Logical Foundations for the ACL2 Theorem Prover

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Joint work with Bob Boyer, J Moore, and the ACL2 community

Presented at JAF 2019

It's a bit odd to be giving a talk about a software system to mathematical logicians.

Once upon a time I was one of you . . .
but I've gone to the dark side.

Now I work on software, ACL2, that proves theorems.

QUESTION: What can I say today that might interest you?

MY ANSWERS:

1. Introduce ACL2 as a **practical application** of logic.
2. Discuss **foundational issues** for ACL2.

OUTLINE

Outline

Overview and Context

Introduction to the ACL2 System

Logical Foundations for ACL2

Conclusion

Overview and Context

The ACL2 home page begins with the following summary.

> *ACL2 is a logic and programming language in which you can model computer systems, together with a tool to help you prove properties of those models. "ACL2" denotes "**A** **C**omputational **L**ogic for **A**pplicative **C**ommon **L**isp".*

But before we talk about ACL2, let's put it in context.

Formal Verification

Formal verification (FV) of hardware and software systems is the use of tools to check their correctness using mathematical methods, notably **proof**.

FV tools include *equivalence checkers*, *model checkers*, various *static checkers*, and (occasionally) *interactive theorem provers* (ITPs) such as Coq, Isabelle, HOL4, PVS, Agda — and **ACL2**.

Interactive Theorem Proving

- Yearly ITP conference
- ITP is typically more scalable than fully automatic tools, but it requires human assistance.
  - **In ACL2**, one proves lemmas that may be used **automatically** to simplify terms in later proofs.

Some strengths of ACL2 among ITPs:

- Proof automation and debugging
- Fast execution of programs
- Documentation in hypertext format (120,000 lines for system; many more for libraries)
- Scalability (see next slide)

On ACL2 Applications

ACL2 has been used not only at universities and the U.S. Government, but also at several companies [4]:

- AMD, ARM, ArterisIP, Battelle, Centaur, GE, IBM, Intel, NXP, Kestrel, Oracle, Rockwell Collins

People are actually *paid* to prove theorems with ACL2.

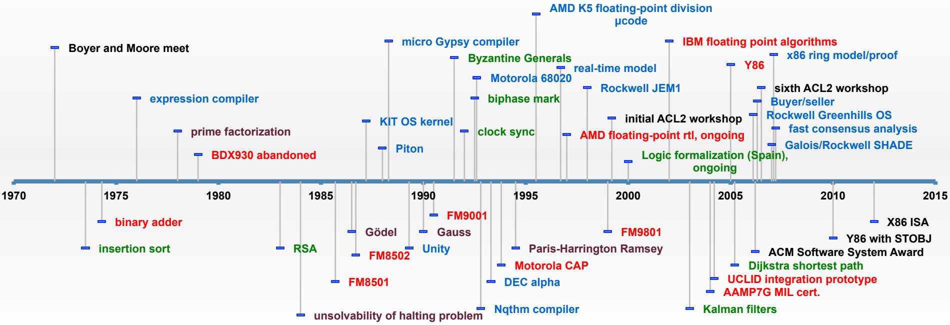
> *"Microprocessor design goes daily through numerous optimizations that affect thousands of lines of code. These optimizations must be proved correct."*

— Anna Slobodova, verification manager, Centaur Technology

A recent example of an ACL2 formalization at UT Austin:
An *efficient* checker for Boolean satisfiability (SAT) proofs

- Used in recent international SAT competitions
- Has checked 2-petabyte SAT proof of longstanding open problem (Schur number 5) [3]; $\sim$16 CPU years

Partial timeline



Outline

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Introduction to the ACL2 System

- ACL2 is freely available with libraries of *certifiable books*.
  - Available from the ACL2 home page and Github
  - Libraries provide more than 500,000 *events* (theorems, definitions, other).
- ACL2 is written mostly in itself (!).
  - About 11 MB of source files
- ACL2 community holds workshops: #15 held Nov. 2018
- History of the ACL2 *system*
  - Bob Boyer and J Moore started ACL2 in 1989. I joined in 1993; Bob stopped in 1995. J and I continue the work.
  - *Boyer-Moore Theorem Provers* go back to their collaboration starting in 1971. [10]
  - The ACL2 community contributes with feature requests and (on occasion) prototype implementations.

USING ACL2

Let's get familiar with ACL2 (and its syntax):
first demo **programming**, then **theorem proving**.

- ACL2 programming and evaluation
  [DEMO]: file `demo-1.lsp`
  (log `demo-1-log.txt`)

- ACL2 as an automated theorem prover
  [DEMO]: file `demo-2.lsp`
  (log `demo-2-log.txt`)
  - ACL2 provides **automation** for induction, linear arithmetic, Boolean reasoning, rule application, ...
  - During a proof, each goal is replaced by a list of subgoals (possible empty) such that if they are all theorems, then that goal is a theorem.

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LOGICAL FOUNDATIONS (1)

The ACL2 logic is a first-order logic with ε_0-induction.

Probably weaker induction would usually suffice **in practice**;
maybe only ω^ω;
maybe only each of ω, ω^ω, ω^{ω^ω}, etc., iterated through only **standard** natural numbers . . .

- . . . but it hasn't been a priority to consider this, let alone to consider effects on the implementation.

(Anyhow, it's nice to have Ken Kunen's Nqthm proof of the Paris-Harrington theorem. [9])

Logical Foundations (2)

Restriction: ACL2 theories extend the *ground-zero* theory: essentially PA with ε_0-induction, extended with data types.

- numbers (complex rationals);
- characters;
- strings;
- symbols; and
- closure under an ordered pair operation, `cons`.

`Cons` provides lists, with the symbol `nil` for the empty list.

```
ACL2 !>(cons 3 nil)
(3)
ACL2 !>(cons 2 (cons 3 nil))
(2 3)
ACL2 !>(cons 1 (cons 2 (cons 3 nil)))
(1 2 3)
ACL2 !>
```

Logical Foundations (3)

Theory extensions made with ACL2 are *conservative* (no new theorems in the existing language).

- . . . This holds even for recursive definitions, since "termination" must be provable.
- We will see the importance of introducing new concepts **locally**: justified by conservativity.
- Theories *evolve* by introducing new function symbols using the *extension principles*. [6]

EXTENSION PRINCIPLE: DEFINITIONS

A definition extends the *current theory* with the axiom equating the call with the body. **Example** (from first demo):

```
(defun fact (n) ; factorial
  (if (posp n) ; n is a positive integer
      (* n (fact (- n 1)))
    1))
```

This adds the following axiom (and of course induction axioms):

```
(fact n) =
(if (posp n) ; n is a positive integer
    (* n (fact (- n 1)))
  1)
```

A definition may be recursive if some *measure* into ε_0 is proved to decrease on each recursive call.

Extension principle: Choice (and $\exists$)

Quantification is implemented using a choice operator. When asked to define
$P(\vec{x}) = \exists \vec{y} A(\vec{x}, \vec{y})$
then ACL2 generates the following.
Conservatively introduce a Skolem (witness) function $w(\vec{x})$ and a predicate $P(\vec{x})$:
$w(\vec{x}) = \varepsilon \vec{y} A(\vec{x}, \vec{y})$ [If any $\vec{y}$ satisfies $A(\vec{x}, \vec{y})$, then $w(\vec{x})$ does.]
$P(\vec{x}) = A(\vec{x}, w(\vec{x}))$

```
(defun-sk fermat-counterex (n)
  (exists (i j k)
    (and (posp i) (posp j) (posp k)
         (equal (+ (expt i n) (expt j n))
                (expt k n)))))
(defthm fermat
  (implies (and (integerp n) (< 2 n))
           (not (fermat-counterex n))))
```

Extension principle: Choice (and $\exists$) (2)

This sort of thing is clearly conservative (we have countable theories, so we don't even need Choice)...

... IF we ignore induction!

Conservativity *with* induction follows from a model-theoretic forcing argument.

EXTENSION PRINCIPLE: CONSTRAINTS

It is also legal to introduce *constrained* functions, using axioms that are *proved* about *local witnesses*.
Example:

```
(encapsulate ((fn (x y) t))
  (local (defun fn (x y)
           (+ x y)))
  (defthm fn-commutative
    (equal (fn x y) (fn y x))))
```

A derived inference rule, *functional instantiation* [2], is often useful with constrained functions.
Example:

```
(defun map2-fn (lst1 lst2)
  (if (consp lst1)
      (cons (fn (first lst1) (first lst2))
            (map2-fn (rest lst1) (rest lst2)))
    nil))
(defthm map2-fn-commutative
  (implies (equal (len lst1) (len lst2)) ; same length
           (equal (map2-fn lst2 lst1)
                  (map2-fn lst1 lst2))))
(defun map2-* (lst1 lst2)
  (if (consp lst1)
      (cons (* (first lst1) (first lst2))
            (map2-* (rest lst1) (rest lst2)))
    nil))
(defthm map2-*-commutative
  (implies (equal (len lst1) (len lst2))
           (equal (map2-* lst2 lst1)
                  (map2-* lst1 lst2)))
  :hints (("Goal" :by (:functional-instance
                       map2-fn-commutative
                       (fn *) (map2-fn map2-*)))))
```

CONSERVATIVITY AND LOCAL

Fun **example** in ACL2(r), a variant of ACL2 that supports the real numbers, due to Ruben Gamboa:
The Overspill Principle of non-standard analysis.
Informally:
If internal predicate $P(n, x)$ holds for all standard natural numbers n, then $P(n, x)$ holds for some non-standard natural number n.

- `overspill.lisp`: Relatively concise formalization (which I'll flash on the next slide)
  25 lines
- `overspill-proof.lisp`: Ugly proof (shows need for human assistance), but `LOCAL` to the main proof, by conservativity
  256 lines

Using `LOCAL` can dramatically speed up book inclusion!

```
(local ; Hence skipped when including this top-level book!
  (include-book "overspill-proof"))

(defstub overspill-p (n x) t)

(defun overspill-p* (n x)
  (if (zp n)
      (overspill-p 0 x)
    (and (overspill-p n x)
         (overspill-p* (1- n) x))))

(defchoose overspill-p-witness (n) (x)
  (or (and (natp n) (standardp n)
           (not (overspill-p n x)))
      (and (natp n) (i-large n)
           (overspill-p* n x))))

(defthm overspill-p-overspill
  (let ((n (overspill-p-witness x)))
    (or (and (natp n) (standardp n)
             (not (overspill-p n x)))
        (and (natp n) (i-large n)
             (implies (and (natp m)
                           (<= m n))
                      (overspill-p m x)))))
  :rule-classes nil)
```

META-THEORETIC REASONING (1)

In ACL2, you can [1, 5]:

- code a simplifier,
- prove that it is sound, and
- direct its use during later proofs.

Efficient execution can be important for meta-theoretic reasoning!

A comment in the ACL2 sources, the "Essay on Correctness of Meta Reasoning", works out the correctness argument.

ITERATION

Useful for programming, with reasoning support. **Examples**:

```
ACL2 !>(loop$ for i in '(3 5 7) sum (* i i))
83
ACL2 !>
```

ACL2 gives the following semantics to the second of these.

```
(sum$ '(lambda (i) (* i i))
      '(3 5 7))
```

where `sum$` is defined essentially as follows.

```
(defun sum$ (fn lst)
  (if (endp lst) ; lst is empty
      0
    (+ (apply$ fn (list (first lst)))
       (sum$ fn (rest lst)))))
```

"HIGHER-ORDER" Apply$ (1)

We cannot employ the usual two-sorted, weak second-order approach. **Example**: Not a theorem without the `defun`!

```
(local (defun f (x) x))
(thm (equal (apply$ 'f (list x)) x))
```

Example successful use of `apply$`:

```
(include-book "projects/apply/top" :dir :system)
(defun$ norm^2 (x y) (+ (* x x) (* y y)))
(assert-event (equal (norm^2 3 4) 25))
(thm (equal (norm^2 3 4) 25))
(assert-event (equal (apply$ 'norm^2 (list 3 4))
                     25))
```

But the following fails, as it should:

`apply$` is a constrained function with trivial constraints.

```
(thm (equal (apply$ 'norm^2 (list 3 4))
            25))
```

"HIGHER-ORDER" `Apply$` (2)

However, the proof succeeds for the `thm` below, where the *warrant hypothesis*, `(warrant norm^2)`, asserts:

```
(∀ x y) (equal (apply$ 'norm^2 (list x y))
               (norm^2 x y)).
```

```
(thm (implies (warrant norm^2)
              (equal (apply$ 'norm^2 (list 3 4))
                     25)))
```

Warrant hypotheses are not vacuous!
There is a natural *evaluation theory* where every warrant is *attached* to the constant "true" function. [8]

DEFATTACH (1)

`Defattach` provides a way to evaluate constrained functions by giving them new definitions. But it allows extensions that are **not** conservative. **Example:**

- **Constraint for** a "specification" function, `spec`:
  $x \in \mathbb{Z} \implies \texttt{spec}(x) \in \mathbb{Z}$
- **Define** function `f`: $\texttt{f}(x,y) = \texttt{spec}(x+y)$
- **Define** an "implementation" function, `impl`:
  $\texttt{impl}(x) = 10 * x$
- Attach `impl` to `spec`: `(defattach spec impl)`
  Meaning: $(\forall x)(\texttt{spec}(x) = \texttt{impl}(x))$

Result not provable from axioms for `f` and `spec`:

```
ACL2 !>(f 3 4) ; = spec(7) = impl(7)
70
ACL2 !>
```

DEFATTACH (2)

Issues to consider:

- Is `(local (defattach ...))` supported?
  YES, `local` is supported.
- Then how do we deal with conservativity?
  Two theories: The *current theory* for reasoning and a stronger *evaluation theory*, extended using `defattach`:

$$(\forall x)(\mathtt{spec}(x) = \mathtt{impl}(x))$$

- Ah, but what about this?

```
(thm (equal (f 3 4) 70))
```

  The proof fails! (Good!)
- Is the evaluation theory consistent?
  Yes, where the attachment relation must be acyclic.

Details: see *Essay on Defattach* comment in the ACL2 sources.

Some More Logical Challenges

Practical considerations create some more logical challenges.

- *Packages* are a programming convenience but introduce axioms such as the following: *not conservative*!
  `symbol-package-name('PKG1::F) = "PKG1"`
  Hence packages must be recorded.
- One can specify a *measure* in order to admit a recursive definition. But what if the measure is defined in terms of a function whose definition is `LOCAL`?
- *Congruence-based reasoning* allows replacing one subterm by another that is equivalent but not necessarily equal. [7]

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- ACL2 has a 29 (or 48) year history and is used in industry.
- As an ITP system, it relies on user guidance for large problems but enjoys scalability.
- Logic provides critical foundational support for practical theorem proving software.
- For more information, see the ACL2 home page, in particular links to The Tours and Publications, which links to introductory material.

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J Strother Moore. Milestones from The Pure Lisp Theorem Prover to ACL2. Submitted; see
http://www.cs.utexas.edu/users/moore/publications/milestones.pdf.

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Slides for this talk are available via links from my home page:
`http://www.cs.utexas.edu/users/kaufmann`

THANK YOU!

EXTRA SLIDES

We can go on, time permitting....

Some ACL2 features *not* discussed further today:

- Prover algorithms
  - Waterfall, linear arithmetic, Boolean reasoning, . . .
  - Rewriting: Conditional, congruence-based, rewrite cache, syntaxp, bind-free, . . .
- Using the prover effectively
  - The-method and introduction-to-the-theorem-prover
  - Theories, hints, rule-classes, . . .
  - Accumulated-persistence, brr, proof-checker, dmr, . . .
- Programming support, including (just a few):
  - Guards
  - Hash-cons and function memoization
  - Packages
  - Mutable State, stobjs, arrays, applicative hash tables, . . .
- System-level: Emacs support, books and certification, abbreviated printing, parallelism (ACL2(p)), . . .

META-THEORETIC REASONING (2)

ACL2 supports a notion of "evaluation", together with this sort of *meta* theorem, directing the use of `fn` to transform terms that are calls of `nth` or of `foo`.

```
(defthm fn-correct-1
  (equal (evl x a)
         (evl (fn x) a))
  :rule-classes ((:meta :trigger-fns (nth foo))))
```

More complex forms are supported, including:

- extended-metafunctions that take `STATE` and contextual inputs;
- transformations at the goal level; and
- hypotheses that extract known information from the logical world.

For details, including issues pertaining to evaluation, see the *Essay on Correctness of Meta Reasoning* comment in the ACL2 sources. *Attachments* provide a challenge.

On Efficient Execution

Efficient execution is a key design goal.

- ACL2 definitions are actually programs in the Common Lisp programming language.
- *Guards* specify intended domains of functions and support sound, efficient Common Lisp evaluation.
- Several features support efficient computation by reusing storage, yet with a first-order logic foundation.
  - *Single-threaded objects* including *state*
  - *Arrays*
  - *Function memoization* (reuse of saved results)
  - *Fast alists* (applicative hash tables)