Reductions of Hilbert's 10^{th} problem to undecidable fragments of set theory

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Abstract

Various fragments of ZF—the Zermelo-Fraenkel first-order set theory with regularity axiom—are shown to have *un*decidable satisfiability problems. To wit, if Φ is among those sublanguages of ZF, no algorithm can establish whether or not any given formula ψ in Φ becomes true under suitable assignments of sets to its free variables.

For each Φ taken into account, the undecidability result stems from a uniform translation method which turns every instance $D = 0$ of Hilbert's 10^{th} problem into a formula ψ of Φ so that ψ is satisfiable if and only if the polynomial equation $D = 0$ has an integer solution. Through this translation, the algorithmic unsolvability of Hilbert's 10^{th} problem carries over to the satisfiability problem for Φ.

Some of the undecidable Φ's are slight extensions of a core language consisting of all conjunctions of literals of the forms $x = y \cup z$, $x = y \otimes z$, $x \cap y = \varnothing$, and $|x| = |y|$, where x, y, z stand for variables, $y \otimes z$ is a variant of Cartesian product consisting of singletons and unordered doubletons, and $|x| = |y|$ designates equinumerosity between x and y. Additional conjuncts entering into play can be, e.g.: one literal of the form $\mathsf{Finite}(x)$, stating that x has finitely many elements, taken along with one literal of the form $x \neq \varnothing$. Another option would be to extend the said syntactic core by allowing three *negated* equinumerosity literals of the form $|x| \neq |y|$ to appear in the conjunction. Our interest in these, and similar, slim undecidable set-theoretic languages is increased by the fact that they lie extremely

close to fragments of ZF which are either known, or strongly conjectured, to be decidable.

One can recast the undecidable fragments of ZF under study in a set-theoretic language entirely devoid of function symbols (such as $\cup, \otimes$, etc.), which only involves the relators $\in$ and $=$, propositional connectives, and also the constructs $\forall\, x \in y\ \varphi$ and $\exists\, x \in y\ \varphi$ involving bounded quantifiers, subject to a very restrained use. The fact that somewhat sophisticated notions like "being an ordinal", "being the first limit ordinal ω", "having a finite cardinality", "being a hereditarily finite set", can be specified inside this collection of formulae, dubbed $(\forall\exists)_0$ formulae, gives evidence of the high expressive power of bounded quantification in the context of ZF.

In a full-fledged language for set theory, $(\forall\exists)_0$ specifications are only seldom used in order to define mathematical notions. Hence, to support the correctness of the proposed $(\forall\exists)_0$-specifications of ω, equinumerosity, and finitude, we have proved that they are equivalent to more direct and practical characterizations of the same notions. As we report, this formal accomplishment was carried out with the aid of a proof-checker embodying a computational version of ZF.

References

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