Betweenness in order-theoretic trees

Intermédiarité dans les structures arborescentes dénombrables

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Betweenness (in this talk) :

B(x,y,z) : $\Leftrightarrow$ y is between x and z.

in linear orders, in trees,

in *order-theoretic trees* : partial orders such that the elements larger than any one are linearly ordered.

In *join-trees* : those where two elements have a least upper-bound, called their *join*.

In *topological trees* : trees of straight lines in the plane.

Except topological trees, all structures are countable.

Betweenness has also been studied in partial orders, and in graphs with respect to shortest paths or to induced paths.

Objectives : axiomatizations in first-order (FO) or monadic second- order (MSO) logic of several betweenness relations in order-theoretic trees.

Initial motivation and previous works :

Rank-width of countable graphs $\rightarrow$ Quasi-trees (JCT B 2017)

Algebraic and MSO characterizations of join-trees (LMCS 2017)

Rank-width of countable graphs

Rank-width is a complexity measure of finite graphs defined by Oum and Seymour, based on a layout of graph G : a tree with nodes of degree 1 (leaves) or 3 ; nodes are at the leaves.

Each edge e has a weight defined as the rank of the matrix of adjacencies between the nodes in the two subtrees separated by e.

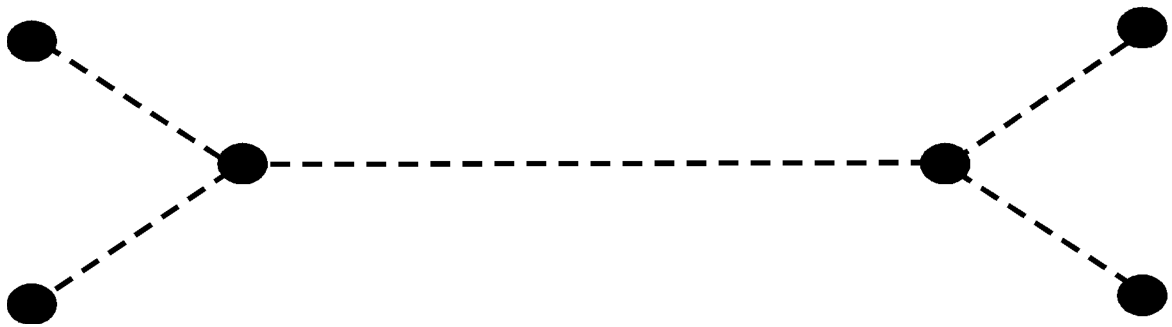
The weight of the tree is the maximum weight of an edge and the rank-width rk(G) is the minimum weight of a layout. If G is an induced subgraph of H then :

$$\mathrm{rk}(G) \leq \mathrm{rk}(H).$$

For a countable graph G, the rank-width rk(G) is the least upper-bound of the rank-widths of its finite induced subgraphs,

but only if we use generalized trees called quasi-trees where the unique "path" between two nodes may be infinite. They are to infinite trees what **Q** is to **Z**

Here is a quasi tree. There is no notion of edge or neighbour node, as in **Q** , there is no successor. The "path" between two nodes may be a dense linear order: dashed lines on picture.



Betweenness is a ternary relation

Linear orders : B(x,y,z) :⇔ x<y<z *or* z<y<x

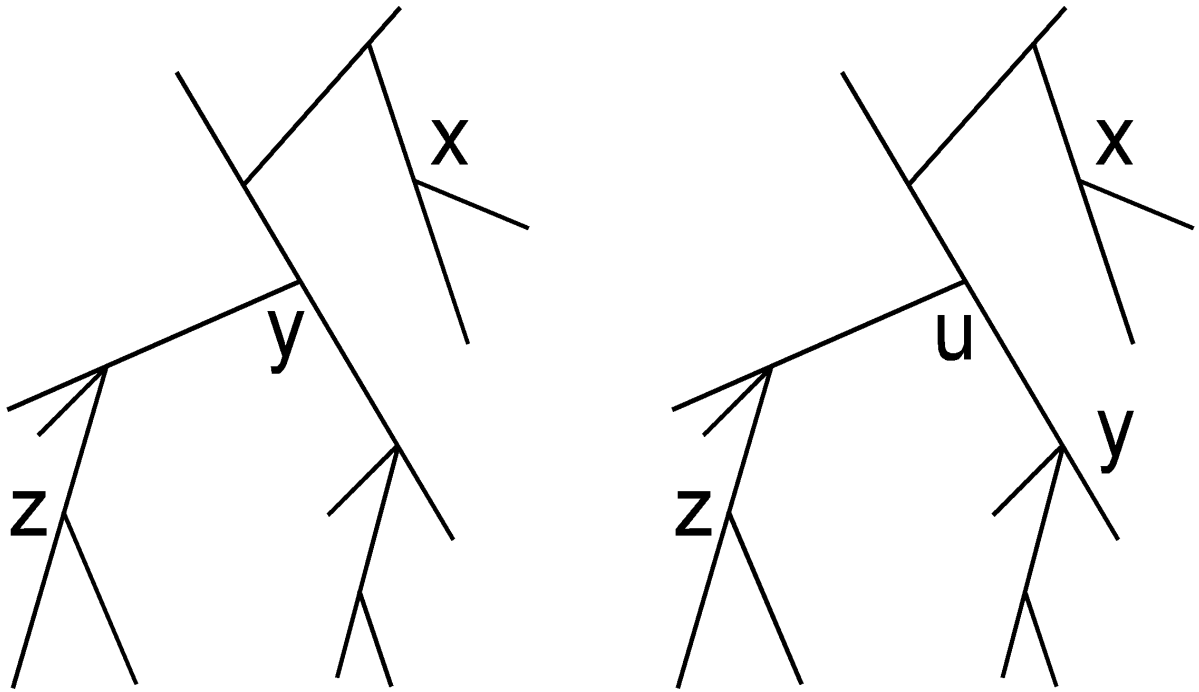
Trees : B(x,y,z) :⇔ y is on the unique path between x and z.

Proposition : Betweenness is axiomatized in finite trees by the following conditions :

$$\begin{aligned}
&\mathrm{A1}: B(x,y,z) \Rightarrow x \neq y \neq z \neq x.\\
&\mathrm{A2}: B(x,y,z) \Rightarrow B(z,y,x).\\
&\mathrm{A3}: B(x,y,z) \Rightarrow \neg B(x,z,y).\\
&\mathrm{A4}: B(x,y,z) \wedge B(y,z,u) \Rightarrow B(x,y,u) \wedge B(x,z,u).\\
&\mathrm{A5}: B(x,y,z) \wedge B(x,u,y) \Rightarrow B(x,u,z) \wedge B(u,y,z).\\
&\mathrm{A6}: B(x,y,z) \wedge B(x,u,z) \Rightarrow\\
&y = u \vee (B(x,u,y) \wedge B(u,y,z)) \vee (B(x,y,u) \wedge B(y,u,z)).\\
&\mathrm{A7}: x \neq y \neq z \neq x \Rightarrow\\
&B(x,y,z) \vee B(x,z,y) \vee B(y,x,z) \vee (\exists u.B(x,u,y) \wedge B(y,u,z) \wedge B(x,u,z)).
\end{aligned}$$

For linear orders : Replace A7 by A7' defined as A7 without $\exists$ u.

The meaning of A7 :



In a rooted tree $(N,\leq)$ where $\leq$ is the ancestor relation :

$$B(x,y,z) \Leftrightarrow ((x < y \leq x \vee z) \;\&\; y \neq z) \text{ or } ((z < y \leq x \vee z) \;\&\; y \neq x)$$

where $\vee$ denotes the join (least common ancestor)

Order-theoretic tree

Definition : A partial order $T=(N,\leq)$ such that, for each x, the set $\{y \;/\; y \geq x\}$ is linearly ordered. If it has a maximal element this element is the root. T is a join-tree if any two elements have a least upper-bound, called their join, denoted by $\vee$.

Betweenness in a join-tree is *defined* as it is characterized in rooted trees:

$$B(x,y,z) \Leftrightarrow ((x < y \leq x \vee z) \;\&\; y \neq z) \text{ or } ((z < y \leq x \vee z) \;\&\; y \neq x)$$

Definition : A quasi-tree (QT) is a ternary structure S = (N,B) that satisfies properties A1-A7. Finite ones are just trees by previous proposition.

Theorem : S = (N,B) is a quasi-tree if and only if it is the betweenness relation of a join-tree.

Proof : Let S=(N,B) . Choose a root r in N (any) and define :

$$x \leq y \quad :\Leftrightarrow \quad x = y \text{ or } y = r \text{ or } B(x,y,r).$$

If S satisfies A1-A6, then (N, $\leq$) is an order-theoretic tree with root r

If S satisfies A1-A7, then (N, $\leq$) is a join-tree, and B is its betweenness relation. The existing node u in A7 defines the join.

In an order-theoretic tree $(N,\leq)$ ($\leq$ is the ancestor relation),

we *define* betweenness :

$B(x,y,z) :\Leftrightarrow ((x < y \leq x \vee z)$ & $y \neq z)$ *or* $((z < y \leq x \vee z)$ & $y \neq x)$

$\vee$ denotes the join (least common ancestor) that *may not exist.*

If $x \vee z$ does not exist, $B(x,y,z)$ holds for no y.

We have two classes of infinite betweenness relations :

QT : quasi-trees, i.e. betweenness in join-trees, and

BO : betweenness in *order-theoretic trees* (O-trees).

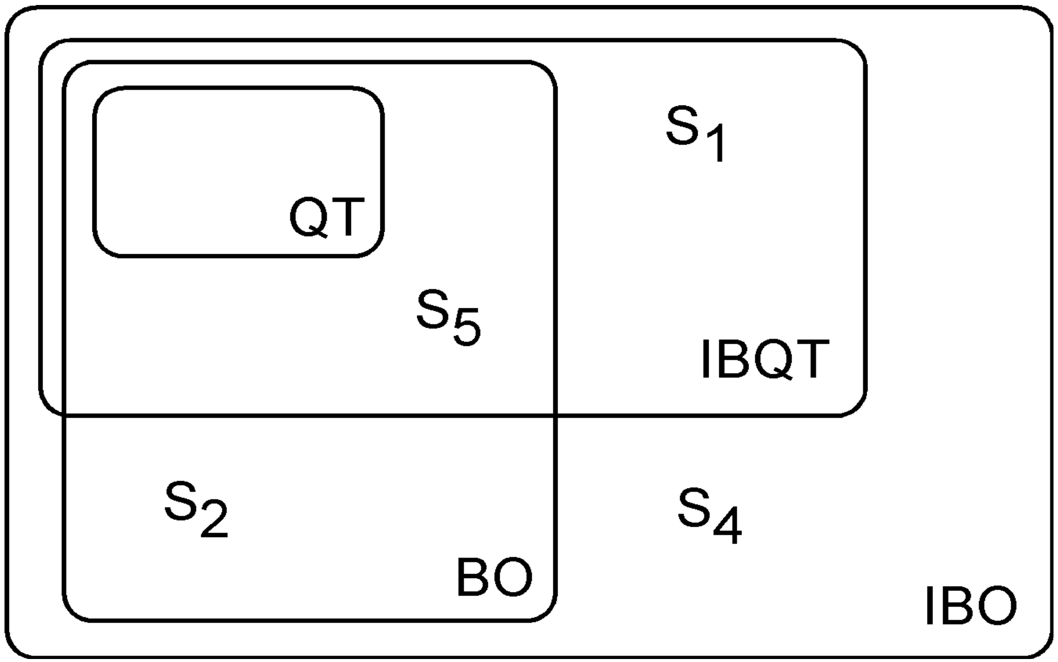
Induced betweenness :

If S = (N,B) is in **QT** or **BO**, an induced bewteenness is

S[X] := (X, B[X]) where $X \subseteq N$;

S[X] is respectively in **IBQT** or **IBO**.

Four classes are related as follows :



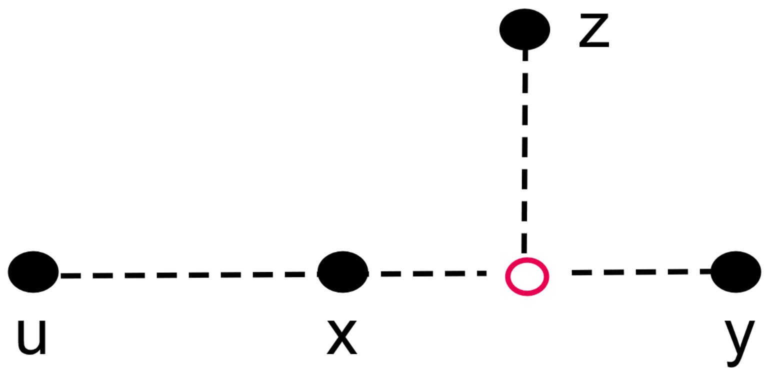
Induced betweenness in quasi-trees, equivalently, in join-trees.

A7 is no longer valid, but we have :

A8 : $\neg A(x,y,z)$ & $B(u,x,y) \Rightarrow B(u,x,z)$

where $A(x,y,z)$ means ; $B(x,y,z)$ or $B(y,x,z)$ or $B(x,z,y)$,

i.e., x, y, z are on a line, in any order.



Theorem : A1-A6 and A8 axiomatize the class **IBQT**.

Proof : Let S=(N,B) satisfy A1-A6, A8. Choose a root r $\in$ N (any). Then define, as for **QT** :

$$x \leq y :\Leftrightarrow x = y \text{ or } y = r \text{ or } B(x,y,r).$$

A1-A6 and A8 are universal, hence satisfied in induced substructures.

S satisfies A1-A6 $\Rightarrow$ T=(N, $\leq$) is an order-theoretic tree (O-tree) with root r.

We must expand T into a join-tree W = (N $\cup$ M, $\leq$) such that

$B = B_W[N]$ where B_W is the betweenness of W.

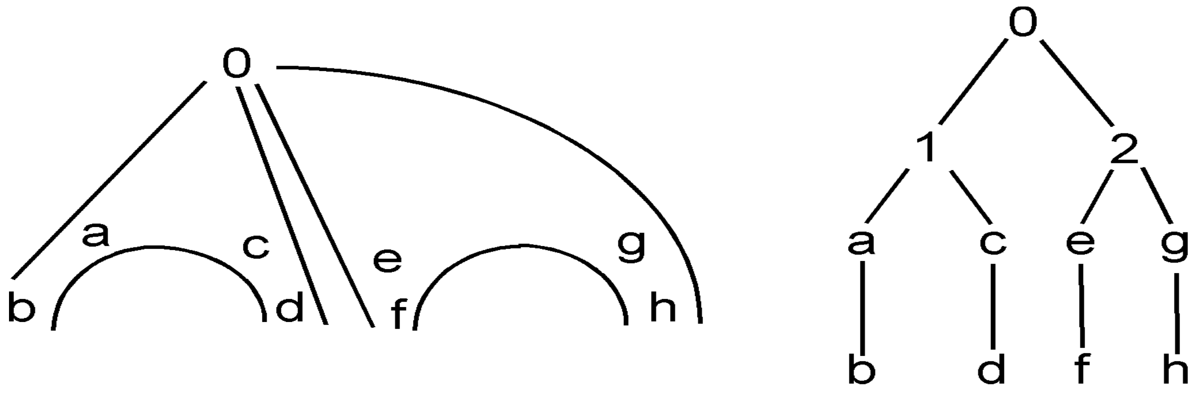
Next slide shows an example.

Let S in (a) : We have B(0,a,b), B(0,c,d) , B(0,e,f), B(0,g,h)

B+(b,a,c,d), B+(f,e,g, h) (avoiding 0)

B+(b,a,0,e,f), B+(b,a,0, g,h), B+(d,c,0,e,f), B+(d,c,0,g,h),

B+(x,y,z,t,u) means : B(x,y,z) & B(y,z,t) & B(z,t,u).



(a) (b)

The chosen r is 0. The added nodes 1 and 2 prevent B+(b,a,0,c,d), B+(f,e,0,g,h), but B_W +(b,a,1,c,d) and B_W +(f,e,2,g,h) (join-tree W in (b))

We have B+(b,a,c,d) and B+(f,e,g, h) in the restriction to N.

Directions relative to a line L : To identify the nodes to be added

Let $L = L(x,y)$ = the nodes $> \{x,y\}$ in T , x and y are incomparable.

$u < L$ and $v < L$ are in the *same direction* relative to L if :

$u < w < L$ and $v < w < L$ for some w.

This is an equivalence relation. Its classes are the *directions* of L.

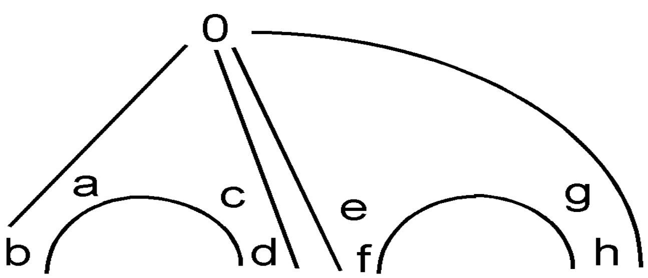
Lemma : A1-A6 $\Rightarrow$ If $m \in L$ and D,D' are directions relative to L,

then, for all $u,u' \in D$ and $v,v' \in D'$: $B(u, m, v) \Leftrightarrow B(u', m, v')$

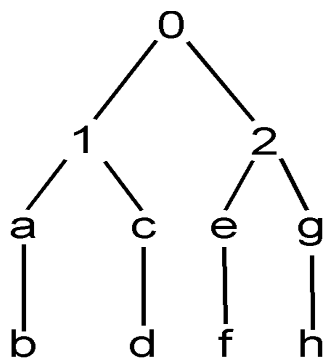
We can write $B(D,m,D')$.

Lemma : A1-A6 , A8 $\Rightarrow$ for fixed L, $m \in L$, the relation on directions

defined by $\neg B(D,m,D')$ is an equivalence denoted by $\approx$



(a)



(b)

Example : Directions relative to {0} are {a,b}, {c,d}, {e,f}, {g,h}. We have {a,b} $\approx$ {c,d} and {e,f } $\approx$ {g,h}.

Final proof : To build W, we add a common upper-bound to unions of equivalent directions. We obtain a join-tree W as desired.

Example : We add 1 for {a,b,c,d} and 2 for {e,f,g,h}.

Remark 1 :

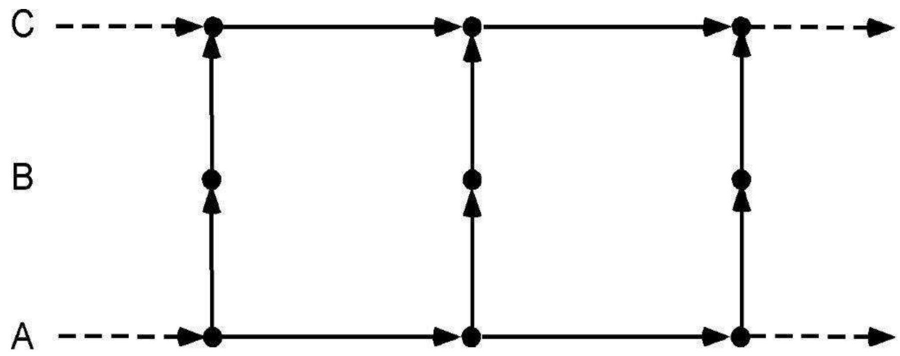
Structures in the classes **QT** and **IBQT** are « unoriented » :

Any node r can be chosen as root in the constructions.

This will not be the same for the next two betweenness relations in order-theoretic trees (O-trees).

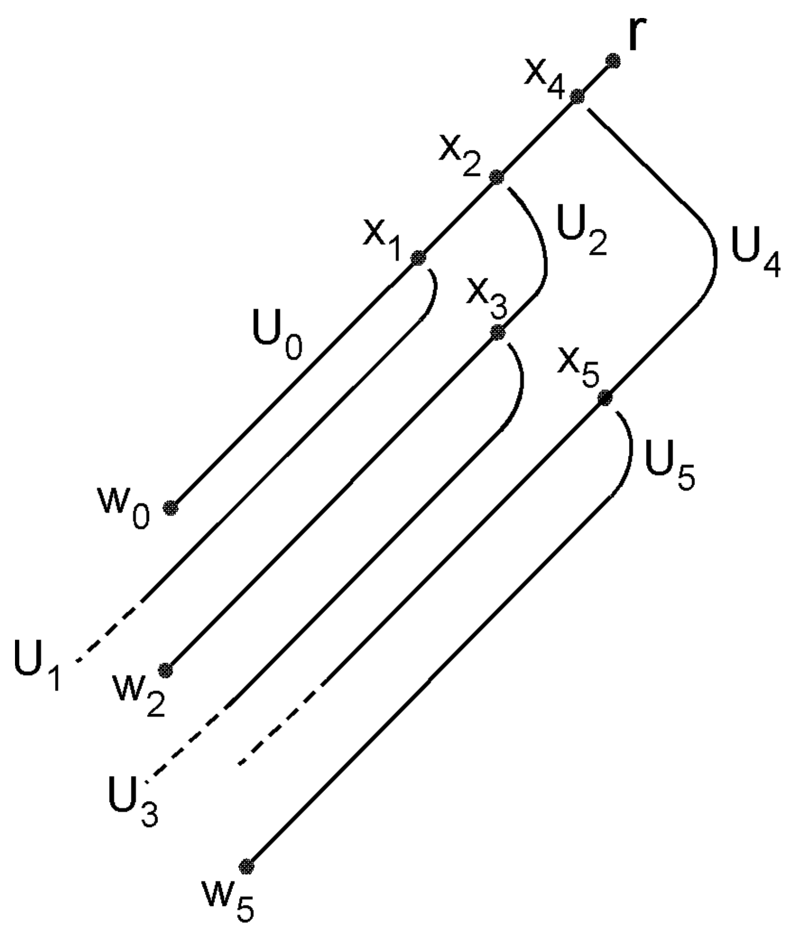
Remark 2 : If a class $\mathcal{C}$ of relational structures is FO axiomatizable, the class Ind($\mathcal{C}$) of its induced substructures is *not necessarily* FO (or even MSO) axiomatizable.

Example: An FO sentence can describe unions of infinite ladders and rings based on the following pattern where each vertex is labelled by A, B or C.



If the induced substructures are FO definable, those with one rectangle are MSO definable. But No : one cannot check in MSO equal lengths of the A- and C- paths.

Remark 3 : The transformation of S into W is a monadic second-order transduction : the set of nodes M to be added is MSO definable by using a notion of structuring of order-theoretic trees.



Any $L(x,y)$ is $L^+(z)$ for some z.

Here $L(x_1,w_2) = L^+(x_3)$. Each union of equivalent directions is specified by a single node, *not a triple of nodes.*

Topological tree :

Definition : A connected union **L** of countably many straight half-lines that dos not contain homeomorphic images of circles.

Every two points are linked by a unique path, homeomorphic image of the real interval [0,1].

Betweenness (yet another notion !) :

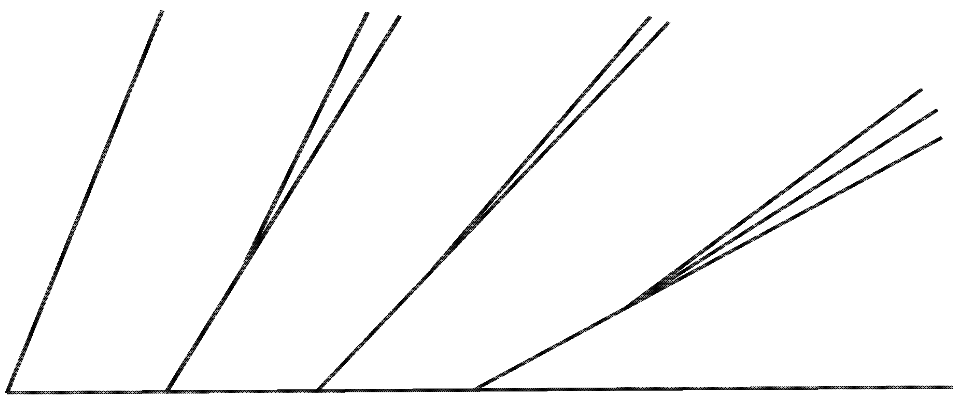
B**L**(x,y,z) : y is on the unique path between x and z.

Proposition : S=(N,B) is in **IBQT** $\Leftrightarrow$ B is B**L**[N] for a countable subset of a topological tree **L**.

Proposition : Every join-tree can be embedded into a tree of lines

Main observation : If L and K are straight half-lines with same origin O, one can draw inside the sector they define countably many half-lines with origin O.

Proof idea : If the angle between L and K is α, we choose angle $\alpha/2n$ between consecutive lines L_n and L_{n+1}.

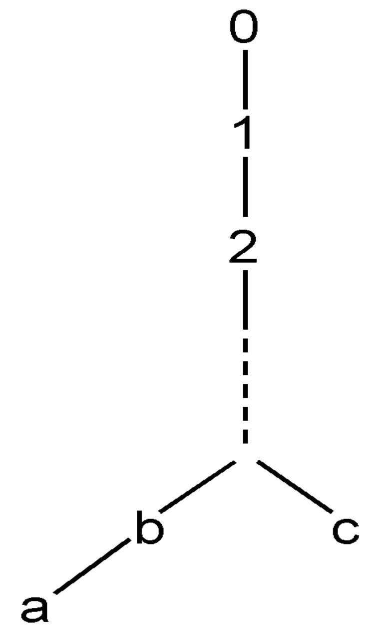


Betweenness in order-theoretical trees (O-trees)

This notion depends on the « orientation » : changing the root may change betweenness.

For the O-tree (b), we have neither B(a,b,c) nor B(a,c,0).

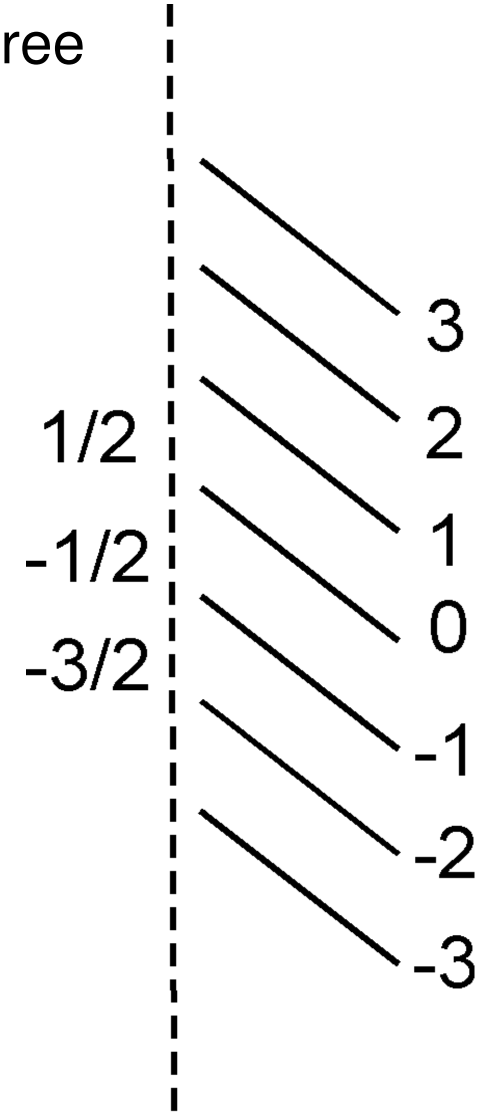
By taking c as root, we have them in the betweenness relation..



(b)

Other example : The betweenness of this O-tree is not defined from any rooted order-theoretic tree.

We have **Q**-**Z** on the main dense branch .



Proposition : The class **BO** (betweenness in O-trees) is MSO axiomatizable.

Case 1 : S=(N,B) defined from a rooted O-tree. One guesses a root r, one defines T(S,r) as before and one checks that its betweenness is the given relation B.

All this with FO formulas.

Case 2 : S=(N,B) defined from an O-tree but not a rooted one.

To guess an O-tree, we choose a maximal line in S (a set L satisfying A1-A7' that is maximal for inclusion) and a, b $\in$ L.

There is a unique linear order on L such that a < b and whose betweenness is B[L]. It is quantifier-free definable.

Then, assume there is an adequate O-tree $T=(N, \leq)$, and L is a maximal line in T that is upwards closed.

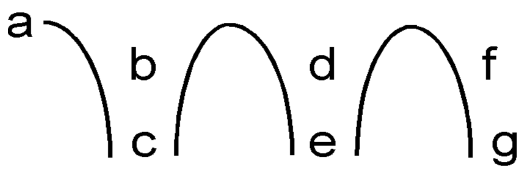
Let a, b $\in$ L with a<b. Then the order < is FO definable in S=(N,B) in terms of L, a and b.

We get an O-tree $T=(N,\leq)$ defined from L, a, b.

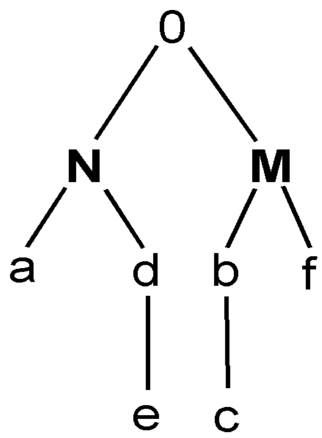
It remains to check that its betweenness is the given relation B.

Easy in FO. But MSO is needed for choosing L.

Induced betweenness in O-trees.



(*a*)



(*b*)

Structure (a) is not in **IBO** (long proof with case analysis).

Without node g, it is ; it is defined from the O-tree (b) where **N,M** are infinite decreasing chains.

Conjecture : The class **IBO** is MSO axiomatizable.

Related facts and future work

Conjecture : Induced betweenness in O-trees is MSO axiomatizable.

Article in progress : Algebraic and MSO characterizations of O-trees.

The case of join-trees is studied in : *Algebraic and logical descriptions of generalized trees. Logical Methods in Computer Science,* 13 (2017). Join-trees and O-trees can be generated by finitely many operations via infinite terms. The regular such terms define exactly the MSO-definable join-trees and O-trees.

Betweenness in partial orders is axiomatizable by a countable set of FO sentences (Lihova, 2000). For *finite* partial orders, it is by a single MSO sentence. Extension to infinite partial order is unsolved.

Betweenness in graphs has been studied by many authors: Chvatal, Mulder, Nebesky and many others.

The case of directed graphs has not been much considered (to my knowledge).