Arithmetical theories and the automation of induction

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This talk is about the relationship between (weak) arithmetical theories and methods for automated inductive theorem proving. Automating the search for proofs by induction is an imporant topic in computer science with a history that stretches back decades. A variety of different approaches, algorithms and systems has been developed.

Connections between these two areas can be established in the following way: a method M for automatically finding proofs by induction is analysed by finding a theory T s.t. every theorem provable by M is also provable in T. This yields an upper bound on, and sometimes even a characterisation of, the strength of the method M. The primary application of such a connection is to carry over unprovability results: if σ is a sentence s.t. $T \nvdash \sigma$, then $M \nvdash \sigma$.

Usually, there are trivial choices for T and σ: since most methods for the automation of induction deal with (very) weak fragments of Peano arithmetic (PA), one can often choose $T = \mathrm{PA}$ and, e.g., $\sigma = \mathrm{Con}_{\mathrm{PA}}$. However, we are interested in *practically meaningful unprovability results*, i.e., in such sentences σ which are sufficiently simple so that in computer science they would be considered as being in the scope of automated inductive theorem proving. In order to obtain such unprovability results it is necessary to tighten the upper bound T on M considerably which leads to the consideration of weaker arithmetical theories.

In this talk I will describe which arithmetical theories and what kind of unprovability results are relevant for this endeavour. In particular, I will talk about induction on literals, i.e., atoms and negated atoms. In [2] we have shown that induction on literals covers some methods implemented in the automated theorem prover Vampire [4] and obtained unprovability results for simple statements such as "every number is even or odd". I will also talk about $\exists_1^-$-induction which, as shown in [1], is related to clause set cycles. A variant of clause set cycles underlies the n-clause calculus [3]. I will present unprovability results for simple formulas such as $x + 0 = x + x \rightarrow x = 0$ for $\exists_1^-$ induction and hence for clause set cycles.

I will mention a number of open problems and conjectures arising from this connection between arithmetical theories and computer science.

References

[1] Stefan Hetzl and Jannik Vierling. Unprovability results for clause set cycles. *Theoretical Computer Science*, 935:21–46, 2022.

[2] Stefan Hetzl and Jannik Vierling. Induction and Skolemization in saturation theorem proving. *Annals of Pure and Applied Logic*, 174(1):103167, 2023.

[3] Abdelkader Kersani and Nicolas Peltier. Combining superposition and induction: A practical realization. In Pascal Fontaine, Christophe Ringeissen, and Renate A. Schmidt, editors, *9th International Symposium on Frontiers of Combining Systems (FroCoS)*, volume 8152 of *Lecture Notes in Computer Science*, pages 7–22. Springer, 2013.

[4] Giles Reger and Andrei Voronkov. Induction in saturation-based proof search. In Pascal Fontaine, editor, *27th International Conference on Automated Deduction (CADE)*, volume 11716 of *Lecture Notes in Computer Science*, pages 477–494. Springer, 2019.