Order-Invariance in the Two-Variable Fragment of First-Order Logic

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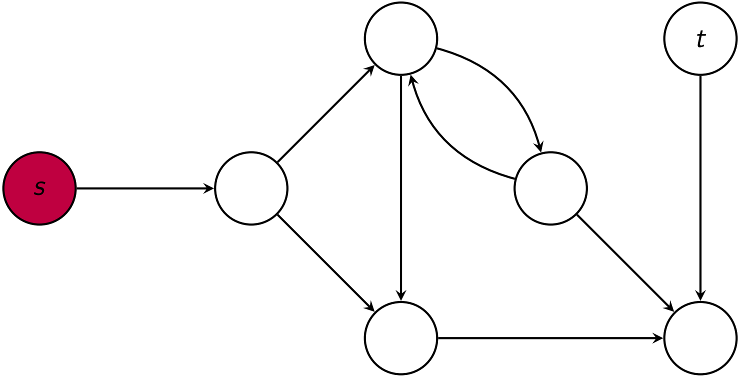
Reach(s,t) via BFS



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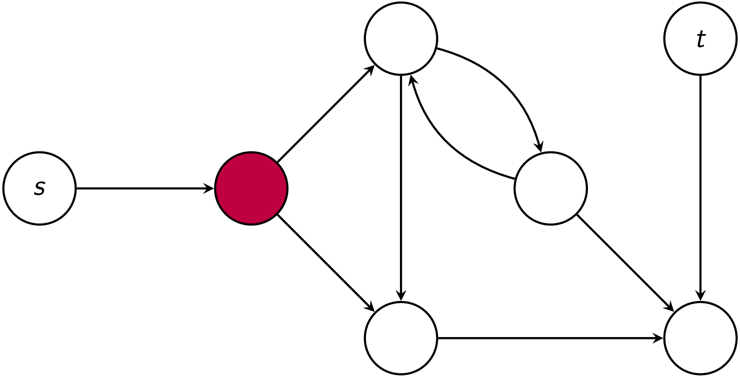

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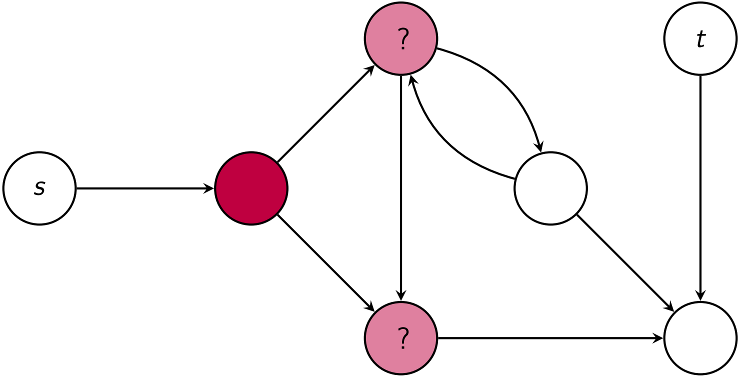

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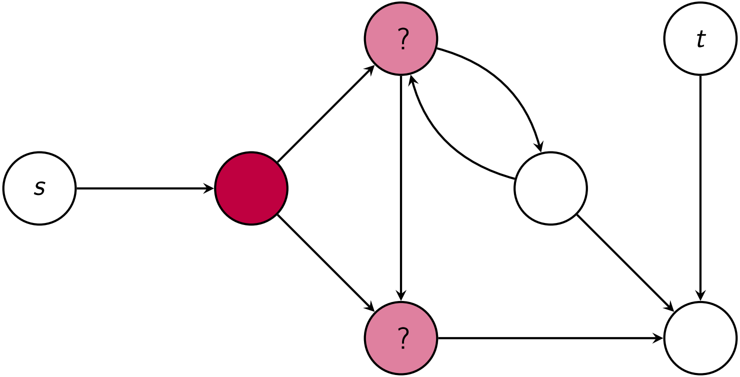

- An order on vertices lets us make choices
- The end result does not depend on the order

Definition of $<$-inv FO

$\varphi \in \mathsf{FO}(\Sigma, <)$ is **order-invariant** over a finite Σ-structure $\mathcal{A}$ if

$$\forall <_1, <_2, \qquad (\mathcal{A}, <_1) \models \varphi \quad \leftrightarrow \quad (\mathcal{A}, <_2) \models \varphi$$

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$<$-inv FO $:= \{\varphi : \varphi$ is order-invariant over every finite $\mathcal{A}\}$

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Theorem (Immerman-Vardi)

PTIME $=$ $<$-inv LFP

`same_parity` : All the branches have the same length parity

Complete unordered binary tree,
with the descendant relation.

$(aa)^\star \notin$ FO :
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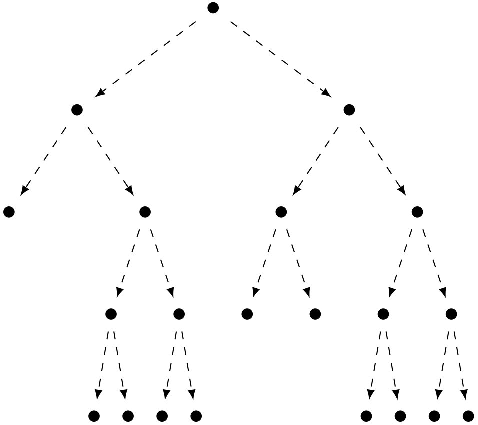

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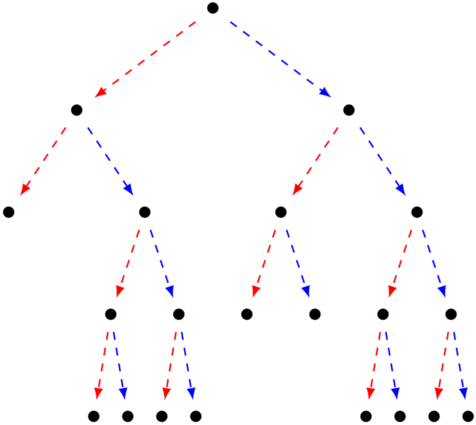

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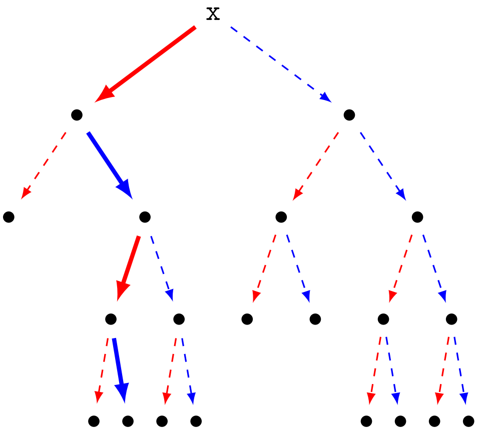

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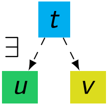
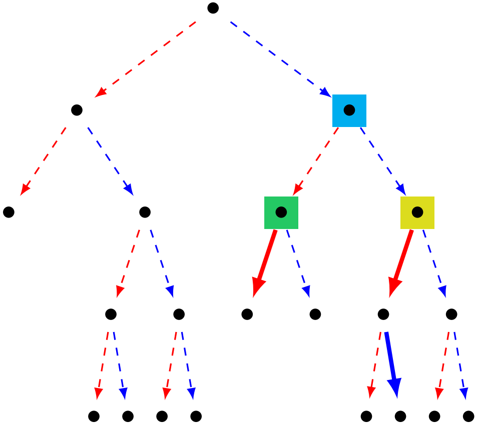
iff $\exists$ 

, $\begin{cases} \neg\texttt{zigzag\_even}(u) \\ \texttt{zigzag\_even}(v) \end{cases}$



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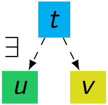
iff $\exists$ 

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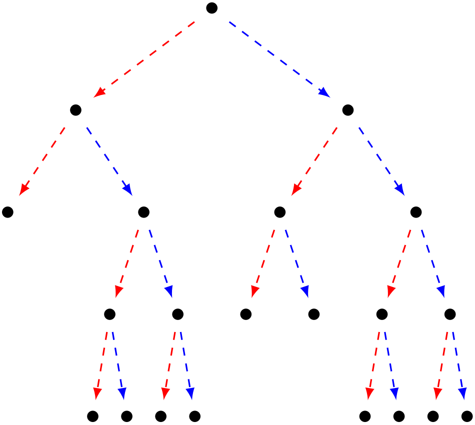
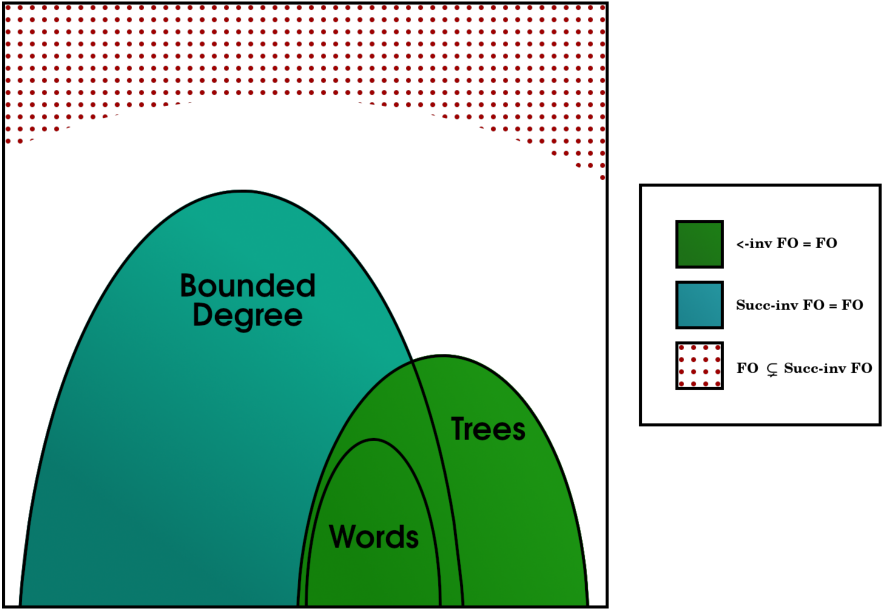
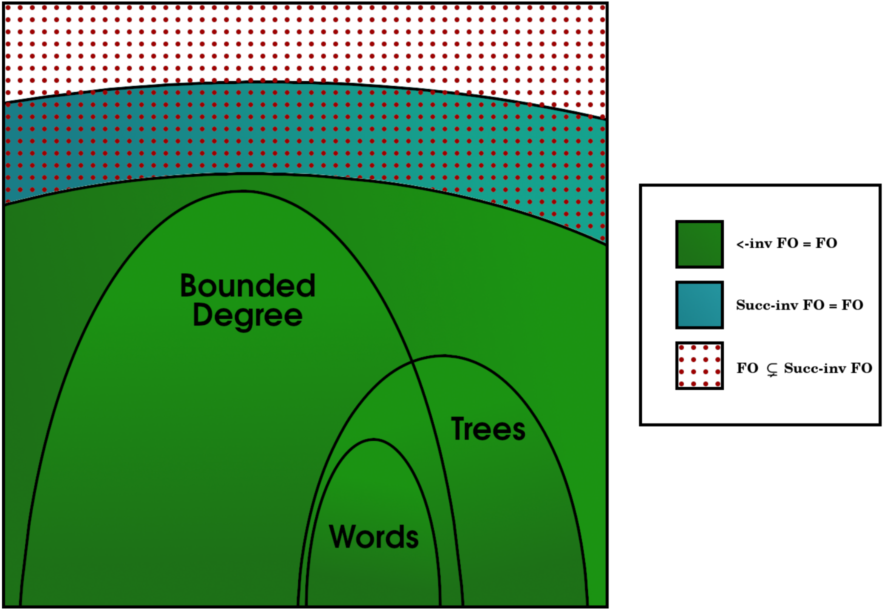
same_parity $\in$ $<$-inv FO



The world as we know it [Gur.] [Ross.] [Ben., Seg.] [G.]



The world as I think it is



Order-Invariant FO^2

As a first step towards

Conjecture

$<$-inv FO $=$ FO *when the degree is bounded*

we show that

Theorem

$<$-inv $FO^2 \subseteq$ FO *when the degree is bounded*

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FO^2 is the restriction of FO to two variables

- $<$-inv FO^2 has a recursive syntax [Harwath, Zeume '16]
- $FO^2 \subsetneq <$-inv FO^2 since $<$-inv FO^2 can count

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Elements of proof

Theorem

$<$-inv $\mathsf{FO}^2 \subseteq \mathsf{FO}$ *when the degree is bounded*

It's all about burying the lead : given

$$\mathcal{A}_0 \equiv^{\mathsf{FO}} \mathcal{A}_1 \,,$$

we construct $<_0, <_1$ such that

$$(\mathcal{A}_0, <_0) \equiv^{\mathsf{FO}^2} (\mathcal{A}_1, <_1) \,.$$

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That way, given $\varphi \in <$-inv FO^2,

$$\mathcal{A}_0 \models \varphi \leftrightarrow (\mathcal{A}_0, <_0) \models \varphi \leftrightarrow (\mathcal{A}_1, <_1) \models \varphi \leftrightarrow \mathcal{A}_1 \models \varphi \,,$$

and $<$-inv FO^2 cannot make a distinction when FO can't.

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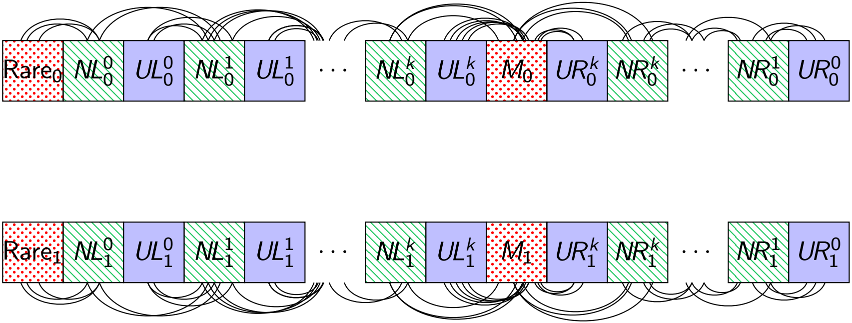
- **universal** (U) segments, realizing every possible ordered neighborhood
- **neighbor** (N) segments, to slow the traversal



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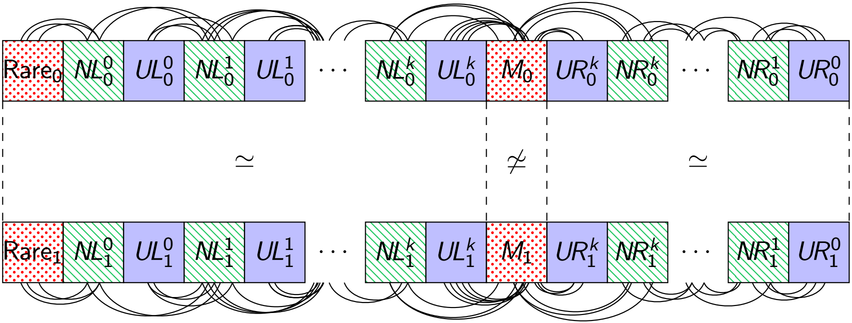

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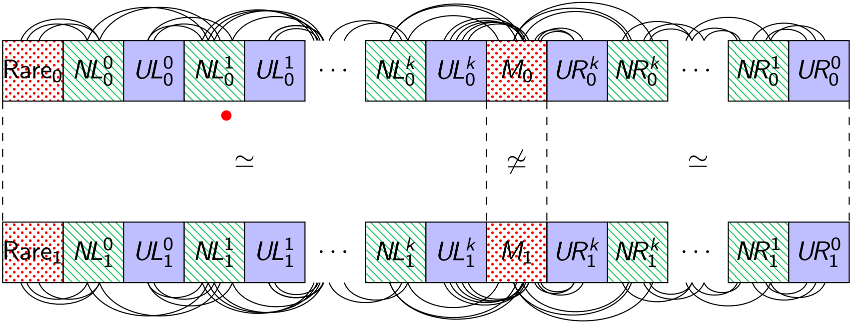

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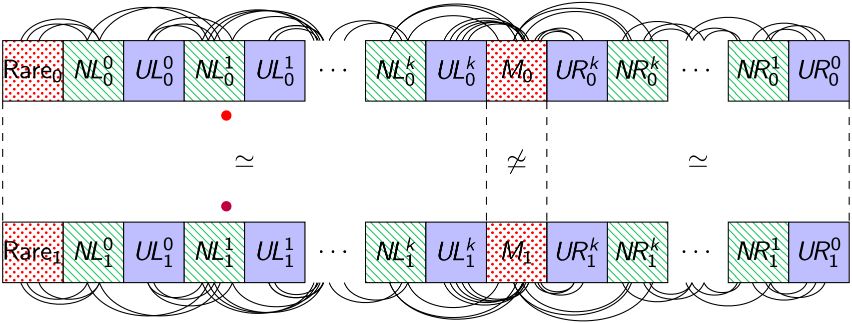

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- tit-for-tat on $\text{Rare} \cup \bigcup_{j \leq i} \text{NL/R}^j \cup \text{UL/R}^j$

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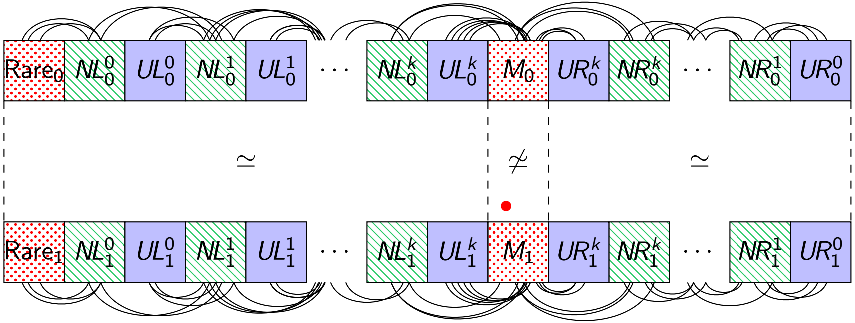

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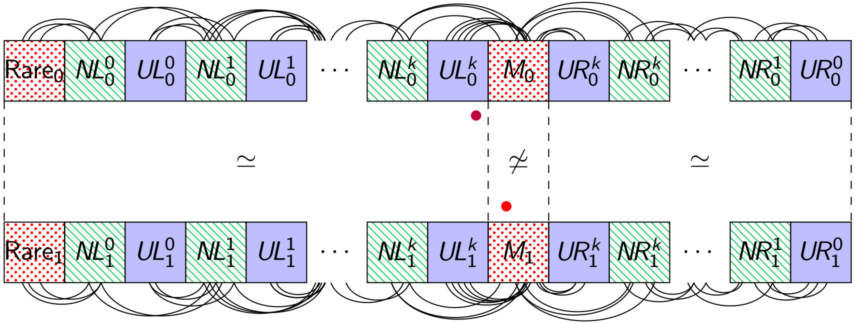

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- tit-for-tat on $\text{Rare} \cup \bigcup_{j \leq i} \text{NL/R}^j \cup \text{UL/R}^j$
- play in some UL/R^j otherwise

Future work

About $<$-inv FO^2 :

- Exact expressive power of $<$-inv FO^2 when the degree is bounded ?
- Does the inclusion $<$-inv $FO^2 \subseteq FO$ hold in general ? If not, find a separating example.

About $<$-inv FO :

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