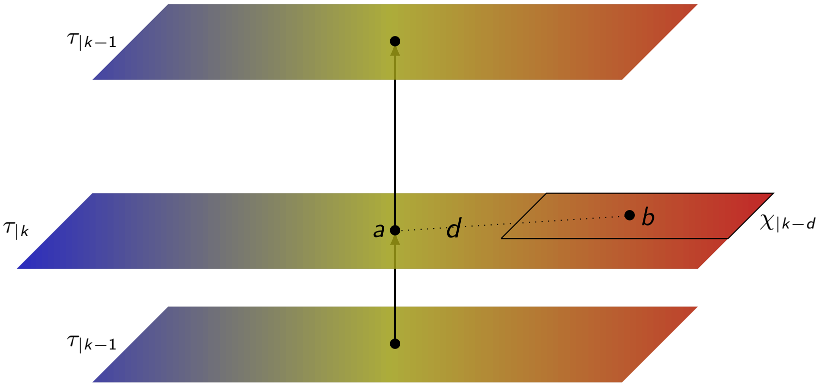

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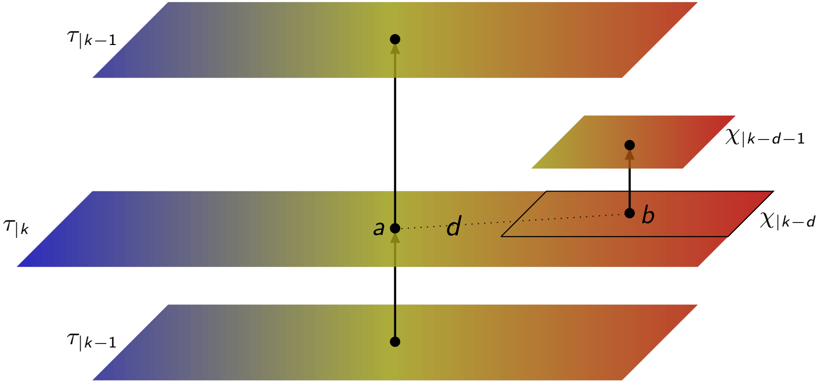

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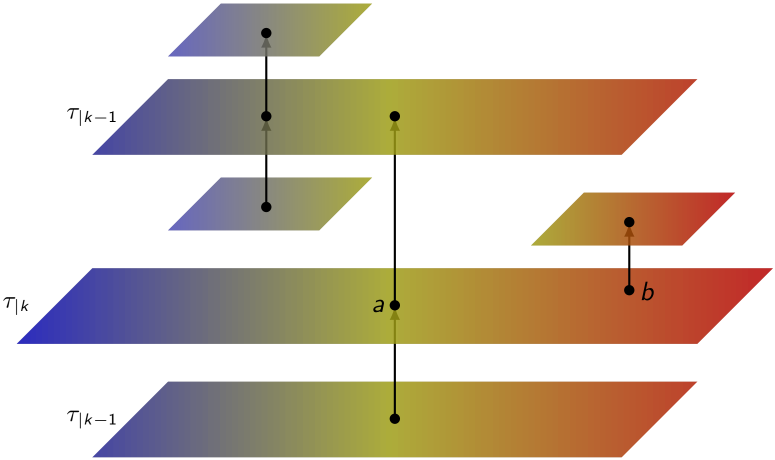

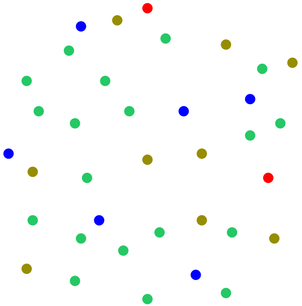

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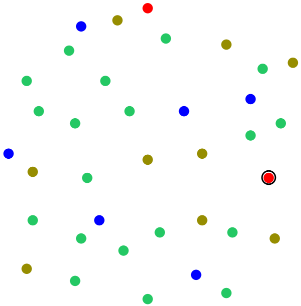

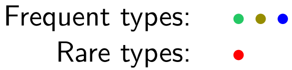
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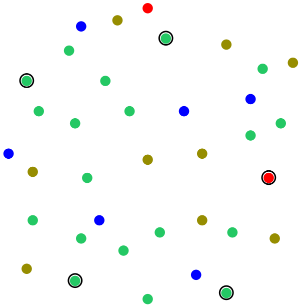

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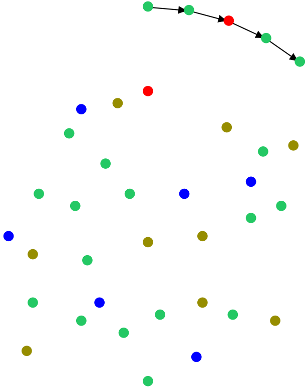

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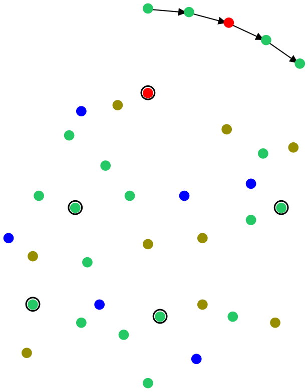

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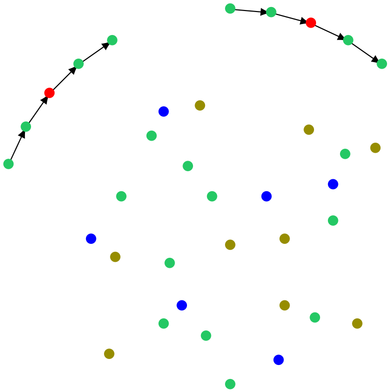

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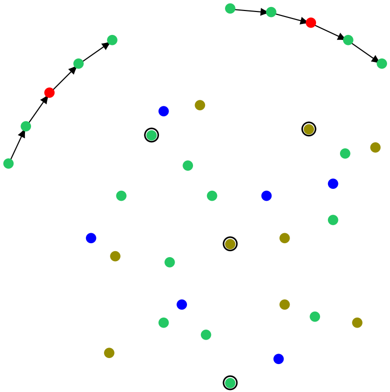

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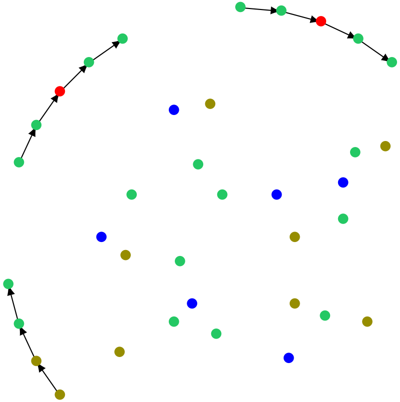

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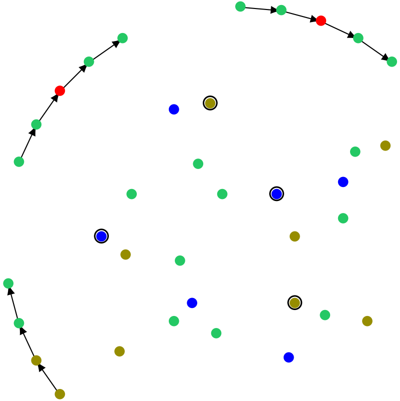

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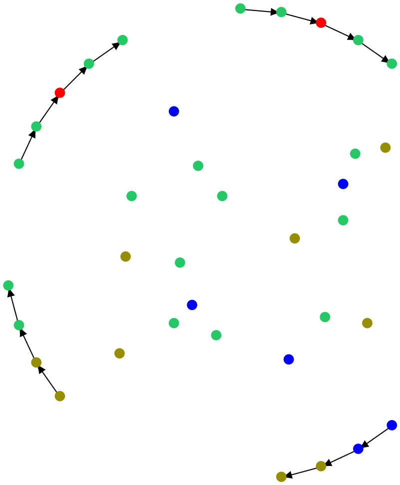

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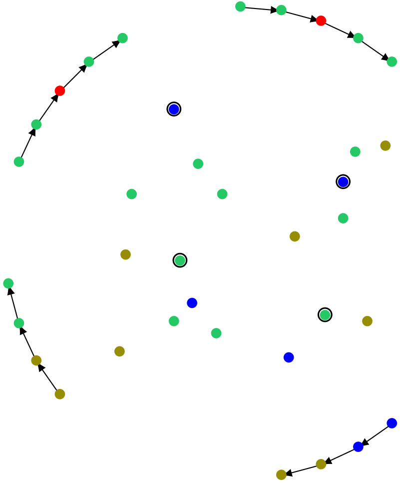

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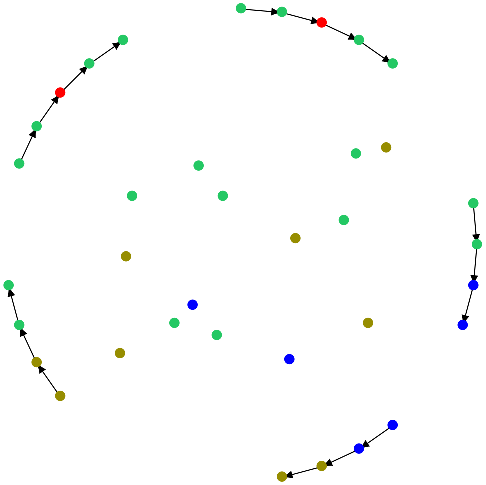

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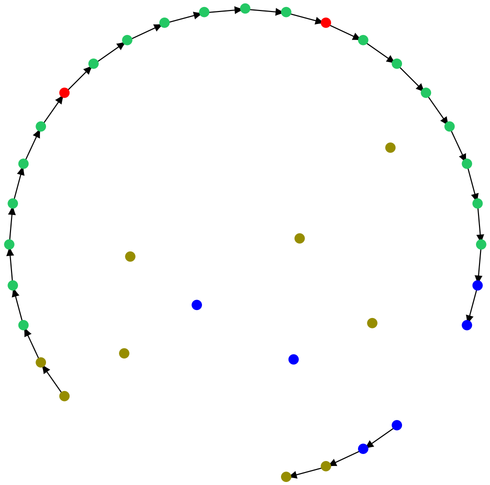

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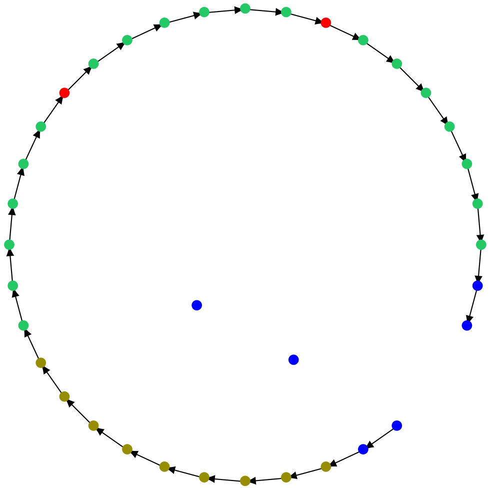

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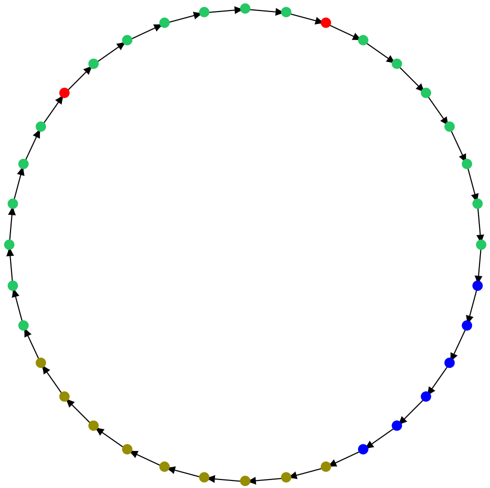

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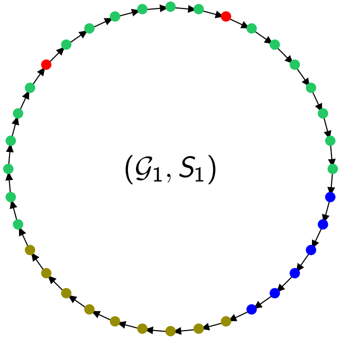
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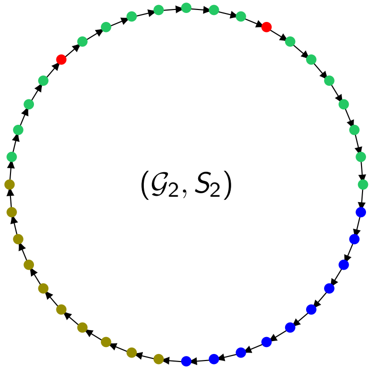
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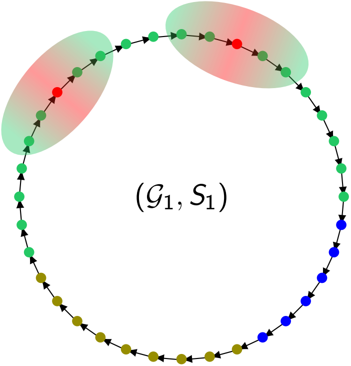
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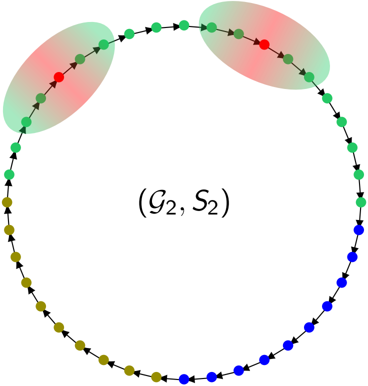
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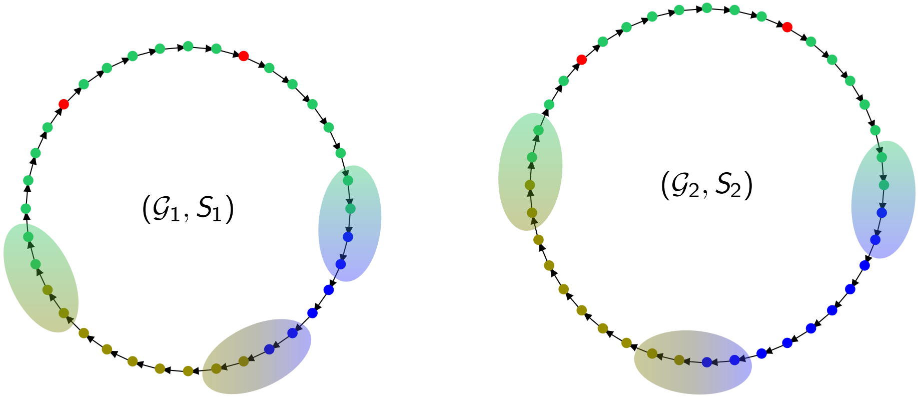
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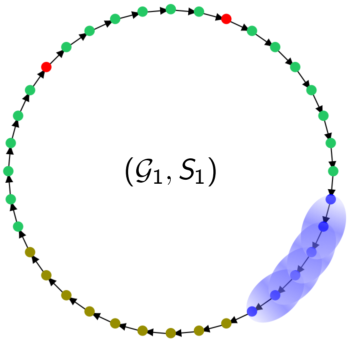
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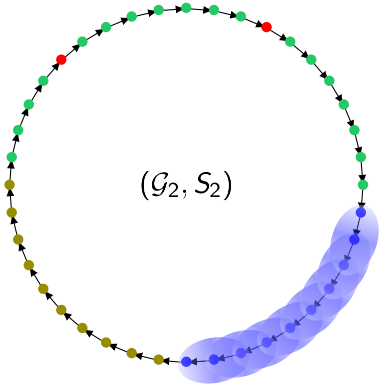
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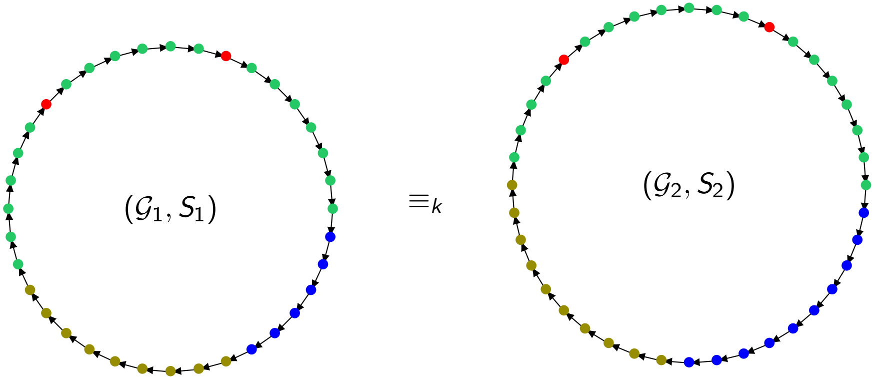

$\square$

Conclusion

Theorem

Succ-inv $\mathrm{FO} = \mathrm{FO}$ *on classes of bounded degree*

Follow-up questions

Succ-inv $\mathrm{FO} = \mathrm{FO}$ on other sparse classes?
$<$-inv $\mathrm{FO} = \mathrm{FO}$ on classes of bounded degree?
What about Succ-inv $\mathcal{L}$, where $\mathcal{L}$ is a fragment of FO (e.g. CQ)?