The first-order parts of Weihrauch degrees

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Classical reverse mathematics

Reverse mathematics

Measures the strengths of (countable versions, or countable representations of) theorems of ordinary mathematics.

Subsystems of second-order arithmetic (Z_2) serve as benchmarks.

Base subsystem. RCA_0 consists of:

- PA^-;
- recursive comprehension axiom (Δ^0_1 comprehension);
- Σ^0_1 induction.

Stronger subsystems.

- $WKL_0 = RCA_0 +$ Weak König's lemma (WKL);
- $ACA_0 = RCA_0 +$ arithmetical comprehension (ACA).

Some principles

Second-order statements.

- Weak König's lemma (WKL) : every infinite tree $T \subseteq 2^{\mathbb{N}}$ has an infinite branch.
- Weak weak König's lemma (WWKL) : every infinite tree $T \subseteq 2^{\mathbb{N}}$ of positive measure has an infinite branch.
- Ramsey's theorem (RT^n_k) : every coloring $c : [\omega]^n \to k$ has an infinite homogeneous set.

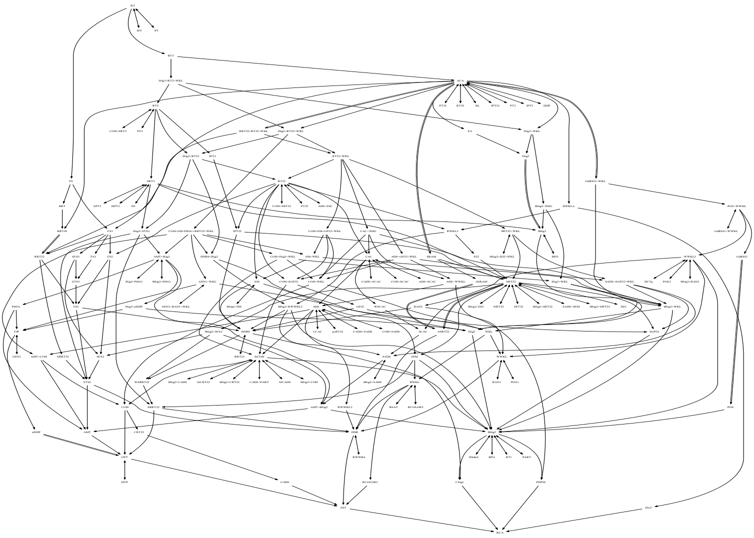
Kirby-Paris hierarchy

- $\mathsf{B}\Gamma$ is the following scheme: for every formula $\phi \in \Gamma$,

$$(\forall k)[(\forall x < k)(\exists y)\phi(x, y) \to (\exists j)(\forall x < k)(\exists y < j)\, \phi(x, y)].$$

- $\mathsf{I}\Sigma^0_1 < \mathsf{B}\Sigma^0_2 < \mathsf{I}\Sigma^0_2 < \mathsf{B}\Sigma^0_3 < \mathsf{I}\Sigma^0_3 < \cdots$.

Reverse mathematics zoo



First-order parts

Defn. Let T be a statement in the language of Z_2.
The first-order part of T is the set of arithmetical consequences of $\mathsf{RCA}_0 + T$.

Examples.

- The first-order part of RCA_0 and WKL_0 is Σ^0_1-PA.
- The first-order part of ACA_0 is PA.

A combinatorial example.

Consider $(\forall k)\ \mathsf{RT}^1_k$, i.e., the infinitary pigeonhole principle,

$$(\forall k)(\forall c : \omega \to k)(\exists H)[H \text{ infinite and } c \restriction H \text{ constant}].$$

Thm (Hirst 1987). $\mathsf{RCA}_0 \vdash \mathsf{RT}^1 \leftrightarrow \mathsf{B}\Sigma^0_2$.

The first-order part(s) of Ramsey's theorem

RT^n_k : Every $c : [\mathbb{N}]^n \to k$ has an infinite homogeneous set.

Thm.

- **(Jockusch 1972).** For all k and all $n \geq 3$, $\mathsf{RCA}_0 \vdash \mathsf{RT}^n_k \leftrightarrow \mathsf{ACA}_0$.
- **(Liu 2011).** $\mathsf{RCA}_0 + \mathsf{RT}^2_2 \nvdash \mathsf{WKL}$.

Thm (Hirst 1987). $\mathsf{RCA}_0 + \mathsf{RT}^2_2 \vdash \mathsf{B}\Sigma^0_2$ and $\mathsf{RCA}_0 + (\forall k)\ \mathsf{RT}^2_k \vdash \mathsf{B}\Sigma^0_3$.

Thm (Cholak, Jockusch, and Slaman 2001). $\mathsf{RCA}_0 + (\forall k)\ \mathsf{RT}^2_k$ is Π^1_1-conservative over $\mathsf{I}\Sigma^0_3$.

Thm (Slaman and Yokoyama 2016). $\mathsf{RCA}_0 + \mathsf{RT}^2_2$ is Π^1_1-conservative over $\mathsf{B}\Sigma^0_3$.

Thm (Chong, Slaman, and Yang 2017). $\mathsf{RCA}_0 + \mathsf{RT}^2_2 \nvdash \mathsf{I}\Sigma^0_2$.

Reverse math, the reboot

Instance-solution problems

Typical theorems studied in reverse mathematics have the canonical form

$$(\forall X)[\phi(X) \to (\exists Y)\psi(X, Y)],$$

where ϕ and ψ are arithmetical predicates of reals.

We view this as a problem: given X such that $\phi(X)$, find Y such that $\psi(X, Y)$.

Defn. A problem is a partial multifunction $\mathsf{P} :\subseteq \omega^\omega \rightrightarrows \omega^\omega$.

The P-instances are the elements of $\mathrm{dom}(\mathsf{P})$.

For each $X \in \mathrm{dom}(\mathsf{P})$ the P-solutions to X are the elements of $\mathsf{P}(X)$.

Example. In RT^2_2, the instances are the colorings $c : [\omega]^2 \to 2$, and the solutions to such a c are all the infinite homogeneous sets.

Computable reducibility

Defn (D. 2013). Let P and Q be problems.
P is computably reducible to Q, $\mathsf{P} \leq_c \mathsf{Q}$, if:

- every P-instance X computes a Q-instance $\widehat{X}$,
- for every Q-solution $\widehat{Y}$ to $\widehat{X}$, we have that $X \oplus \widehat{Y}$ computes a P-solution Y to X.

$$\begin{tikzcd} X \arrow[r, "\text{computes}"] \arrow[d, dashed, "\text{is solved by}"'] & \widehat{X} \arrow[d, dashed, "\text{is solved by}"] \\ Y & \widehat{Y} \arrow[l, "X\text{-computes}"] \end{tikzcd}$$

Weihrauch reducibility

Defn (Weihrauch 1990). Let P and Q be problems.
P is Weihrauch reducible to Q, $\mathrm{P} \leq_{\mathrm{W}} \mathrm{Q}$, if there are Turing functionals Φ, Ψ s.t.:

- for every P-instance X, we have that $\Phi(X)$ is a Q-instance, and
- for every Q-solution $\widehat{Y}$ to $\Phi(X)$, we have that $\Psi(X \oplus \widehat{Y})$ is a P-solution Y to X.

$$\begin{tikzcd}
X \arrow[rr, "\Phi"] \arrow[d, dashed, "\text{is solved by}"'] & & \Phi(X) \arrow[d, dashed, "\text{is solved by}"] \\
Y & & \widehat{Y} \arrow[ll, "\Psi(X \oplus \cdot)"]
\end{tikzcd}$$

Equivalence classes under $\leq_{\mathrm{W}}$ form the Weihrauch degrees, denoted $\mathcal{W}$.

The Weihrauch lattice

Thm (Pauly 2010; Brattka and Gherardi 2011). Under suitable operations of $\vee$ and $\wedge$, $(\mathcal{W}, \leq_{\mathrm{W}}, \vee, \wedge)$ is a lattice.

Let P_0 and P_1 be problems.

- $\mathsf{P}_0 \times \mathsf{P}_1$ is the problem with domain $\mathrm{dom}(\mathsf{P}_0) \times \mathrm{dom}(\mathsf{P}_1)$, with the solutions to (X_0, X_1) being all pairs (Y_0, Y_1) such that Y_i is a P_i-solution to X_i.
- $\mathsf{P}_0^2 = \mathsf{P}_0 \times \mathsf{P}_0$; $\mathsf{P}_0^{n+1} = \mathsf{P}_0^n \times \mathsf{P}_0$; $\mathsf{P}_0^* = \bigcup_n \mathsf{P}_0^n$.
- P_0' is the problem with domain all $f : \omega^2 \to \omega$ such that $\lim_s f(x, s) \downarrow$ for all x, $X = \lim_s f \in \mathrm{dom}(\mathsf{P})$, and the solutions to f are all the P_0-solutions to X.
- $\mathsf{P}_0^{(2)} = \mathsf{P}_0''$; $\mathsf{P}_0^{(n+1)} = (\mathsf{P}_0^{(n)})'$.
- $\mathsf{P}_0 \star \mathsf{P}_1$ is the composition product of P_1 followed by P_0. Intuitively: "solve P_1 first, then use your solution to create an instance of problem P_0."

A refinement of reverse mathematics

Implications over RCA_0 between Π^1_2 principles tend to be formalizations computable or Weihrauch (or stronger) reductions.

Example.

- For all n, j, k, we have $\mathsf{RCA}_0 \vdash \mathsf{RT}^n_k \leftrightarrow \mathsf{RT}^n_j$.
- **(Patey 2015.)** If $j < k$ then $\mathsf{RT}^n_k \not\leq_{\mathrm{c}} \mathsf{RT}^n_j$.

Defn. A coloring $c : [\omega]^2 \to 2$ is stable if $(\forall x)\ \lim_y c(x, y)$ exists.
A set X is limit-homogeneous for c if $(\exists i)(\forall x \in X)\ \lim_y c(x, y) = i$.

SRT^2_2 is the restriction of RT^2_2 to stable colorings.
D^2_2 : Every stable coloring has an infinite limit-homogeneous set.

- **(Chong, Lempp, and Yang 2011.)** $\mathsf{RCA}_0 \vdash \mathsf{SRT}^2_2 \leftrightarrow \mathsf{D}^2_2$.
- **(D. 2016.)** $\mathsf{D}^2_2 \leq_{\mathrm{W}} \mathsf{SRT}^2_2$ but $\mathsf{SRT}^2_2 \not\leq_{\mathrm{W}} \mathsf{D}^2_2$.

First-order Weihrauch problems

First-order problems

Defn. A problem P is first-order if $\mathsf{P}(X) \subseteq \mathbb{N}$ for all $X \in \mathrm{dom}(\mathsf{P})$.

Denote the collection of first-order problems by $\mathcal{FO}$.

Examples.

- LPO : instances: $0^n 1^\omega \in 2^\omega$ for all $n \geq 0$;
  solutions: 0 if $n = 0$ and 1 otherwise.
- $\lim_{\mathbb{N}}$: instances: convergent sequences $\langle x_i : i \in \mathbb{N} \rangle \subseteq \mathbb{N}$;
  solutions: $\lim_i x_i$.
- $\mathsf{C}_{\mathbb{N}}$: instances: (co-enumerations of) non-empty sets $X \subseteq \mathbb{N}$;
  solutions: points in X.
- $\mathsf{K}_{\mathbb{N}}$: instances: (co-enumerations of) non-empty bounded sets $X \subseteq \mathbb{N}$;
  solutions: points in X.

Brattka's question

$\mathsf{C}_{\mathbb{N}}$ can be viewed as corresponding to $\mathrm{I}\Sigma^0_1$, and $\mathsf{K}_{\mathbb{N}}$ as corresponding to $\mathrm{B}\Sigma^0_1$.

Defn.

- max $:\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$, $p \mapsto \max\{p(n) : n \in \mathbb{N}\}$.
- min $: \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$, $p \mapsto \min\{p(n) : n \in \mathbb{N}\}$.

Prop (Brattka). $\max \equiv_W \mathsf{C}_{\mathbb{N}}$ and $\min \equiv_W \mathsf{K}_{\mathbb{N}}$.

We have the following hierarchy,

$$\mathsf{K}_{\mathbb{N}} <_W \mathsf{C}_{\mathbb{N}} <_W \mathsf{K}'_{\mathbb{N}} <_W \mathsf{C}'_{\mathbb{N}} <_W \mathsf{K}''_{\mathbb{N}} <_W \mathsf{C}''_{\mathbb{N}} <_W \cdots$$

which can thus be viewed as an analogue of the Kirby-Paris hierarchy.

First-order parts of Weihrauch degrees

Defn. Let P be a problem. The first-order part of P, denoted ${}^1\mathsf{P}$, is

$$\sup_{\leq_W}\{\mathsf{R} \in \mathcal{FO} : \mathsf{R} \leq_W \mathsf{P}\}.$$

Prop (DSY). ${}^1\mathsf{P}$ exists, for every P.

Proof. Let Q to be the following problem:

- the instances are all pairs (X, Ψ) such that $X \in \mathrm{dom}(\mathsf{P})$ and $\Psi(X, Y)(0)\downarrow$ for all P-solutions Y to X;
- the solutions to (X, Ψ) are all $y \in \mathbb{N}$ such that $\Psi(X, Y)(0)\downarrow= y$ for some P-solution Y to X.

Then $\mathsf{Q} \equiv_W {}^1\mathsf{P}$.

Basic facts

Obs. If $\mathrm{P} \in \mathcal{FO}$ then ${}^1\mathrm{P} \equiv_{\mathrm{W}} \mathrm{P}$.

Defn. Let P be a problem. Then P is

- computably true if $P \leq_{\mathrm{c}} \mathrm{Id}$.
- uniformly computably true if $P \leq_{\mathrm{W}} \mathrm{Id}$.

Prop (DSY). If ${}^1\mathrm{P}$ is uniformly computably true then ${}^1(\mathrm{P} \times \mathrm{Q}) \equiv_{\mathrm{W}} {}^1\mathrm{Q}$.

Prop (DSY). A problem P is computably true iff $\mathrm{P} \leq_{\mathrm{W}} \mathrm{Q}$ for some $\mathrm{Q} \in \mathcal{FO}$.

Proof. Clearly if $P \leq_{\mathrm{W}} \mathrm{Q}$ for some $\mathrm{Q} \in \mathcal{FO}$ then P is computably true. Conversely, suppose P is computably true. Let Q be the problem whose instances are the same as those of P, and the solutions are all (indices of) Turing functionals Φ such that $\Phi(X)$ is a P-solution to X. Then $\mathrm{Q} \in \mathcal{FO}$ and $\mathrm{P} \leq_{\mathrm{W}} \mathrm{Q}$.

Non-diagonalizable problems

Defn (Hirschfeldt and Jockusch 2016). A problem P is non-diagonalizable if there is a $\{0,1\}$-valued Turing functional Δ such that for every P-instance X and every $\sigma \in \omega^{<\omega}$,

$$\Delta(X,\sigma) = \begin{cases} 1 & \text{if } \sigma \text{ is extendible to a P-solution to } X, \\ 0 & \text{otherwise.} \end{cases}$$

Prop (DSY). If P is non-diagonalizable then ${}^{1}\mathsf{P}$ is uniformly computably true.

The converse fails.

TS^1_3 : Every $c : \omega \to 3$ omits at least one color on some infinite set.

This is uniformly computable true, but not Weihrauch reducible to any non-diagonalizable problem (Hirschfeldt and Jockusch 2016).

Case studies

ACA

Defn. $\mathsf{J} : \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$, $p \mapsto p'$.

Note: the models of ACA_0 are the subsets of $\mathbb{N}^\mathbb{N}$ closed under J.

Defn.

- Σ^0_n-Tr : instances: indices of Σ^0_n statement of second-order arithmetic;
  solutions: 1 if the statement is true, 0 otherwise.
- Use : instances: pairs (X, Γ), $X \in \mathbb{N}^\mathbb{N}$, Γ a Turing functional s.t. $\Gamma(X)(0)\downarrow$;
  solutions: all $\ell \geq \mathrm{use}(\Gamma(X)(0))$.

Prop (DSY). ${}^1\mathsf{J}^{(n)} \equiv_\mathrm{W} (\Sigma^0_n\text{-}\mathsf{Tr}) \star \mathsf{Use}^{(n)}$.

(Recall: $\star$ denotes the compositional product.)

In particular, ${}^1\mathsf{J}^{(m)} \not\leq_\mathrm{W} {}^1\mathsf{J}^{(n)}$ whenever $m > n$.

WKL

Obs. ${}^1\mathsf{WKL} \equiv_W {}^1\mathsf{WWKL}$.

C_2 : instances: (co-enumerations of) non-empty $X \subseteq \{0, 1\}$;
solutions: points in X.

Thm (DSY).

- ${}^1\mathsf{WKL} \equiv_W (\mathsf{C}_2)^*$.
- ${}^1\mathsf{WKL}^{(n)} \equiv_W (\mathsf{C}_2^{(n)})^* \star \mathsf{Use}^{(n)}$.

Jumps are combinatorially natural:

- The principle COH is (provably in RCA_0, and as a Weihrauch equivalence) the jump inversion of WKL'. (More on COH below.)
- The Rainbow Ramsey's theorem for bounded colorings is the jump of DNR, a close relative of WKL (J. Miller, unpublished).

Ramsey's theorem

Obs. $\mathsf{RT}^1_2 \equiv_{\mathrm{W}} {}^1\mathsf{RT}^1_2$.

Prop. $\mathsf{RT}^1_2 \equiv_{\mathrm{W}} C_2'$.

Thm (DSY). ${}^1(\forall k)\,\mathsf{RT}^1_k \equiv_{\mathrm{W}} {}^1(\mathsf{RT}^{1}_2{}^*) \equiv_{\mathrm{W}} (\forall k)\,\mathsf{RT}^1_k \equiv_{\mathrm{W}} \mathsf{RT}^{1}_2{}^* \equiv_{\mathrm{W}} (C_2')^*$.

For higher exponents, we use the observation that $(\mathsf{RT}^1_k)^{(n-1)} \leq_{\mathrm{W}} \mathsf{RT}^n_k$.

Thm (DSY). $(C_2^{(n)})^* \leq_{\mathrm{W}} {}^1(\forall k)\,\mathsf{RT}^n_k \leq_{\mathrm{W}} (C_2^{(n)})^* \star \mathsf{Use}^{(n)}$.

Recall SRT^2_k, the restriction of RT^2_k to stable colorings.

Thm (DSY). $(C_2'')^* \leq_{\mathrm{W}} {}^1(\forall k)\,\mathsf{SRT}^2_k \leq_{\mathrm{W}} (C_2'')^* \star \mathsf{Use}''$.

So our best bounds on the first-order parts of $(\forall k)\,\mathsf{RT}^2_k$ and $(\forall k)\,\mathsf{SRT}^2_k$ agree.

Bounded first-order parts

Bounding first-order parts

Defn.

Let $\mathsf{P} \in \mathcal{FO}$.

${}^{b}\mathsf{P}$: same instances as P, with the solutions to an instance X being all $n \in \mathbb{N}$ such that there is a P-solution $y \leq n$ to X.

Obs.

Obviously, ${}^{1}\mathsf{P} \leq_{\mathrm{W}} {}^{b}\mathsf{P}$ for all problems P.

Conversely, consider $\mathsf{C}_2 \in \mathcal{FO}$.

- $\mathsf{C}_2 \equiv_{\mathrm{W}} {}^{1}\mathsf{C}_2$ is not uniformly computably true.
- ${}^{b}\mathsf{C}_2$ is uniformly computably true.

SRT^2_2 and COH

COH : for every sequence $\langle c_0, c_1, \ldots \rangle$ of colorings $c_i : \omega \to 2$ there exists an infinite set X s.t. for all i, X is almost homogeneous for c_i.

Thm (Cholak, Jockusch, and Slaman 2001). $\mathsf{RCA}_0 \vdash \mathsf{RT}^2_2 \leftrightarrow \mathsf{SRT}^2_2 + \mathsf{COH}$.

The implication $\mathsf{SRT}^2_2 + \mathsf{COH} \to \mathsf{RT}^2_2$ is a formalization of a Weihrauch reduction: $\mathsf{RT}^2_2 \leq_\mathrm{W} \mathsf{SRT}^2_2 \star \mathsf{COH}$.

Thm (D., Hirschfeldt, Patey, Pauly 2019). $\mathsf{SRT}^2_2 \star \mathsf{COH} \not\leq_\mathrm{W} \mathsf{RT}^2_2$.

As mentioned, our best bounds on the first-order parts of Ramsey's theorem for pairs and the stable Ramsey's theorem agree. But they are not sharp.

Thm (DSY). ${}^b((\forall k)\ \mathsf{SRT}^2_k \star \mathsf{COH}) \equiv_\mathrm{W} {}^b(\forall k)\ \mathsf{RT}^2_k \equiv_\mathrm{W} {}^b\ (\forall k)\ \mathsf{SRT}^2_k$.

Thanks for your attention!