An Introduction to invariant logics

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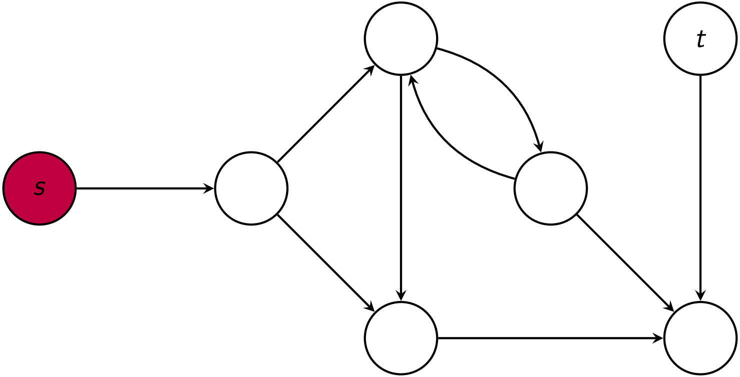
Reach(s,t) via BFS



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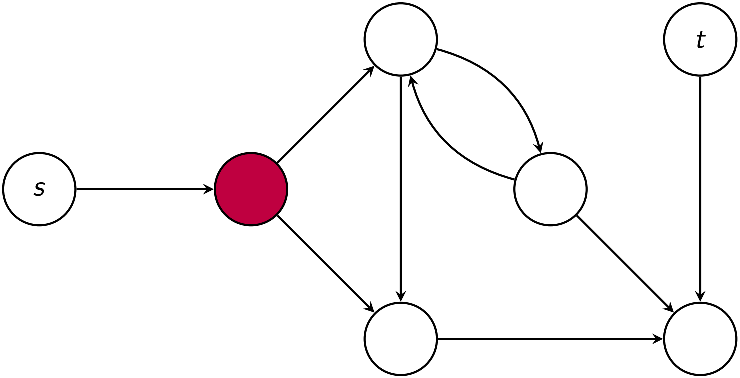

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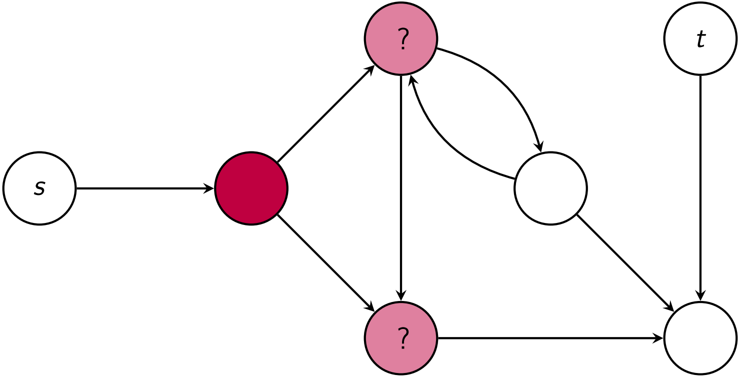

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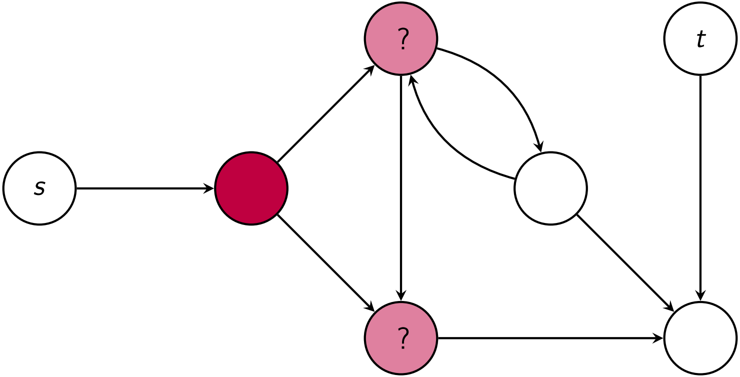

- An order on vertices lets us make choices
- The end result does not depend on the order

Definition of $<$-inv $\mathcal{L}$

$\varphi \in \mathcal{L}(\Sigma, <)$ is **order-invariant** over a finite Σ-structure $\mathcal{A}$ if

$$\forall <_1, <_2, \qquad (\mathcal{A}, <_1) \models \varphi \quad \leftrightarrow \quad (\mathcal{A}, <_2) \models \varphi$$

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Proof. Add $U^{(1)}$ to Σ, and reduce sentence $\varphi \in \mathcal{L}$ to

$$\widehat{\varphi} := U(\texttt{first}) \wedge \varphi\,.$$

Then $\widehat{\varphi} \in <$-inv $\mathcal{L}$ iff φ has no model of size at least two. □

Expressive power

- If $\mathcal{L}$ (e.g. $\exists$SO) can define an order, $<$-inv $\mathcal{L} = \mathcal{L}$.
- For $\mathcal{L} \in \{\mathsf{MSO}, \mathsf{LFP}\}$,

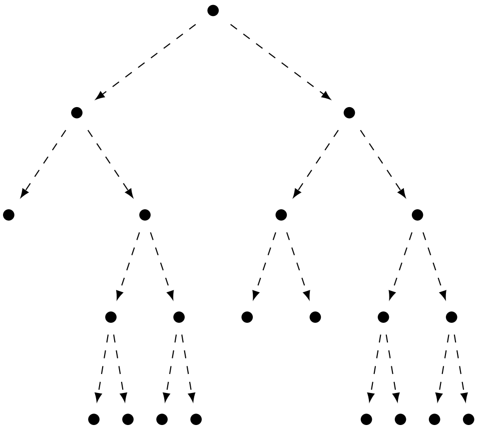
$$\begin{cases} \texttt{even\_size} \in <\text{-inv } \mathcal{L} \\ \texttt{even\_size} \notin \mathcal{L} \end{cases}$$

- What about $<$-inv FO ?

same_parity : All the branches have the same length parity

Complete unordered binary tree,
with the descendant relation.

$(aa)^\star \notin$ FO :
same_parity $\notin$ FO



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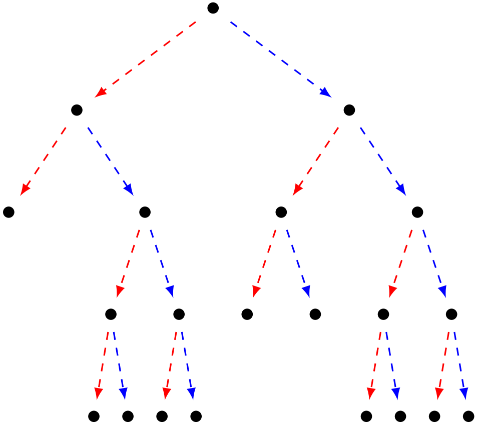
$({\color{red}l}{\color{blue}r})^\star \in$ FO :



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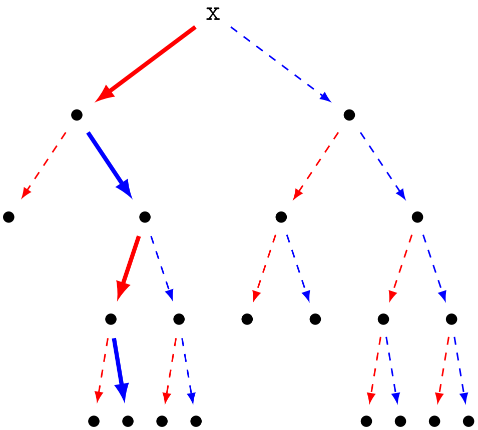

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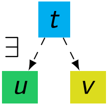
iff $\exists$ 

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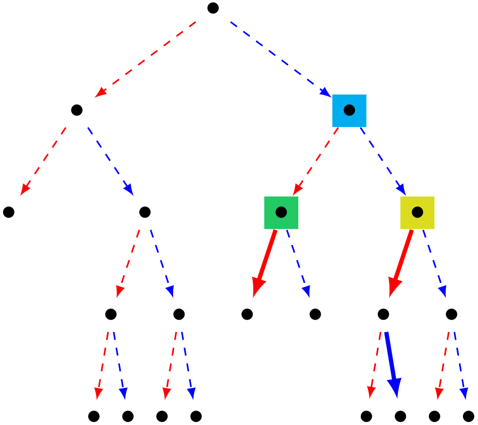

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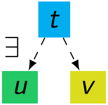
iff $\exists$ 

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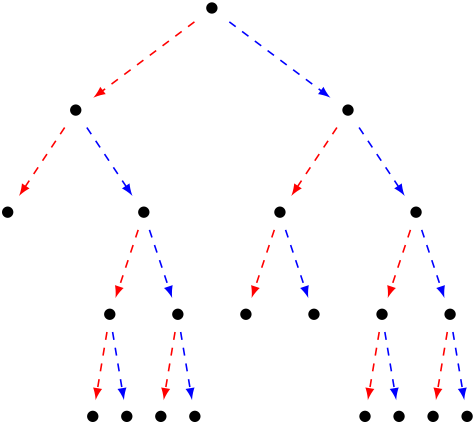
same_parity $\in$ $<$-inv FO



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Beyond order-invariance

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$+$-inv $\mathcal{L}$: addition on an initial segment of $\mathbb{N}$

Arb-inv $\mathcal{L}$: arbitrary numerical predicates

Descriptive complexity

Theorem (Fagin '74)

$\exists$SO *describes* NP

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Theorem (Makowsky '97)

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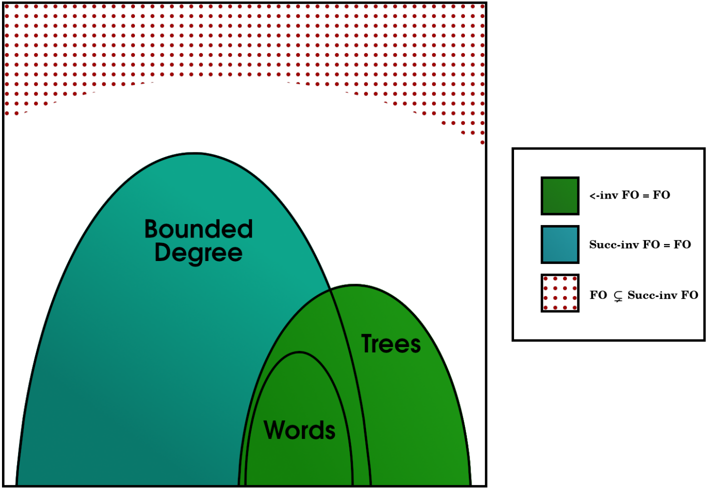
Main lines of research :

- find decidable syntax for invariant logics
- understand their expressive power

Zoom on FO (expressive power)



Gurevich, [Rossman '07], [Benedikt, Segoufin '09], [G. '20]

Zoom on FO (locality)

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Theorem (Anderson et al. '11)

Arb-inv FO *is local with a polylogarithmic radius.*

Brains needed !

Some interesting open questions :

- Which density is needed for $<$-inv FO/Succ-inv FO to go beyond FO ?
- Is $+$-inv FO local with constant radius ?

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Thanks !