On Countable Short Recursively Saturated Models of PA and their Automorphisms

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Recursive Saturation

- Definition: A model is recursively saturated if it realizes every finitely realizable recursive type with a finite number of parameters.

- Theorem (Barwise and Schlipf (1976) and Ressayer (1977)): Every consistent theory in a finite language has a countable recursively saturated model.

Countable Recursively Saturated Models of PA

- There are continuum many different theories in the language of Arithmetic extending PA (Gödel).
- Moreover, each theory extending PA has continuum many countable recursively saturated models with different standard systems.
- Theorem: Two countable recursively saturated models, M and N are isomorphic iff Th(M)=Th(N), SSy(M)=SSy(N), .

- Let M be a countable recursively saturated model of PA. Then,
  - M has continuum many elementary initial segments.
  - All but countably many are recursively saturated.
  - The elementary initial segments that are not recursively saturated are all short (boundedly) recursively saturated.

Short Recursive Saturation

- Definition: A type $p(v,a)$ is ***bounded*** if it contains the formula $v<t(a)$ for some Skolem term t.

- Definition: A model M is ***short (boundedly) recursively saturated*** if it realizes every finitely realizable *bounded* recursive type with a finite number of parameters.
- Notice that every recursively saturated model is short (boundedly) recursively saturated.
- Question: Which short recursively saturated models of PA are not recursively saturated? Answer…

Short Models

- Let *M(a)* be the smallest elementary initial segment of *M* containing a. It can be shown that

$$M(a) = \{x \in M : x \leq t(a) \text{ for some Skolem term } t\}$$

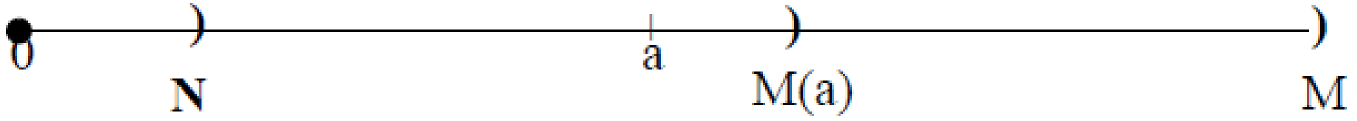
- Models of this form are called <u>short models</u>.

- Fact: If *M* is recursively saturated then *M(a)* is short (boundedly) recursively saturated, but not recursively saturated.

- Theorem: (Smorynski 1981) two countable short recursively saturated models, M(a) and N(b) are isomorphic iff Th(M)=Th(N), SSy(M)=SSy(N), and there are c in gap(a) and d in gap(b) such that tp(c)=tp(d).
- Corollary: (follows from above and results of Gaifman) Every countable recursively saturated model of PA has countably many non isomorphic short elementary initial segments.

- If M is short (boundedly) recursively saturated but not recursively saturated, then $M=M(a)$ for some $a \varepsilon M$.

- Fact (Smorynski 1981 (countable case), Kossak 1983): Every short recursively saturated model of PA has a recursively saturated elementary end extension of the same cardinality.
- Therefore, when we study (nonstandard) short recursively saturated models which are short, we can think of them as in the diagram below.

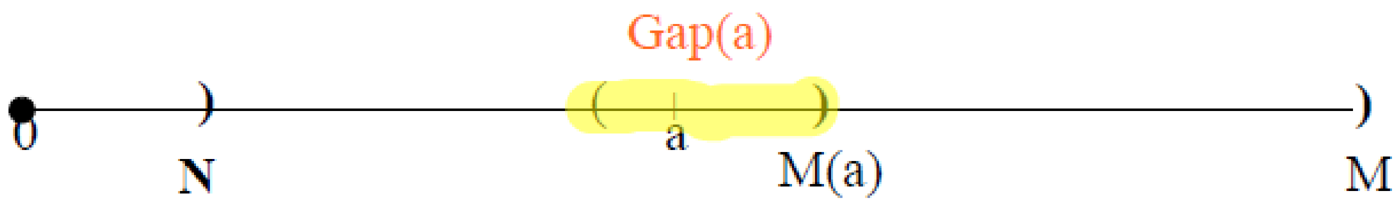


For the rest of the talk:

- Let *M* be a countable recursively saturated model of PA. Let $a \varepsilon M$. Thus, *M(a)* is a short recursively saturated model which is not recursively saturated.

- Let

$$Gap(a) = \{b \in M : t(a) > b \text{ and } t(b) > a \text{ for some Skolem term t}\}$$



Automorphisms

- Let g be an automorphism of M which fixes Gap(a) setwise. Then the restriction of g to $M(a)$, $g|M(a)$ is an automorphism of $M(a)$.

- Fact: Aut($M(a)$) contains continuum many automorphisms of $M(a)$ which can be extended to automophisms of M.

- Question: Are there any automorphisms of $M(a)$ which cannot be extended to automorphisms of M?

- Yes!
- Theorem: (S. 2006) There are continuum many automorphisms of M(a) which do not extend to an automorphism of M.

- Sketch of Proof:
  - Construct a set *X* which is a cofinal ω-sequence in *M(a),* and is coded by some element of *M*. This guarantees that X has at most countably many automorphic images under the action of Aut(M).
  - In addition: require that *X* will have the property that whenever *bεM(a),* there are *x* in *X* and *y* not in *X* such that $$(M(a),b,x)\equiv(M(a),b,y).$$

    This guarantees that X has continuum many automorphic image under the action of Aut(M(a)) (By a result of Kueker and Reyes (75')).
  - Thus, continuum many automorphisms of M(a) do not extend to an automorphism of M.

Automorphism Groups

- We define a topology on the automorphism group of a model of PA, M, by taking the basic open subgroups to be stabilizers of elements, that is, all the subgroups of the form:

  $G_{(a)} = \{g \in G : g(a)=a\}$ For some a in M.

Closed Normal Subgroups

- Theorem: (Kaye 1994) The closed normal subgroups of the automorphism group of a countable recursively saturated model of PA are exactly the pointwise stabilizers of invariant initial segments.
- Theorem: (S. 2006) The closed normal subgroups of the automorphism group of a countable <u>short</u> recursively saturated model of PA are exactly the pointwise stabilizers of invariant initial segments.

Invariant Initial Segments

- If a countable short recursively saturated model of PA realizes a rare type in its last gap (type that is realized only once in the gap), then it will have continuum many invariant initial segments in its last gap (cofinally high).
- If a countable short recursively saturated model of PA does not realize a rare type in its last gap, then it will have no invariant initial segments in its last gap.

This implies:

- Theorem: (S. 2006) Countable short recursively saturated models of PA which realize rare types in their last gap, and countable short recursively saturated models of PA which do not realize rare types in their last gap, have non-isomorphic automorphism groups (as topological groups).

Open Questions

1. Do ctble S.R.S models have the Small Index Property?
2. Do different standard systems imply different automorphism groups?

   Yes for short arithmetically saturated models vs. just short recursively models. (S. 2010)
3. Do different theories imply different automorphism groups?

   Yes for models of TA vs. models of FA (in some cases)

Thank You!