Order-Invariant First-Order Logic over Hollow Trees

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01/13/2020

Introduction

Databases stored on disk come with an order

- Useful to scan the database
- Query results shouldn't depend upon it

Example of query using the order

Example (Order-dependent query)

$\mathcal{Q}_{\text{red}} :=$ "The first node is red"

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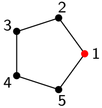
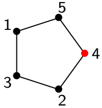


$\models \quad \mathcal{Q}_{\text{red}}$



$\models \quad \neg\mathcal{Q}_{\text{red}}$

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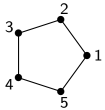

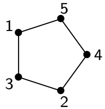

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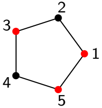

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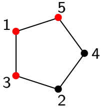

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Definition of $<$-inv FO

$\varphi \in \mathrm{FO}(\Sigma, <)$ is **order-invariant** over a finite Σ-structure $\mathcal{A}$ if:

$$\forall <_1, <_2 \qquad (\mathcal{A}, <_1) \models \varphi \quad \leftrightarrow \quad (\mathcal{A}, <_2) \models \varphi$$

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Definition (Order-invariant First-Order Logic)

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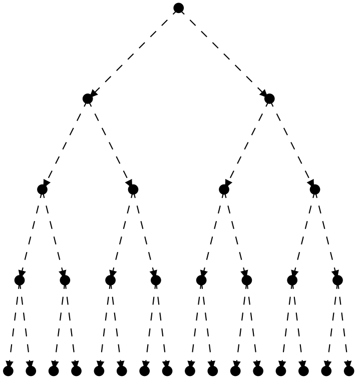
$<$-inv FO doesn't have a recursive syntax.

Potthoff's example (1994)

Complete unordered binary tree,
with the descendant relation.

$(aa)^\star \notin \mathrm{FO}$:
$\texttt{even\_height} \notin \mathrm{FO}$



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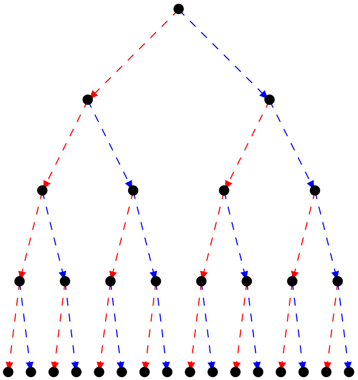

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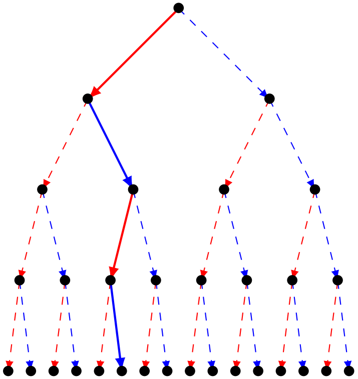

Known results

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- $<$-inv $\mathrm{FO} \subseteq \mathrm{MSO}$ on
  - graphs of bounded degree
  - graphs of bounded treewidth
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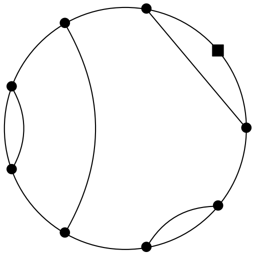
Conjecture

$<$-inv $\mathrm{FO} = \mathrm{FO}$ on graphs of bounded treewidth

From Pathwidth 2 to Hollow Trees



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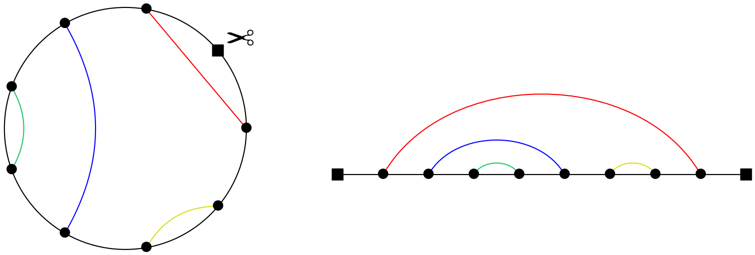

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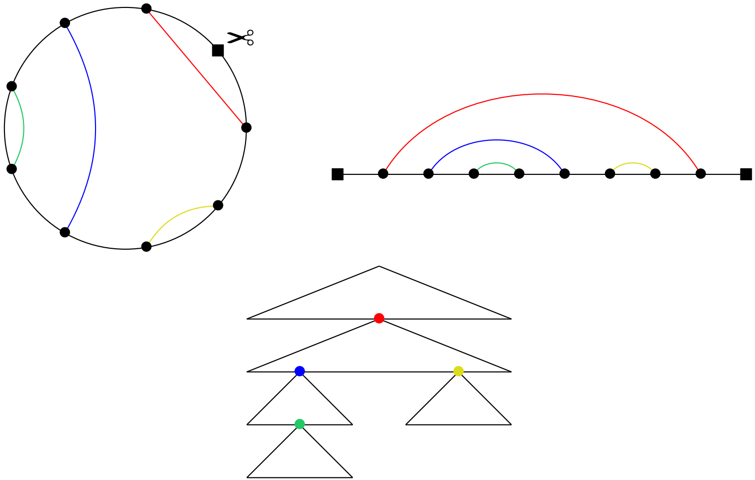
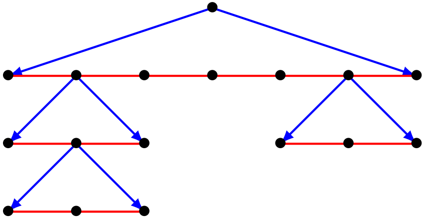


Definition of Hollow Trees

Two binary relations :

- S (oriented)
- E (symmetric)

Nodes are coloured with unary predicates.



$<$-inv FO collapses to FO on Hollow Trees

Theorem

$<$-inv $\mathrm{FO} = \mathrm{FO}$ *on Hollow Trees*

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Theorem

$<$-inv FO $=$ FO *on Hollow Trees*

$$\varphi \in \, <\text{-inv FO}$$

$$\downarrow$$

$$\exists \psi \in \text{FO}, \quad \psi \leftrightarrow \varphi \text{ on Hollow Trees}$$

Proof of the collapse

$$\mathcal{T} \equiv_{f(k)} \mathcal{T}'$$

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$\varphi \in$ $<$-inv FO,
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$\mathcal{T} \models \varphi$

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$\exists \psi \in$ FO of qr. $f(k)$, $\psi \leftrightarrow \varphi$ on Hollow Trees

Proof of the collapse

Proposition

There exist operations $(o_i)_i$ such that

$$\mathcal{A} \xrightarrow{o_i} \mathcal{B} \quad \rightarrow \quad \exists <_A, <_B, \quad (\mathcal{A}, <_A) \equiv_k (\mathcal{B}, <_B)$$

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and $\forall \mathcal{T} \equiv_{f(k)} \mathcal{T}'$,

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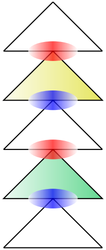
Vertical swap



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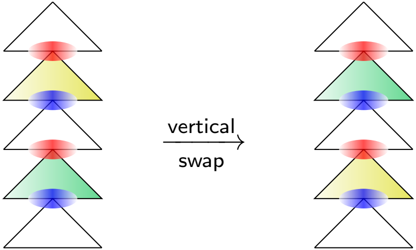

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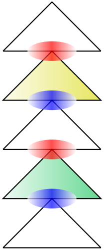
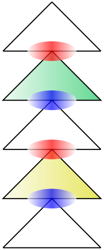
( $, <_1)$ $\equiv_k$ ( $, <_2)$

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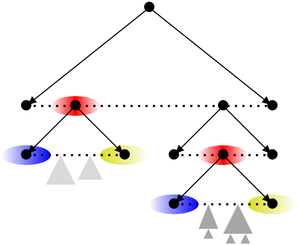

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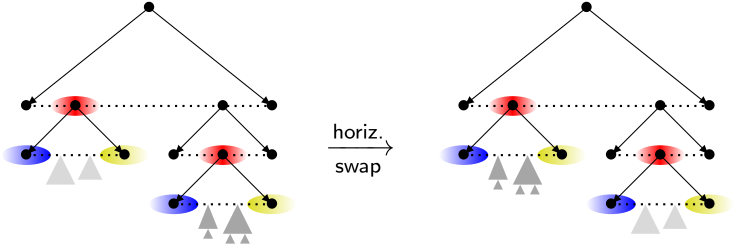

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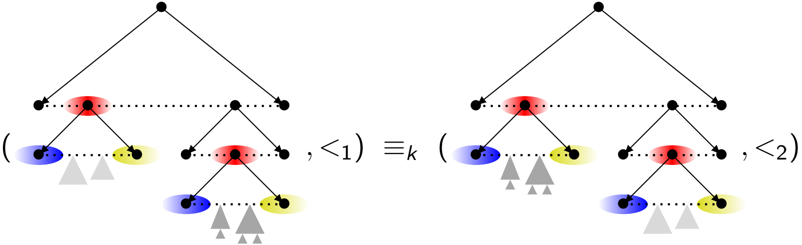
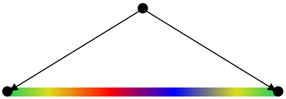


Mirror swap



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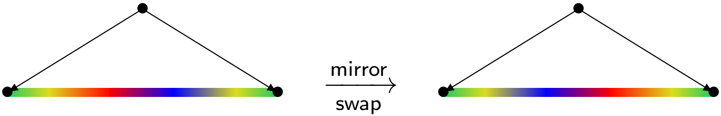

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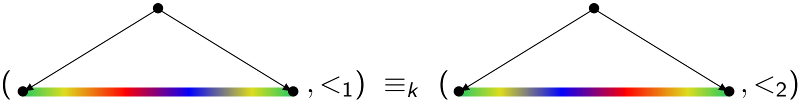
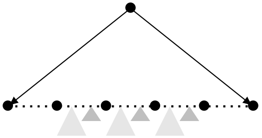


Aperiodicity



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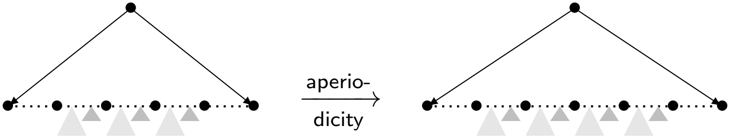

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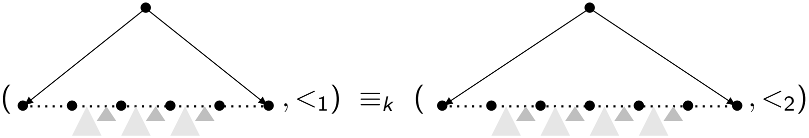

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Proposition

With $(o_i) = (\{\text{vertical, horizontal, mirror}\}\ \text{swap, aperiodicity}, \ldots)$,
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$$\begin{array}{ccccccccc}
\mathcal{T} & \xrightarrow{o_1} & \mathcal{A}_1 & & & & & & \\
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Conclusion

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Open questions

$<$-inv $\mathrm{FO} = \mathrm{FO}$ on graphs of pathwidth 2

$<$-inv $\mathrm{FO} = \mathrm{FO}$ on graphs of bounded treewidth

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- $(E, S) \rightarrow E \cup S \cup S^{-1}$
- unbounded degree