Satisfaction classes with the full collection scheme

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Satisfaction classes are subsets of models of Peano arithmetic which satisfy Tarski's compositional clauses. Alternatively, we can view a satisfaction class as an extension of a certain fresh predicate $S(x)$ which satisfies certain axiomatic theory, postulating precisely that S satisfies compositionality conditions. We call this theory of compositional satisfaction CS^- in analogy to the theory of compositiona truth CT^-, investigated in the field of the axiomatic theories of truth.

It is easy to see that CS^- extended with a full induction scheme is not conservative over PA, since it can prove, for instance, the uniform reflection over arithmetic. By a nontrivial argument of Kotlarski, Krajewski, and Lachlan, the sole compositional axioms of CS^- in fact form a conservative extension of PA. Moreover, in order to obtain non-conservativity it is enough to add induction axioms for the Δ_0 formulae contaning the truth predicate.

Answering a question of Kaye, we will show that the theory of compositional satisfaction, CS^- with the full collection scheme is a conservative extension of Peano Arithmetic. Following the initial suggestion of Kaye, we will in fact show that any countable recursively saturated model M of PA has an elementary ω_1-like end extension M' such that M' carries a full satisfaction class. This in turn will rely on our main technical result that and countable model of CT^- with a very limited amount of induction, called internal induction, can be end=extended to a model of the same theory.