END EXTENSIONS OF MODELS OF Σ_1-COLLECTION

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As usual, $I\Sigma_1$ (respectively, $I\Delta_0$) denotes the theory of Σ_1 (respectively, Δ_0) induction, $B\Sigma_1$ the theory of Σ_1-collection, *IOpen* the theory of open induction, *exp* denotes the axiom $\forall x, y \exists z (z{=}x^y)$ (where $z{=}x^y$ is a Δ_0 formula representing the graph of the exponential function) and e_p denotes the p-th axiom à la Thapen (see [5]), i.e. $\forall x \exists y (x{<}p(y) \wedge$ "x^y exists"), where p denotes any primitive recursive function.

One of the most well-known problems concerning fragments of Peano arithmetic is the following.

Problem 1 (Paris-Wilkie, Fundamental problem F in [2], page 11)**.** *Does every countable model of $B\Sigma_1$ have a proper end extension satisfying $I\Delta_0$?*

An extensive study of this problem was carried out in [6], which led to, among other results, the following.

Theorem 1. *Every countable model of $B\Sigma_1{+}exp$ has a proper end extension satisfying $I\Delta_0$.*

A question arising naturally, in view of Theorem 1, is whether or not the result holds for models of any cardinality, i.e., whether or not the following problem can be solved.

Problem 2. *Does every model of $B\Sigma_1{+}exp$ have a proper end extension satisfying $I\Delta_0$?*

An apparently easier question has been recently answered in the positive; namely, the next result has been proved by A. Enayat and T. L. Wong (see [4], [7]) and, independently, us (see [3]).

Theorem 2. *Every model of $I\Sigma_1$ has a proper end extension satisfying $I\Delta_0$.*

In this talk, we will discuss attempts to solve Problem 2; in particular, we will show how one can modify the method used in [3] to prove Theorem 2, in order to give a partial solution to Problem 2 as follows.

Theorem 3. *Every model of $B\Sigma_1{+}exp$ has a proper end extension to a model satisfying IOpen.*

We also discuss how, by refining the same method, one can do a little better, namely prove the following result.

Theorem 4. *Every model of $B\Sigma_1{+}e_p$ has a proper end extension to a model satisfying IOpen.*

Theorem 4 is by no means the best one can obtain; indeed, the following result due to S. Boughattas (see [1]) is far stronger.

Theorem 5. *Every model of IOpen has a proper end extension to a model satisfying IOpen.*

However, we will argue that the approach we followed for proving Theorem 4 could be used, modulo suitable modifications, towards settling Problem 2 or variants of it.

References

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