Absolute Undefinability in Arithmetic

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A program?

Problem

When is a countable nonstandard model of ... expandable to a model of ..., and if there is an expansion, how hard is it to find it?

We will consider expansions of

1. models of $\mathrm{Th}(\mathbb{N}, S)$ to models of $\mathrm{Th}(\mathbb{N}, <)$, where S is a successor relation;
2. models $\mathrm{Th}(\mathbb{N}, <)$ to models of Presburger arithmetic Pr;
3. models of Pr to models of PA;
4. models of PA to models axiomatic theories of truth or satisfaction.

Expansions of the standard model

Theorem

Let S be the successor relation in the set of natural numbers $\mathbb{N}$.

1. $(\mathbb{N}, S)$ *and* $(\mathbb{N}, <)$ *are* *minimal*, *i.e., every definable subset of* $\mathbb{N}$ *is either finite or cofinite.*
2. $(\mathbb{N}, <)$ *is a proper expansion of* $(\mathbb{N}, S)$
3. *Even numbers are definable in* $(\mathbb{N}, +)$; *hence,* $(\mathbb{N}, +)$ *is a proper expansion of* $(\mathbb{N}, <)$.

Theorem (Ginsburg-Spanier)

All subsets of $\mathbb{N}$ *that are definable in* $(\mathbb{N}, +)$ *are* *ultimately periodic*, *i,e., for each definable* X *there is a* p *such that for sufficiently large* x

$$x \in X \iff x + p \in X.$$

Corollary

Squares are definable in $(\mathbb{N}, \times)$; *hence* $(\mathbb{N}, +, \times)$ *is proper expansion* $(\mathbb{N}, +)$.

Multiplication is not definable from addition

Observation

S is not definable in $(\mathbb{N}, \times)$. There is $f \in \mathrm{Aut}(\mathbb{N}, \times)$ such that $(2) = 3$ and $f(3) = 2$. However,

$$x + y = z \Leftrightarrow (zx + 1)(zy + 1) = z^2(xy + 1) + 1.^a$$

Hence, $+$ is definable in $(\mathbb{N}, \times, S)$.

[a] Tarski-Robinson Identity. I found it in *Axiomatic (and Non-Axiomatic) Mathematics* by Saeed Salehi, Rocky Mountain Journal of Mathematics 52:4 (2022).

Truth and partial truth

Theorem (Tarski)

$\mathsf{Tr} = \{\ulcorner\varphi\urcorner : (\mathbb{N}, +, \times) \models \varphi\}$ *is undefinable. Hence* $(\mathbb{N}, +, \times, \mathsf{Tr})$ *is a proper expansion of* $(\mathbb{N}, +, \times)$*.*

Theorem (Kleene et al.)

For each n*,* $\mathsf{Tr}_n = \{\ulcorner\varphi\urcorner : \varphi \in \Sigma_n \& (\mathbb{N}, +, \times) \models \varphi\}$ *is definable in* $(\mathbb{N}, +, \times)$*.*

More expressive power: infinite conjunctions and disjunctions

Definition

$\mathcal{L}_{\omega_1,\omega}$ is an extension of $\mathcal{L}_{\omega,\omega}$ with one additional rule: if Φ is a countable set of formulas with a fixed finite number of free variables, then $\bigwedge \Phi$ and $\bigvee \Phi$ are formulas.

Example

Let $\varphi_0(x) = \forall y \neg S(y,x)$ and for all n, let $\varphi_{n+1}(x) = \exists y[\varphi_n(y) \wedge S(y,x)]$. Then, for every $X \subseteq \mathbb{N}$,

$$\{X = \{x : (\mathbb{N}, S) \models \bigvee_{n \in X} \varphi_n(x)\}.$$

In particular, addition is defined by

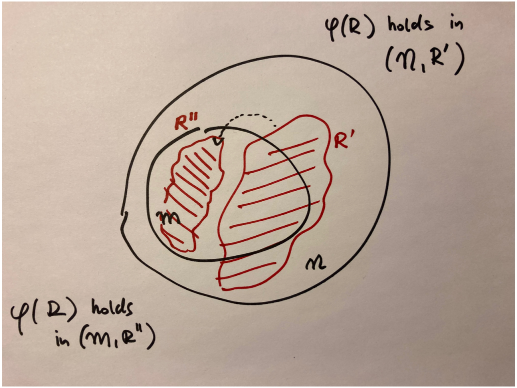
$$\bigvee\{\varphi_m(x) \wedge \varphi_n(y) \wedge \varphi_k(z) : m+n=k\}.$$

Example

$\mathsf{Tr}(x) = \bigvee\{\mathsf{Tr}_n(x) : n \in \mathbb{N}\}$.

Definition

A structure $\mathfrak{M}$ is resplendent if for any first-order sentence $\varphi(R)$ with a new relation symbol R, if $\mathfrak{M}$ has an elementary extension that is expandable to a model of $\varphi(R)$, then $\mathfrak{M}$ is expandable to a model of $\varphi(R)$.



Resplendence is relevant

Theorem (Presburger)

Satisfaction of additive reducts is definable in models of PA*; hence, if* $(M, +, \times)$ *is a nonstandard countable model of* PA*, then* $(M, +)$ *is resplendent.*

Theorem (Cegielski, Nadel)

Satisfaction of multiplicative reducts is definable in models of PA*; hence, if* $(M, +, \times)$ *is a nonstandard countable model of* PA*, then* $(M, \times)$ *is resplendent.*

Theorem (Kotlarski, Krajewski, Lachlan)

A countable nonstandard model of PA *carries a full satisfaction class if and only if it is resplendent.*

Counting automorphic images

Theorem (Scott)

For every countable structure $\mathfrak{M} = (M, \dots)$ and every $X \subseteq M^n$, t.f.a.e.

1. X *is preserved by all automorphisms of* $\mathfrak{M}$*, i.e.,* $f(X) = X$ *for every automorphism* f.
2. X *is* $\mathcal{L}_{\omega_1,\omega}$*-definable in* $\mathfrak{M}$.

Theorem (Kueker)

For every countable structure $\mathfrak{M} = (M, \dots)$ and every $R \subseteq M^n$, t.f.a.e.

1. R *has at most* $\aleph_0$ *automorphic images.*
2. R *has less than* $2^{\aleph_0}$ *automorphic images.*
3. R *is parametrically* $\mathcal{L}_{\omega_1,\omega}$*-definable in* $\mathfrak{M}$.

Corollary

If $|\operatorname{Aut}(\mathfrak{M})| < 2^{\aleph_0}$*, then every relation on* $\mathfrak{M}$ *is parametrically* $\mathcal{L}_{\omega_1,\omega}$*-definable.*

Absolute undefinability

Corollary

If a relation R on a ct $\mathfrak{M}$ is parametrically $\mathcal{L}$ definable, for some logic $\mathcal{L}$, the R is parametrically $\mathcal{L}_{\omega_1,\omega}$ definable.

Definition

A relation on the domain of a countable $\mathfrak{M}$ is absolutely undefinable if it has $2^{\aleph_0}$ automorphic images.[a]*.*

[a] Athanassios Tzouvaras, in A note on real subsets of a recursively saturated model, Z. Math. Logik Grundlag. Math. 37 (1991) called such R imaginary

Lemma (Kueker-Reyes Lemma)

Let $\mathfrak{M} = (M, \dots)$ be countable. If for for every tuple $\bar{a}$ in $M^{<\omega}$ there are $b \in R$ and $c \notin R$ such that $\mathrm{tp}(\bar{a}, b) = \mathrm{tp}(\bar{a}, c)$, then R is absolutely undefinable.

Absolute undefinability cannot be avoided

Theorem (Barwise, Schlipf)

Every countable resplendent model has continuum many automorphisms.

Theorem (Schlipf)

If $(\mathfrak{M}, R)$ is countable, resplendent, and R is not parametrically definable in $\mathfrak{M}$, then has $2^{\aleph_0}$ automorphic images.

Corollary

If $\mathfrak{M}$ is countable, resplendent, and there is a parametrically undefinable R such that $(\mathfrak{M}, R) \models \varphi(R)$, then there is an absolutely undefinable R such that $(\mathfrak{M}, R) \models \varphi(R)$.

Absolutely undefinable expansions

1. A model of $\mathrm{Th}(\mathbb{N}, S)$ to a model of $\mathrm{Th}(\mathbb{N}, <)$. Always exist. All expansions are absolutely undefinable when (M, S) is resplendent; otherwise they are all $\mathcal{L}_{\omega_1,\omega}$ definable.
2. A model $\mathrm{Th}(\mathbb{N}, <)$ to a model of Pr. Exist if an only if $(M, <)$ is resplendent and they are all absolutely undefinable (Emil Jeřábek).
3. A model of Pr to a model of PA. Exist if an only if $(M, +)$ is resplendent and they are all absolutely undefinable (Alfred Dolich, Simon Heller, based on the work of David Llewellyn-Jones on automorphisms of models of Pr.)
4. A model of PA a model of one of the axiomatic theories of truth or satisfaction. Exist if an only if $(M, +, \times)$ is resplendent and they are all absolutely undefinable... a longer story.

Let $\mathfrak{M}$ be a countable resplendent model of PA. The following sets are absolutely undefinable in $\mathfrak{M}$:

- (RK, Kotlarski) Inductive partial satisfaction classes.
- (Schmerl) Undefinable classes.

  $X \subseteq M$ is a class if for every a, $\{x \in X : x < a\}$ is parametrically definable. If (M, X) is a model of PA(X), we call X inductive. All inductive sets are classes; hence all undefinable classes absolutely undefinable.

- (RK, Wcisło) Full satisfaction classes. Bartosz Wcisło, *Full satisfaction classes, definability, and automorphisms*, arXiv:2104.09969.
- (RK, Kotlarski) Graphs of nontrivial automorphisms.
- (Schmerl) Cofinal elementary submodels.