Bounded recursion and Δ_0 definability

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A function f is defined by bounded recursion in such a way:

$$\begin{cases} \bar{f}(\vec{x},0) = g(\vec{x}) \\ \bar{f}(\vec{x},i+1) = h(\vec{x},\bar{f}(\vec{x},i),i) \\ \bar{f}(\vec{x},y) \leq \beta(\vec{x},y) \\ f(\vec{x}) = \bar{f}(\vec{x},\lambda(\vec{x})) \end{cases}$$

The log-shortened counting function $f_R^{ls}(\vec{x}) = Card\{i < lh_2(x_0); R(\vec{x},i)\}$ is so defined with $g(\vec{x}) = 0$, $h(\vec{x},z,i) = z + \chi_R(\vec{x},i)$, $\beta(\vec{x},y) = x_0$ and $\lambda(\vec{x}) = lh_2(x_0)$ is the length of the binary expansion of x_0 (so that $lh_2(0) = 1$).

A **major open question** is the following: *suppose that the graphs of g and h are Δ_0-definable, and $\beta(\vec{x},y)$ is a polynomial and $\lambda(\vec{x}) = x_0$. Is the graph of f Δ_0-definable?*

A few partial results are available (see [2]). The following one is provable by use of standard coding of the history of the recursive computation:
Theorem. (Woods 1981). Suppose that the relation R is Δ_0-definable. The graph of f_R^{ls} is Δ_0-definable.

It has some corollaries concerning sums $\sum_{i=0}^{i=lh_2(x_0)} \varphi(\vec{x},i)$ or products $\prod_{i=0}^{i=lh_2(x_0)} \varphi(\vec{x},i)$ which are Δ_0-definable under reasonnable assumptions (see [1], [3], [4]).

Here we extend this result to families of functions recursively defined with transition functions h that are Δ_0-piecewise linear and $\lambda(\vec{x}) = lh_2(x_0)$. This study leads to focus on recursion where the index i appears through the corresponding binary digit $(\bar{x})_i$ of one *control* parameter x:

$$\begin{cases} \bar{F}(\vec{u},0) = g(\vec{u}) \\ \bar{F}(\vec{u},x,i+1) = H(\vec{u},(\bar{x})_i,\bar{F}(\vec{u},x,i)) \\ F(\vec{u},x,y) = \bar{F}(\vec{u},x,lh_2(y)) \end{cases}$$

[1] A. Berarducci, P. d'Aquino, Δ_0-*complexity of the relation* $y = \prod_{i<n} F(i)$, Annals of Pure and Applied Logic, 75, 1995, pp. 49-56
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[3] J.B. Paris, A.J. Wilkie and A.R. Woods, *Provability of the pigeonhole principle and the existence of infinitely many primes*, J. Symbolic Logic 53 (1988) 1235-1244
[4] A. Woods, *Some problems in logic and number theory and their connections*, Ph.D. thesis, University of Manchester, Manchester, 1981. Studies in Weak Arithmetics, Volume 2, Edited by P. Cégielski, C. Cornaros, and C. Dimitracopoulos, CSLI Lecture Notes 211, Stanford 2013