On end extensions of models of open induction

Ch. Cornaros & C. Dimitracopoulos

42ème Journées sur les Arithmétiques Faibles
University of Aegean, Karlovassi
25 September 2023

Theorem 1 (MacDowell-Specker, 1961). Every model of PA has a proper elementary end extension.

Aim. Miniaturize the MacDowell-Specker theorem

$I\Sigma_n$: induction for Σ_n formulas (plus base theory)

$B\Sigma_n$: $I\Delta_0$ + collection for Σ_n formulas

Theorem 2 (Paris & Kirby 1978).
(a) For all $n \in \mathbb{N}$, $I\Sigma_{n+1} \Rightarrow B\Sigma_{n+1} \Rightarrow I\Sigma_n$
and the converse implications are false.
(b) For $n \geq 2$, if M is a countable model of $B\Sigma_n$, then M has a proper Σ_n-elementary end extension K satisfying $I\Delta_0$.

Problem 1. For $n \geq 2$, does every model of $B\Sigma_n$ have a proper Σ_n-elementary end extension satisfying $I\Delta_0$?

“Theorem” (Clote 1986). For any $n \geq 2$, if M is a model of $B\Sigma_n$, then M has a proper Σ_n-elementary end extension K satisfying $I\Delta_0$.

Theorem 3 (Clote 1998). For any $n \geq 2$, if M is a model of $I\Sigma_n$, then M has a proper Σ_n-elementary end extension K satisfying $I\Delta_0$.

Problem 2. (a) Does every model of $I\Sigma_1$ have a proper Σ_1-elementary end extension satisfying $I\Delta_0$?
(b) Does every model of $I\Delta_0$ have a proper Δ_0-elementary end extension satisfying $I\Delta_0$?

Fact 1. If $M \subset_e K$ (i.e., K is a proper end extension of M) and K satisfies $I\Delta_0$, then M is a Δ_0-elementary substructure of K.

Problem 3. Does every model of $I\Delta_0$ have a proper end extension satisfying $I\Delta_0$?

Fact 2. If $M \subset_e K$ and K satisfies $I\Delta_0$, then M satisfies $B\Sigma_1$.

Fact 3. $I\Delta_0 \not\Rightarrow B\Sigma_1$. (recall Theorem 2(a))

Problem 4. Does every model of $B\Sigma_1$ have a proper end extension satisfying $I\Delta_0$? (Fundamental problem F in the list of open problems published by Clote & Krajíček in 1993)

Theorem 4 (Wilkie & Paris 1989). If M is a countable model of $B\Sigma_1 + exp$, then there exists K such that $M \subset_e K$ and K satisfies $I\Delta_0$. (*exp* expresses "exponentiation is total")

Theorem 5 (Hilbert-Bernays 1939 - ACT). Let M be a model of PA and T be a theory definable in M. If M satisfies $Con(T)$, then there exists a model K of T such that K is "strongly definable" in M.

Lemma 6. If M, K satisfy PA and K is strongly definable in M, then M is isomorphic to an initial segment of K.

Theorem 7 (Enayat & Wong 2016-7). Every model of $I\Sigma_1$ has a proper end extension satisfying $I\Delta_0$.

Remark 1. Theorem 7 gives a positive solution to a variant of Problem 4.

C. Dimitracopoulos and V. Paschalis (2016 & 2020). Alternative proofs of Theorems 3 and 7, using variants of ACT. The main ideas for the proofs are:

- using induction **in the metalanguage**, construct a consistent theory T in an extension of LA (in the given model), via a lemma on the possibility of witnessing bounded existential quantifiers with appropriate constants and
- take as universe of the required extension an appropriate set of elements definable in a model of T.

Theorem 8 (Boughattas 1991). Every model of *IOpen* has a proper end extension to a model of *IOpen*.

Problem 5. (a) Does every model of $B\Sigma_1 + exp$ have a proper end extension satisfying $I\Delta_0$?
(b) Does every model of $I\Delta_1 + exp$ have a proper end extension satisfying $I\Delta_0$? ($I\Delta_1$: induction for provably Δ_1 formulas)

Theorem 9 (Slaman 2004). $B\Sigma_1 + exp \Leftrightarrow I\Delta_1 + exp$.
(Slaman proved ($\Leftarrow$), while the converse had been known to hold, even without *exp*, by a result of R. Gandy)

Remark 2. A positive solution to Problem 5(b), would
(i) combined with Fact 2, imply ($\Leftarrow$) of Theorem 9
(ii) give a positive solution to Problem 5(a), thus generalizing Theorem 4 (Wilkie & Paris 1989).