Π^0_1-computable quotient presentations of nonstandard models of arithmetic

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joint work with T. Slaman and L. Harrington

Computable quotient presentations

Definition

A ***computable quotient presentation*** of a structure $\mathcal{A}$ (an E-structure isomorphic to $\mathcal{A}$) consists of:

1. a computable structure on the natural numbers $\langle \mathbb{N}, \star, *, \ldots \rangle$, meaning that the operations and relations of the structure are computable,
2. an equivalence relation E on $\mathbb{N}$ (not necessarily computable) which is a congruence with respect to this structure,

such that:
the quotient $\langle \mathbb{N}, \star, *, \ldots \rangle / E$ is isomorphic to the given structure $\mathcal{A}$.

Motivations for studying quotient presentations

Theorem (Homomorphism Theorem)

For any countable algebra $\mathbb{A}$ there exists a surjective homomorphism $h : F \to \mathbb{A}$ from the term algebra $\mathcal{F}$ into $\mathbb{A}$ Hence, the algebra $\mathbb{A}$ is isomorphic to $\mathcal{F}/E$, where E is the kernel of the homomorphism:

$$E = \{(x, y) \mid h(x) = h(y)\}.$$

Every countable algebra (a structure in a language with no relations) arises as the quotient of the term algebra on a countable number of generators.

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For any countable algebra $\mathbb{A}$ *there exists a surjective homomorphism* $h : F \to \mathbb{A}$ *from the term algebra* $\mathcal{F}$ *into* $\mathbb{A}$ *Hence, the algebra* $\mathbb{A}$ *is isomorphic to* $\mathcal{F}/E$*, where* E *is the kernel of the homomorphism:*

$$E = \{(x, y) \mid h(x) = h(y)\}.$$

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Observation

Every consistent c.e. theory T *in a functional language admits a computable quotient presentation by an equivalence relation* E *of low Turing degree.*

Khoussainov's conjectures

Question: can nonstandard models of arithmetic be realized as E-structures (do they have computable quotient presentations) for sufficiently non-complex E?

In a joint work with J.D. Hamkins we prove several generalizations of Tennebaum's theorem for computable quotient presentations of models of PA:

Theorem

No nonstandard model of arithmetic has a computable quotient presentation by a c.e. equivalence relation, that is: there is no computable structure $\langle \mathbb{N}, \oplus, \odot \rangle$ and a c.e. equivalence relation E, which is a congruence with respect to this structure, such that the quotient $\langle \mathbb{N}, \oplus, \odot \rangle / E$ is a nonstandard model of arithmetic.

Theorem

There is no computable structure $\langle \mathbb{N}, \oplus, \odot \rangle$ and a co-c.e. equivalence relation E, which is a congruence with respect to this structure, such that the quotient $\langle \mathbb{N}, \oplus, \odot \rangle / E$ is a Σ_1-sound nonstandard model of arithmetic, or even merely a nonstandard model of arithmetic with $0'$ in the standard system of the model.

Π^0_1-recursively presentable nonstandard model of arithmetic

Theorem (G., Harrington, Slaman)

There exists a nonstandard model $M \models PA$ *s.t.* $M \cong \langle \mathbb{N}, \oplus, \otimes, S, 0, 1\rangle / E$, *where* $\langle \mathbb{N}, \oplus, \otimes, S, 0, 1\rangle$ *is computable and* E *is* Π^0_1.

Proof...

Let $\mathcal{L}^+ = \mathcal{L}_{PA} + \{c_i : i \in \omega\}$ and let $T^+ = PA + \neg Con_{PA}$.
We simulate the Henkin construction via finite injury priority argument, doing two things:

1. building a Henkin tree,
2. enumerating inequalities, which will give us a c.e. complement of E, making E co-c.e.

The Construction

Let $\{\varphi_n(\overline{c})\}_{n\in\omega}$ be a recursive enumeration of all sentences of the langugae $\mathcal{L}^+_{PA}$, assuming each $\varphi_n(\overline{c})$ to be in a prenex normal form.
Stage $s+1$:
We are given a sequence (T_s, A_s, E_s), where:

1. $T_s =$

$$= T_0 + \left(\varphi^*_1, \psi^*_1(c_{i_1}), \varphi^*_2, \psi^*_2(c_{i_2}), \ldots, \varphi^*_{k_s}, \psi^*_{k_s}(c_{i_{k_s}})\right),$$

where for each $j \leq k_s$ φ^*_j is of the form $\exists x\, \psi_j(x)$ or $\forall x\, \neg\psi_j(x)$, and

$$\psi^*_j(c_{i_j}) = \begin{cases} \psi_j(c_{i_j}) \text{ if } \varphi^*_j = \exists x\, \psi_j(x) \\ \neg\psi_j(c_{i_j}) \text{ if } \varphi^*_j = \forall x\, \neg\psi_j(x), \end{cases}$$

The Construction

2. $A_s(b_s)$ is the set of inequalities enumerated by the stage s with number b_s being the highest index of a Henkin constant that occurs in any formula in the set A_s.
   and
3. $E_s = \{\tau(\overline{c}) = \sigma(\overline{c}) :$

   $$\overline{c} \subseteq \{c_{i_1}, c_{i_2}, \ldots, c_{i_{k_s}}\},\ \tau, \sigma \in Trm(\mathcal{L}_{PA}),\ T_s + A_s \vdash_s \tau(\overline{c}) = \sigma(\overline{c})\},$$

   i.e. E_s is the set of equalities in constants of T_s that are known provable from $T_s \cup A_s$ by the end of stage s,

The Construction

Given (T_s, A_s, E_s), we are given a pair of formulas

$$\left(\varphi_{k_s+1}, \psi^*_{k_s+1}(c_{i_{k_s+1}})\right).$$

Let $T_{s+1} = T_s + \varphi_{k_s+1}$. We begin by considering the theory

$$U_{s+1} := T_{s+1} + A_s,$$

and the finite set of *short* proofs associated with this theory:

$$\{x \leq s+1 : Prf_{U_{s+1}}(x, \ulcorner 0 = 1 \urcorner)\} = \{x_0, x_1, \dots, x_m\}.$$

If the set above is non-empty, we apply the Release Protocole.

The Release Protocol

Define a function f that associates with each Gödel code $x_i \leq s+1$ of a proof of contradiction from U_{s+1} the least index of an initial segment T_a of T_{s+1} such that the proof x_i uses only the axioms from T_a.
We now pick the minimum of the image of f, i.e. let:

$$a = min(f[\{x_0, \dots, x_m\}])$$

be the index of the shortest initial segment of T_{s+1} that allows for a proof (with the Gödel number bounded by $s+1$) of inconsistency. Consider the theory T_a.

The Release Protocol

There are two cases:

1. there is $i \leq m$ such that

$$f(x_i) = a \text{ and } \forall j \leq l_i \; \alpha_{i,j} \neq \psi^*_{k_a}(c_{i_{k_a}}),$$

   which means that $\psi^*_{k_a}(c_{i_{k_a}})$ is not necessary in deriving a contradiction from T_a. This just means that it is $\varphi^*_{k_a}$ that is the source of the problem.

The Release Protocol

In this case we

- *release* all the Henkin constants used in the construction between T_a and T_{s+1}, i.e. forget about all the constants with indices higher than i_{k_a} and consider them candidates for being *fresh*,
- change the truth value of $\varphi^*_{k_a}$, i.e. we define

$$S_a := T_a \setminus \{\varphi^*_{k_a}\} \cup \{\neg\varphi^*_{k_a}\}$$

  and update T_a to S_a,
- if $\neg\varphi^*_{k_a}$ is an inequality, enumerate it into A_s, i.e.

$$A_{s+1} := A_s \cup \{\neg\varphi^*_{k_a}\},$$

- if φ_{k_a} is existential, keep $\neg\psi^*(c_{i_{k_a}})$ in S_a, otherwise keep $\psi^*(c_{i\,k_a})$ in S_a

The Release Protocol

The second case:

2. T_a ends with the formula $\psi^*_{k_a}(c_{i_{k_a}})$ - formally:
   there is $i \leq m$ such that

$$f(x_i) = a \text{ and } \exists j \leq l_i \, \alpha_{i,j} = \psi^*_{k_a}(c_{i_{k_a}}),$$

   which means that $\psi^*_{k_a}(c_{i_{k_a}})$ is necessary in deriving a contradiction from T_a (i.e. it is the source of the problem).

The Release Protocol

In this case:

- replace $\psi^*_{k_a}(c_{i_{k_a}})$ with $\psi^*_{k_a}(\tilde{c})$, where $\tilde{c}$ is a fresh constant.
- figure out the equalities E'_s that are $\leq s+1$-provable (possibly with the new constant $\tilde{c}$), i.e. a set such that

$$T_a(c_{i_0}, \ldots, \tilde{c}) \vdash_{s+1} E'_s.$$

- since T_a was inconsistent, it must have been inconsistent with the set A_s, so we need to handle it now before we proceed to the next stage.

Decidability Lemma and The Extension Protocol

Lemma

Let $I = (p_1, \ldots, p_n)$ be a finitely generated ideal in the ring of polynomials with integer coefficients. Then the set

$$\{q(x_1, \ldots, x_k) : \mathbb{Z}[x_1, \ldots, x_k]/I \models q(x_1, \ldots, x_k) = 0\}$$

is decidable.

Check if A_s is satisfiable in $\mathbb{Z}[c_{i_0}, \ldots, \tilde{c}]/(E'_s)$. By the Lemma, this property is decidable. There are two cases again:

1. A_s is satisfiable in $\mathbb{Z}[c_{i_0}, \ldots, \tilde{c}]/(E'_s)$: then use $T_a(c_{i_0}, \ldots, \tilde{c})$ (i.e. with $c_{i_{k_a}}$ replaced by $\tilde{c}$) and proceed to the next stage
2. A_s is unsatisfiable in $\mathbb{Z}[c_{i_0}, \ldots, \tilde{c}]/(E'_s)$: we found out we were wrong - it is rather $\varphi^*_{k_a}$ that was the source of the problem, but we had not checked for the new ideal before: change the Boolean value of $\varphi^*_{k_a}$ and update T_a as before.

The construction works

Proposition

1. *Injury Lemma:*
$$T := \lim_{s \to \infty} T_s$$
exists, i,.e. there is a theory T such that for any $\varphi \in \mathcal{L}^+_{PA}$ it holds that $\varphi \in T$ iff $\exists t \forall s > t\ \varphi \in T_s$,
2. *T is complete, Hekinized, consistent (with $PA^+ + \neg Con(PA)$),*
3. *For any inequality γ we have that $\gamma \in T$ iff γ has been enumerated during the construction,*
4. *The construction yields a model for T:*
$$\{c_n : n \in \omega\}/E_\infty \models T,$$
where E_∞ denotes all the equalities provable in T.

Notes on the Injury Lemma

What happens when we discover an inconsistency and apply the Release Protocol?

1. $\mathbb{Z}[c_{i_0}, \ldots, c_{i_{k_a}}]/(E_a) \models \exists x_{i_{k_a+1}} \ldots \exists x_{i_{k_b}} A_s(\overline{x})$,
2. $T_a(\overline{c}) \vdash \forall \overline{x} \neg A_s(\overline{x})$,
   from which it follows that:

$$\mathbb{Z}[c_{i_0}, \ldots, c_{i_{k_a}}]/(E_a) \not\models T_a.$$

Thus: we can extract new (via a *product method*) polynomials (from $\mathbb{Z}[\overline{c}]$) $p_1, \ldots p_n \notin (E_a)$ such that

$$T_a \vdash \forall j \leq n \; p_j \equiv 0.$$

Product Method for extracting polynomials

The inconsistency given by $T_a(\overline{c}) \vdash \forall \overline{x} \neg A_s(\overline{x})$ means that there must be an identity (provbable) of the form

$$\prod_{p \in A_s} p = 0.$$

But the product is a polynomial itself: $\prod p = q(c_{i_0}, \ldots, c_{i_{k_a}}) = 0$.
We can rewrite it as a polynomial in $\mathbb{Z}[c_{i_0}, \ldots, c_{i_{k_a}}]$, and its coefficients are polynomials that were not in the ideal (E_a).
But since $q = 0$, its coefficients all have to be 0, so they must be put into the ideal.

Proof of the Injury Lemma

By the remarks above, we have that every time we apply the Release Protocol, we have a new ideal:

$$J := (E_a \cup \{p_1, \ldots, p_n\}).$$

We check if

$$\mathbb{Z}[c_{i_0}, \ldots, c_{i_{k_a - 1}}, \tilde{c}]/J \models A_s(\overline{x}).$$

1. If no, it means $T_a(\overline{c}) + \exists x \psi_{k_a}(x) \vdash \forall x \neg A_s(\overline{x})$.
   Then, since $\tilde{c}$ is a new constant, it actually follows that we have to put $\forall x \neg \psi_{k_a}(x)$ into T_a.
2. If yes, we proceed (as in the Release Protocol) - but we can do so only finitely often. Why?

Proof of the Injury Lemma

Hilbert's Nullensatz

Suppose $p_1, \ldots, p_n, \ldots$ are polynomials in a given ring. Then for an ideal generated by them, i.e. $I = ((p_n)_{n \in \omega})$ there exists a natural number n such that $I = (p_1, \ldots, p_n)$.

Therefore the injury of the strategy for φ_{k_a} cannot happen infinitely often:

Summary of the Injury Lemma.

Every stage t that we *discover* an inconsistency at, there is a new equality of the form $\tau(c_{i_0}, \ldots, c_{i_{k_a-1}}, x) = 0$ provable from $T_a + A_t$ and $\tau \notin \mathcal{I}_t$ in $\mathbb{Z}[c_{i_1}, \ldots, c_{i_{k_a-1}}]$, and we put τ into this ideal, i.e. the ideal generated at stage t by polynomials in the ring $\mathbb{Z}[c_{i_1}, \ldots, c_{i_{k_a-1}}]$.
If this happened ∞-often, we would get back ∞-often to the ring $\mathbb{Z}[c_{i_1}, \ldots, c_{i_{k_a-1}}][x]$ and we would have (in this ring) an infinite sequence:

$$\mathcal{I}_{t-1} \subsetneq \mathcal{I}_t \subsetneq \ldots \subsetneq \mathcal{I}_{t+k-1} \subsetneq \mathcal{I}_{t+k} \subsetneq \ldots \subsetneq \ldots$$

which would contradict Hilbert's Basis Theorem (that every ring is Noetherian).

Remarks and Question

1. We can begin with any finite set of sentences unprovable in *PA* as we wish (as long as they guarantee nonstandardness of the resulting model): we can construct infinitely many unequivalent models.
2. Open problem: is it possible to construct infinitely many equivalent, but nonisomorphic such models?
3. Our models have to be Σ_1-unsound for general reasons.

Thank You and Go Warriors!