COMPUTABLE ANALYSIS ON THE SPACE OF MARKED GROUPS

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Abstract. We begin the systematic study of decision problems for finitely generated groups given by a solution to their word problem. We relate this to the study of computable analysis on the space of marked groups. We point out that several distinct approaches to computable analysis, some of which are sometimes considered obsolete, yield relevant results. In particular, we give necessary and sufficient conditions in terms of Banach-Mazur computability for the existence of a finitely presented group with solvable word problem but whose subgroups with a certain property cannot be recognized.

We classify group properties in different effective Borel hierarchies. For most common group properties, the classical and effective Borel classifications coincide. However, we show that the set of LEF groups is a closed set that is computably a G_δ, but not computably closed.

Finally, we show that the space of marked groups is a Polish space which is not *computably Polish,* because it does not admit a dense and computable sequence. This poses several interesting problems in terms of computable topology. The space of marked groups is the first natural example of this kind.

Contents

1. Introduction

When Max Dehn introduced in 1911 [Deh11, Deh87] the isomorphism problem for groups, he was concerned solely with finitely presented groups. The problem he considered was thus: given two finite presentations, decide whether or not they define isomorphic groups.

Max Dehn likely hoped for a positive solution to the isomorphism problem for finitely presented groups.

By the Adian-Rabin Theorem [Rab58, NB22], not only the isomorphism problem for finitely presented groups, but even the problem of deciding whether a finite presentation defines the trivial group are undecidable.

Because of this theorem, group theorists have learned to expect that "most properties" of finitely presented groups are undecidable, and that "almost nothing can be said of a group" given a finite presentation of it.

Consider however the following theorem of Groves and Wilton:

Theorem 1.1 ([GW09])**.** *There exists an algorithm that, given as input a presentation for a group G and a solution to the word problem in G, determines whether or not G is free.*

This theorem is non-trivial, the proof given in [GW09] relies on a deep understanding of the universal theory of free groups and of its models, the limit groups. Another proof of that result was given by Touikan in [Tou18].

Our interpretation of this theorem is the following:

Claim 1.2. The correct description to study decision problems for finitely generated groups is the one associated to "finite presentations together with a solution to the word problem".

We will discuss at length how this sentence can be formalized.

In light of Claim 1.2, the Adian-Rabin Theorem does not show that infinite groups cannot be manipulated by algorithms, but simply that we should consider finite presentations as incomplete descriptions, lacking a full solution to the word problem. In [Rau21], we show that the usual proof of the Adian-Rabin Theorem provides strictly no information about decision problems for groups when a solution to the word problem is available.

The description of finitely presented groups associated to Claim 1.2 is a composite one: it can be expressed as the *conjunction* of two simpler descriptions: groups given by finite presentations, and groups given by their solutions to the word problem. (We will see that this "conjunction" has a precise meaning in the lattice of subnumberings types of marked groups, it corresponds to the *join of two subnumbering types*.) A natural and necessary step towards understanding this composite description is to understand both of its halves. While plenty is known about decision problems for finitely presented groups, virtually nothing is known about decision problems for groups given by their solution to the word problem.

The present article is the first to address this problem.

Our main method is to use tools that were developed in computable analysis and to apply them to the space of marked groups.

However, it turns out that many interesting phenomena occur in this context, non-trivial interactions between computability, topology and group theory, that do not naturally occur in computable analysis. In particular, the space of marked groups is the first natural example of a Polish space which is not computably Polish with respect to its natural *representation*, i.e. with respect to the natural notion of computability that we have on it.

1.1. **Groups given by a solution to their word problem.** The first step of our study is to formalize the concept of "a group being given by a solution to its word problem".

For $k \in \mathbb{N}$, a *k-marked group* is a countable group G together with a generating tuple $(s_1, ..., s_k) \in G^k$. Two k-marked groups $(G, (s_1, ..., s_k))$ and $(H, (s'_1, ..., s'_k))$ are isomorphic if the function $s_i \mapsto s'_i$ extends to a group isomorphism. Most finite descriptions of finitely generated groups (and in particular finite presentations) are actually descriptions of marked groups.

By fixing a free group $\mathbb{F}_k$ with basis $(x_1, ..., x_k)$, we can identify any marked group $(G, (s_1, ..., s_k))$ with the kernel of the morphism $\mathbb{F}_k \to G$ defined by $x_i \mapsto s_i$. In other words: a marked group is uniquely determined by its set of *relations*.

Consider a bijection $\theta_k : \mathbb{N} \to \mathbb{F}_k$. The set of relations satisfied by the k-marked group (G, S) can be seen as a binary sequence $(u_n)_{n\in\mathbb{N}} \in \{0, 1\}^{\mathbb{N}}$, defined by

$$u_n = 1 \iff \theta_k(n) \text{ is a relation in } (G, S).$$

We call $(u_n)_{n\in\mathbb{N}}$ the *binary expansion* of (G, S).

It now appears that, in order to define functions that are "computable for groups given by a solution to the word problem", it is necessary and sufficient to be able to define computability on the Cantor space $\{0, 1\}^{\mathbb{N}}$.

We will thus turn our attention to the mathematical field where computability on Cantor space and on other spaces with cardinality that of the continuum is studied: *computable analysis*.

Defining computable functions on the Cantor space and on the set of real numbers is a problem that goes back to Turing himself, who, in [Tur36], right after having defined the machines that now bear his name, introduced those real numbers whose decimal expansion can be output by such a machine, the *computable real numbers*, and proposed a notion of a computable function from the computable reals to the computable reals.

Several other authors have proposed notions of computable functions on the real numbers or on the Cantor space. Contrary to what happens for functions defined on the natural numbers, for which there is a single unanimously acclaimed notion of computable function, in the case of functions defined on $\{0, 1\}^{\mathbb{N}}$ or on $\mathbb{R}$, there isn't a single definition that we will be able to choose and use throughout.

We will present here three notions: Banach-Mazur computability, Markov computability (also called Type 1 computability), and Type 2 computability. A detailed historical account can be found in [AB14].

Type 2 computability. Type 2 computability is a concept that goes back to Kleene. The general framework we now present is the one developed by Weihrauch between 1985 and 2000 [KW85, Wei00], it was later on extended and rendered more robust by Schröder [Sch01]. Modern references are [Pau16, IK21, Sch21, BH21].

To define Type 2 computability, we start by defining computability on Baire space $\mathbb{N}^{\mathbb{N}}$.

A partial function $f :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ is called *Type 2 computable* if there is a partial functions $F :\subseteq \mathbb{N}^* \to \mathbb{N}^*$ (where $\mathbb{N}^*$ designates the set of words over $\mathbb{N}$) such that

- F is computable (in the usual Church-Turing sense);
- F is prefix increasing: if u is a prefix of v, $F(u)$ is a prefix of $F(v)$;

- And $F(u_0...u_n) \underset{n\to\infty}{\to} f((u_n)_{n\in\mathbb{N}})$.

Notice that it follows immediately from the definition that a Type 2 computable function $f :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ is continuous in the Baire space topology, and even effectively continuous: there is a Turing machine which, given a finite sequence of natural numbers $w \in \mathbb{N}^*$, produces a sequence $(u_i)_{i\in\mathbb{N}} \in (\mathbb{N}^*)^{\mathbb{N}}$ such that

$$f^{-1}(w\mathbb{N}^{\mathbb{N}}) = \mathrm{dom}(f) \cap \bigcup_{i\in\mathbb{N}} u_i\mathbb{N}^{\mathbb{N}}.$$

Thus the preimage of a basic clopen set of Baire space can be computably written as a union of basic clopen sets.

Once computability is defined on Baire space, we can extend it to other sets by encoding their elements using sequences of natural numbers. A *representation* of a set X is a partial surjection $\rho :\subseteq \mathbb{N}^{\mathbb{N}} \to X$. A function $f : X \to Y$ between represented spaces (X, ρ) and (Y, τ) is called *computable,* or (ρ, τ)*-computable,* if there is a computable function $F :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ such that for all $p \in \mathrm{dom}(\rho)$, $f(\rho(p)) = \tau(F(p))$. The situation is summed up in the following diagram:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle\rho}\uparrow & & \uparrow{\scriptstyle\tau} \\ \mathbb{N}^{\mathbb{N}} & \xrightarrow[F]{} & \mathbb{N}^{\mathbb{N}} \end{array}$$

In this case, the function F is called a *computable realizer* of f. When $\rho(p) = x$, p is called a ρ*-name* of x.

Type 2 computability is strongly related to topology, because, as was said before, computable functions on Baire space are continuous, and conversely, continuous functions on Baire space are computable modulo some oracle. This relationship (i.e. continuity being "computability modulo some oracle") extends to all topological spaces that admit an *admissible representation*, these were characterized by Schröder [Sch01, Sch21] as the T_0 quotients of countably based spaces, they include all second countable spaces.

In particular, every represented set (X, ρ) is naturally equipped with the final topology of the representation ρ, and we get the following relation between computability and topology: the open sets of the final topology of ρ are exactly the sets that are *semi-decidable modulo some oracle.*

Markov computability. Markov computability, sometimes also called Type 1 computability, is the notion of computability that was considered by Turing in [Tur36]. It is named after Andreï Andreïevitch Markov[1], who was the leader of the "Russian school of constructivism" who considered that a mathematical object exists only if we can exhibit a Turing machine that can, in some sense, represent this object.

The idea of Markov computability is very similar to that of Type 2 computability, except that we only allow finite descriptions of objects. We thus consider as base set $\mathbb{N}$, with computability notion that of Church and Turing, and extend computability to other sets via *numberings.* A *numbering* of a set X is a partial surjection $\nu :\subseteq \mathbb{N} \to X$. As for representations, if $\nu(n) = x$, we say that n is a ν-name of x. A function $f : X \to Y$ between numbered sets (X, ν) and (Y, μ) is called *Markov computable,* or (ν, μ)*-computable,* if there is a computable function $F :\subseteq \mathbb{N} \to \mathbb{N}$ such that for all $n \in \mathrm{dom}(\nu)$, $f(\nu(n)) = \mu(F(n))$.

While, in Type 2 computable analysis, computable functions are by definition effectively continuous, some of the most important theorems in Markovian computable analysis are effective continuity theorems: theorems that state that Markov computable functions between sets with some specified properties are automatically effectively continuous. These include results due to Myhill and Shepherdson [MS55], Kreisel, Lacombe and Schoenfield [KLS57], Ceitin [Cei67], Moschovakis [Mos64] and Spreen [Spr98].

The easiest one to quote is the following one:

Theorem 1.3 (Kreisel-Lacombe-Schoenfield, [KLS57])**.** *Let* Tot *be the subset of* $\mathbb{N}^{\mathbb{N}}$ *consisting of total computable functions. Any Markov computable function* $f :$ Tot $\to$ Tot *is effectively continuous. And this statement is uniform: from the code of a computable function* f*, one can recover a code for a function that witnesses effective continuity of* f*.*

The theorem of Ceitin extends this result to Type 1 computable Polish spaces, that of Moschovakis to spaces that admit a certain Computable Choice Axiom (see Definition 5.44). Finally, Spreen gives in [Spr98] a unified proof of the results of Ceitin and of Myhill and Shepherdson, generalizing them both.

Other references on Markov computability are: [Abe80, Kus84, Mar63, Mar54].

[1]Markov was the son of the famous probabilist after whom Markov chains are named. Both are called Andreï Andreïevitch Markov.

Banach-Mazur computability. Banach-Mazur computability was invented by Banach and Mazur in Lwów before the second world war, but their results were not published before much later (see [Maz63]). A modern account of their results is given in [Her01].

The general setting of Banach-Mazur computability is the same one as that of Markov computability: the basic notion of computability is given by Church-Turing computability on $\mathbb{N}$, and is then transferred to other sets via numberings. However, the definition of a computable function changes: a function $f : X \to Y$ between numbered sets (X, ν) and (Y, μ) is called *Banach-Mazur computable* if it maps ν-computable sequences to μ-computable sequences. In other words: for every ν-computable sequence[2] $(u_n)_{n\in\mathbb{N}}$, the sequence $(f(u_n))_{n\in\mathbb{N}}$ is computable.

Markov computable functions are Banach-Mazur computable. Proving that the converse does not hold is difficult, three examples were given, one by Friedberg [Fri58b] (for a function Tot $\to \mathbb{N}$), two by Hertling in [Her05] and [Her06] (for functions $\mathbb{R}_c \to \mathbb{R}_c$ defined on the computable reals), and Bauer and Simpson have shown in [BS04] how to extend these results to computable Polish spaces that are "effectively without isolated points".

1.2. **Choosing a framework.** It is by now widely accepted that Type 2 computability is the better approach to computable analysis. In particular, it can be used to study computability on sets with cardinality that of the continuum, without having to restrict one's attention to points that have finite descriptions. Type 2 computability literature is much more abundant than Markov and Banach-Mazur literature.

On the other hand, Banach-Mazur computability is by many aspects not a good notion. Contrary to what happens in Type 2 computability and in Type 1 computability, we cannot define a Cartesian closed category using numbered sets as objects and Banach-Mazur computable functions as morphisms: Banach-Mazur computable functions are not associated to finite descriptions. Even worse: there exist some sets on which continuously many Banach-Mazur computable functions exist. Consider an infinite subset A of $\mathbb{N}$ which contains no infinite c.e. set. Then every function from A to $\mathbb{N}$ is Banach-Mazur computable: a computable sequence in A has finitely many elements. In this case, the notion of Banach-Mazur function is vacuous.

In accordance, much less has been written about Banach-Mazur computability than about Markov computability.

However, we prove the following result:

Theorem A. *Let P be a property of marked groups. The following are equivalent:*

- *There exists a finitely presented group G with solvable word problem, but where the problem of determining if a finitely generated subgroup of G has P is not semi-decidable;*
- *P is not Banach-Mazur semi-decidable (with respect to the numbering associated to groups given by word problem algorithms).*

This result is very interesting in that it shows that Banach-Mazur computability, despite all of its flaws, remains a valid notion that has to be studied.

This is the first general theorem of its kind, it deals at once with a wide range of group properties. This result can be used in conjunction with Markov's Lemma (Lemma G), it then applies to all properties that are "effectively not open" in the space of marked groups. Such a result was asked for in [DI22]. In [DI22], a finitely presented group in which amenability of subgroups is not semi-decidable was obtained thanks to an application of results in the theory of intrinsically computable relations [Ash98]. The fact that we obtain necessary and sufficient conditions shows that computable analysis methods are the correct tools to tackle such problems.

1.3. **Relationship between the three frameworks.** Type 2, Markov and Banach-Mazur computability are related via the following implications:

$$\text{Type 2 computable} \implies \text{Markov computable} \implies \text{Banach-Mazur computable.}$$

Notice that because of this sequence of implications, it is natural to expect that proving a function Type 2 computable is harder than proving it Type 1 computable which should be harder than proving it Banach-Mazur computable. However, this is not the case, and what happens in practice is that proving a function Markov computable without in fact proving that it is Type 2 computable is very hard. The main tool to do this is to use Kolmogorov complexity, see Theorem 1.8 of this introduction. Similarly, proving a function Banach-Mazur computable without in fact proving that it is Markov computable is difficult, this was achieved by Friedberg [Fri58b] and Hertling [Her05].

Let us now look at the contrapositive implications:

$$\text{Not Banach-Mazur computable} \implies \text{Not Markov computable} \implies \text{Not Type 2 computable.}$$

[2] A computable sequence is a $(\mathrm{id}_{\mathbb{N}}, \nu)$-computable function $g : \mathbb{N} \to X$. There is only only one notion of computable sequence, whether working in Type 2, Markov or Banach-Mazur computability.

Looking at this sequence of implications, we expect that proving that a function is not Banach-Mazur computable is harder than proving that it is not Markov computable which is harder than proving that it is not Type 2 computable. In this case, what happens is the following: most of the time, when a function is proved not Markov computable, it is in fact also proved not Banach-Mazur computable. The reason for this is simple: proving that a function is not computable by using a reduction to the halting problem amounts to finding a computable sequence on which this function is not computable, thereby proving that it is not Banach-Mazur computable. However, proving that a function is not Type 2 computable is not only in theory but also in practice easier than proving that it is not Markov computable. Indeed, Type 2 computable functions being continuous, topological arguments are sufficient to prove that a function is not Type 2 computable.

A typical instance of this phenomenon is the problem of recognizing a fixed group G. By compactness, as soon as G is infinite, there must be a group that is different from G which is adherent to the set of markings of G. This is sufficient to say that the function which is 1 on markings of G and 0 elsewhere is not Type 2 computable. However, this argument far from proves that that same function is not Markov computable: for it to be useful, we have to know that in the closure of the set of markings of G, there is a group, different from G, *and which has solvable word problem.* There is no automatic way of effectivizing this topological argument. Problem J gives another instance for which establishing a Type 1/Banach-Mazur undecidability result is harder than establishing its Type 2 counterpart.

1.4. **Partial conclusion.** The conclusion of the above considerations is the following. In the study of decision problems for groups given by their solution to the word problem,

- decidability results have to be established in terms of Type 2 computability,
- undecidability results have to be established in terms of Banach-Mazur computability.

In both cases, we ask for the stronger result. We stress that obtaining Banach-Mazur undecidability results is important because of Theorem A.

Despite the above conclusion, the present article is mostly written in the context of Markov computability: this is a middle ground between Type 2 computability and Banach-Mazur computability which provides a good solution in order to not work with several frameworks at the same time. But it will be clearly indicated throughout which undecidability results hold for Banach-Mazur computability, and all of our decidability results in fact hold for Type 2 computability.

We now present in more details our results. We start by introducing more precisely the objects we will talk about.

1.5. **Representation and numbering associated to word problems.** For each k, consider a free group $\mathbb{F}_k$ with basis $(x_1, ..., x_k)$. Denote by $\theta_k : \mathbb{N} \to \mathbb{F}_k$ the bijection associated to the shortlex order on $\mathbb{F}_k$.

In this context, as explained already, each marked group $(G, (s_1, ..., s_k))$ is determined by a unique binary expansion $(u_n)_{n\in\mathbb{N}} \in \{0,1\}^{\mathbb{N}}$, given by

$$u_n = 1 \iff \theta_k(n) \text{ is a relation of } (G, S).$$

We thus define a representation $\rho_{WP} :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathcal{G}$ of the set $\mathcal{G}$ of isomorphism classes of marked groups as follows:

$$\rho_{WP}((v_n)_{n\in\mathbb{N}}) = (G, (s_1, ..., s_k)) \text{ if and only if } v_0 = k \text{ and } (v_n)_{n\geq 1} \text{ is the binary expansion of } (G, (s_1, ..., s_k))$$

Let $(n, m) \mapsto \langle n, m\rangle$ denote Cantor's pairing function. Denote by $(\varphi_0, \varphi_1, \varphi_2, ...)$ a standard enumeration of all partial computable functions [Rog87]. We now define the numbering induced by the representation ρ_{WP}, denoted ν_{WP}, and which will interest us throughout.

We define ν_{WP} by

$$\nu_{WP}(\langle k, i\rangle) = (G, S) \iff (k, \varphi_i(0), \varphi_i(1), \varphi_i(2)...) \in \mathbb{N}^{\mathbb{N}} \text{ is a } \rho_{WP}\text{-name of } (G, (s_1, ..., s_k)).$$

We denote by $\mathcal{G}^+$ the set of marked groups with solvable word problem.

1.6. **Topology of the space of marked groups.** As we have seen in the previous paragraph, the set of k-marked groups can be seen as a subset of $\{0,1\}^{\mathbb{N}}$. It thus inherits the product topology of the Cantor space. This topology comes from the ultrametric distance d given by: for $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ elements of $\{0,1\}^{\mathbb{N}}$, let $n_0 = \inf(n, u_n \neq v_n) \in \mathbb{N} \cup \{+\infty\}$, and put $d((u_n)_{n\in\mathbb{N}}, (v_n)_{n\in\mathbb{N}}) = 2^{-n_0}$.

We denote $\mathcal{G}_k$ the topological space defined this way, and $\mathcal{G}$ the disjoint union of the $\mathcal{G}_k$, $k \geq 1$, equipped with the disjoint union topology. The topology of $\mathcal{G}$ is also metrizable, as the distance d can be extended to $\mathcal{G}$ by imposing that groups marked by families of different cardinalities be far away[3], say at distance exactly 2.

[3]Note that we do not embed $\mathcal{G}_n$ into $\mathcal{G}_{n+1}$ by identifying a marking with the marking obtained by adding the identity as a redundant generator, as is usually done, see for instance [CG05]. This is inconsequential.

The topology of the space of marked groups admits a natural basis. If $(r_1, ..., r_m;\ s_1, ..., s_{m'})$ is a pair of tuples of elements of $\mathbb{F}_k$, we denote by $\Omega^k_{r_1,...,r_m;s_1,...,s_{m'}}$ the set of k-marked groups in which $(r_1, ..., r_m)$ are indeed relations while $(s_1, ..., s_{m'})$ are not relations. Sets of this form are called the *basic clopen sets.*

We call a group an *abstract group* when we want to emphasize the fact that it is not a marked group.

1.7. **A non-computably Polish space.** The space of marked groups is a Polish space, i.e. it has a complete metric and is separable. It is also σ-compact, as the set of k-marked groups is compact.

By equipping the space of marked groups with a representation and an induced numbering, we have equipped it with a notion of computability. We can then ask which of the above facts hold effectively: is the space of k-marked groups computably compact? Is its metric computable? Is it computably separable? Is it computably complete?

In fact, each of these properties is easily seen to hold computably, except for separability: the space of marked groups is not computably separable.

Let us first quickly define the effective terms above, precise definitions are found in Section 5:

- A represented set has a *computable metric* if there is an algorithm that, given the names of two points x and y and some $n \in \mathbb{N}$, produces a rational approximation of $d(x, y)$ within 2^{-n}.
- A represented set is *computably complete* if there is a program which, given a Cauchy sequence $(x_n)_{n\in\mathbb{N}}$ and a *regulator* for it, i.e. a function g which, given n, produces N such that $\forall p, q > N$, $d(x_p, x_q) < 2^{-n}$, produces a name for its limit.
- A second countable space X with numbered basis $(B_i)_{i\in\mathbb{N}}$ is *computably compact* if there is a program that, given a sequence of basic sets whose union covers X, outputs a finite subset which already covers X. (See [Pau16] for a general definition that does not rely on second countability.)
- A represented space is *computably separable* if it admits a computable and dense sequence.

The above definitions are set in Type 2, the corresponding definition in Type 1 computability are obtained by replacing represented spaces by numbered sets. Note however that there is no convincing notion of computable compactness that works well in Type 1 computability. In particular, the set of computable elements of $\{0, 1\}^{\mathbb{N}}$ is not closed, thus not classically compact. We could however have expected a result such as "a *computable* union of basic clopen sets that covers the set of computable points $\{0, 1\}^{\mathbb{N}}$ must contain a finite sub-cover", but this is contradicted by a famous example of Kleene: there is a computable union of basic clopen sets whose union contains all computable points of $\{0, 1\}^{\mathbb{N}}$, but whose complement is infinite. See also [BL12] where constructive notions of compactness for metric spaces are discussed.

Following [Mos64], a numbered set with a computable metric is called *a recursive metric space.* We could maybe also call this a "Type 1 computable metric space". However, the commonly used Type 2 notion of *computable metric space* [BP03, IK21] asks strictly more than a computable metric: a computable metric space is a computably separable subset of a computably Polish space.

Lemma 1.4. *The represented space $(\mathcal{G}, \rho_{WP})$ has a computable metric, it is computably complete, and $\mathcal{G}_k$ is a computably compact subset of it.*

The following lemma gives the corresponding Type 1 statement, which is an immediate consequence of the above.

Lemma 1.5. *The triple $(\mathcal{G}^+, d, \nu_{WP})$ is a computably complete recursive metric space.*

The Boone-Novikov Theorem, which states that there exists a finitely presented group with unsolvable word problem, easily translates to the following:

Theorem 1.6 (Boone-Novikov, reformulated)**.** *No ρ_{WP}-computable sequence is dense in $\mathcal{G}$, and thus $(\mathcal{G}, d, \rho_{WP})$ is not a computable Polish space.*

Note that the above result is a Type 2 computability result. In the vocabulary of [dBPS20], the represented space $(\mathcal{G}, \rho_{WP})$ is a precomputable quasi-Polish space which is not overt. The corresponding result in Type 1 computability is obtained by reformulating the Boone-Rogers Theorem [BR66], which says that the word problem is not uniformly solvable on the set of finitely presented groups with solvable word problem.

Theorem 1.7 (Boone-Rogers, reformulated)**.** *No ν_{WP}-computable sequence is dense in $\mathcal{G}^+$, and thus $(\mathcal{G}^+, d, \nu_{WP})$ is not a recursive Polish space.*

Theorem 1.6 relies on the fact that the problem "is $\Omega^k_{R,T}$ the empty set" is not co-semi-decidable, Theorem 1.7 on the fact that the problem "is $\Omega^k_{R,T} \cap \mathcal{G}^+$ the empty set" is not co-semi-decidable. However, we show that while "is $\Omega^k_{R,T}$ the empty set" is semi-decidable, "is $\Omega^k_{R,T} \cap \mathcal{G}^+$ the empty set" is not. This appears as Theorem 6.8 in the text.

Theorem B. *No algorithm can stop exactly on the basic clopen sets $\Omega^k_{R,T}$ that do not contain a group with solvable word problem.*

A standard approach to studying computable Polish spaces is via notions of "computable presentations of Polish spaces". The first of these is due to Moschovakis [Mos80]. There are several essentially equivalent notions of computable presentations, see [GKP16] for the different definitions and their equivalence. We will use the following definition: a *computable presentation* of a Polish space X with distance d is a dense sequence $(u_n)_{n\in\mathbb{N}} \in X^{\mathbb{N}}$ for which the map

$$\begin{aligned} d : \mathbb{N}\times\mathbb{N} &\to \mathbb{R} \\ (n,m) &\mapsto d(u_n,u_m) \end{aligned}$$

is computable. In other words, there is an algorithm that, given $(n,m,p)\in\mathbb{N}^3$, produces a rational approximation of $d(u_n,u_m)$ within 2^{-p}. The use of presentations can be understood as a means of defining computable Polish spaces without going through the steps of defining spaces with a computable metric, computably complete and computably separable: all this is done in one single concise step. This approach is thus not adapted to our setting, since the space of marked groups, while being not computably separable, has a computable metric which is computably complete: we are making distinctions that are finer than what notions of computable presentations would allow us to do.

Let us quote a sentence of Moschovakis from [Mos80, Chapter 3], who, after defining a recursive presentation of a Polish space, states:

> "Not every Polish space admits a recursive presentation—but every interesting space certainly does."

We prove in Theorem 6.5:

Theorem C. *The space of marked groups $\mathcal{G}$ associated to its ultrametric distance d does not have a recursive presentation.*

Moschovakis' statement should be understood as the belief that the only examples of Polish spaces that do not admit recursive presentations will be artificially built counterexamples, while the Polish spaces that occur naturally in mathematics, and whose definitions do not involve Turing machines, always have those presentations. Thus the fact that a naturally arising Polish space exists for which this fails is very interesting.

The difference between Theorem C and Theorems 1.6 and 1.7 is that for this last result we do not demand a priori that groups should be given by their solutions to the word problem. We do not know how to obtain the above result with respect to any metric that induces the topology of the space of marked groups, so as to have a result that concerns the space of marked groups purely as a topological space. Note however that this question has no bearing on the present study: whether or not the topological space of marked groups has a certain representation ρ that makes it computably Polish has no consequence at all on the algorithmic problem "what can be said about a group, given a solution to its word problem". The representation ρ_{WP} is the only *realistic* one, in the sense that it is the only one which tells us what can on cannot be performed on actual computer algebra systems like GAP or Magma.

1.8. **Continuity problems on the space of marked groups.** We will now discuss what we know about continuity and effective continuity of computable functions on the space of marked groups.

A function $f : X \to Y$ between numbered sets with computable metrics is called *effectively metric continuous*[4] if there is a program that, given $x\in X$ and $n\in\mathbb{N}$, produces $p\in\mathbb{N}$ such that if $d(y,x) < 2^{-p}$, $d(f(x),f(y)) < 2^{-n}$.

Type 2 computability. In Type 2 computability, computable functions are by definition effectively continuous, and thus this remains true on the space of marked groups.

Markov computability. We ask both whether Markov computable functions are continuous or effectively metric continuous on the space of marked groups.

Markov computable functions are effectively metric continuous on Type 1 Computable Polish spaces by Ceitin's theorem, but by Theorem 1.7 this theorem does not apply to $(\mathcal{G}^+,d,\nu_{WP})$.

In [Mos64], Moschovakis proved a generalization of Ceitin's Theorem, which gives, as of today, the weakest known conditions on a recursive metric space, that are sufficient in order for the functions defined on this space to be effectively metric continuous. The hypotheses of this theorem are detailed in Section 5.4.

[4]Note that in the context of recursive Polish spaces, this notion was shown by Moschovakis [Mos64, Theorem 11] to be equivalent to the one obtained by effectivizing the idea that the preimage of an open set is open. In general it could be weaker. This is investigated in detail in [Rau23].

We however prove that the hypotheses of Moschovakis' Effective Continuity Theorem are not satisfied by $(\mathcal{G}^+, d, \nu_{WP})$. This a consequence of the following result (which appears as Theorem 6.9 in the text):

Theorem D (Failure of an Effective Axiom of Choice for groups)**.** *No algorithm can, on input a basic clopen set $\Omega^k_{r_1,...,r_m;s_1,...,s_{m'}}$ such that $\Omega^k_{r_1,...,r_m;s_1,...,s_{m'}} \cap \mathcal{G}^+ \neq \emptyset$, produce a word problem algorithm for a group in this basic clopen set.*

The following theorem, given to us by Mathieu Hoyrup, gives necessary and sufficient conditions for existence of a discontinuous Markov computable function defined on the space of marked groups. It is based on results of [HR16] and appears as Theorem 5.41 in the text.

Theorem 1.8. *There exists a ν_{WP}-computable and discontinuous function $f : \mathcal{G}^+ \to \{0,1\}$ if and only if there is a marked group (G,S) with solvable word problem, which is not isolated in $\mathcal{G}^+$, and a computable function $g : \mathbb{N} \to \mathbb{N}$ such that for each $n \in \mathbb{N}$ bigger than the smallest ν_{WP}-name of (G,S), (G,S) is the only marked group in $B((G,S), 2^{-g(n)})$ to have a ν_{WP}-name inside $\{0,1,...,n\}$.*

Or again:

$$\forall n \in dom(\nu_{WP}),\ \nu_{WP}(n) \neq (G,S) \implies d(\nu_{WP}(n),(G,S)) > 2^{-g(n)}.$$

In other words, (G,S), while not isolated, can only be approached by groups of very high Kolmogorov complexity. This is a strong negation of "(G,S) is the limit of a computable sequence of marked groups that are distinct from it".

Banach-Mazur computability. Banach-Mazur computable functions defined on a recursive Polish space are continuous [Maz63, Her01].

However, even on the computable reals or on the computable points of $\{0,1\}^{\mathbb{N}}$, continuous but not effectively metric continuous Banach-Mazur functions can exist. In [Her06], Hertling has even constructed a Banach-Mazur computable function on the computable reals which is at no point effectively metric continuous. We extend this to the space of marked groups by using a computable retraction on $\{0,1\}^{\mathbb{N}}$. The following appears as Corollary 6.2:

Proposition E. *There exists a Banach-Mazur computable function $\mathcal{G}^+ \to \mathbb{N}$ which is continuous but not effectively continuous.*

Whether or not Banach-Mazur computable functions must be continuous on the space of marked groups remains an open problem. There is no known characterization similar to that of Theorem 1.8 for continuity of Banach-Mazur computable functions. It is not clear whether the methods of [HR16] can be applied to Banach-Mazur computability.

Note also that there are countably many Banach-Mazur computable functions $\mathbb{R}_c \to \mathbb{R}_c$: by continuity, each such function is determined by its behavior on $\mathbb{Q}$, and because $\mathbb{Q}$ can be computably enumerated, each Banach-Mazur computable function f on $\mathbb{Q}$ has a finite description: the code of a machine that produces a computable enumeration of $f(\mathbb{Q})$. On the other hand, if A is an infinite subset of $\mathbb{N}$ with no infinite c.e. subset, there are continuously many Banach-Mazur computable functions $f : A \to \mathbb{N}$. What of Banach-Mazur functions of the space of marked groups?

Summary. We summarize the situation in the following table:

Notion of computability:	**Banach-Mazur**			**Markov**			**Type 2**		
	C^0	Eff. C^0	Cardinality	C^0	Eff. C^0	Cardinality	C^0	Eff. C^0	Cardinality
Arbitrary set	×	×	Can be $2^{\aleph_0}$	×	×	$\aleph_0$	✓	✓	$\aleph_0$
Space of marked groups	?	×	?	?	?	$\aleph_0$	✓	✓	$\aleph_0$
Computable Polish space	✓	×	$\aleph_0$	✓	✓	$\aleph_0$	✓	✓	$\aleph_0$

The main problem arising from our study is thus the following one:

Problem F (Main Problem)**.** *Are Banach-Mazur computable functions defined on $\mathcal{G}^+$ continuous? How many of these are there? Are Markov computable functions defined on $(\mathcal{G}^+, \nu_{WP})$ continuous or effectively continuous?*

A very interesting consequence of a positive answer to the first question would be that it would be possible to apply Theorem A to any non-clopen property of $\mathcal{G}^+$.

1.9. **Markov's Lemma.** As discussed above, the more advanced continuity theorems of Type 1 and of Banach-Mazur computable analysis cannot be applied to the space of marked groups.

However, the fact that $(\mathcal{G}^+, d, \nu_{WP})$ is an effectively complete recursive metric space provides us with some basic results coming from computable analysis that can be used to obtain group theoretical results. The most useful result in this regard is Markov's Lemma [Mar63, Theorem 4.2.2], as applied to the space of marked groups.

Lemma G (Markov's Lemma for groups)**.** *If $((G_n, S_n))_{n\in\mathbb{N}}$ is a ν_{WP}-computable sequence of marked groups that converges to a marked group (H, S) with solvable word problem, with $(G_n, S_n) \neq (H, S)$ for each n, then there is a ν_{WP}-computable sequence $((\Gamma_n, T_n))_{n\in\mathbb{N}} \in (\{(H, S)\} \cup \{(G_n, S_n), n \in \mathbb{N}\})^{\mathbb{N}}$ such that $\{n \in \mathbb{N}, (\Gamma_n, T_n) = (H, S)\}$ is not a c.e. subset of $\mathbb{N}$.*

In other words: if (H, S) is the limit of a ν_{WP}-computable sequence of groups in some set A with $(H, S) \notin A$, then $\{(H, S)\}$ is not a Banach-Mazur ν_{WP}-semi-decidable subset of $\{(H, S)\} \cup A$.

This result is elementary, and in fact one can find in the literature results that are obtained by "manual" applications of this lemma to different effectively converging sequences. See for instance [McC70] and [Loc81].

Markov's Lemma is our main tool to study decidability on the space of marked groups.

1.10. **Effective Borel hierarchies.** The study of the Borel hierarchy on $\mathcal{G}$ is a difficult topic related to many deep questions. For instance the fact that the set of groups with polynomial growth is open cannot, to the best of our knowledge, be proved without Gromov's Theorem. And the problem of deciding which identities of the form $\forall w,\ w^n = 1$ define a clopen set is related to the Burnside problem: for $n \in \{1, 2, 3, 4, 6\}$ the set of groups satisfying $\forall w,\ w^n = 1$ is clopen, and for $n \gg 1$ the construction of infinite Burnside groups shows that it is closed but not open. The remaining cases are not classified.

Some results about the Borel hierarchy on $\mathcal{G}$ were obtained in [BK22], but, again, many group properties remain unclassified.

By using Type 2 computability and the representation of $\mathcal{G}$, we can refine its Borel hierarchy. Indeed, a set is open in $\mathcal{G}$ if and only if it is ρ_{WP}-semi-decidable modulo "some oracle". In concrete examples, it is natural to ask which oracle is required. (Note that, by Theorem C, Moschovakis' approach to defining an effective Borel hierarchy [Mos80] is not applicable to the space of marked groups.)

Because of Theorem A, we must also investigate Markov and Banach-Mazur computability. This is done by restricting our attention to the set $\mathcal{G}^+$ of groups with solvable word problem, and by using the numbering ν_{WP} instead of the representation ρ_{WP}.

And thus each group property has four classifications: in the classical Borel hierarchy, in the effective Borel hierarchy for ρ_{WP}, in terms of the arithmetical hierarchy with respect to ν_{WP} (Type 1 classification), and in terms of a possible "Banach-Mazur arithmetical hierarchy". It seems that such an object has yet to be defined, but this is not a problem for us since we will only consider the first levels of such a hierarchy: semi-decidable, co-semi-decidable, and both or neither of these.

Properties are classified as clopen, open, closed, or "above", and similarly, properties are classified as decidable, semi-decidable, co-semi-decidable, or "above". Note that the main reason why we restrict our attention to these levels is that we are *unaware of differences (or conjectured differences) between the Borel hierarchy and its effective counterparts that occur above the very first levels.*

Borel hierarchy	Effective Borel hierarchy on $\mathcal{G}$	Type 1 hierarchy on $\mathcal{G}^+$	Banach-Mazur hierarchy on $\mathcal{G}^+$
Clopen	ρ_{WP}-decidable	ν_{WP}-decidable	Banach-Mazur ν_{WP}-decidable
Open	ρ_{WP}-semi-decidable	ν_{WP}-semi-decidable	Banach-Mazur ν_{WP}-semi-decidable
Closed	ρ_{WP}-co-semi-decidable	ν_{WP}-co-semi-decidable	Banach-Mazur ν_{WP}-co-semi-decidable

In most explicit cases, the four classifications will perfectly coincide. We thus expect a natural property $P \subseteq \mathcal{G}$ to be open if and only if it is ρ_{WP}-semi-decidable if and only if it is ν_{WP}-semi-decidable if and only if it is Banach-Mazur ν_{WP}-semi-decidable. Notice that establishing such correspondences will also involve proving negative results: the correspondence works only if a non-open property is also not ν_{WP}-semi-decidable.

The implications that are known to hold between the four classifications are represented below:

$$\text{clopen} \Longleftarrow \rho_{WP}\text{-decidable} \Longrightarrow \nu_{WP}\text{-decidable} \Longrightarrow \text{Banach-Mazur } \nu_{WP}\text{-decidable}$$
$$\text{open} \Longleftarrow \rho_{WP}\text{-semi-decidable} \Longrightarrow \nu_{WP}\text{-semi-decidable} \Longrightarrow \text{Banach-Mazur } \nu_{WP}\text{-semi-decidable}$$
$$\text{closed} \Longleftarrow \rho_{WP}\text{-co-semi-decidable} \Longrightarrow \nu_{WP}\text{-co-semi-decidable} \Longrightarrow \text{Banach-Mazur } \nu_{WP}\text{-co-semi-decidable}$$

Partial solutions to Problem F could add more implications. Indeed, a non-effective continuity theorem, either for Banach-Mazur or for Markov-computable functions, would permit to establish the corresponding implication:

$$\text{Banach-Mazur } \nu_{WP}\text{-decidable} \implies \text{clopen in } \mathcal{G}^+,$$

$$\nu_{WP}\text{-decidable} \implies \text{clopen in } \mathcal{G}^+.$$

An effective continuity theorem for Markov-computable functions would permit to establish the implication:

$$\nu_{WP}\text{-decidable} \implies \rho_{WP}\text{-decidable in } \mathcal{G}^+.$$

There are no other actual implications between the four notions. (In particular, establishing a Markov effective continuity result would *not* imply that a ν_{WP}-semi-decidable property is open in $\mathcal{G}^+$, this is known to be false.)

However, there is the following heuristic:

- A natural ν_{WP}-semi-decidable property is (*very much*) expected to be ρ_{WP}-semi-decidable and open in $\mathcal{G}^+$, and a natural ν_{WP}-co-semi-decidable property is (*very much*) expected to be ρ_{WP}-co-semi-decidable and closed in $\mathcal{G}^+$.

Semi-decidable properties that are not open do exist, but they are built using Kolmogorov complexity, those are sets one does not expect to run into when dealing with properties defined by algebraic or geometric constructions. (See Section 5.3 for an example.)

- A natural open property in $\mathcal{G}$ is (*a little*) expected to be ρ_{WP}-semi-decidable, and a natural closed property in $\mathcal{G}$ is (*a little*) expected to be ρ_{WP}-co-semi-decidable.

This second fact is rather of an empirical nature, and is justified by the results given in Section 7.2. Indeed, there is given a table containing around 30 group properties for which we prove that the four classifications coincide perfectly. Theorem A can be used with all those that are not clopen/decidable.

1.11. **Differences between the Borel hierarchy and its effective counterpart.** The main interest of studying effective versions of the Borel hierarchy on the space of marked groups lies in the fact that there are natural group properties which, while open, are not computably open, or while closed, are not computably closed. The main tool to obtain such properties is the study of universal theories of different classes groups, thanks to the following theorem.

Theorem H. *Let $\mathcal{C}$ be a set of groups, $\mathcal{S}(\mathcal{C})$ the set of subgroups of groups in $\mathcal{C}$, and $\overline{\mathcal{S}(\mathcal{C})}$ its closure. Let $T_\forall(\mathcal{C})$ be the universal theory of groups in $\mathcal{C}$, i.e., the set of first order universal sentences in the language of groups that are true in all groups of $\mathcal{C}$. We have the equivalence*

$$T_\forall(\mathcal{C}) \text{ is c.e.} \iff \overline{\mathcal{S}(\mathcal{C})} \text{ is computably closed.}$$

Theorem I. *The following sets are computably closed:*

- *The set of limit groups,*
- *The set $\overline{\mathcal{S}(\{H\})}$, where H is a hyperbolic group.*

The following sets are closed, but not computably so:

- *The set of LEF groups,*
- *More generally the closure of any set of finite groups that contains all finite three-solvable groups,*
- *The closure of the set of finite nilpotent groups,*
- *The closure of the set $\mathcal{G}^+$ of groups with solvable word-problem,*
- *The closure of the set of hyperbolic groups.*

Let us illustrate the meaning of the above theorem in the case of LEF groups.

By definition, a marked group (G, S) is LEF if every ball in the labeled Cayley graph of (G, S) is also a ball in the labeled Cayley graph of a certain finite group. This condition is naturally expressed by a $\forall\exists$ statement:

$$\forall n \in \mathbb{N}, \exists (F, S') \text{ finite marked group}, B_{(G,S)}(n) = B_{(F,S')}(n).$$

Upon closer examination, one realizes that the existential quantifier above is a bounded one: there are finitely many labeled graphs of diameter at most $2n$ and with edges having degree at most $|S|$, some of them are a part of the Cayley graph of a finite group, others are not, and there is a certain number $g(n)$ which is the least number such those that do belong to some finite group belong to a finite group of size at most $g(n)$. And thus the existential quantifier, being bounded, defines a finite union of clopen sets, itself clopen, the universal quantification then gives an intersection of clopen sets, which is thus closed.

However, Slobodskoi's Theorem [Slo81] on the unsolvability of the universal theory of finite groups implies, in particular, that the function g defined above cannot be computable (or even majored by a computable function).

And thus from the point of view of effective mathematics, the $\forall\exists$ statement that defines the set of LEF groups cannot be reduced to a single $\forall$ statement. The fact that g cannot be majored by a computable function was remarked by Bradford in [Bra22].

Confront Theorem I with the following problems:

Problem 1.9 (Gromov)**.** Is there a non-LEF hyperbolic group?

Problem 1.10 (Zelmanov)**.** Are infinite free Burnside groups LEF?

Computably closed properties are exactly those which it is algorithmically possible to refute. Theorem I can thus be seen as explaining why these two problems are so difficult.

Note finally that Theorem I is a Type 2 result: it says that the set of LEF groups is not ρ_{WP}-co-semi-decidable. We do not know how to obtain the corresponding Type 1 or Banach-Mazur results.

Problem J. *Extend the negative results of Theorem I to Type 1 and Banach-Mazur computability results.*

Other candidates for properties that would be open/closed but not computably so are given in the following conjecture:

Conjecture K. *The sets of finitely presented simple groups and of isolated groups are open but not computably open, and also not ν_{WP}-semi-decidable or Banach-Mazur ν_{WP}-semi-decidable.*

1.12. **Contents of this paper.** In Section 2, we describe the space of marked groups, and give a few results related to computability: impossibility of deciding whether or not a basic clopen set is empty, impossibility of deciding whether a basic clopen set contains a group with solvable word problem.

In Section 3, we fix the vocabulary about numberings that is required to present concepts from Type 1 and Banach-Mazur computable analysis.

In Section 4, we give several equivalent definitions that formalize the concept that a group is described by a word problem algorithm, or by its labeled Cayley graph.

In Section 5, we describe some results of Markovian computable analysis, giving proofs for some of them and references for the others. We prove Markov's Lemma and Mazur's Continuity Theorem, and quote the Kreisel-Lacombe-Schoenfield-Ceitin Theorem, and Moschovakis' extension of this theorem. We also include examples of computable but discontinuous functions built using Kolmogorov complexity.

In Section 6, we start investigating the space of marked groups as a recursive metric space. We prove that none of the continuity results given in the previous section can be applied to the space of marked groups.

In Section 7, we give a wide range of examples of group properties whose classical and effective Borel classifications coincide.

In Section 8, we relate the effective Borel hierarchy on the space of marked groups to the universal theory of groups, and give several examples of *closed but not computably closed* properties.

In Section 9, we prove Theorem A and give examples of its applications.

2. The topological space of marked groups

2.1. **Definitions.** Let k be natural number. A *k-marked group* is a finitely generated group G together with a k-tuple $S = (s_1, ..., s_k)$ of elements of G that generate it. We call S a *generating family*. Note that repetitions are allowed in S, the order of the elements matters, and S could contain the identity element of G. A *morphism of marked groups* between k-marked groups $(G, (s_1, ..., s_k))$ and $(H, (t_1, ..., t_k))$ is a group morphism φ between G and H that additionally satisfies $\varphi(s_i) = t_i$. It is an *isomorphism of marked groups* if φ is a group isomorphism. Marked groups are considered up to isomorphism. We call a group an *abstract group* when we want to emphasize the fact that it is not a marked group.

It is in fact convenient, when studying k-marked groups, to fix a free group $\mathbb{F}_k$ of rank k, together with a basis S for $\mathbb{F}_k$. A k-marking of a group G can then be seen as an epimorphism $\varphi : \mathbb{F}_k \to G$, the image of S by φ defines a marking with respect to the previous definition. Two k-marked groups are then isomorphic if they are defined by morphisms with identical kernels: the isomorphism classes of k-marked groups are classified by the normal subgroups of a rank k free group. The set S can be thought of as a set of generating symbols, and we often consider that all groups are generated by those letters.

Remark that a word problem algorithm for a group G is thus a description of a *marking* of G, and similarly, a presentation of a group defines a marked group.

We note $\mathcal{G}_k$ the set of isomorphism classes of k-marked groups, and $\mathcal{G}$ the disjoint union of the $\mathcal{G}_k$.

Note that some authors consider that each set $\mathcal{G}_k$ is embedded in the set $\mathcal{G}_{k+1}$, identifying a marking $(G, (g_1, ..., g_k))$ with the same marking where the identity e_G of G is added as a last generator: the k-marking $(G, (g_1, ..., g_k))$ is

identified with the $k+1$-marking $(G,(g_1,...,g_k,e_G))$. We do not adhere to this convention, for reasons that appear clearly in [Rau21, Proposition 59]: adding generators that define the identity to a marking can change whether or not a marked group is recognizable from finite presentations. It is thus detrimental in the study of decision problems for groups to identify a marking to the markings obtained by adding trivial generators.

For an abstract group G, we denote $[G]_k$ the set of all its markings in $\mathcal{G}_k$, and $[G]$ the set of all its markings in $\mathcal{G}$ (as in [CG05]). If (G,S) is a marking of G, we also define $[(G,S)]_k$ and $[(G,S)]$, those are identical to $[G]_k$ and $[G]$ respectively.

2.2. **Topology on $\mathcal{G}$.** We define a topology on $\mathcal{G}$ by equipping each separate space $\mathcal{G}_k$ with a topology, the topology we then consider on $\mathcal{G}$ is the disjoint union topology of the $\mathcal{G}_k$.

For each k, consider a finite set $\{s_1,...,s_k\}$, choose arbitrarily an order on the set $\{s_1,...,s_k\}\cup\{s_1^{-1},...,s_k^{-1}\}$, and enumerate by length and then lexicographically the elements of the free group $\mathbb{F}_k$ over S (i.e. following the *shortlex order*).

Denote by $\theta_k(n)$ the nth element obtained in this enumeration, θ_k is thus a bijection between $\mathbb{N}$ and $\mathbb{F}_k$.

To a normal subgroup N of $\mathbb{F}_k$ we can associate its characteristic function $\chi_N:\mathbb{F}_k\to\{0,1\}$, and composing it with the bijection θ_k, we obtain an element of the Cantor space $\mathcal{C}=\{0,1\}^{\mathbb{N}}$. This defines an embedding:

$$\Phi_k:\begin{cases}\mathcal{G}_k & \longrightarrow & \{0,1\}^{\mathbb{N}}\\ N\lhd\mathbb{F}_k & \longmapsto & \chi_N\circ\theta_k\end{cases}$$

of the space of k-marked groups into the Cantor space. We call the image $\Phi_k((G,S))$ of a k-marked group (G,S) the *binary expansion* of (G,S).

With the Cantor set being equipped with its usual product topology, the topology we will study on $\mathcal{G}_k$ is precisely the topology induced by this embedding.

It is easy to see that $\Phi_k(\mathcal{G}_k)$ is a closed subset of $\{0,1\}^{\mathbb{N}}$ of empty interior. It is thus compact.

The product topology on $\{0,1\}^{\mathbb{N}}$ admits a basis which consists of clopen sets: given any finite set $A\subseteq\mathbb{N}$ and any function $f:A\to\{0,1\}$, define the set Ω_f by: $(u_n)_{n\in\mathbb{N}}\in\Omega_f\iff\forall n\in A;\ u_n=f(n)$. Sets of the form Ω_f are clopen and form a basis for the topology of $\{0,1\}^{\mathbb{N}}$.

Sets of the form $\mathcal{G}_k\cap\Omega_f$ thus define a basis for the topology of $\mathcal{G}_k$.

The set $\mathcal{G}_k\cap\Omega_f$ is defined as a set of marked groups that must satisfy some number of imposed relations, while on the contrary a fixed set of elements must be different from the identity.

We fix the following notation. For m and m' natural numbers, and elements $r_1,...,r_m;\ s_1,...,s_{m'}$ of $\mathbb{F}_k$, we note $\Omega^k_{r_1,...,r_m;s_1,...,s_{m'}}$ the set of k-marked groups that satisfy the relations $r_1,...,r_m$, while they do not satisfy $s_1,...,s_{m'}$. We call $s_1,...,s_{m'}$ *irrelations.*

The sets $\Omega^k_{r_1,...,r_m;s_1,...,s_{m'}}$ are called the *basic clopen sets.*

The following lemma from [dCGP07] is very useful in many situation. For instance, it implies that if A is a set of marked groups closed under group isomorphism, then so are its closure and its interior.

Lemma 2.1 ([dCGP07], Lemma 1)**.** *Consider two markings $(G,S_1)\in\mathcal{G}_{m_1}$ and $(G,S_2)\in\mathcal{G}_{m_2}$ of a same group. Then there are clopen neighborhoods V_i, $i=1,2$ of (G,S_i) in $\mathcal{G}_{m_i}$ and a homeomorphism $\phi:V_1\to V_2$ mapping (G,S_1) to (G,S_2) and preserving isomorphism classes, i.e. such that, for every $(H,T)\in V_1$, $\phi((H,T))=(H,T')$ for some T'.*

In what follows, we call the set $r_1,...,r_m;\ s_1,...,s_{m'}$ of relations and irrelations *coherent* if $\Omega^k_{r_1,...,r_m;s_1,...,s_{m'}}$ is not empty. The Boone-Novikov theorem, which implies that there exists a finitely presented group with unsolvable word problem, directly implies the following:

Theorem 2.2 (Boone-Novikov reformulated)**.** *No algorithm can decide whether or not a given finite set of relations and irrelations is coherent. More precisely, there is an algorithm that stops exactly on incoherent sets of relations and irrelations, but no algorithm can stop exactly on coherent sets of relations and irrelations.*

Proof. It is always possible, given a finite set $r_1,...,r_m$ of relations, to enumerate their consequences, thus it can be found out if one of the irrelations $s_1,...,s_{m'}$ is a consequence of the relations $r_1,...,r_m$, in which case $r_1,...,r_m;\ s_1,...,s_{m'}$ is incoherent.

A finite set of relations and irrelations $r_1,...,r_m;\ s_1,...,s_{m'}$ is coherent if and only if $\Omega^k_{r_1,...,r_m;s_1,...,s_{m'}}$ is non-empty, but $\Omega^k_{r_1,...,r_m;s_1,...,s_{m'}}$ is non-empty if and only if it contains the finitely presented group $\langle S\,|\,r_1,...,r_m\rangle$, i.e. if and only if the relations $s_1,...,s_{m'}$ are not satisfied by $\langle S\,|\,r_1,...,r_m\rangle$.

Thus solving the word problem in $\langle S \,|\, r_1, ..., r_m\rangle$ is equivalent to producing an algorithm that stops on non-empty basic clopen sets of the form $\Omega^k_{r_1,...,r_m;w}$, as w varies. By the Boone-Novikov Theorem there exists a group for which no such algorithm exists. □

We will call a set $r_1, ..., r_m;\ s_1, ..., s_{m'}$ of relations and irrelations *word problem coherent*, or *wp-coherent*, if the basic clopen set $\Omega^k_{r_1,...,r_m;s_1,...,s_{m'}}$ contains a group with solvable word problem. The remarkable fact that the notions of coherence and wp-coherence differ follows from a theorem of Miller [Mil92, Corollary 3.9] which we present in details in Section 6.4 (see Theorem 6.11).

The fact that coherence and wp-coherence differ can be equivalently formulated as: "groups with solvable word problem are not dense in $\mathcal{G}$".

This remark calls for the following theorem:

Theorem 2.3 (Boone-Rogers reformulated)**.** *No algorithm can stop exactly on wp-coherent sets of relations and irrelations.*

Proof. This follows from the Boone-Rogers theorem [BR66] which states that there is no uniform solution to the word problem on the set of finitely presented groups with solvable word problem. Indeed, an effective way of recognizing wp-coherent sets of relations and irrelations would provide a uniform algorithm for the word problem on finitely presented groups with solvable word problem, by an argument similar to that of Theorem 2.2. □

We show in Theorem 6.8 that detecting wp-coherence is strictly more complex than detecting coherence: it is neither semi-decidable nor co-semi-decidable.

We finally include a lemma which will be useful in Section 8:

Lemma 2.4. *Inclusion between basic clopen sets is semi-decidable. And inclusion between finite unions of basic clopen sets is semi-decidable.*

This means that there is an algorithm that takes as inputs two finite sets of tuples of relations and irrelations, and stops if and only if the union of the basic clopen sets defined by the first set is a subset of the union of the basic clopen sets defined by the second set.

Proof. This lemma is in fact a simple corollary of the fact that emptiness of basic open sets is semi-decidable. We first give an example.

Consider two generators a and b. To determine whether the basic open set $\Omega_{(ab)^2\neq 1}$ is a subset of $\Omega_{b^2=1}$, we write both basic open sets as disjoint unions of basic open sets with the same elements of the free groups appearing as relations and irrelations:

$$\Omega_{(ab)^2\neq 1} = \Omega_{b^2=1;\,(ab)^2\neq 1} \sqcup \Omega_{b^2\neq 1,\,(ab)^2\neq 1},$$

$$\Omega_{b^2=1} = \Omega_{b^2=1;\,(ab)^2\neq 1} \sqcup \Omega_{b^2=1,\,(ab)^2=1}.$$

It is then clear that $\Omega_{(ab)^2\neq 1} \subseteq \Omega_{b^2=1}$ if and only if $\Omega_{(ab)^2\neq 1,\,b^2\neq 1}$ is empty.

In general, given two tuples of basic open sets, define the total support to be the set of all elements of the free group that appear either as relations or irrelations in one of these basic open sets. Decompose all the basic open sets as disjoint union as above. Inclusion of the union of the first tuple inside the second tuple is then equivalent to emptiness of a certain finite number of basic sets, which is semi-decidable by Theorem 2.2. □

2.3. **Different distances.** The topology defined above on the space of marked groups is metrizable. We describe here two possible distances which generate this topology, and which we may most of the time use interchangeably throughout this paper.

Those distances are defined on each $\mathcal{G}_k$ separately, the distances between groups marked by generating families that have different cardinalities can be fixed arbitrarily, as long as there remains a strictly positive lower bound to those distances. For convenience, we will adopt throughout the convention that the distance between groups marked by families with different cardinalities is exactly 2.

2.3.1. *Ultrametric distance.* For sequences $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ in $\{0,1\}^{\mathbb{N}}$ that are different, denote n_0 the least number for which $u_{n_0} \neq v_{n_0}$, and set $d((u_n)_{n\in\mathbb{N}}, (v_n)_{n\in\mathbb{N}}) = 2^{-n_0}$. If the sequences $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ are equal, set $d((u_n)_{n\in\mathbb{N}}, (v_n)_{n\in\mathbb{N}}) = 0$.

This defines an ultrametric distance on $\{0,1\}^{\mathbb{N}}$ which generates its topology, and which induces a distance on $\mathcal{G}_k$ via the embedding $\mathcal{G}_k \hookrightarrow \{0,1\}^{\mathbb{N}}$.

2.3.2. *Cayley Graph distance.* Yet another way to define a distance on $\mathcal{G}_k$ is by using labeled Cayley graphs. A labeled Cayley graph defines uniquely a marked group. And a word problem algorithm can be seen as an algorithm that produces arbitrarily large (finite) portions of the labeled Cayley graph of a given group, this is explained precisely in Section 4. We can define a new distance d_{Cay} as follows.

For two k-marked groups (G, S) and (H, S'), consider the respective labeled Cayley graphs, Γ_G and Γ_H. The natural bijection between S and S' induces a relabeling of Γ_G, denoted Γ'_G, replacing edge labels which belong to S by the corresponding elements in S'. The balls centered at the identity in Γ'_G and Γ_H agree up to a certain radius, call r the radius for which the balls of radius r of Γ'_G and Γ_H are identical, while their balls of radius $r+1$ differ. Then put $d_{\text{Cay}}((G, S), (H, S)) = 2^{-r}$. If Γ'_G and Γ_H are identical, r is infinite, and we put $d_{Cay}((G, S), (H, S)) = 0$. It is easy to check that d_{Cay} is an ultrametric distance which induces the topology of the space of marked groups.

The distance d_{Cay} could be preferred to d, as it brings a visual dimension to proofs, however in most cases it is just less precise than d: the only difference between d and d_{Cay} is that, in the computation of d_{Cay}, the relations are considered "in packs", corresponding to their length as elements of the free groups, while they are considered each one by one when using d. The choice of an order of the free group is precisely what allows to give more or less importance to relations of the same length.

3. Vocabulary about numberings

3.1. Numberings, subnumberings and numbering types.

3.1.1. *Paring functions.* Throughout, we denote by $(n, m) \mapsto \langle n, m\rangle$ Cantor's computable bijection between $\mathbb{N}^2$ and $\mathbb{N}$. We denote by fst and snd functions that define its inverse: for all n and m natural numbers, we have: $\text{fst}(\langle n, m\rangle) = n$ and $\text{snd}(\langle n, m\rangle) = m$. We extend this bijection to tuples by the formula $\langle n_1, ..., n_m\rangle = \langle n_1, \langle n_2, \langle ..., n_m\rangle ...\rangle$.

3.1.2. *First definitions.* We will now introduce numbered spaces and numbering types, which will be useful throughout this paper. For more details, see the chapter on numberings in Weihrauch's book [Wei87]. The expressions "numbering type" and "subnumbering" are ours.

Definition 3.1. Let X be a set. A *numbering* of X is a surjective function ν that maps a subset A of $\mathbb{N}$ onto X. A *subnumbering* of X is a numbering of a subset of X. We denote this by: $\nu :\subseteq \mathbb{N} \to X$.

The pair (X, ν) is a *subnumbered set*. The domain of ν is a subset of $\mathbb{N}$ denoted by $\text{dom}(\nu)$. The set of all subnumberings of X is denoted $\mathcal{N}_X$.

The image $\nu(\text{dom}(\nu))$ of ν is called the set of ν-computable points of X, and denoted X_ν. Given a point x in X, an integer n such that $\nu(n) = x$ is called a *ν-name* of x.

Definition 3.2. Let (X, ν) and (Y, μ) be subnumbered spaces. A function $f : X \to Y$ is called (ν, μ)-computable if there exists a partial recursive function $F :\subseteq \mathbb{N} \to \mathbb{N}$ such that for all n in the domain of ν, $f \circ \nu(n) = \mu \circ F(n)$. That is to say, there exists F which renders the following diagram commutative:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle\nu}\uparrow & & \uparrow{\scriptstyle\mu} \\ \mathbb{N} & \xrightarrow{F} & \mathbb{N} \end{array}$$

Whether a function f between subnumbered spaces (X, ν) and (Y, μ) is computable only depends on its behavior on the set of ν-computable points of X. The following is a simple consequence of the fact that the composition of computable functions (defined on $\mathbb{N}$) is computable:

Lemma 3.3. *If (X, ν), (Y, μ) and (Z, τ) are subnumbered sets, the composition of a (ν, μ)-computable function with a (μ, τ)-computable function is (ν, τ)-computable.*

The identity function $\text{id}_{\mathbb{N}}$ on $\mathbb{N}$ defines its most natural numbering.

Definition 3.4. If (X, ν) is a subnumbered set, a *ν-computable sequence* is an $(\text{id}_{\mathbb{N}}, \nu)$-computable function from $\mathbb{N}$ to X.

It was explained in the introduction why we need Banach-Mazur computability.

Definition 3.5. Let (X, ν) and (Y, μ) be subnumbered spaces. A function $f : X \to Y$ is called *Banach-Mazur (ν, μ)-computable* if the image of any ν-computable sequence is a μ-computable sequence.

We consider a partial order on the subnumberings of a set X:

Definition 3.6. A subnumbering ν of a space X is *stronger* than a subnumbering μ of this same space if the identity on X is (ν, μ)-computable. We denote this $\nu \succeq \mu$. Those subnumberings are *equivalent* if $\nu \succeq \mu$ and $\mu \succeq \nu$ both hold. We denote this by $\nu \equiv \mu$.

The relation $\nu \succeq \mu$ exactly means that there is an algorithm that, given a ν-name of a point in X, produces a μ-name of it, and thus can be interpreted as: a ν-name of a point x contains more information about it than a μ-name of this same point.

The following lemma is a direct consequence of the fact that composition preserves computability of functions.

Lemma 3.7. *The relation $\succeq$ is transitive and reflexive. In particular, the relation $\equiv$ is an equivalence relation.*

Definition 3.8. The *subnumbering types* on X are the equivalence classes of $\equiv$.

If (X, ν) and (Y, μ) are subnumbered spaces, and if $f : X \to Y$ is a (ν, μ)-computable function between X and Y, then f will be computable with respect to any pair of subnumberings of X and Y which are $\equiv$-equivalent respectively to ν and μ. Thus f can be considered computable with respect to the subnumbering types associated to ν and μ. Subnumbering types are in fact the objects that we want to be studying, rather than subnumberings. We denote $\mathcal{NT}_X$ the set of subnumbering types on X. Given a subnumbering, we denote $[\nu]$ its $\equiv$-equivalence class. We usually denote subnumbering types by capital greek letters, Λ, Δ, ...

The relation $\succeq$ defines an order on the set of subnumbering types, whose description is an important part of the study of the different subnumberings of a set.

Definition 3.9. Let (X, ν) be a subnumbered set. Then the equivalence relation defined on $\mathrm{dom}(\nu)$ by $n \sim m \iff \nu(n) = \nu(m)$ is called the *subnumbering equivalence induced by* ν [Ers99], and denoted η_ν.

The subnumbering ν is called *positive* [Mal71] when equality on X is a ν-semi-decidable relation, i.e. when there is a recursively enumerable set $L \subseteq \mathbb{N} \times \mathbb{N}$ such that $\eta_\nu = L \cap \mathrm{dom}(\nu) \times \mathrm{dom}(\nu)$.

The subnumbering ν is called *negative* when equality on X is a ν-co-semi-decidable relation, i.e. when there is a co-recursively enumerable set $L \subseteq \mathbb{N} \times \mathbb{N}$ such that $\eta_\nu = L \cap \mathrm{dom}(\nu) \times \mathrm{dom}(\nu)$.

It is called *decidable* when if it is both positive and negative.

Proposition 3.10. *If $\nu \succeq \mu$, and if μ is any of negative, positive or decidable, then so is ν.*

Proof. As ν-names provide more information than μ-names, if μ-names allow to (partially) distinguish points, then so do ν-names. □

3.1.3. *Constructions and examples.* There are several useful constructions that allow one to build subnumberings of sets using subnumberings of simpler sets.

Definition 3.11. Given a subnumbered set (X, ν) and a subset Y of X, define *the restriction of ν to Y* to be the subnumbering $\nu_{|Y}$ defined by the following:

$$\mathrm{dom}(\nu_{|Y}) = \mathrm{dom}(\nu) \cap \nu^{-1}(Y),$$

$$\forall n \in \mathrm{dom}(\nu_{|Y}),\ \nu_{|Y}(n) = \nu(n).$$

We define the product of two subnumberings.

Definition 3.12. If (X, ν) and (Y, ν) are subnumbered sets, *the product of the subnumberings ν and μ* is the subnumbering $\nu \times \mu$ of $X \times Y$ defined by the following:

$$\mathrm{dom}(\nu \times \mu) = \{\langle n, m\rangle \in \mathbb{N},\ n \in \mathrm{dom}(\nu),\ m \in \mathrm{dom}(\mu)\};$$

$$\forall n \in \mathrm{dom}(\nu \times \mu),\ \nu \times \mu(\langle n, m\rangle) = (\nu(n), \mu(m)).$$

We do not prove the following easy proposition:

Proposition 3.13. *If $\nu_1 \equiv \nu_2$ and if $\mu_1 \equiv \mu_2$ then $\nu_1 \times \mu_1 \equiv \nu_2 \times \mu_2$.*

Finally, we give the definition of the natural numbering of the set of computable functions between numbered sets. Denote by $(\varphi_0, \varphi_1, \varphi_2...)$ a standard Gödel enumeration of all recursive functions [Rog87]. (We do not describe here how to obtain such an enumeration: numberings only allow to translate any enumeration of the partial recursive functions to an enumeration of the computable functions between numbered sets.)

Definition 3.14. If (X, ν) and (Y, μ) are subnumbered sets, we define a subnumbering $\mu^\nu :\subseteq \mathbb{N} \to Y_\mu^{X_\nu}$ of the set of functions that map ν-computable points of X to μ-computable points of Y as follows:

$$\begin{aligned}\mathrm{dom}(\mu^\nu) = \{i \in \mathbb{N} |\ & \mathrm{dom}(\nu) \subseteq \mathrm{dom}(\varphi_i),\\ & \forall n, m \in \mathrm{dom}(\nu),\ \nu(n) = \nu(m) \implies \mu(\varphi_i(n)) = \mu(\varphi_i(m))\},\end{aligned}$$

$$\forall i \in \mathrm{dom}(\mu^\nu), \forall x \in X_\nu, \forall k \in \mathrm{dom}(\nu),\ (x = \nu(k)) \implies (\mu^\nu(i))(x) = \mu(\varphi_i(k)).$$

The image of μ^ν is exactly the set of (ν, μ)-computable functions. The following commutative diagram renders this definition clearer:

$$\begin{array}{ccc} X_\nu & \xrightarrow{\mu^\nu(i)} & Y_\mu \\ {\scriptstyle\nu}\uparrow & & \uparrow{\scriptstyle\mu} \\ \mathbb{N} & \xrightarrow{\varphi_i} & \mathbb{N} \end{array}$$

Several examples follow from those constructions:

- The Baire space $\mathcal{N} = \mathbb{N}^\mathbb{N}$ is naturally equipped with the subnumbering $\mathrm{id}_\mathbb{N}^{\mathrm{id}_\mathbb{N}}$, which we usually denote $c_\mathcal{N}$. Denote by $\mathcal{N}_c = \mathbb{N}_c^\mathbb{N}$ the set of computable points of the Baire space.
- The Cantor space $\mathcal{C} = \{0,1\}^\mathbb{N}$ admits a subnumbering induced by its natural embedding into $\mathcal{N}$, we denote it $c_\mathcal{C}$. Denote by $\mathcal{C}_c = \{0,1\}_c^\mathbb{N}$ the set of computable points of the Cantor space.

3.2. **Lattice operations on $\mathcal{NT}_X$.** We now introduce the lattice structure on $\mathcal{NT}_X$. Here, by lattice, we mean a partially ordered set that admits meet and join operations: we will thus show that any pair of subnumbering types in $\mathcal{NT}_X$ admits both a greatest lower bound and a least upper bound for the order $\succeq$.

The lattice operations of $\mathcal{NT}_X$ are the conjunction and the disjunction. Given two subnumberings ν and μ of a set X, we will define new subnumberings $\nu \wedge \mu$ and $\nu \vee \mu$ by saying respectively that a $\nu \wedge \mu$-name for a point x is a ν-name for x together with a μ-name for it, and that a $\nu \vee \mu$-name for a point y is either a ν-name for it, or a μ-name for it. Those definitions are explained below.

Definition 3.15. Let ν and μ be subnumberings of X. Define a subnumbering $\nu \vee \mu$ (the *disjunction* of ν and μ) by setting, for any natural number k, $\nu \vee \mu(2k) = \nu(k)$ and $\nu \vee \mu(2k+1) = \mu(k)$. The domain of $\nu \vee \mu$ is the set $\{2k, k \in \mathrm{dom}(\nu)\} \cup \{2k+1, k \in \mathrm{dom}(\mu)\}$.

Thus a $\nu \vee \mu$-name for a point x of X is either a ν-name for it, or a μ-name for it.

Proposition 3.16. *Let ν and μ be subnumberings of X. Then $\nu \succeq \nu \vee \mu$ and $\mu \succeq \nu \vee \mu$, and for any τ in $\mathcal{N}_X$, if $\nu \succeq \tau$ and $\mu \succeq \tau$, then $\nu \vee \mu \succeq \tau$.*

Proof. Left to the reader. □

Proposition 3.17. *Let ν, μ and τ be subnumberings of X.*

- *If $\nu \equiv \mu$, then $\nu \vee \tau \equiv \mu \vee \tau$,*
- *$\nu \vee \mu \equiv \mu \vee \nu$,*
- *$\nu \vee \nu \equiv \nu$*

Proof. Left to the reader. □

As a corollary to this proposition we can define the following:

Definition 3.18. Let Λ and Θ be subnumbering types of X. The subnumbering type $\Lambda \vee \Theta$ is defined as being the $\equiv$-class of $\nu \vee \mu$ for any ν in Λ and μ in Θ.

We now define the subnumbering obtained by giving as a name for a point in x both a ν-name and a μ-name for it, the "conjunction" of the subnumberings ν and μ. Recall that $(n, m) \mapsto \langle n, m\rangle$ denotes a pairing function. The decoding functions are denoted fst and snd.

Definition 3.19. Let ν and μ be numberings of X. Define a subnumbering $\nu \wedge \mu$ (the *conjunction* of ν and μ) by the following:

$$\begin{aligned}\mathrm{dom}(\nu \wedge \mu) = \{\langle n, m\rangle \in \mathbb{N}, n \in \mathrm{dom}(\nu), m \in \mathrm{dom}(\mu), \nu(n) = \mu(m)\},\\ \forall \langle n, m\rangle \in \mathrm{dom}(\nu \wedge \mu),\ \nu \wedge \mu(\langle n, m\rangle) = \nu(n).\end{aligned}$$

As we have already said, a $\nu \wedge \mu$-name for a point x of X is constituted of both a ν-name and a μ-name for it.

Proposition 3.20. *Let ν and μ be subnumberings of X. Then $\nu \wedge \mu \succeq \nu$ and $\nu \wedge \mu \succeq \mu$, and for any τ in $\mathcal{N}_X$, if $\tau \succeq \nu$ and $\tau \succeq \mu$, then $\tau \succeq \nu \wedge \mu$.*

Proof. We first show that $\nu \wedge \mu \succeq \nu$ and $\nu \wedge \mu \succeq \mu$. But given x in X and $\langle n, m\rangle$ in $\mathbb{N}$ such that $\nu \wedge \mu(\langle n, m\rangle) = x$, by definition of $\nu \wedge \mu$ one must have $x = \nu(n) = \mu(m)$, and thus the functions fst and snd which form the inverse of the pairing function are computable witnesses respectively for $\nu \wedge \mu \succeq \nu$ and $\nu \wedge \mu \succeq \mu$.

Suppose now that $\tau \in \mathcal{N}_X$ is such that $\tau \succeq \nu$ and $\tau \succeq \mu$. This means that there are computable functions $F :\subseteq \mathbb{N} \to \mathbb{N}$ and $G :\subseteq \mathbb{N} \to \mathbb{N}$ such that $\forall n \in \mathrm{dom}(\tau), \tau(n) = \nu(F(n))$ and $\forall n \in \mathrm{dom}(\tau), \tau(n) = \mu(G(n))$. Let n be a τ-name for a point x in X. Then $\langle F(n), G(n)\rangle$ is a $\nu \wedge \mu$-name for x, since $\nu(\mathrm{fst}(\langle F(n), G(n)\rangle)) = \nu(F(n)) = \tau(n)$ and $\mu(\mathrm{snd}(\langle F(n), G(n)\rangle)) = \mu(G(n)) = \tau(n)$. Thus $\tau \succeq \nu \wedge \mu$. □

The following proposition is straightforward.

Proposition 3.21. *Let ν, μ and τ be subnumberings of X.*

- *If $\nu \equiv \mu$, then $\nu \wedge \tau \equiv \mu \wedge \tau$;*
- *$\nu \wedge \mu \equiv \mu \wedge \nu$;*
- *$\nu \wedge \nu \equiv \nu$;*

Proof. Left to the reader. □

This proposition allows us to define the conjunction of subnumbering types:

Definition 3.22. Let Λ and Θ be subnumbering types of X. The numbering type $\Lambda \wedge \Theta$ is defined as being the $\equiv$-class of $\nu \wedge \mu$ for any ν in Λ and μ in Θ.

We thus obtain the following result:

Theorem 3.23. *$(\mathcal{NT}_X, \succeq, \wedge, \vee)$ is a lattice.*

3.3. **Computably enumerable and semi-decidable sets.** Let (X, ν) be a numbered set.

Definition 3.24. A subset Y of X is called *ν-computably enumerable* (ν-c.e.) if there is a computably enumerable subset A of $\mathbb{N}$, such that $A \subseteq \mathrm{dom}(\nu)$ and $Y = \nu(A)$.

Remark that a code for the c.e. set A that appears in this definition constitutes via ν a description of the ν-c.e. set Y. This allows us to define a subnumbering $\nu_{c.e.} :\subseteq \mathbb{N} \to \mathcal{P}(X)$ of the powerset of X.

Recall that if $(\varphi_0, \varphi_1, \varphi_2, ...)$ is a standard enumeration of the partial computable functions, we obtain an effective enumeration $(W_0, W_1, W_2, ...)$ of computably enumerable subsets of $\mathbb{N}$ by setting $W_i = \mathrm{dom}(\varphi_i)$. Define $\nu_{c.e.}$ by the following:

$$\mathrm{dom}(\nu_{c.e.}) = \{i \in \mathbb{N},\ W_i \subseteq \mathrm{dom}(\nu)\};$$
$$\forall i \in \mathrm{dom}(\nu_{c.e.}),\ \nu_{c.e.}(i) = \nu(W_i).$$

Of course we have the following lemma:

Lemma 3.25. *If $\nu \equiv \mu$, then $\nu_{c.e.} \equiv \mu_{c.e.}$.*

Proof. This follows from the standard fact that the image of a c.e. subset of $\mathbb{N}$ by a computable function is again c.e., and that this statement is effective: for each computable f, there is a computable map g such that for all $i \in \mathbb{N}$, $f(W_i) = W_{g(i)}$. This follows directly from the smn theorem. □

We now define decidable, semi-decidable and co-semi-decidable subsets.

Definition 3.26. A set Y is *ν-semi-decidable* if there is a computably enumerable subset A of $\mathbb{N}$ such that $A \cap \mathrm{dom}(\nu) = \nu^{-1}(Y)$. The set A is *ν-co-semi-decidable* if there is a co-computably enumerable subset B of $\mathbb{N}$ such that $B \cap \mathrm{dom}(\nu) = \nu^{-1}(Y)$.

A set is *ν-decidable* if it is both ν-semi-decidable and ν-co-semi-decidable.

An equivalent formulation is that a set Y is ν-decidable if there exists a procedure that, given a ν-name of an element in X, decides whether or not it belongs to Y. It is ν-semi-decidable if there exists a procedure that stops exactly on the ν-names of elements of Y, and ν-co-semi-decidable if there exists a procedure that stops exactly on the ν-names of elements that do not belong to Y.

Each of the definitions above allows to define subnumberings of the set $\mathcal{P}(X_\nu)$ of all subsets of the ν-computable points of X. We detail those definitions.

Associated to the subnumbering ν of X, we define three subnumberings $\nu_D :\subseteq \mathbb{N} \to \mathcal{P}(X_\nu)$, $\nu_{SD} :\subseteq \mathbb{N} \to \mathcal{P}(X_\nu)$ and $\nu_{co-SD} :\subseteq \mathbb{N} \to \mathcal{P}(X_\nu)$ associated respectively to ν-decidable, ν-semi-decidable and ν-co-semi-decidable subsets of X_ν.

As before, denote $W_i = \mathrm{dom}(\varphi_i)$.

- Define ν_D by
$$\mathrm{dom}(\nu_D) = \{i \in \mathbb{N} |\ \mathrm{dom}(\nu) \subseteq W_i, \varphi_i(\mathrm{dom}(\nu)) \subseteq \{0,1\},$$
$$\forall n, m \in \mathrm{dom}(\nu), \nu(n) = \nu(m) \implies (\varphi_i(n) = \varphi_i(m))\};$$
$$\forall i \in \mathrm{dom}(\nu_D),\ \nu_D(i) = \nu(\varphi_i^{-1}(\{1\})).$$
- Define ν_{SD} by
$$\mathrm{dom}(\nu_{SD}) = \{i \in \mathbb{N} |\ \forall n, m \in \mathrm{dom}(\nu), \nu(n) = \nu(m) \implies (n \in W_i \iff m \in W_i)\};$$
$$\forall i \in \mathrm{dom}(\nu_{SD}),\ \nu_{SD}(i) = \nu(\mathrm{dom}(\nu) \cap W_i).$$
- Define ν_{co-SD} by
$$\mathrm{dom}(\nu_{co-SD}) = \{i \in \mathbb{N} |\ \forall n, m \in \mathrm{dom}(\nu), \nu(n) = \nu(m) \implies (n \notin W_i \iff m \notin W_i)\};$$
$$\forall i \in \mathrm{dom}(\nu_{co-SD}),\ \nu_{co-SD}(i) = \nu(\mathrm{dom}(\nu) \setminus W_i).$$

We do not prove the following easy lemmas:

Lemma 3.27. *If $\nu \equiv \mu$, then $\nu_D \equiv \mu_D$, $\nu_{SD} \equiv \mu_{SD}$ and $\nu_{co-SD} \equiv \mu_{co-SD}$.*

Lemma 3.28. *The subnumberings ν_D, ν_{SD} and ν_{co-SD} satisfy $\nu_D \equiv \nu_{SD} \wedge \nu_{co-SD}$.*

A ν_{SD}-name of a ν-semi-decidable set is usually simply called its "code", the same goes for c.e., decidable and co-semi-decidable sets, and the four subnumberings $\nu_{c.e.}$, ν_D, ν_{SD} and ν_{co-SD} defined here are usually not explicitly called upon: the sentence "given a decidable subset of X" is usually used instead of "given the ν_D-name of a subset of X".

4. The word problem subnumbering type

We will give four equivalent definitions of the word problem subnumbering type, which we denote Λ_{WP}. Λ_{WP} is the $\equiv$-equivalence class of the numbering ν_{WP} defined in the introduction. It is in fact the object we want to study: the definition of ν_{WP} relies on arbitrary choices that are not $\equiv$-invariants.

We present four definitions to illustrate the fact that the definition of Λ_{WP} is robust. One is the usual definition, one uses the notion of computable group of Malcev and Rabin, one uses labelled Cayley graphs, the last one uses the embedding of the space of marked group in a disjoint union of countably many Cantor spaces. These definitions and their equivalence are well known, although this fact is seldom expressed in terms of subnumberings.

Before defining subnumberings of marked groups, we first include a paragraph on the numberings of the elements of fixed a marked group.

4.1. Numberings of the elements of a marked group.

In a marked group (G, S), it is customary to describe group elements by words on $S \cup S^{-1}$. This description is in fact canonical, in a sense that can be made precise using numberings.

We denote by $\Lambda_{(G,S)}$ the numbering type of (G, S) associated to the idea that elements are described by words on $S \cup S^{-1}$, we define it formally in the next definition. Denote by $(p_n)_{n\in\mathbb{N}}$ the sequence of prime numbers.

Definition 4.1. If (G, S) is a marked group, and $S = (s_0, s_2, ..., s_{k-1})$, we define the numbering $\nu_{(G,S)}$ on $\mathbb{N}$ as follows. For i between k and $2k-1$, denote by s_i the element s_{i-k}^{-1} of G. Given a natural number n, decompose it as a product of primes $n = p_0^{\alpha_0}...p_m^{\alpha_m}$. Then, for each i between 1 and m, denote $\widetilde{\alpha_i}$ the remainder in the Euclidian division of α_i by $2k$. We then put:
$$\nu_{(G,S)}(n) = s_{\widetilde{\alpha_1}} s_{\widetilde{\alpha_2}} ... s_{\widetilde{\alpha_m}} \in G.$$
The numbering type $\Lambda_{(G,S)}$ is the $\equiv$-equivalence class of $\nu_{(G,S)}$.

The arbitrary choices that are made in this definition are unimportant, as is shown by the following proposition:

Proposition 4.2. *The numbering type $\Lambda_{(G,S)}$ is the greatest subnumbering type in the lattice $\mathcal{NT}_G$ which satisfies the following conditions:*

- *All elements of G are $\Lambda_{(G,S)}$-computable;*
- *The group law and the inverse function on G are respectively $(\Lambda_{(G,S)} \times \Lambda_{(G,S)}, \Lambda_{(G,S)})$ and $(\Lambda_{(G,S)}, \Lambda_{(G,S)})$ computable.*

(In particular, any subnumbering type which satisfies these conditions can be compared to $\Lambda_{(G,S)}$ for the order $\succeq$.)

Note that the first condition of this proposition could be replaced equivalently by: "The elements of the generating tuple S are $\Lambda_{(G,S)}$-computable".

Proof. Suppose that μ is any numbering of G for which the group law and the inverse function are computable.

We show that $\nu_{(G,S)} \succeq \mu$, where $\nu_{(G,S)}$ is the numbering which was used to define the numbering type $\Lambda_{(G,S)}$. The generating set of G is denoted $S = (s_0, s_2, ..., s_{k-1})$. As μ is surjective, there are numbers u_0, ...,u_{k-1} such that $\mu(u_i) = s_i$.

As the group law is be computable for μ, there is a computable function F such that $\mu(F(i,j)) = \mu(i)\mu(j) \in G$, and a computable function I that computes the inverse for μ.

Given an integer n, which we decompose as a product of primes, $n = p_0^{\alpha_0}...p_m^{\alpha_m}$, recall that $\nu_{(G,S)}(n) = s_{\widetilde{\alpha_1}} s_{\widetilde{\alpha_2}} ... s_{\widetilde{\alpha_m}}$. Consider the function $H : \mathbb{N} \to \mathbb{N}$ defined as follows: map an integer α to its reminder modulo $2k$, which we denote $\widetilde{\alpha}$, then, if $\widetilde{\alpha} \leq k-1$, map it to $u_{\widetilde{\alpha}}$, otherwise, if $\widetilde{\alpha} \geq k$, map it to $I(u_{\widetilde{\alpha}-k})$.

Then $F(H(\alpha_1), F(H(\alpha_2), ...)))$ gives a μ-name for $\nu_{(G,S)}(n)$, and the procedure which produces this name from n is clearly computable. □

This proposition is in fact a simple application of a well known fact about subnumberings: if (X, Δ) is a subnumbered set, and if $\{f_1, ..., f_n\}$ are functions defined on cartesian powers of X to X, then there is a greatest subnumbering type Λ, that is below Δ ($\Delta \succeq \Lambda$), and for which all the functions f_i are computable. This was first detailed in [Wei87, Section 2.2]. The numbering type $\Lambda_{(G,S)}$ is obtained following this principle, using for Δ the equivalence class of the subnumbering ν_0 that describes only S: ν_0 is defined on $\{0, ..., k-1\}$ and it maps i to s_i.

Proposition 4.2 shows that describing group elements as products of the generators is the description that gives as much information as can be given on group elements if we want the group operations to be computable. The previous proposition thus has an easy corollary:

Corollary 4.3. *The numbering type $\Lambda_{(G,S)}$ has a decidable equality if and only if there exists a numbering type Δ of G which has decidable equality and which makes the group operations of G computable.*

Proof. Suppose that Δ is as in the hypotheses of the corollary. Then, by Proposition 4.2, we have $\Lambda_{(G,S)} \succeq \Delta$. Thus if Δ has a decidable equality, then so has $\Lambda_{(G,S)}$. □

Another corollary of Proposition 4.2 is:

Corollary 4.4. *If S and S' are two generating families of G, then $\Lambda_{(G,S)} = \Lambda_{(G,S')}$.*

Proof. Proposition 4.2 gives a characterization of $\Lambda_{(G,S)}$ which is independent of S. □

We can finally define solvability of the word problem:

Definition 4.5. A marked group (G, S) is said to have *solvable word problem* if equality is decidable for $\Lambda_{(G,S)}$.

In this case, what we call a *word problem algorithm* is the computable function that witnesses for the fact that $\Lambda_{(G,S)}$ is decidable.

4.2. **First definition of Λ_{WP}.** We will now define the subnumbering type associated to word problem algorithms. Our first definition is based on the numbering $\nu_{(G,S)}$ defined above.

The numbering $\nu_{(G,S)}$ has a decidable equality if and only if there is a computable function of two variables H such that $H(i,j) = 0$ if $\nu_{(G,S)}(i) = \nu_{(G,S)}(j)$ and $H(i,j) = 1$ otherwise. In this case, H is said to witness for the fact that $\nu_{(G,S)}$ has decidable equality.

Let $(\varphi_0, \varphi_1, ...)$ be a standard enumeration of all partial computable functions. We can consider that those functions depend on two variables using an encoding of pairs of natural numbers.

Define as follows a subnumbering ν_{WP}.

Pose $\nu_{WP}(n) = (G, S)$ if and only if n encodes a pair (k, m), i.e. $n = \langle k, m \rangle$, S is a family with k elements, $\nu_{(G,S)}$ is decidable, and φ_m is a computable function that witnesses for the fact that $\nu_{(G,S)}$ is decidable.

To check that this is a correct definition, we must check that the marked group (G, S) is uniquely defined by any of its ν_{WP}-names. This is to say: we must check that a word problem algorithm defines uniquely a marked group.

But this is straightforward: in the definition above, if $n = \langle k, m \rangle$ codes for two marked groups (G, S) and (H, S'), first it must be that S and S' have the same cardinality k, and, secondly, that φ_m is a computable function that witnesses for the decidability of both $\nu_{(G,S)}$ and $\nu_{(H,S')}$, but then (G, S) and (H, S') satisfy exactly the same relations, and thus they are isomorphic as marked groups.

We then define Λ_{WP} to be the $\equiv$-equivalence class of ν_{WP}.

4.3. **Labelled Cayley graph definition of Λ_{WP}.** Describing a marked group by a word problem algorithm is equivalent to describing it by its labelled Cayley graph. And thus, as we investigate decision problems for groups described by word problem algorithms, we are also studying "what can be deduced about a marked group, given its labeled Cayley graph". The graph should be suitably encoded into a finite amount of data. We detail this now.

Definition 4.6. If $(G, S = (s_1, ..., s_n))$ is a marked group, the labelled Cayley graph associated to it is the graph whose vertexes are elements of G, and whose (directed) edges are defined as follows: there is an edge with label s_i from the vertex $g_1 \in G$ to the vertex $g_2 \in G$ if and only if $g_1 s_i = g_2$.

We can then use the standard way to encode infinite graphs (computable graphs) to define a subnumbering of Cayley graphs. This again uses a standard enumeration $(\varphi_0, \varphi_1, \varphi_2, ...)$ of partial computable functions.

An oriented edge-labelled graph is a quadruple $(V, E, C, c : E \to C)$, where: V is a set, the set of vertices, E is the set of oriented edges, i.e. a subset of $V \times V$, C is a set of colors, which here we suppose finite, and c is a function which defines the color of a given edge.

Such a graph $\Gamma = (V, E, C, c : E \to C)$ is called computable if it is isomorphic to a graph $\Gamma_1 = (V_1, E_1, C_1, c_1 : E_1 \to C_1)$, which satisfies additionally that: V_1 is a computable subset of $\mathbb{N}$, E_1 is a computable subset of $\mathbb{N} \times \mathbb{N}$, $C_1 = \{1, ..., k\}$ for some $k \in \mathbb{N}$, and $c_1 : E_1 \to C_1$ is a computable function. In this case, Γ_1 is called a *computable model* of Γ.

Notice that each element of the tuple $(V_1, E_1, C_1, c_1 : E_1 \to C_1)$ is associated to some finite data that can be encoded: the characteristic function of V_1, denoted χ_{V_1}, which is computable, the characteristic function of E_1, denoted χ_{E_1}, the natural number k such that $C_1 = \{1, ..., k\}$, and the code of the function c_1.

We can thus define a subnumbering ν_Γ of edge-labelled graphs by saying that a computable model $\Gamma_1 = (V_1, E_1, C_1, c_1 : E_1 \to C_1)$ of a graph Γ is encoded by a tuple (i, j, k, l), where: $\varphi_i = \chi_{V_1}$, $\varphi_j = \chi_{E_1}$, $C_1 = \{1, ..., k\}$, $\varphi_l = c_1$.

One easily checks that the tuple (i, j, k, l) defines a unique graph, and thus the definition given above is sound.

We now have a subnumbering ν_Γ of edge-labelled graphs, we can restrict it to the set of Cayley graphs, and, because a labelled Cayley graph defines uniquely a marked group, we can consider that this new subnumbering is a subnumbering of the set of marked groups (we compose the subnumbering of graphs to the function that maps a labelled Cayley graph to the group it defines).

Note that, in the computable model of a Cayley graph, we can always suppose that there is a vertex at 0, and that it is associated to the identity element of the group whose graph it is.

This defines a subnumbering that we denote ν_{Cay}, which is associated to the idea "a marked group is described by algorithms that describe its Cayley graph".

We can now show:

Theorem 4.7. *The subnumbering ν_{Cay} is $\equiv$-equivalent to ν_{WP}.*

Proof. Given a ν_{Cay}-name for a marked group (G, S), i.e. given a Cayley graph Γ_1 for it, we can solve the word problem in (G, S) by following edges along a word: given a word $w = a_1 a_2 ... a_n$ on $S \cup S^{-1}$, starting from any vertex v_1 in Γ_1, we can find (by an exhaustive search) a sequence of vertices v_2, ..., v_{n+1} such that $v_{i+1} = v_i a_i$. We then solve the word problem by checking whether $v_1 = v_{n+1}$, i.e. by checking whether the word w defines a loop in the Cayley graph of G.

Conversely, given a word problem algorithm for (G, S), we can build a computable model $\Gamma_1 = (V_1, E_1, C_1, c_1 : E_1 \to C_1)$ of the Cayley graph of (G, S) as follows:

- Consider an enumeration of all words on $S \cup S^{-1}$ following a given order, say by length and then lexicographically. We can then delete, using the word problem algorithm of G, any element that is redundant in this list. We obtain a list w_0, w_1, w_2, ... which contains a single word on $S \cup S^{-1}$ for each element of G.
- Define a numbering μ of G by saying that $\mu(i) = w_i$. We put $V_1 = \text{dom}(\mu)$.
- V_1 is $\mathbb{N}$ if G is infinite, it is $\{0, ..., \text{card}(G) - 1\}$ otherwise. A computable characteristic function for V_1 can be obtained as follows: the list w_0, w_1, ... can be enumerated, and thus, given some number i, if it was found that G contains more than $i + 1$ elements, i belongs to V_1. On the contrary, while the list w_0, w_1, w_2, ... is enumerated, we can search for an initial segment of it of length less than i, and that is stable by multiplication by any generator. If such a segment is found, G must be finite, and we know it has cardinality less than i. In this case, we can conclude that $i \notin V_1$.
- We define E_1 and c_1 by saying that (i, j) is an edge labelled by $s \in S$ if and only if $w_i s = w_j$. This can be effectively checked thanks to the word problem algorithm for (G, S), and thus we can produce the computable functions that define E_1 and c_1. □

4.4. **Computable groups definition of Λ_{WP}.** Another point of view on Λ_{WP} follows (more or less) the point of view of Malcev in [Mal71, Chapter 18] and Rabin in [Rab60].

Definition 4.8. A countable group G is *computable* if there are computable function $P : \mathbb{N}\times\mathbb{N} \to \mathbb{N}$ and $I : \mathbb{N} \to \mathbb{N}$ such that either $(\mathbb{N}, P, I, 0)$ or $(\{0, .., \text{card}(G)-1\}, P, I, 0)$ (if G is finite) is a group that is isomorphic to $(G, \cdot, ^{-1}, e)$.

If G is a finitely generated group, an isomorphism $\Theta : (G, \cdot, ^{-1}, e) \to (\mathbb{N}, P, I, 0)$ can be described by giving the images of a generating family of G in $\mathbb{N}$.

We define a new subnumbering of marked groups, denoted ν_{MR}, as follows.

The description of a marked group (G, S) for ν_{MR} is an encoded quintuple (p, k, m, i, j), such that:

- The function φ_p has domain $\mathbb{N}$ if G is infinite, $\{0, ..., \text{card}(G) - 1\}$ otherwise;
- There exists a group isomorphism $\Theta : (G, \cdot, ^{-1} e) \to (\text{dom}(\varphi_p), P, I, 0)$, where P and I are computable functions;
- k gives the cardinality of S;
- m can be decoded as a k-tuple of elements of $\mathbb{N}$, which give the images of the elements of S by Θ;
- i and j define the computable functions P and I, i.e. $\varphi_i = P$ and $\varphi_j = I$.

As before, one can easily check that this definition is sound by checking that there is no ambiguity as to which isomorphism Θ is encoded by a number n.

The following theorem is well known (for instance it is contained in Theorem 7.1 in [Can66], or it is Theorem 4 of [Rab60]).

Theorem 4.9. *The subnumberings ν_{WP} and ν_{MR} are $\equiv$-equivalent.*

Proof. The proof is essentially the same as Theorem 4.7. □

4.5. **Subnumberings of $\mathcal{G}^+$ induced by subnumberings of the Cantor space.** Recall that in Section 2 we defined an embedding Φ_k of the space $\mathcal{G}_k$ of k-marked groups into the Cantor space.

The natural subnumbering $\nu_{\mathcal{C}}$ of the Cantor space is obtained by seeing it as the set of functions from $\mathbb{N}$ to $\{0, 1\}$.

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We can extend the subnumberings $\nu_{\mathcal{C}}$ to a countable disjoint union of Cantor spaces. Consider a countable set of Cantor spaces, denoted $\mathcal{C}_i$, $i \in \mathbb{N}$.

Define a subnumbering $\hat{\nu_{\mathcal{C}}}$ of $\bigcup_{n\in\mathbb{N}} \mathcal{C}_i$ by the following:

$$\hat{\nu_{\mathcal{C}}}(\langle i, j\rangle) \in \mathcal{C}_i,$$

$$\hat{\nu_{\mathcal{C}}}(\langle i, j\rangle) = \nu_{\mathcal{C}}(j).$$

The following proposition proves that this definition gives yet another way to introduce the subnumbering type Λ_{WP}.

Proposition 4.10. *The restriction of $\hat{\nu_{\mathcal{C}}}$ to the space of marked groups (seen as a subset of a countable union of Cantor spaces via the maps Φ_k) is equivalent to ν_{WP}.*

(The embeddings $\Phi_k : \mathcal{G}_k \longrightarrow \{0, 1\}^{\mathbb{N}}$ were chosen so that this proposition would hold: so that Φ_k would be $(\nu_{WP}, \nu_{\mathcal{C}})$-computable. Those embeddings were defined thanks to bijections $\theta_k : \mathbb{N} \to \mathbb{F}_k$.

Proof. We show that ν_{WP} is equivalent to $\hat{\nu_{\mathcal{C}}}$ on $\mathcal{G}$.

If $\nu_{WP}(n) = (G, S)$, then n encodes both the cardinality of S and the code for a function that, given two natural numbers, decides whether or not they encode the same element in G with respect to the encoding of the elements of a marked group defined in Section 4.1. Denote here $c_k : \mathbb{N} \to \mathbb{F}_k$ this encoding applied to the elements of the free groups.

If $\hat{\nu_{\mathcal{C}}}(n) = (G, S)$, then n encodes the cardinality k of S and the code for a function that, given a natural number m, indicates whether the elements $\theta_k(m)$ of $\mathbb{F}_k$ is a relation satisfied by (G, S).

The result then follows from the easy fact that each function $c_k \circ \theta_k^{-1} : \mathbb{N} \to \mathbb{N}$ is a computable surjection which has a computable right inverse, that this holds uniformly in k, and from the standard fact that the code of the composition of two functions can be computed given the codes for those functions. □

4.6. **Numbering and representation associated to presentations.** We now introduce the presentation representation. For each k, we have fixed a free group $\mathbb{F}_k$ and a basis $(x_0, ..., x_k)$. We use again the shortlex bijection $\theta_k : \mathbb{N} \to \mathbb{F}_k$.

We then define the representation $\rho_{pres} : \mathbb{N}^{\mathbb{N}} \to \mathcal{G}$ associated to presentations, via:

$$\rho_{pres}(p) = (\mathbb{F}_{p_0}/\langle\langle\{\theta_{p_0}(p_i), i \geq 1\rangle\rangle, (x_0, ..., x_{p_0})).$$

In words: the first term in the name of a marked group gives the arity of its marking, and the rest gives an infinite presentation via the corresponding bijection θ_k.

We denote by $\nu_{pres} :\subseteq \mathbb{N} \to \mathcal{G}$ the associated numbering:

$$\nu_{pres}(i) = \rho_{pres}(\varphi_i)$$

whenever φ_i is a total computable function. Groups in the image of ν_{pres} are the recursively presented groups. And ν_{pres} is the numbering that allows to discuss decision problems for groups given by recursive presentations.

This representation will be useful in Section 7.1.6. Note also that the final topology of ρ_{pres} is the Scott topology on the poset $(\mathcal{G}, \to)$ of marked groups. The Scott topology on $(\mathcal{G}^k, \to)$ is connected, this was used in [Rau21] to prove the Rice Theorem for recursively presented groups.

5. Introduction on some results in Markovian computable analysis

5.1. **Recursive Polish spaces.** This introduction follows mostly Kushner [Kus84], but it is hopefully more accessible, since the constructivist setting adds technical complications. Note that Section 5.2 follows closely Hertling [Her01], who studies Banach-Mazur computable functions.

5.1.1. *The computable reals.* A precise definition of the set $\mathbb{R}_c$ of computable reals first appeared in Turing's famous 1936 article [Tur36], but the numbering type of computable real numbers which is best suited to developing computable analysis was introduced one year later in the corrigendum [Tur37].

We start by defining a numbering $c_{\mathbb{Q}}$ of the set of rationals. The numbering $c_{\mathbb{Q}}$ is defined on $\mathbb{N}$. Given a natural number n, seen as encoding a triple $n = \langle a, b, c\rangle$ via a pairing function, we put $c_{\mathbb{Q}}(n) = (-1)^a \frac{b}{c+1}$.

We now define the Cauchy subnumbering $c_{\mathbb{R}}$ of $\mathbb{R}$. Denote again by $(\varphi_0, \varphi_1, \varphi_2...)$ a standard enumeration of all partial computable functions.

Definition 5.1. The Cauchy subnumbering of $\mathbb{R}$ is defined by the formulas:

$$\mathrm{dom}(c_{\mathbb{R}}) = \{i \in \mathbb{N}, \exists x \in \mathbb{R}, \forall n \in \mathbb{N}, |c_{\mathbb{Q}}(\varphi_i(n)) - x| < 2^{-n}\};$$

$$\forall i \in \mathrm{dom}(c_{\mathbb{R}}), c_{\mathbb{R}}(i) = \lim_{n\to\infty}(c_{\mathbb{Q}}(\varphi_i(n))).$$

Thus the description of a real number x is a Turing machines that produces a sequence $(u_n)_{n\in\mathbb{N}}$ of rationals with exponential convergence speed.

Definition 5.2. The set of $c_{\mathbb{R}}$-computable real numbers is denoted $\mathbb{R}_c$, the $c_{\mathbb{R}}$-computable reals are simply called the *computable real numbers.*

Several other definitions of the real numbers (decimal expansions, Dedekind cuts), when rendered effective, yield subnumbering types that define the same set of computable real numbers, but that are not $\equiv$-equivalent to the Cauchy subnumbering type -they are strictly stronger. See for instance [Mos79].

Proposition 5.3 (Rice, [Ric54])**.** *Addition, multiplication and divisions are $(c_{\mathbb{R}} \times c_{\mathbb{R}}, c_{\mathbb{R}})$-computable functions defined respectively on $\mathbb{R}_c \times \mathbb{R}_c$, $\mathbb{R}_c \times \mathbb{R}_c$ and $\mathbb{R}_c \times (\mathbb{R}_c \setminus \{0\})$.*

The following is a well known proposition which follows easily from Markov's Lemma, see Lemma 5.29, this result is implicit in [Tur36].

Proposition 5.4. *Equality is undecidable for computable reals. There is no algorithm that, given two computable reals x and y, chooses one of $x \leq y$ or $y < x$ which is true.*

5.1.2. *Recursive metric spaces.* We can now define what is a recursive metric spaces. We stick to the old terminology that uses the term "recursive" instead of "computable" because the term "computable metric space" has a different meaning, and it is not the case that every recursive metric space is a computable metric space. In particular recursive metric spaces are countable sets. *Computable metric spaces* as studied in Type 2 computable analysis [BP03] or *recursively presented Polish spaces* as studied in effective descriptive set theory [Mos80] can be uncountable sets.

Definition 5.5. *A recursive metric space (RMS)* is a countable metric space (X, d) equipped with a numbering $\nu :\subseteq \mathbb{N} \to X$, and such that the distance function $d : X \times X \to \mathbb{R}$ is $(\nu \times \nu, c_{\mathbb{R}})$-computable.

Whether a triple (X, d, ν) is a RMS depends only on the $\equiv$-equivalence class of the numbering ν, we can thus define a RMS to be a metric space equipped with a compatible numbering type.

Definition 5.6. A function f between recursive metric spaces (X, d, ν) and (Y, d, μ) is called *effectively metric continuous* if given a ν-name n of a point x in X and the $c_{\mathbb{R}}$-name of a computable real number $\epsilon > 0$, it is possible to compute the $c_{\mathbb{R}}$-name of a number $\eta > 0$ such that

$$\forall y \in X;\ d(x, y) < \eta \implies d(f(x), f(y)) < \epsilon.$$

Note that the number η is allowed to depend not only on x and ϵ, but also on the given names for those points. A program that computes the $c_{\mathbb{R}}$-name for η given x and ϵ is said to *witness for the effective continuity of* f.

We also introduce effective continuity for computably second countable spaces.

Definition 5.7. A function f between second countable spaces (X, ν) and (Y, μ) with bases $(B_i^X)_{i\in\mathbb{N}}$ and $(B_i^Y)_{i\in\mathbb{N}}$ is called *effectively continuous* if there is a program that given $i \in \mathbb{N}$ produces the code of a c.e. set $A \subseteq \mathbb{N}$ such that

$$f^{-1}(B_i^X) = \bigcup_{t\in A} B_t^Y.$$

Proposition 5.8. *If (X, d, ν) is a RMS, if $Y \subseteq X$, then $(Y, d, \nu_{|Y})$ is also a RMS.*

Proof. Obvious. □

Example 5.9. The following spaces, equipped with their usual distances and numberings, are recursive metric spaces: $\mathbb{N}$, $\mathbb{R}_c$, the set $\{0, 1\}_c^{\mathbb{N}}$, the set $\mathbb{N}_c^{\mathbb{N}}$. Although details can be found in [Kus84, Chapter 9], we still include a proof for the Cantor space.

Proposition 5.10. *The set $\{0, 1\}_c^{\mathbb{N}}$ of computable points of the Cantor space equipped with its ultrametric distance d and its usual numbering $\nu_{\mathcal{C}}$ is a RMS.*

Proof. Given the $\nu_{\mathcal{C}}$ name of two sequences $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$, we show that we can compute arbitrarily good approximations of their distance. To compute the distance between $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ with an error of at most 2^{-n}, it suffices to enumerate the first n digits of $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$. Then, if $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ agree on their first n digits, it must be that $d((u_n)_{n\in\mathbb{N}}, (v_n)_{n\in\mathbb{N}}) < 2^{-n}$, and so 0 constitutes a good approximation of $d((u_n)_{n\in\mathbb{N}}, (v_n)_{n\in\mathbb{N}})$. Otherwise, the first index after which the sequences $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ differ can be computed exactly, and thus also the distance $d((u_n)_{n\in\mathbb{N}}, (v_n)_{n\in\mathbb{N}})$.

In both cases, the desired approximation of $d((u_n)_{n\in\mathbb{N}}, (v_n)_{n\in\mathbb{N}})$ can be computed. □

Corollary 5.11. *Let $\hat{\nu_{\mathcal{C}}}$ denote the natural numbering of a disjoint union of Cantor spaces $\bigcup_{i\in\mathbb{N}} \{0, 1\}^{\mathbb{N}}$ ($\hat{\nu_{\mathcal{C}}}$ was defined in Section 4.5). Consider the metric d that puts different copies of $\{0, 1\}^{\mathbb{N}}$ at distance exactly 2. Then the set $(\bigcup_{i\in\mathbb{N}} \{0, 1\}^{\mathbb{N}})_{\hat{\nu_{\mathcal{C}}}}$ of computable points of $\bigcup_{i\in\mathbb{N}} \{0, 1\}^{\mathbb{N}}$, equipped with the distance d and numbering $\hat{\nu_{\mathcal{C}}}$, is a RMS.*

Proof. Given the $\hat{\nu_{\mathcal{C}}}$ -names of two sequences, we decide whether or not they belong to the same copy of the Cantor space. If they don't, we know that their distance is 2. Otherwise apply Proposition 5.10. □

The following proposition is fundamental for us. Recall that $\mathcal{G}^+$ is the set of marked groups with solvable word problem.

Corollary 5.12. *The space of marked groups with solvable word problem $\mathcal{G}^+$ equipped with its ultrametric distance d and with the numbering type Λ_{WP} is a recursive metric space.*

Proof. This follows directly from the previous corollary, together with Proposition 5.8, which states that a subset of a RMS with the induced numbering remains a RMS, and with the fact that the numbering type Λ_{WP} is the numbering induced on the space of marked groups by the numbering type $[\hat{\nu_{\mathcal{C}}}]$ defined on a disjoint union of Cantor spaces, as detailed in Section 4.5. □

The space of marked groups equipped with the distance d_{Cay} defined in Section 2 and with the numbering type Λ_{WP} is also a RMS. We leave it to the reader to prove this easy fact.

5.1.3. *Effective completeness and effective separability.* Since the space of marked groups is a Polish space, it is natural to ask whether $\mathcal{G}^+$ is a recursive Polish space, that is, whether it is effectively complete and effectively separable.

Here, we define those two notions, and give some properties that follow from them. The importance of these notions lies in the facts that Ceitin's Theorem is set on recursive Polish spaces.

Definition 5.13. A sequence $(u_n)_{n\in\mathbb{N}}$ of computable points in X is called *effectively convergent* if it converges to a point $y \in X$, and if there exists a computable function $f : \mathbb{N} \to \mathbb{N}$ such that:

$$\forall (n, m) \in \mathbb{N}^2; n \geq f(m) \implies d(u_n, y) \leq 2^{-m}.$$

A sequence $(u_n)_{n\in\mathbb{N}}$ of computable points in X is called *effectively Cauchy* if there exists a computable function $f : \mathbb{N} \to \mathbb{N}$ such that:

$$\forall (p, q, m) \in \mathbb{N}^3; p, q \geq f(m) \implies d(u_p, u_q) \leq 2^{-m}.$$

In both cases the function f is called a *regulator* for the sequence $(u_n)_{n\in\mathbb{N}}$.

Definition 5.14. Let (X, d, ν) be a recursive metric space. An *algorithm of passage to the limit* (the name is from [Kus84] and [Spr98]) is an algorithm that takes as input a computable Cauchy sequence together with a regulator for it, and produces the ν-name of a point towards which this sequence converges.

A recursive metric space (X, d, ν) is *effectively complete* if it admits an algorithm of passage to the limit.

Note that even though we will in this paper focus exclusively on effectively complete spaces when stating continuity theorems, a weaker condition is sufficient to apply Markov's Lemma and the theorems of Ceitin and Moschovakis: that there exist an algorithm of passage to the limit which works only on converging sequences. Thus for instance the open interval $(0, 1)$ admits such an algorithm for $c_{\mathbb{R}}$, even though it is not effectively complete, as the sequence $n \mapsto 2^{-n}$ does not converge in $(0, 1)$. As the space of marked groups is effectively complete, we are not concerned here with theorems that do not rely on effectively completeness.

It is easy to see that any recursive metric space can be effectively completed into an effectively complete metric space, we describe here the construction.

Recall that $(\varphi_0, \varphi_1, \varphi_2, ...)$ denotes an effective enumeration of all partial computable functions.

Definition 5.15. Let (X, d, ν) be a recursive metric space. Denote by $(\hat{X}, d)$ the classical completion of (X, d). Denote by $j : X \hookrightarrow \hat{X}$ the embedding of X into $\hat{X}$. Define a subnumbering $\hat{\nu}$ of $\hat{X}$ by

$$\mathrm{dom}(\hat{\nu}) = \{i \in \mathbb{N}, \forall p \in \mathbb{N}, \forall q > p,\ d(\nu(\varphi_i(p)), \nu(\varphi_i(q))) < 2^{-p}\};$$
$$\forall i \in \mathrm{dom}(\hat{\nu}),\ \hat{\nu}(i) = \lim_{n\to\infty} j(\nu(\varphi_i(n))).$$

We then obtain the effective completion of (X, d, ν) by restricting our attention to the $\hat{\nu}$-computable points of $\hat{X}$: the *effective completion* of (X, d, ν) is the triple $(\hat{X}_{\hat{\nu}}, d, \hat{\nu})$.

We do not prove the following well known result (see for instance [Spr98, Corollary 2.13]):

Lemma 5.16. *The numbering $\hat{\nu}$ defined above is the supremum for $\succeq$ of the sets of subnumberings of $\hat{X}$ that satisfy the following conditions:*

- *the injection $j : X \hookrightarrow \hat{X}$ is $(\nu, \hat{\nu})$-computable;*
- *the triple $(\hat{X}_{\hat{\nu}}, d, \hat{\nu})$ is effectively complete.*

Proposition 5.17. *Let (X, d, ν) be an effectively complete recursive metric space. A closed subset Y of X, together with the numbering induced by ν, is also an effectively complete metric space.*

Proof. It suffices to notice that algorithm of passage to the limit for X also works for Y. □

The following proposition can be found in [Kus84]:

Proposition 5.18. *The set $\{0, 1\}^{\mathbb{N}}_c$ is effectively complete.*

This has the immediate corollary:

Corollary 5.19. *The recursive metric space $(\mathcal{G}^+, d, \nu_{WP})$ is effectively complete.*

This last fact could easily have been proved directly. We now describe the effective notion associated to separability.

Definition 5.20. A recursive metric space (X, d, ν) is called *effectively separable* if there exists a ν-computable sequence $(u_n)_{n\in\mathbb{N}}$ of points in X that is dense in X.

Note that the numbering $\nu :\subseteq \mathbb{N} \to X$ is surjective, but not total. An equivalent formulation of the above statement is the following: (X, d, ν) is effectively separable if there exists a c.e. subset A of $\mathbb{N}$ such that $A \subseteq \mathrm{dom}(\nu)$ and $\nu(A)$ is dense in X. As the set $\mathrm{dom}(\nu)$ could be very complicated, there is no reason a priori for it to contain even an infinite c.e. set.

We can finally define recursive Polish spaces.

Definition 5.21. A recursive metric space (X, d, ν) which is both effectively complete and effectively separable is a *recursive Polish space*.

Note that the above definition cannot be found in the modern literature on computability of Polish spaces (see for instance [IK21]), because the use of notions of "presentations of metric spaces" (see Definition 5.27) allows for a direct definitions of computably Polish spaces. This approach does not permit to discuss effective separability, because all defined spaces are effectively separable.

Example 5.22. The following spaces, equipped with their usual distances and numberings, are recursive Polish spaces: $\mathbb{N}$, $\mathbb{R}_c$, the set of computable points of the Cantor space $\{0,1\}^{\mathbb{N}}_c$, the set of computable points of Baire Space $\mathbb{N}^{\mathbb{N}}_c$.

The following proposition, while obvious, shall be very useful in its group theoretical version.

Proposition 5.23. *Let (X, d, ν) be a recursive metric space, and Y be a ν-c.e. set in X. Then any point in the closure $\overline{Y}$ of Y is the limit of a computable sequence of points of Y that converges effectively.*

This proposition could have been phrased: any computable point in the closure of Y is automatically in its "effective closure".

Proof. Given a point x adherent to Y, we can define a computable sequence $(v_n)_{n\in\mathbb{N}}$ by: v_n is the first element, in a fixed enumeration of Y, which is proven to satisfy $d(x, v_n) < 2^{-n}$. □

This proposition has the immediate corollary:

Corollary 5.24. *In a recursive Polish space with a dense and computable sequence $(u_n)_{n\in\mathbb{N}}$, each point is the limit of an effectively Cauchy and computable sequence extracted from $(u_n)_{n\in\mathbb{N}}$.*

The following result is important in that it relates the less general approach based on notions of recursive presentations of Polish spaces (see Definition 5.27) to the approach based on numberings. Indeed, using numberings, it is possible to define spaces that are not necessarily effectively separable, and not effectively complete. The idea behind the use of recursive presentations of Polish spaces is to use the fact that the computable structure on a recursive Polish space is entirely defined by the distance function between elements of its dense sequence to define it in one single step. The following theorem shows that we arrive to the same result via both approaches.

Theorem 5.25. *A recursive Polish space is computably isometric to the effective completion of any of its computable and dense sequences.*

What this theorem means is in fact the following. Let (X, d, ν) be a recursive Polish space, and $(u_n)_{n\in\mathbb{N}}$ a ν-computable and dense sequence. One can restrict ν to the dense sequence, to obtain a recursive metric space $(\{u_n, n \in \mathbb{N}\}, d, \nu_{|(u_n)_{n\in\mathbb{N}}})$. One can then consider the effective completion of $(\{u_n, n \in \mathbb{N}\}, d, \nu_{|(u_n)_{n\in\mathbb{N}}})$, following Definition 5.15.

One thus obtains a recursive Polish space $(A, d, (\hat{\nu_{|(u_n)_{n\in\mathbb{N}}}}))$. The set A can be seen as a subset of the abstract completion $\hat{X}$ of X. The numbering ν can also be seen as a subnumbering of $\hat{X}$ via the embedding of X into its classical completion. What we then claim is:

$$(\hat{\nu_{|(u_n)_{n\in\mathbb{N}}}}) \equiv \nu.$$

Proof. Let (X, d, ν) be a recursive Polish space, and $(u_n)_{n\in\mathbb{N}}$ any ν-computable and dense sequence. Denote $(\varphi_0, \varphi_1, \varphi_2, ...)$ an effective enumeration of all partial recursive functions.

Denote by $\hat{X}$ the abstract completion of X, X is seen as a subset of $\hat{X}$.

The effective completion (Definition 5.15) of $(u_n)_{n\in\mathbb{N}}$ is given by a subnumbering μ of the abstract completion of the metric space $(\{u_n, n \in \mathbb{N}\}, d)$. The embedding $\{u_n, n \in \mathbb{N}\} \hookrightarrow X$ induces an embedding of this completion inside $\hat{X}$. We thus see μ as a subnumbering of $\hat{X}$.

The fact that (X, d, ν) is effectively complete directly implies that the image of μ in $\hat{X}$ lies in fact in X. We thus consider that μ is a subnumbering of X.

We show now that the subnumberings ν and μ are equivalent, i.e. the identity on X is both (ν, μ)-computable and (μ, ν)-computable.

A μ-name of a point x in X is the description of a computable Cauchy sequence that converges to x, with $g : n \mapsto 2^{-n}$ being a regulator for this sequence. The algorithm of passage to the limit of (X, d, ν) can thus be applied to this description with g as regulator, and it yields precisely a ν-name of x. This shows that the identity on X is (μ, ν)-computable.

To show that it is also (ν, μ)-computable, one only has to notice that the procedure described in the proof of Proposition 5.23 is uniform, in that it allows, given a ν-name of a point x, to produce a computable sequence extracted from $(u_n)_{n\in\mathbb{N}}$ which converges to x with the desired speed. This is precisely a μ-name for x. □

The following corollary to Theorem 5.25 shows that when a Polish space can be equipped with a recursive Polish space structure, this structure is unique.

Corollary 5.26. *Given an abstract Polish space (X, d) with a dense sequence $(u_n)_{n\in\mathbb{N}}$, there is at most one subnumbering type Λ of $\mathcal{NT}_X$ which makes of (X_Λ, d, Λ) a recursive Polish space with $(u_n)_{n\in\mathbb{N}}$ as a computable and dense sequence.*

Proof. This follows directly from Theorem 5.25. □

Note that this corollary starts with an actual Polish space, possibly uncountable. The countable set of points that are the Λ-computable points is uniquely given by the theorem. Theorem 5.25 allows for a very simple definition of what is an "effective Polish space", that relies only on the distance between the elements of a dense sequence. Weihrauch and Moschovakis both used such definitions. We give here a definition that mimics that of Moschovakis, see [GKP16] for the complete definitions of Weihrauch and Moschovakis, and their differences. Note however that the definition that we give is weaker than the ones given by Weihrauch and Moschovakis -a space that admits a recursive presentation in this sense also admits one following the definitions of Weihrauch and Moschovakis. Note also that the following definition depends on a choice of a distance for the considered Polish space, and not only on its topology.

Definition 5.27. A *recursive presentation of a Polish space* (X, d) is a dense sequence $(u_n)_{n\in\mathbb{N}}$ of points of X such that the function

$$\begin{aligned} \phi : \mathbb{N} \times \mathbb{N} &\to \mathbb{R}_c \\ (n, m) &\mapsto d(u_n, u_m) \end{aligned}$$

is $(\mathrm{id}_\mathbb{N} \times \mathrm{id}_\mathbb{N}, c_\mathbb{R})$-computable.

Note that the concept above applies to an actual Polish space, and not to a countable set of points.

The term presentation is from [Mos80], and has no relation the notion of a presentation for a group. The following proposition renders explicit the link between recursive presentations and recursive Polish spaces.

Proposition 5.28. *A Polish space (X, d) admits a recursive presentation if and only if it admits a numbering ν with dense image that makes of (X_ν, d, ν) a recursive Polish space.*

Proof. If ν is a numbering of X that makes of (X_ν, d, ν) a recursive Polish space, it means that there exists a ν-computable dense sequence $(u_n)_{n\in\mathbb{N}}$. The function ϕ defined by $\phi(n, m) = d(u_n, u_m)$ is then computable, and thus $(u_n)_{n\in\mathbb{N}}$ defines a recursive presentation of (X, d).

Conversely, if $(u_n)_{n\in\mathbb{N}}$ is a sequence which is dense in X, the condition that $(n, m) \mapsto d(u_n, u_m)$ be computable exactly asks that the function

$$\begin{aligned} \mu : \mathbb{N} &\to X \\ n &\mapsto u_n \end{aligned}$$

define a numbering of X which makes of (X_μ, d, μ) a recursive metric space. Then, the effective completion of μ defines a numbering ν of X, for which (X_ν, d, ν) is a recursive Polish space. And X_ν is dense in X. □

In Section 6, we prove that the space of marked group, associated to its usual ultrametric distance, does not have a recursive presentation. We will also prove that the space of marked groups does not contain dense and computable sequences of groups described by word problem algorithms, but the former result is more general, because we do not suppose *a priori* that a dense sequence should consist in groups described by word problem algorithms.

5.2. **Markov's Lemma and abstract continuity.** We will give here a proof of the fact that Banach-Mazur computable functions on a recursive Polish space are continuous, starting with Markov's Lemma, which is both very useful and very simple to use, and which will remain our main tool in the space of marked groups, since the effective continuity theorems that we present in Section 5.4 are not applicable there.

We fix a recursive metric space (X, d, ν) which we suppose effectively complete. Denote by $\mathcal{A}_{lim}$ an algorithm of passage to the limit for it.

Lemma 5.29 (Markov, [Mar54], English version: [Mar63], Theorem 4.2.2)**.** *Suppose that a ν-computable sequence $(u_n)_{n\in\mathbb{N}}$ effectively converges in X to a point x. Suppose additionally that for any n, $u_n \neq x$. Then there is a ν-computable sequence $(w_p)_{p\in\mathbb{N}}$ of $X^{\mathbb{N}}$ such that: for each p, $w_p \in \{u_n, n \in \mathbb{N}\} \cup \{x\}$, and the set $\{p, w_p = x\} \subseteq \mathbb{N}$ is co-c.e. but not c.e..*

Proof. Consider an enumeration of all Turing machines M_0, M_1, ... To the machine M_p, we associate a computable sequence $(x_n^p)_{n\in\mathbb{N}}$ of points in X. To define $(x_n^p)_{n\in\mathbb{N}}$, start a run of the machine M_p with no input. While it lasts, the sequence $(x_n^p)_{n\in\mathbb{N}}$ is identical to the sequence $(u_n)_{n\in\mathbb{N}}$. If, at some point, the machine M_p stops, the sequence $(x_n^p)_{n\in\mathbb{N}}$ becomes constant.

To sum this definition up, $(x_n^p)_{n\in\mathbb{N}}$ is defined as follows:

While (M_p does not stop) enumerate $(u_n)_{n\in\mathbb{N}}$.

If (M_p stops in k computation steps), set $x_n^p = u_k$ for $n \geq k$.

Each sequence $(x_n^p)_{n\in\mathbb{N}}$ is Cauchy, and in fact it converges at least as fast as the original sequence $(u_n)_{n\in\mathbb{N}}$. Thus the algorithm of passage to the limit $\mathcal{A}_{lim}$ can be applied to any sequence $(x_n^p)_{n\in\mathbb{N}}$, using the regulator of convergence of $(u_n)_{n\in\mathbb{N}}$.

The sequence $(w_p)_{p\in\mathbb{N}}$ is the sequence obtained by using the algorithm of passage to the limit on each sequence $(x_n^p)_{n\in\mathbb{N}}$, for $p \in \mathbb{N}$.

If follows directly from our definitions that if the machine M_p is non-halting, the sequence $(x_n^p)_{n\in\mathbb{N}}$ is identical to $(u_n)_{n\in\mathbb{N}}$, and thus w_p, which is its limit, is equal to x. On the other hand, if M_p halts in k computation steps, we have $w_p = u_k$ and thus w_p is different from x. □

Lemma G is an immediate corollary of the above together with Corollaries 5.12 and 5.19.

Corollary 5.30. *Let f be a Banach-Mazur computable function between recursive metric spaces X and Y, suppose that X is effectively complete, and let $(x_n)_{n\in\mathbb{N}}$ be a computable sequence that effectively converges to a point x in X. Then the sequence $(f(x_n))_{n\in\mathbb{N}}$ converges to $f(x)$.*

Proof. We proceed by contradiction. Suppose that the sequence $(f(x_n))_{n\in\mathbb{N}}$ does not converge to $f(x)$. Then there must exist a subsequence $(x_{\phi(n)})_{n\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ and a rational $r > 0$ such that

$$\forall n \in \mathbb{N},\ d(f(x_{\phi(n)}), f(x)) > r.$$

The existence of such a sequence, which need not a priori be computable, implies that there must also exist such a sequence where, additionally, the function $\phi : \mathbb{N} \to \mathbb{N}$ is computable.

This follows from Proposition 5.23, as the set of terms of the sequence $(x_n)_{n\in\mathbb{N}}$ for which $d(f(x_n), f(x)) > r$ holds is a c.e. set.

Finally, the function f can be used to distinguish between the elements of the sequence $(x_{\phi(n)})_{n\in\mathbb{N}}$ and its limit x, as, given a computable point u in X, it is possible to chose one which is true between $d(f(u), f(x)) > r$ and $d(f(u), f(x)) < r$, if we know a priori that $d(f(u), f(x))$ is not equal to r. This contradicts Markov's Lemma, and thus $(f(x_n))_{n\in\mathbb{N}}$ must converge to $f(x)$. □

We can use the previous corollary to prove that, under the additional assumption that the space X be a recursive Polish space, any Banach-Mazur computable function $f : X \to Y$ is continuous -not necessarily effectively so. This corollary of Markov's Lemma was first proven by Mazur for functions defined on intervals of $\mathbb{R}_c$ (see [Maz63]). See [Her01].

Corollary 5.31 (Mazur's Continuity Theorem)**.** *Consider a Banach-Mazur computable function $f : X \to Y$ between a recursive Polish space X and a recursive metric space Y. Then f is continuous.*

Proof. Denote $(u_n)_{n\in\mathbb{N}}$ an effective and dense sequence of X.

Suppose that f is not continuous at a point x of X. This means that there exists a sequence $(x_n)_{n\in\mathbb{N}}$ and a real number $r > 0$ such that $(x_n)_{n\in\mathbb{N}}$ converges to x, but for any $n \in \mathbb{N}$, $d(f(x_n), f(x)) > r$. Any point of $(x_n)_{n\in\mathbb{N}}$ is the limit of an effective subsequence of $(u_n)_{n\in\mathbb{N}}$, by Proposition 5.23. And thus, by Corollary 5.30, for each point x_k of the sequence $(x_n)_{n\in\mathbb{N}}$, there must exist a point $u_{\phi(k)}$ in the sequence $(u_n)_{n\in\mathbb{N}}$, such that both inequalities $d(u_{\phi(k)}, x_k) < 2^{-k}$ and $d(f(u_{\phi(k)}), f(x)) > r$ hold.

Thus there exists a subsequence $(u_{\phi(n)})_{n\in\mathbb{N}}$ of $(u_n)_{n\in\mathbb{N}}$, which converges to x and such that for any $n \in \mathbb{N}$, $d(f(u_{\phi(n)}), f(x)) > r$.

This subsequence is a priori not computable, but by Proposition 5.23, the abstract fact that such a sequence exists automatically implies that there must also exist such a subsequence that is, in addition, both computable and effectively converging to x.

Finally, we conclude by applying Corollary 5.30 again. □

5.3. **Computable but discontinuous functions and Kolmogorov complexity.** We give here an example of a Markov computable function that is not continuous and of a semi-decidable set in a recursive Polish space which is not open. The fact that those exist is well known, we explain it here in terms of Kolmogorov complexity, following Hoyrup and Rojas [HR16].

We also render explicit how the main theorem of [HR16] gives necessary and sufficient conditions for a computable discontinuous Markov computable function to exist on a computably second countable space.

5.3.1. *Brief definition of the Kolmogorov complexity.* Kolmogorov complexity formalizes the idea of "size of the minimal description of a string". This is in fact not well defined for a single string, but it is defined "up to a constant".

See Chapter 1 of [SVU17] for a better introduction to Kolmogorov complexity. We here just give the definitions and results we need. Denote by $\{0,1\}^*$ is the set of binary strings. Denote by $l(x)$ the length of an element of $\{0,1\}^*$.

A partial computable map $F :\subseteq \{0,1\}^* \to \{0,1\}^*$ is called a *decompressor.*

We then define *the complexity with respect to* F:

$$C_F(x) = \min\{l(y),\, y \in \{0,1\}^* \,\&\, F(y) = x\}.$$

It is infinite if x is not in the image of F.

A decompressor F is called *not worse* than another decompressor G if there exists a constant k such that

$$\forall x \in \{0,1\}^*,\, C_F(x) \leq C_G(x) + k.$$

Theorem 5.32 (Solomonoff–Kolmogorov, see [SVU17], Theorem 1, p. 3)**.** *There is a decompressor not worse than every other decompressor.*

Such a decompressor is called *optimal.* Let $(x, y) \mapsto \langle x, y\rangle$ denote a computable bijection between $\{0,1\}^*$ and $\{0,1\}^* \times \{0,1\}^*$. One obtains an optimal decompressor simply by using a universal Turing machine: if U is the function computed by a universal Turing machine (which means that $U(x, y)$ interprets x as the code of a Turing machine and applies this machine to y), then $\langle x, y\rangle \mapsto U(x, y)$ is an optimal decompressor.

We fix an optimal decompressor F_0 and define *the Kolmogorov complexity,* denoted C, to be C_{F_0}, the complexity with respect to this optimal decompressor.

The identity $\mathrm{id}_{\{0,1\}^*}$ of $\{0,1\}^*$ is a decompressor (which is not optimal), the complexity $C_{\mathrm{id}_{\{0,1\}^*}}$ is just the length function. Thus there exists k such that for any x, $C_{F_0}(x) \leq l(x) + k$. This shows that the length is an upper bound to Kolmogorov complexity.

Conversely, strings of maximal Kolmogorov complexity do exit.

Theorem 5.33 (See [SVU17], Theorem 5, p.8)**.** *For any $n > 0$, there is a string of length at most n and of Kolmogorov complexity n.*

The sequence given by this theorem cannot be computable.

Theorem 5.34 (See [SVU17], Theorem 10, p.22)**.** *Kolmogorov complexity is not computable. What's more, there is no computable sequence of strings $(x_n)_{n\in\mathbb{N}}$ such that $C(x_n) > \frac{l(x_n)}{2}$.*

Applying computable functions does not change the asymptotic Kolmogorov complexity of sequences:

Theorem 5.35 (See [SVU17], Theorem 3, p.5)**.** *For any partial computable function $g :\subseteq \{0,1\}^* \to \{0,1\}^*$, there is k such that $C(g(x)) \leq C(x) + k$ for each x in $dom(g)$.*

We will finally need the following result:

Theorem 5.36 (See [SVU17], Theorem 8, p.19)**.** *Kolmogorov complexity is upper semi-computable: there is a computable function $F : \{0,1\}^* \times \mathbb{N} \to \mathbb{N}$ such that for any x in $\{0,1\}^*$, $n \mapsto F(x, n)$ is decreasing and converges to $C_{F_0}(x)$.*

5.3.2. *Computable but discontinuous function.* We first build a computable but discontinuous function. This is done by considering a function defined on a peculiar domain. We set ourselves in the Cantor space $\{0,1\}^{\mathbb{N}}$, we will consider a certain subset of it.

Consider a sequence $(u_n)_{n\in\mathbb{N}}$ of finite strings of zeroes and ones, such that the length of u_n is n, and which has linear asymptotic Kolmogorov complexity: $K(u_n) \underset{n\to\infty}{\sim} n$. This is given by Theorem 5.33. Consider now the sequence $v_n = 0^n 1 u_n 00000.....$ of elements of $\{0,1\}^{\mathbb{N}}$. This sequence also has linear asymptotic Kolmogorov complexity: a single Turing Machine can transform any element v_n into the corresponding u_n, thus we can apply Theorem 5.35.

We call $\mathcal{A}$ the subset of $\{0,1\}^{\mathbb{N}}$ consisting of the null sequence (which we denote by 0^ω) and of the set $\{v_n, n\in\mathbb{N}\}$.

Proposition 5.37. *The function $\delta_0 : \mathcal{A} \to \{0,1\}$ which sends the null sequence to 1 and all other sequences to 0 is computable on $\mathcal{A}$. However, it is discontinuous.*

Proof. Because $K(v_n) \underset{n\to\infty}{\sim} n$, there must exist an integer $b\in\mathbb{Z}$ such that for all n, $K(v_n) > \frac{n}{2} + b$.

We now show how to compute δ_0.

An element x of $\mathcal{A}$ is given as input of δ_0 via a finite description, which is the code k of a Turing machine that enumerates x. This code precisely constitutes an upper bound to the Kolmogorov complexity of x. Thus either x it is the null sequence, or, if it can be written v_n for some n, we have $k > \frac{n}{2} + b$, and thus $n < 2(k-b)$.

But since the element v_n agrees with the null sequence only on its first n terms, this means that if x is not the null sequence, one of its first $2(k-b)$ digits must be a one. This can be easily checked, using the Turing Machine coded by k until it has written the first $2(k-b)$ digits of x. □

5.3.3. *A semi-decidable set that is not open.* The previous example was obtained by considering a function defined on a set with bad properties. On a recursive Polish space, the decidable sets must be clopen, since their characteristic functions must be continuous. By Markov's Lemma, the semi-decidable sets cannot be "effectively not-open": if x is a point of a semi-decidable set Y, and if $(u_n)_{n\in\mathbb{N}}$ is a computable sequence that effectively converges to x, then infinitely many elements of this sequence must belong to Y. One might wonder whether this result can be strengthened to: "the ν-semi-decidable sets on a recursive Polish space (X,d,ν) are open". An example of Friedberg [Fri58a] shows that this is not the case, we reproduce here the account of this result from [HR16], which renders explicit the role of Kolmogorov complexity in the construction of this example.

This example is set in the Cantor space $\{0,1\}^{\mathbb{N}}$. For w an element of $\{0,1\}^*$, denote by $[w]$ the clopen set of all sequences that start by w.

Theorem 5.38 (Friedberg, see [HR16], Theorem 4.1)**.** *On the Cantor space, the set*

$$A = \{0^\omega\} \cup \bigcup_{\{n \mid K(0^n) < \frac{\log(n)}{2}\}} [0^n 1]$$

is semi-decidable but not open.

Proof. A is not open, as it does not contain a neighborhood of 0^ω, because infinitely often in n one has $K(0^n) \geq \frac{\log(n)}{2}$ (by Theorem 5.33, noting that the complexity of the string 0^n is equivalent to the complexity of the binary expansion of n, since it is possible to computably go from 0^n to the binary expansion of n, and conversely).

We now show that A is semi-decidable.

There exists a program T that maps any element x of the Cantor space that is different from 0^ω to the number of zeroes that appear at the beginning of x.

We are now given a computable point x of $\{0,1\}^{\mathbb{N}}$.

The description of x by a Turing machine that produces it gives an upper bound K_0 on the Kolmogorov complexity of x. Noting l the length of the program T defined above, one has that either x is 0^ω, or, if x can be decomposed as $x = 0^n 1 x'$, it must be that $K(0^n) \leq K(x) + l \leq K_0 + l$.

Start enumerating x. If x starts with more than $2^{2(K_0+l)}$ zeros, then either it is the sequence 0^ω, or it can be written as $x = 0^n 1 x'$, with $\frac{\log(n)}{2} > K_0 + l$, and thus with $\frac{\log(n)}{2} > K(0^n)$. In any case, we know that x belongs to A, without having to compute n.

If x starts with less than $2^{2(K_0+l)}$ zeroes, a number n such that x can be rewritten as $x = 0^n 1 x'$ can be effectively found. From this, to determine whether x belongs to A, one only needs to check whether $K(0^n) < \frac{\log(n)}{2}$ holds. This last inequality is semi-decidable, as the Kolmogorov complexity is upper-semi-computable (Theorem 5.36). □

Other examples of non-open but semi-decidable sets can be found in [HR16]. It is however clear that those examples are artificially built, and this justifies the heuristic which says that a *natural* semi-decidable property can be expected to be open.

Note finally that although we have just seen that a semi-decidable subset of a recursive Polish space does not have to be open, it must share the following property of open sets: it meets any computable and dense sequence.

Proposition 5.39 (Moschovakis, [Mos64], Theorem 4)**.** *Let $(X, d, \nu, (u_n)_{n\in\mathbb{N}})$ be a recursive Polish space. A non-empty ν-semi-decidable subset of X must intersect the dense sequence $(u_n)_{n\in\mathbb{N}}$.*

The proof of this result is in fact very close to that of Mazur's Continuity Theorem, Corollary 5.31. A consequence of this fact, pointed out in [HR16], is the following:

Corollary 5.40 (Hoyrup, Rojas, [HR16], Section 4, Proposition 3)**.** *In a recursive Polish space $(X, d, \nu, (u_n)_{n\in\mathbb{N}})$, there is an algorithm that takes as input the code for a ν-semi-decidable set D and stops if and only if this set is non-empty.*

In case the set D is non-empty, this algorithm will produce the ν-name for a point in it.

Proof. Just enumerate the sequence $(u_n)_{n\in\mathbb{N}}$ in search of a point in D, by Proposition 5.39, D is non-empty if and only if it contains a point from $(u_n)_{n\in\mathbb{N}}$. □

Note that the previous proposition contains an effective version of the Axiom of Choice for recursive Polish spaces: a possible formulation of AC is that for any set X, there exists a choice function that maps a non-empty subset of X to a point in this subset. The above corollary shows that a computable choice function exists for semi-decidable subsets of a recursive Polish space. We will give in Theorem 6.9 a strong negation of such an effective Axiom of Choice for the space of marked groups: there does not exist a computable function that, given a non-empty basic clopen set $\Omega^k_{r_1,\dots,r_m;s_1,\dots,s_{m'}}$, can produce a point in this set.

5.3.4. *Necessary and sufficient condition for discontinuous functions to exist.* We show how the example given in Section 5.3.2 above is essentially the only possible one on spaces that satisfy a certain form of "effective second countability", which is easily seen to be satisfied by the space of marked groups.

Consider a countable T_0 topological space X, which is second countable with a totally numbered basis $(B_i)_{i\in\mathbb{N}}$. We suppose w.l.o.g. that $(B_i)_{i\in\mathbb{N}}$ is stable by finite intersection and that the intersection map is computable.

Equip X with the numbering ν induced by the basis $(B_i)_{i\in\mathbb{N}}$:

$$\nu(i) = x \iff \mathrm{dom}(\varphi_i) = \{n \in \mathbb{N},\ x \in B_n\}.$$

In other words, the name of a point in X is a program that recognizes the basic open sets to which this point belongs. These assumptions amount to a form of "effective second countability".

We say that points in X have *effective neighborhood bases of co-semi-decidable sets* if for each point x there is a neighborhood basis $(A_i)_{i\in\mathbb{N}}$ of x consisting of co-semi-decidable sets, and a computable map F_x such that if $x \in B_i$, then $x \in A_{F_x(i)} \subseteq B_i$. The typical example of this is metric spaces, where closed balls are co-semi-decidable, and where the map F_x can be chosen to map the open ball of radius r centered at x to the closed ball of radius $r/2$ centered at x. In the space of marked groups the basic clopen sets are naturally decidable, so F_x can be taken as the identity.

The following result was indicated to me by Mathieu Hoyrup. It follows from [HR16] but does not appear there.

Theorem 5.41. *If there exists a ν-computable and discontinuous function $f : X \to \{0, 1\}$, then there exists a non-isolated point x and a computable function $g : \mathbb{N} \to \mathbb{N}$ such that for each n, x is the only point of Kolmogorov complexity less than n in $B_{g(n)}$. If furthermore points in X have effective neighborhood bases of co-semi-decidable sets, then this is actually an equivalence: any such g yields a computable discontinuous function $f : X \to \{0, 1\}$.*

Proof. By the main result of [HR16], the function f is ν-computable if and only if it is computable in a Type 2 model where points are given with, as additional information, an upper bound on the Kolmogorov complexity of the input point. This means that there is a machine M_f that computes f which, instead of taking ν-names as input (these would be elements of $\mathbb{N}$), takes pairs (k, p), $k \in \mathbb{N}$ and $p \in \mathbb{N}^{\mathbb{N}}$ as input, where a pair (k, p) describes a point x if:

- k is such that x admits a ν-name i with $i \leq k$;
- p is a sequence consisting of the set $\{n \in \mathbb{N},\ x \in B_n\}$ in some order. This sequence is given on an oracle tape of the Turing machine.

Suppose that f is discontinuous at x. Fix a computable sequence $(p_n)_{n\in\mathbb{N}}$ which is such that $\{p_n, n \in \mathbb{N}\} = \{n \in \mathbb{N}, x \in B_n\}$. Let k_0 be a ν-name of x.

Now consider what the machine M_f does on input a pair $(k, (p_n)_{n\in\mathbb{N}})$, for $k > k_0$. This pair is a description of x, thus M_f will eventually answer $f(x)$. However, when doing so, the machine M_f inspects only a finite portion of the infinite sequence $(p_n)_{n\in\mathbb{N}}$.

The extent to which this sequence is inspected is a computable function of k: there is g_1, computable, which is such that $g_1(k) = N$ if on input $(k, (p_n)_{n\in\mathbb{N}})$, M_f inspects only the first N elements of the sequence $(p_n)_{n\in\mathbb{N}}$.

It follows from this that if a point y has Kolmogorov complexity at most k, then either it is x, or it does not belong to $B_{p_1} \cap ... \cap B_{p_{g_1(k)}}$. We have supposed that intersections are computable for the basis $(B_i)_{i\in\mathbb{N}}$, and thus we can replace g_1 by g as in the statement of the theorem.

For the converse, suppose that such a g exists, and that x has a computable neighborhood basis of co-c.e. sets. Then the function $\delta_x : X \to \{0, 1\}$ which maps x to 1 and other points to 0 is computable. Indeed, given $y = \nu(n)$, we know that either $y = x$, or $y \notin B_{g(n)}$. By hypothesis we can compute a co-c.e. set A_m such that $x \in A_m \subseteq B_{g(n)}$. Then either $y \in B_{g(n)}$ or $y \notin A_m$, both conditions are semi-decidable, and thus it can be determined which one of them holds. □

The following is the above result reformulated for metric spaces:

Theorem 5.42. *Suppose that X has a computable metric, and that the basis $(B_i)_{i\in\mathbb{N}}$ generates the metric topology (without necessarily being a set of open balls). Suppose furthermore every point has a c.e. neighborhood basis $(B_{\psi(i)})_{i\in\mathbb{N}}$ with $diam(B_{\psi(i)}) < 2^{-n}$.*

Then there exists a ν-computable and discontinuous function $f : X \to \{0, 1\}$ iff there is a point x, which is not isolated, and a computable function g s.t. for all $y \neq x$, if $y = \nu(n)$, then $d(y, x) > 2^{-g(n)}$.

Proof. Similar to the proof of Theorem 5.41, using as neighborhood basis of x the sequence $(B_{\psi(i)})_{i\in\mathbb{N}}$. □

The hypotheses of the above Theorem are true of the space of marked groups. The basic clopen sets $\Omega^k_{R,S}$ are not open balls in the space of marked groups (since they can be empty), yet an upper bound to their diameter can easily be computed.

Note that despite being far reaching, Theorem 5.41 does not completely solve the continuity problem. Indeed:

- It relies on a hypothesis of effective second countability;
- It does not provide necessary and sufficient conditions for effective continuity of the considered functions (see [Sli64] for a continuous but not effectively continuous computable function);
- It does not *a priori* apply to Banach-Mazur continuity. It is however possible that the methods of [HR16] do in fact apply also to Banach-Mazur computability.

5.4. Kreisel, Lacombe and Schoenfield and Ceitin Theorems, Moschovakis' addendum.

5.4.1. *Theorems of Kreisel, Lacombe, Schoenfield, Ceitin.* The following theorem is one of the most important theorems in Markovian computable analysis. It was first proved by Kreisel, Lacombe and Schoenfield in 1957 in [KLS57] in the case of functions defined on the Baire space $\mathbb{N}^{\mathbb{N}}$, and obtained independently by Ceitin in 1962 (English translation in [Cei67]), in the more general setting of recursive Polish spaces.

Theorem 5.43 (Kreisel-Lacombe-Schoenfield, Ceitin)**.** *A computable function defined on a recursive Polish space (with images in any RMS) is effectively continuous.*

Moreover, for each pair constituted of a recursive Polish space and a RMS, there is an algorithm that takes as input the description of a computable function defined between those spaces, and produces a program that will attest for the effective continuity of this function.

5.4.2. *Moschovakis' Theorem.* In 1964, Moschovakis gave a new proof of Ceitin's Theorem, at the same time providing the only known effective continuity result for metric spaces set in a more general context than that of recursive Polish spaces.

In what follows, (X, d, ν) denotes a recursive metric space.

Definition 5.44. We say that (X, d, ν) satisfies *Moschovakis' condition (B)* if there is an algorithm that, given the code of a ν-semi-decidable set $A \subseteq X$ and an open ball $B(x, r)$ which intersects it, will produce the ν-name of a point y in the intersection $A \cap B(x, r)$.

Note that in this definition, the algorithm is always given as input a pair of intersecting sets, it then produces a point in the intersection. This algorithm is not supposed to be able to determine whether or not two given sets

intersect. This definition asks for an effective Axiom of Choice, similar to the one described in Proposition 5.39. And an easy consequence of Proposition 5.39 is the following:

Proposition 5.45. *A recursive Polish space satisfies Moschovakis' condition (B).*

We can now state Moschovakis' Theorem on the effective continuity of computable functions.

Theorem 5.46 (Moschovakis, [Mos64], Theorem 3)**.** *A computable function defined on an effectively complete RMS that satisfies Moschovakis' condition (B) is effectively metric continuous.*

Moreover, for each pair constituted of such a space and of any RMS, there is an algorithm that takes as input the description of a computable function defined between those spaces, and produces a program that will attest for the effective continuity of this function.

We will prove in Corollary 6.14 that the hypotheses of this theorem fail for the space of marked groups, leaving open the conjecture which says that computable functions on $\mathcal{G}^+$ are effectively continuous.

Note that several other results that can be found in [Mos64] can give rise to conjectures for the space of marked groups, in particular the characterization of effective open sets as Lacombe sets, which is Theorem 11 of [Mos64]. See for instance Section 8.1.

6. The space of marked groups as a recursive metric space

6.1. Banach-Mazur computable but not Markov computable functions.

Lemma 6.1. *For every $k \geq 2$, there exists a computable retraction from $\mathcal{G}_k$ onto a Cantor space: there exists a computable injection $i : \{0,1\}^{\mathbb{N}} \to \mathcal{G}_k$ which admits a computable right inverse $r : \mathcal{G}_k \to \{0,1\}^{\mathbb{N}}$.*

Proof. Many constructions are possible. We use Philip Hall's 2-generated center-by-metabelian group which admits $\bigoplus_{\mathbb{N}} \mathbb{Z}$ as center, which gave the first constructions of continuously many 2 generated solvable groups [Hal54]. Let a and b be its generators. For each $n \in \mathbb{N}$, denote by w_n a word of $\{a, b, a^{-1}, b^{-1}\}^*$ that defines in G a generator of the n-th central copy of $\mathbb{Z}$. The map $n \mapsto w_n$ can be taken computable (an explicit formula is given in [Hal54]). Then map any marked group (G, S) to the sequence $(p_i)_{i\in\mathbb{N}}$ given by

$$p_i = 1 \iff w_i \text{ is a relation in } (G, S).$$

Because the relations w_i are independent, this defines a surjection $r : \mathcal{G}_k \to \{0,1\}^{\mathbb{N}}$ onto the Cantor space. Because $n \mapsto w_n$ is computable, it is a computable map. And the computable injection $i : \{0,1\}^{\mathbb{N}} \to \mathcal{G}_k$ is given by Hall's construction. □

Corollary 6.2. *For every $k \geq 2$, there exists a Banach-Mazur computable function defined on $\mathcal{G}_k$ which is not Markov computable.*

Proof. If $f : \{0,1\}^{\mathbb{N}}_c \to \mathbb{N}$ is a Banach-Mazur computable function which is not Markov computable, as given in [BS04], and if r is the map given in the previous lemma, then $f \circ r$ is a Banach-Mazur computable function $\mathcal{G}^+ \to \mathbb{N}$. It is not Markov computable, otherwise $f \circ r \circ i$ would be as well, but this is f. □

6.2. Non effective separability of $\mathcal{G}^+$.

Theorem 6.3. *No sequence of marked groups can be both computable and dense in $\mathcal{G}^+$.*

Proof. This is a simple application of Theorem 2.3, together with Corollary 5.40. Corollary 5.40 states that in a recursive Polish space, there is an algorithm that stops exactly on semi-decidable sets that are non-empty.

The basic clopen sets $\Omega^k_{r_i;s_j}$ are obviously semi-decidable in $\mathcal{G}^+$, but a program that recognizes those basic clopen sets that are non-empty would allow one to recognize wp-coherent sets of relations and irrelations, contradicting Theorem 2.3. □

It is interesting to interpret this proof as a variation on McKinsey's theorem [McK43] which states that finitely presented residually finite groups have solvable word problem. Notice that if X is a set of marked groups which is dense in $\mathcal{G}^+$, then every finitely presented group with solvable word problem is "residually-X", and a proof similar to McKinsey's would then contradict the Boone and Rogers Theorem [BR66].

This proposition directly implies the following:

Corollary 6.4. *The recursive metric space $(\mathcal{G}^+, d, \nu_{WP})$ is effectively complete but not effectively separable, and thus it is not a recursive Polish space.*

As we have already seen in the previous section, Mazur's Continuity Theorem and Ceitin's Effective Continuity Theorem both apply to recursive Polish spaces. This corollary thus shows that those theorems cannot be directly applied to the space of marked groups.

The following result is more general than the previous one, because it does not necessarily applies to group given by word-problem algorithms. Note that it is a result about $(\mathcal{G}, d)$ up to isometry, and not up to homeomorphism.

Theorem 6.5. *The metric space $(\mathcal{G}, d)$ does not have a recursive presentation in the sense of Definition 5.27.*

Proof. Recall that a recursive presentation of $(\mathcal{G}, d)$ consists in a sequence $(u_n)_{n\in\mathbb{N}}$, dense in $\mathcal{G}$, and for which the distance between the n-th and m-th terms is computable.

Suppose that $\mathcal{G}$ admits a recursive presentation. Then so does $\mathcal{G}_2$.

Recall that we have defined an embedding $\Phi_2 : \mathcal{G}_2 \to \{0,1\}^{\mathbb{N}}$ by fixing a computable order on the rank two free group. Call a set $(r_1, ..., r_m;\ s_1, ..., s_{m'})$ of relations and irrelations *initial* if the $m + m'$ elements of the free group it contains are exactly the first $m + m'$ elements of this order. The number $m + m'$ is called the length of this set of relations and irrelations.

We will prove that the recursive presentation of $\mathcal{G}_2$ allows to compute, given an integer n, the number of initial coherent sets of relations and irrelations that contain n relations.

We first show that this is sufficient to obtain a contradiction.

There are exactly 2^n possible initial sets of relations and irrelations of length n. Since the incoherent sets of relations and irrelations form a c.e. set, if we had access to the number of initial coherent sets of relations and irrelations of length n, we would be able to compute exactly those sets, by starting with the 2^n possible initial sets, and deleting incoherent ones until the number of coherent sets is attained.

But being able to compute the *initial* coherent sets of relations and irrelations in fact also allows one to compute all coherent sets of relations and irrelations, because a set $(r_1, ..., r_m;\ s_1, ..., s_{m'})$, which is not initial, is coherent if and only if there is an initial and coherent set which contains the elements $(r_1, ..., r_m)$ as relations, and the elements $(s_1, ..., s_{m'})$ as irrelations. Choosing n big enough, it suffices to construct all initial sets of relations and irrelations of length n to determine whether $(r_1, ..., r_m;\ s_1, ..., s_{m'})$ is coherent. And we have seen that this is impossible.

Suppose that $(u_n)_{n\in\mathbb{N}}$ defines a recursive presentation of $\mathcal{G}_2$, we show how to compute the number of initial coherent sets of relations and irrelations that contain n relations. Denote by $\lambda(n)$ this number. Again, because the incoherent sets of relations and irrelations form a c.e. set, λ is an upper semi-computable function: there exists a computable function $\lambda^>$ that, given n, produces a computable and decreasing sequence of integers that converges to $\lambda(n)$. What we show is that the existence of a recursive presentation of $\mathcal{G}_2$ implies that λ is also lower semi-computable, meaning that there exists a computable function $\lambda^<$ that, on input n, produces an increasing sequence of integers which converges to $\lambda(n)$.

Given i and n natural numbers, define x_i^n to be the maximal size of a subset of $\{u_0, u_1, ..., u_i\}$ of which any two elements are at least 2^{-n} apart.

We claim that $(i, n) \mapsto x_i^n$ is a computable function, and that, for any n, $i \mapsto x_i^n$ is an increasing function that converges to $\lambda(n)$.

Setting $\lambda^<(n) = (x_i^n)_{i\in\mathbb{N}}$ will then conclude the proof.

As the distance function d takes values only in $\{0\} \cup \{2^{-n}, n \in \mathbb{N}\}$, given the description of the distance $d(u_i, u_j)$ as a computable real, one can always effectively choose one of $d(u_i, u_j) < 3 \times 2^{-n-2}$ and $d(u_i, u_j) > 3 \times 2^{-n-2}$ which holds, and thus decide whether or not u_i and u_j are 2^{-n} apart. One can thus check every subset of $\{u_0, u_1, ..., u_i\}$ to find one of maximal size, all the elements of which are 2^{-n} apart. Thus $(i, n) \mapsto x_i^n$ is computable.

The function $(i, n) \mapsto x_i^n$ is increasing in i by definition.

Finally, we show that x_i^n goes to $\lambda(n)$ as i goes to infinity. Two points of $\mathcal{G}_2$ are at least 2^{-n} apart if and only if their binary expansion differ on one of their first n digits: those groups must be associated to different initial sets of relations and irrelations of length n. Because the sequence $(u_n)_{n\in\mathbb{N}}$ is supposed to be dense in $\mathcal{G}_2$, for any coherent initial set of relations and irrelations of length n, there is a point of this sequence which satisfies those relations and irrelations.

And thus there must indeed exist a set of $\lambda(n)$ points in the dense sequence which are pairwise 2^{-n} apart, and this number is clearly maximal. □

6.3. **Optimality of the numbering type Λ_{WP}.** A Polish space is a topological space, it is not attached to a particular metric. The notion of recursive presentation of a Polish space is attached to a certain metric. We have shown that $(\mathcal{G}, d)$ does not have a recursive presentation, where d is the usual ultrametric distance of $\mathcal{G}$.

We have no results that state that the space of marked groups, seen purely as a topological space, cannot have a computably Polish model.

In particular the above result on presentations, Theorem 6.5, could easily be extended to concrete examples of other metrics, but it is unclear how to prove the same result for an arbitrary metric that generates the right topology.

What our results show is that if we want to preserve slightly more than just the topology of $\mathcal{G}$, then undecidability occurs. We can obtain undecidability results whenever trying to preserve one of the following three features of $\mathcal{G}$:

- One of the usual explicit metrics of $\mathcal{G}$,
- The explicit embeddings of each $\mathcal{G}_k$ into a Cantor space,
- The numbering of the basis of clopen sets of $\mathcal{G}$ (sets of the form $\Omega^k_{R;S}$, with the obvious numbering: a basic set $\Omega^k_{R;S}$ is given by k and the tuples R and S).

Each one of those points provides $\mathcal{G}$ with an extra structure, in addition to its topology.

The first point was discussed above, in this section we obtain undecidability results concerning the second and third points.

Note that the second and third points are related: the numbering of the basis of $\mathcal{G}$ is the one induced by the embedding of $\mathcal{G}$ into a countable union of Cantor spaces.

The following theorem shows that, if $\hat{d}$ is any distance that generate the topology $\mathcal{G}$, and μ any subnumbering of $\mathcal{G}$, if there is an algorithm that testifies that the open balls of $(\mathcal{G}_\mu, \hat{d}, \mu)$ are effectively open with respect to the numbered basis of clopen sets of $\mathcal{G}$, then μ provides more information than ν_{WP}.

Theorem 6.6. *Suppose that $\hat{d}$ is any distance on $\mathcal{G}$ that generates the same topology as d, and that μ is a subnumbering of $\mathcal{G}$ such that $(\mathcal{G}_\mu, \hat{d}, \mu)$ is a RMS.*

Suppose furthermore that there is an algorithm that takes as input a μ-name for a k-marked group (G, S) and a $c_{\mathbb{R}}$-name for a radius $r > 0$, and produces a basic clopen set $\Omega^k_{R;T}$ such that:

$$(G, S) \in \Omega^k_{R;T};$$

$$\Omega^k_{R;T} \subseteq B_{\hat{d}}((G, S), r).$$

Then one has $\mu \succeq \nu_{WP}$. And thus $\mathcal{G}_\mu \subseteq \mathcal{G}^+$, and if $\mathcal{G}_\mu$ is dense in $\mathcal{G}^+$, then $(\mathcal{G}_\mu, \hat{d}, \mu)$ does not have a μ-computable and dense sequence.

A *discriminating family* of a group G is a subset of G which does not contain the identity element of G, and which intersects any non-trivial normal subgroup of G. We will use Theorem 3.4 from [dCGP07], which is an analysis of Kuznetsov's method for solving the word problem in simple groups [Kuz58]:

Theorem 6.7 (Cornulier, Guyot, Pitsch, [dCGP07])**.** *A group has solvable word problem if and only if it is both recursively presentable and recursively discriminable.*

And this statement is uniform: there is an effective method that allows, given a recursive presentation and an algorithm that enumerates a discriminating family in a marked group (G, S), to find a word problem algorithm for (G, S).

We add the statement about the uniformity of this theorem, but it is easy to see that the proof given in [dCGP07] is uniform.

Proof of Theorem 6.6. Consider a μ-computable marked group (G, S). We show, given a μ-name for (G, S), how to obtain a word problem algorithm for it.

Using the algorithm given by the hypotheses of the theorem, consecutively on each ball $B_{\hat{d}}(G, \frac{1}{n})$, we obtain a computable sequence $(\Omega_{R_n;T_n})_{n\in\mathbb{N}}$ of basic clopen subsets, such that for each n we have:

$$G \in \Omega_{R_n;T_n} \subseteq B_{\hat{d}}(G, \frac{1}{n}).$$

It follows that $\bigcap \Omega_{R_n;T_n} = \{G\}$, and thus that the union $\bigcup_{n\in\mathbb{N}} R_n$ defines a computably enumerable set of relations that defines G, and that the set $\bigcup_{n\in\mathbb{N}} T_n$ defines a computably enumerable discriminating family for G.

We can thus apply Theorem 6.7, which indicates that a word problem algorithm for G can be obtained from this data. □

6.4. **Two applications of a construction of Miller, failure of Moschovakis' (B) condition for the space of marked groups.** In this section, we prove two important theorems that use variations on Miller's example of a finitely presented group that is isolated from groups with solvable word problem.

Theorem 6.8. *No algorithm can stop exactly on those sets of relations and irrelations which are not wp-coherent.*

The following theorem is one of our most important results.

Theorem 6.9 (Failure of an Effective Axiom of Choice for groups)**.** *There is no algorithm that, given a wp-coherent set of relations and irrelations, produces a word problem algorithm for a marked group that satisfies those relations and irrelations.*

The proofs for those results will be similar: they rely on Miller's constructions of a family of groups $L_{P,Q}$ indexed by two subsets P and Q of $\mathbb{N}$. For each of those theorems, we will find some conditions on the sets P and Q that are sufficient for the groups $L_{P,Q}$ to provide proofs for the Theorems 6.8 and 6.9, and then include a lemma to prove that such sets P and Q do exist.

We start by detailing Miller's construction.

6.4.1. *Miller's construction.* We detail the construction of Miller as it was exposed in [Mil92]. This construction was first introduced in [Mil81].

Step 1. Given two subsets P and Q of $\mathbb{N}$, we consider the group $L^1_{P,Q}$ given by the following presentation:

$$\langle e_0, e_1, e_2, ... | e_0 = e_i,\ i \in P,\ e_1 = e_j,\ j \in Q \rangle$$

For simplicity, we shall always assume that P contains 0 and Q contains 1.

Notice that $L^1_{P,Q}$ is recursively presented with respect to the family $(e_i)_{i\in\mathbb{N}}$ if and only if P and Q are c.e. sets, and that $L^1_{P,Q}$ has solvable word problem with respect to the family $(e_i)_{i\in\mathbb{N}}$ if and only if P and Q are computable sets.

In what follows, the sets P and Q will always be computably enumerable, and thus $L^1_{P,Q}$ is recursively presented.

Step 2. Embed the recursively presented group $L^1_{P,Q}$ in a finitely presented group $L^2_{P,Q}$ using some strengthening of Higman's Embedding Theorem. For our purpose, we need to know that:

- A finite presentation of $L^2_{P,Q}$ can be obtained from the recursive presentation of $L^1_{P,Q}$;
- If the group $L^1_{P,Q}$ has solvable word problem with respect to the family $(e_i)_{i\in\mathbb{N}}$, then the group $L^2_{P,Q}$ also has solvable word problem;
- The embedding of $L^1_{P,Q}$ into $L^2_{P,Q}$ is effective, i.e. there exists a computable function that maps a natural number n to a way of expressing the element e_n as a product of the generators of $L^2_{P,Q}$.

Clapham's version of Higman's Embedding Theorem [Cla67] satisfies the required conditions for this step of the construction. Clapham's Theorem is quoted precisely in Section 9.1.

Step 3. Embed the group $L^2_{P,Q}$ into a finitely presented group $L^3_{P,Q}$ with the following property: in any non-trivial quotient of $L^3_{P,Q}$, the image of the element $e_0e_1^{-1}$ is a non-identity element.

This is done as follows.

Consider a presentation $\langle x_1, ..., x_k | r_1, ...r_t \rangle$ for $L^2_{P,Q}$, denote w a word on $\{x_1, ..., x_k, x_1^{-1}, ..., x_k^{-1}\}$ that defines the element $e_0e_1^{-1}$ in $L^2_{P,Q}$. The group $L^3_{P,Q}$ is defined by adding to $L^2_{P,Q}$, in addition to the generators $x_1, ..., x_k$ that are still subject to the relations $r_1, ...r_t$, three new generators a, b and c, subject to the following relations:

(1) $a^{-1}ba = c^{-1}b^{-1}cbc$
(2) $a^{-2}b^{-1}aba^2 = c^{-2}b^{-1}cbc^2$
(3) $a^{-3}\,[w, b]\,a^3 = c^{-3}bc^3$
(4) $a^{-(3+i)}x_iba^{(3+i)} = c^{-(3+i)}bc^{(3+i)}$, $i = 1..k$

End of the construction.

To use Miller's construction, we need to check the following points:

- If $w \neq 1$ in $L^2_{P,Q}$, then $L^2_{P,Q}$ is embedded in $L^3_{P,Q}$ via the map $x_i \mapsto x_i$.
- The presentation of $L^3_{P,Q}$ can be computed from the presentation of $L^2_{P,Q}$ together with the word w.
- If $L^2_{P,Q}$ has solvable word problem, then so does the group $L^3_{P,Q}$.
- The element $e_0e_1^{-1}$ has a non-trivial image in any non-trivial quotient of $L^3_{P,Q}$.

The second point is obvious. The last point is easily proven: remark that the third written relation, together with $w = 1$, implies the relation $b = 1$. This in turn implies that $c = 1$ thanks to the first relation, that $a = 1$ thanks to the second relation, and then that all x_i also define the identity element because of the relations of (4).

The first and third points are proven using the fact that the group $L^3_{P,Q}$ can be expressed as an amalgamated product.

Consider the free product $L^2_{P,Q} * \mathbb{F}_{a,b}$ of $L^2_{P,Q}$ with a free group generated by a and b, and the free group $\mathbb{F}_{b,c}$ generated by b and c. Then, provided that $w \neq 1$ in $L^2_{P,Q}$, the subgroup of $L^2_{P,Q} * \mathbb{F}_{a,b}$ generated by b and the elements that appear to the left hand side in the equations $(1) - (4)$ is a free group on $4 + k$ generators, which we denote A, and so is the subgroup B of $\mathbb{F}_{b,c}$ generated by b and the elements that appear to the right hand side in the equations $(1) - (4)$. (These families are easily seen to be Nielsen reduced [LS77]: any product between distinct elements cancels less than half of both words.)

Thus the given presentation of $L^3_{P,Q}$ shows that it is defined as an amalgamated product of the form:

$$(L^2_{P,Q} * \mathbb{F}_{a,b}) \underset{A=B}{*} \mathbb{F}_{b,c}$$

This proves both the fact that $L^2_{P,Q}$ embeds in $L^3_{P,Q}$, and that the word problem is solvable in $L^3_{P,Q}$ as soon as it is in $L^2_{P,Q}$. Indeed, to solve the word problem in an amalgamated product such as $L^3_{P,Q}$, it suffices to be able to solve the membership problem for A in $L^2_{P,Q} * \mathbb{F}_{a,b}$ and for B in $\mathbb{F}_{b,c}$. This can be done as soon as the word problem is solvable in $L^2_{P,Q}$. Indeed, the fact that the generators of the group A give a Nielsen reduced family means that the word length in $L^2_{P,Q} * \mathbb{F}_{a,b}$ of a product of n generators of A is at least n. This implies immediately that the membership problem to A inside $L^2_{P,Q} * \mathbb{F}_{a,b}$ is solvable.

Finally, we designate by $\Pi_{P,Q}$ the finite set of relations and irrelations that is composed of the relations of $L^3_{P,Q}$, and of a unique irrelation $w \neq 1$, where w is the word that defines the element $e_0 e_1^{-1}$ in $L^3_{P,Q}$.

Note that the finite set $\Pi_{P,Q}$ can be effectively produced from the codes for the c.e. sets P and Q.

6.4.2. *First application: Miller's Theorem.* We include here a proof of Miller's Theorem.

A pair of disjoint subsets P and Q of $\mathbb{N}$ are said to be *computably inseparable* if there cannot exist a computable set H such that $P \subseteq H$ and $Q \subseteq H^c$, where H^c denotes the complement of H in $\mathbb{N}$.

We will need the following well known result (originally due to Kleene):

Lemma 6.10. *There exists a pair (P, Q) of disjoints subsets of $\mathbb{N}$ that are computably enumerable and computably inseparable.*

Proof. Consider a standard enumeration $(\varphi_0, \varphi_1, \varphi_2, ...)$ of all computable functions. Consider the set $P = \{n \in \mathbb{N},\ \varphi_n(n) = 0\}$ and the set $Q = \{n \in \mathbb{N},\ \varphi_n(n) = 1\}$. Those sets are obviously computably enumerable. Suppose now that some computable set H contains P but does not intersect H. Consider an index n_0 such that φ_{n_0} is a total function that computes the characteristic function of H.

If $\varphi_{n_0}(n_0) = 0$, n_0 does not belong to H, but it belongs to P, this is not possible. But if $\varphi_{n_0}(n_0) = 1$, then n_0 belongs to Q and to H, which is also impossible because H does not meet Q.

This is a contradiction, and thus the sets P and Q are indeed computably inseparable. □

Theorem 6.11 (Miller, [Mil81])**.** *Suppose that P and Q are disjoints subsets of $\mathbb{N}$ that are computably enumerable and computably inseparable. Then the set $\Pi_{P,Q}$ is coherent, but not wp-coherent.*

Proof. Suppose that a group K satisfies the relations and irrelations of $\Pi_{P,Q}$, and that is has solvable word problem.

Using the word problem algorithm for K, given an integer i in $\mathbb{N}$, we can solve the questions "is $e_0 = e_i$ in K", since, by the properties of Miller's construction, an expression of the element e_i in terms of the generators of $L^3_{P,Q}$, and thus of K, can be effectively found from i.

The set $\{i \in \mathbb{N},\ e_i = e_0\}$ is thus a computable set that contains P. And it is disjoint from Q, because we have assumed that $e_0 \neq e_1$ in K.

This contradicts the fact that P and Q are computably inseparable. □

6.4.3. *Proof of Theorem 6.8.* We first prove Theorem 6.8:

Theorem. *No algorithm can stop exactly on those sets of relations and irrelations which are not wp-coherent.*

Proof. Given c.e. disjoint sets, we apply Miller's construction to obtain the set $\Pi_{P,Q}$ of relations and irrelations.

By Theorem 6.11, if the sets P and Q are computably enumerable, computably inseparable sets, then $\Pi_{P,Q}$ is not wp-coherent.

On the contrary, if P and Q are both computable sets, we have noted that $L^3_{P,Q}$ itself has solvable word problem, and thus $\Pi_{P,Q}$ is wp-coherent.

Because the set $\Pi_{P,Q}$ can be constructed from the codes for P and Q, an algorithm that stops exactly on those sets of relations and irrelations which are not wp-coherent would produce, through Miller's construction, an algorithm that, given a pair of c.e. sets P and Q that are either computably inseparable or recursive, would stop if and only if those sets are computably inseparable. We prove in the next lemma, Lemma 6.12, that such an algorithm cannot exit, this ends the proof of our theorem. □

Lemma 6.12. *There is no algorithm that, given the code for two computably enumerable and disjoint subsets of* $\mathbb{N}$*, that are either computable or computably inseparable, stops only if they are computably inseparable.*

Proof. Fix two computably enumerable and computably inseparable subsets P and Q of $\mathbb{N}$, that exist by Lemma 6.10.

Consider an effective enumeration M_0, M_1, M_2... of all Turing machines. For each natural number n, define a pair of computably enumerable sets P_n and Q_n defined as follows:

To enumerate P_n, start a run of M_n.

While this run lasts, an enumeration of P gives the first elements of P_n. If M_n halts after k computation steps, stop the enumeration of P.

Thus if M_n halts, the set P_n is a finite set. On the contrary, if M_n does not stop, P_n is identical to P.

The set Q_n is defined similarly, replacing P by Q in its definition.

One then easily sees that the sets P_n and Q_n are uniformly computably enumerable, and that P_n and Q_n are computably inseparable if and only if M_n does not halt.

Since no algorithm can stop exactly on the indices of non-halting Turing machines, the lemma is proved. □

6.4.4. *Proof of Theorem 6.9.* We now prove Theorem 6.9:

Theorem (Failure of an effective Axiom of Choice for groups)**.** *There is no algorithm that, given a wp-coherent set of relations and irrelations, produces a word problem algorithm for a marked group that satisfies those relations and irrelations.*

Proof. We will build in Lemma 6.13 a pair of sequences $(P_n)_{n\in\mathbb{N}}$ and $(Q_n)_{n\in\mathbb{N}}$ such that:

- The sequences $(P_n)_{n\in\mathbb{N}}$ and $(Q_n)_{n\in\mathbb{N}}$ consist only of disjoint computable sets;
- The sequences $(P_n)_{n\in\mathbb{N}}$ and $(Q_n)_{n\in\mathbb{N}}$ are uniformly c.e., but not uniformly computable;
- For any sequence $(H_n)_{n\in\mathbb{N}}$ of uniformly computable sets, there must be some index n_0 such that either H_{n_0} does not contain P_{n_0}, or $H^c_{n_0}$ does not contain Q_{n_0}.

We apply Miller's construction to this sequence to obtain a computable sequence $(\Pi_{P_n,Q_n})_{n\in\mathbb{N}}$ of finite sets of relations and irrelations.

Suppose by contradiction that there is an algorithm $\mathcal{A}$ as in the theorem. For each natural number n, the set Π_{P_n,Q_n} is wp-coherent, because P_n and Q_n are computable. Thus the algorithm $\mathcal{A}$ can be applied to Π_{P_n,Q_n}, to produce the word problem algorithm for a group that satisfies the relations and irrelations of Π_{P_n,Q_n}. Denote G_n the group defined by this algorithm.

For each n, the set $H_n = \{i \in \mathbb{N}, e_i = e_0 \text{ in } G_n\}$ is then a computable set, and this in fact holds uniformly in n.

Finally, one has the inclusions $P_n \subseteq H_n$ and $Q_n \subseteq H_n^c$. This contradicts the properties of the sequences $(P_n)_{n\in\mathbb{N}}$ and $(Q_n)_{n\in\mathbb{N}}$. □

Lemma 6.13. *There exists a pair of sequences* $(P_n)_{n\in\mathbb{N}}$ *and* $(Q_n)_{n\in\mathbb{N}}$ *such that:*

- *The sequences* $(P_n)_{n\in\mathbb{N}}$ *and* $(Q_n)_{n\in\mathbb{N}}$ *consist only of disjoint computable sets;*
- *The sequences* $(P_n)_{n\in\mathbb{N}}$ *and* $(Q_n)_{n\in\mathbb{N}}$ *are uniformly c.e., but not uniformly computable;*
- *For any sequence* $(H_n)_{n\in\mathbb{N}}$ *of uniformly computable sets, there must be a some index* n_0 *such that either* H_{n_0} *does not contain* P_{n_0}*, or* $H^c_{n_0}$ *does not contain* Q_{n_0}*.*

Proof. Fix a pair (P, Q) of computably enumerable but computably inseparable sets[5].

Let $P_n = P \cap \{n\}$ and $Q_n = Q \cap \{n\}$. By construction these sets are disjoint, computable (these are singletons), and the sequences $(P_n)_{n\in\mathbb{N}}$ and $(Q_n)_{n\in\mathbb{N}}$ are uniformly c.e..

Suppose $(H_n)_{n\in\mathbb{N}}$ is a sequence of uniformly computable sets that separate P_n from Q_n.

Consider H given by: $n \in H \iff n \in H_n$. Then $P \subseteq H \subseteq Q^c$, and H is a computable set. This contradicts the fact that P and Q are inseparable. □

[5]Previous versions of this article contained a more complicated example. I thank Laurent Bienvenu for showing me that this simple example works.

Theorem 6.9 allows us to prove that the space of marked groups does not satisfy Moschovakis' condition (B), and thus that Theorem 5.46 cannot be applied to the space of marked groups.

Corollary 6.14. *The triple $(\mathcal{G}^+, d, \nu_{WP})$ does not satisfy Moschovakis' condition (B).*

Proof. A RMS (X, d, ν) satisfies Moschovakis' condition (B) if there exists an algorithm $\mathcal{A}$ that takes as input the description of a ν-semi-decidable set Y and the description of an open ball $B(x, r)$ in X, such that those set intersect, and produces a point that belongs to their intersection.

In $\mathcal{G}_k$, apply such an algorithm to a basic clopen subset $\Omega^k_{r_i;s_j}$ and to an open ball that contains all of $\mathcal{G}_k$ -any open ball of radius $r \geq 1$. This yields a program that takes as input a set of relations and irrelations that is wp-coherent, and produces the ν_{WP}-name of a point that belongs to it. The existence of such an algorithm was proven impossible in Theorem 6.9. □

7. Correspondence between Borel and effective Borel hierarchies

We now apply the results of the previous sections to decision problems for groups described by word problem algorithms.

7.1. List of properties for which the correspondence holds.

Recall from the introduction that we have three distinct effective versions of the Borel hierarchy: the one coming from Type 2 computability, the one coming from Markov computability, and the one coming from Banach-Mazur computability.

We expect that most natural group properties will be given the same classification in these four hierarchies (one classical and three effective). We say that a property *satisfies the correspondence between the different hierarchies* if it is lies at the same position in all four hierarchies.

We will now proceed to list group properties that satisfy the correspondence. These properties are organized into a table, which can be found in Section 7.2.

Undecidability results which are required to establish correspondence results are all obtained by one or two applications of the following version of Markov's Lemma:

Proposition 7.1. *Suppose that a subset A of $\mathcal{G}^+$ is effectively not open, i.e. that there is a computable sequence of marked groups that do not belong to A, which converges to a marked group in A. Then A cannot be Λ_{WP}-semi-decidable.*

In what follows, we will often use this proposition together with the following proposition, which is a reformulation of Proposition 5.23.

Proposition 7.2. *If a marked group (G, S) with solvable word problem is adherent to a c.e. set $\mathcal{C}$ of marked groups, then there is a Λ_{WP}-computable sequence of marked groups in $\mathcal{C}$ that effectively converges to (G, S).*

This proposition, while very easy, is often useful. It can be used with $\mathcal{C}$ being: the class of finite groups, of hyperbolic groups, of all the markings of a given abstract group G, etc.

Those affirmations that appear in Section 7.2 which are not accompanied by either a reference, or for which a proof does not appear in the following paragraphs, are left to the reader.

7.1.1. *Residually finite groups.* A group is residually finite if any non-trivial element of it has a non-trivial image in a finite quotient. Note that the space of marked groups gives a topological characterization of residually finite groups [CG05]: a group is residually finite if and only if some (equiv. any) of its markings is adherent to the set of its marked finite quotients. It is known that the set of residually finite groups is not closed, because the closure of the set of finite groups (which is known as the set LEF groups) strictly contains the set of residually finite groups. The semi-direct product $\mathbb{Z} \ltimes \mathfrak{S}_\infty$ is the limit of the sequence of finite groups $\mathbb{Z}/n\mathbb{Z} \ltimes \mathfrak{S}_n$, as n goes to infinity; it is not residually finite since it contains an infinite simple group. Note that the described sequence effectively converges. (This example comes from [VG98].)

It is also well known that the set of residually finite groups is not open: non-abelian free groups are limits of non-residually finite groups. For instance, it follows from well known results on the Burnside problem that free Burnside groups of sufficiently large exponent are infinite groups that are not residually finite, and that, if S denotes a basis of a non-abelian free group, the sequence $(\mathbb{F}/\mathbb{F}^n, S)$ is Λ_{WP}-computable and converges to $(\mathbb{F}, S)$ as n goes to infinity.

7.1.2. *Amenable groups.* Amenable groups do not form a closed set, since free groups are limits of finite groups. They do not form an open set either. An interesting example that proves this comes from [BE15]: there exists a sequence of markings of $\mathbb{F}_2 \wr \mathbb{Z}$ that converges to a marking of $\mathbb{Z}^2 \wr \mathbb{Z}$. (See Example 7.4 in [BE15].) This sequence is Λ_{WP}-computable.

7.1.3. *Having sub-exponential growth.* The set of finitely generated groups with sub-exponential growth is neither closed nor open. This was proved by Grigorchuk in [Gri85]: there is constructed a family of groups G_ω, $\omega \in \{0,1,2\}^{\mathbb{N}}$, for which it is explained when G_ω has intermediate or exponential growth, and for which it is proved that the convergence in the space of marked groups of a sequence of groups $(G_{\omega_n})_{n\in\mathbb{N}}$ coincides with the convergence in $\{0,1,2\}^{\mathbb{N}}$ (for the product topology) of the sequence $(\omega_n)_{n\in\mathbb{N}}$. The sequence $(G_{\omega_n})_{n\in\mathbb{N}}$ is Λ_{WP}-computable when the sequence $(\omega_n)_{n\in\mathbb{N}}$ is computable for the usual numbering of the Cantor space.

7.1.4. *Orderable groups, locally indicable groups.* A finitely generated group G is *left-orderable* if there exists a total order $\leq$ on G which is compatible with left multiplication:

$$\forall x, a, b \in G,\ a \leq b \implies xa \leq xb.$$

The fact that the set of left-orderable groups is closed was noticed by Champetier and Guirardel in [CG05]. The following well known characterization of left-orderable groups was already used in [BCR19] and [GKEL23] in order to study the complexity of the property "being left-orderable":

Proposition 7.3. *A group G is left-orderable if and only if for any finite set $\{a_1, a_2, ..., a_n\}$ of non-identity elements of G, there are signs $(\epsilon_1, ..., \epsilon_n) \in \{-1,1\}^n$, such that the sub-semi-group generated by $\{a_1^{\epsilon_1}, a_2^{\epsilon_2}, ..., a_n^{\epsilon_n}\}$ does not contain the identity of G.*

It is straightforward to notice that this characterization provides a way of recognizing word problem algorithms for non left-orderable groups, and that it shows that the set of left-orderable groups is closed.

Proposition 7.4. *The set of left-orderable groups is ρ_{WP}-co-semi-decidable and closed in $\mathcal{G}$.*

We note that characterizations similar to the one given in Proposition 7.3 are available for the properties of being bi-orderable and locally indicable, see Fact 3.3.8 of [GKEL23]. These yield:

Proposition 7.5. *The sets of bi-orderable groups and of locally indicable groups are ρ_{WP}-co-semi-decidable and closed in $\mathcal{G}$.*

7.1.5. *Diffuse and unique product groups.* A group G has the *unique product property* if for every finite subsets A and B of G, there is some a in A and b in B whose product ab can be written in a unique way as a product of an element of A by an element of B. This property is relevant in that it implies a positive solution to the unit conjecture for group rings. In [Bow00] Bowditch introduced the following variation on the unique product property. A group is called *diffuse* if for every finite subset $A \subseteq G$ there is some $a \in A$ such that for every $g \in G$ either $ga \notin A$ or $g^{-1}a \notin A$.

We immediately get:

Proposition 7.6. *The sets of diffuse groups and of groups with the unique product property are ρ_{WP}-co-semi-decidable and closed in $\mathcal{G}$.*

7.1.6. *Virtually cyclic groups; virtually nilpotent groups; polycyclic groups.* For the three properties of being virtually cyclic, virtually nilpotent or polycyclic, we use the following lemma. In what follows, denote by ρ_{pres} the representation of $\mathcal{G}$ associated to presentations. The numbering it induces is that associated to recursive presentations. See Section 4.6.

Lemma 7.7. *Suppose that P is a ρ_{WP}-semi-decidable (resp. ρ_{pres}-semi-decidable) subset of $\mathcal{G}$, and that Q is a ρ_{pres}-semi-decidable subset of $\mathcal{G}$.*

Then the set of groups that have a finitely generated normal subgroup in P, and such that the quotient by this normal subgroup is in Q, is ρ_{WP}-semi-decidable (resp. ρ_{pres}-semi-decidable).

Note that in this statement, the normal subgroup should be finitely generated as a group, and not only finitely generated as a normal subgroup.

Proof. Given a word problem algorithm for a group G generated by a family S, we proceed as follows.

Enumerate all finite subsets of G.

There is an effective procedure that recognizes those finite subsets of G that generate a normal subgroup. Indeed, consider a finite set A in G, and the subgroup H it generates. The subgroup H is normal in G if and only if for each a in A and each s in S, the elements $s^{-1}as$ and sas^{-1} both belong to H. An exhaustive search for ways of expressing $s^{-1}as$ and sas^{-1} as products of elements of A will terminate if indeed those elements belong to H.

For each finite subset A which generates a normal subgroup H of G, we can obtain a word problem (resp. a presentation) for H thanks to the word problem for G (resp. thanks the presentation of G), and a presentation for

the quotient G/H. Indeed, an enumeration of the relations of G together with an enumeration of the elements of H yields a presentation of G/H.

The hypotheses of the lemma then allow us to recognize when the group H is in P and the quotient G/H is in Q. □

This lemma can be used directly to show that the properties of being virtually nilpotent or virtually cyclic are ρ_{WP}-semi-decidable. Note that virtually cyclic and virtually nilpotent groups are all finitely presented and have solvable word problem, since the property "being finitely presented and having solvable word problem" is stable under taking finite extensions -indeed, it is even invariant under quasi-isometry (see for instance [Sap11]).

Corollary 7.8. *The set of virtually cyclic groups is open in $\mathcal{G}$ and ρ_{WP}-semi-decidable.*

Proof. Apply Lemma 7.7 with P being the set of cyclic groups and Q the set of finite groups, to prove that the set of virtually cyclic groups ρ_{WP}-semi-decidable.

Analyzing the way the algorithm that stops on word-problem algorithms for virtually cyclic groups thus obtained works, one sees that the proof it produces of the fact that a group is virtually cyclic consists in finitely many relations. Those relations define an open set that contains only virtually cyclic groups. □

Corollary 7.9. *The set of virtually nilpotent groups is open in $\mathcal{G}$ and ρ_{WP}-semi-decidable.*

Proof. Apply Lemma 7.7 with P being the set of nilpotent groups and Q the set of finite groups. □

Note that we have chosen these two properties because they both admit equivalent definitions in terms of asymptotic geometry: by a well-known theorem of Gromov [Gro81], a group is virtually nilpotent if and only if it has polynomial growth, and a group is virtually cyclic if and only if it has zero or two *ends*, in the sense of Stallings [Hop43, Fre44]. It is thus remarkable that both of these properties can be recognized thanks to only finitely many relations.

Corollary 7.10. *The set of polycyclic groups is open in $\mathcal{G}$ and ρ_{WP}-semi-decidable.*

Proof. (Sketch) Iterate Lemma 7.7 with Q being the set of cyclic groups, and P being the set polycyclic groups with a subnormal series of length n, to obtain the result. □

7.1.7. *Groups with infinite conjugacy classes.* A group G has *infinite conjugacy classes* (ICC) if for each non identity element g of G the conjugacy class $\{xgx^{-1},\ x \in G\}$ of g is infinite.

Proposition 7.11. *The set of ICC groups is ρ_{WP}-co-semi-decidable, and it is closed in $\mathcal{G}$.*

Proof. Given a word problem algorithm for a group G over a generating family S and an element g of G, it is possible to prove that the conjugacy class of g is finite: define a sequence of sets $(A_n)_{n\in\mathbb{N}}$ by

$$A_0 = \{g\},$$
$$A_{n+1} = \{s^{-1}xs;\ s \in S \cup S^{-1},\ x \in A_n\}.$$

The conjugacy class of g is finite if and only if there exists an integer n such that $A_n = A_{n+1}$. A blind search for such an integer will terminate if it exists. This shows that the set of non-ICC groups is ρ_{WP}-semi-decidable. □

7.1.8. *Hyperbolic groups.* Let δ be a positive real number. A marked group G is δ-hyperbolic if the triangles (defined thanks to the word metric in G) are δ-thin, that is to say if for any three elements g_1, g_2 and g_3 of G, any geodesic that joins g_1 to g_2 stays in a δ-neighborhood of any pair of geodesics that join respectively g_2 and g_3 and g_1 and g_3.

Proposition 7.12. *Being δ-hyperbolic is ρ_{WP}-co-semi-decidable and thus closed.*

Proof. A marked group (G, S) is not δ-hyperbolic if it admits a triangle that is not δ-thin. The fact that this triangle is not δ-thin can be seen in a sufficiently large ball of the Cayley graph of (G, S), and thus any group whose Cayley graph corresponds to that (G, S) on this ball is also not δ-hyperbolic. It is easy to see that this condition can be effectively checked. □

Remark that being δ-hyperbolic is a marked group property, but not a group property, as can be seen from the fact that there exists a sequence of markings of $\mathbb{Z}$ that converges to a marking of $\mathbb{Z}^2$, which is not hyperbolic. A group is Gromov hyperbolic if any of its marking is δ-hyperbolic, for some δ that can depend on the marking. The set of Gromov hyperbolic groups is neither open nor closed in $\mathcal{G}$, but the previous proposition implies that it is a union of closed sets.

Corollary 7.13. *The set of hyperbolic groups is a F_σ subset of $\mathcal{G}$, and it is effectively not closed and effectively not open.*

7.2. **Table of results.** The following table gathers our examples. Remark that for each subset of $\mathcal{G}$ that appears in this table, eight properties of this set are expressed: whether or not it is open, whether or not it is closed, whether or not it is ρ_{WP}-semi-decidable, whether or not it is ρ_{WP}-co-semi-decidable (Type 2 characterizations), and the corresponding results in terms of Type 1 and of Banach-Mazur computability.

Clopen/decidable properties	**Open/semi-decidable properties**
Being abelian;	Being nilpotent;
Being isomorphic to a given finite group;	Kazhdan's Property (T) ([Sha00, Theorem 6.7], [Oza14]);
Having cardinality at most n, $n \in \mathbb{N}^*$;	Having a non-trivial center;
Being nilpotent of derived length $k > 0$;	Being perfect;
Being a certain marked isolated group.	Having torsion;
	Having rank at most k, $k \in \mathbb{N}^*$;
	Being virtually cyclic;
	Having polynomial growths;
	Being polycyclic;
	Counterexamples to one of Kaplansky's conjectures [Gar21].

Closed/co-semi-decidable properties	**Neither closed nor open properties**
Being infinite;	Being solvable;
Being k-solvable, for a fixed $k > 1$;	Being amenable;
Having finite exponent k, for a fixed $k >> 1$;	Being simple;
Being a limit group (see Section 8.2);	Having sub-exponential growth;
Being left-orderable, bi-orderable or locally indicable;	Being finitely presented;
Having the unique product property;	Being hyperbolic;
Being diffuse;	Being residually finite.
Being δ-hyperbolic, $\delta > 0$;	
Having Infinite Conjugacy Classes (ICC).	

8. Differences between Borel and effective Borel hierarchies

In this section, we translate known results of undecidability of the universal theory of some classes of groups into results that establish differences between the Borel and effective Borel hierarchies on the space of marked groups. We also give several conjectured candidates of properties that could break the correspondence.

8.1. Computably open sets in Type 1 and Type 2 computability.

Definition 8.1. A subset of $\mathcal{G}$ is called *computably open* if it can be written as a computable union of basic open sets. A set is called *computably closed* if its complement is computably open.

By [BR26, Theorem A], we have:

Proposition 8.2. *A subset of $\mathcal{G}$ is computably open if and only if it is Type 2 semi-decidable.*

In [Mos64], Moschovakis established a Type 1 equivalent of this result:

Theorem 8.3 ([Mos64], Theorem 11)**.** *For a recursive Polish space P and a subset U, the following are equivalent:*

- *U can be written as a computable union of basic open set,*
- *U is Type 1 semi-decidable and there is a program which, on input $x \in U$, outputs some radius r such that $B(x, r) \subseteq U$.*

By Theorem 1.7, this theorem cannot be applied to $\mathcal{G}^+$, and thus, while there is a single well-established notion of Type 2 computably open set, there are two competing notions of "Type 1 computably open sets", and it is not clear which one should be preferred.

This fact is a new illustration of the fact that additional difficulties arise when trying to obtain Type 1 (and Banach-Mazur) computability results.

The results of the rest of the present section are all all established in Type 2, and we leave as an open problem:

Problem 8.4. Extend the results of Section 8.2 to Type 1 and Banach-Mazur computability results. In particular, is the set of LEF groups Λ_{WP}-co-semi-decidable?

8.2. **Computably closed sets and universal theory of classes of groups.** The link between results on the topology of the space of marked groups and the study of the elementary theory of classes of groups was first established by Champetier and Guirardel in [CG05], in order to study *limit groups*, i.e. the closure of the set of free groups in $\mathcal{G}$.

We will leverage this link to establish Theorem H: the universal theory of $\mathcal{C}$ is semi-decidable if and only if the closure of the set of subgroups of groups in $\mathcal{C}$ is computably closed.

Let us first introduce some definitions.

We are interested in sentences from first order logic in the signature of groups. A formula is obtained with variables, logical connectors ($\wedge$ is "and", $\vee$ is "or", and $\neg$ is "not"), the equality symbol $=$, the group law $\cdot$, the identity element 1, and the group inverse $^{-1}$, and the two quantifiers $\forall$ and $\exists$. We use shortcuts where it is convenient (as the symbols $\neq$ or $\implies$), and always use implicitly all group axioms. A *sentence* is a formula with no free variables. A *universal sentence* is a sentence that uses only the universal quantifier.

For instance:

$$\forall x \forall y,\ x = y$$

$$\forall x \forall y \forall z,\ xy = yx \wedge yz = zy \wedge y \neq 1 \implies xz = zx$$

For a group G, let $T_\forall(G)$ denote the set of universal sentences that are true in G. For a class $\mathcal{C}$ of group we also write $T_\forall(\mathcal{C})$, meaning the set of universal sentences that hold in *all* groups of $\mathcal{C}$.

For G a group and $\mathcal{T}$ a set of sentences, we denote by $G \models \mathcal{T}$ the fact that G satisfies all sentences in $\mathcal{T}$.

In the space of marked groups, a universal sentence defines a closed set, and the correspondence with the arithmetical hierarchy holds, i.e., from a word problem algorithm, it is possible to prove that a group does not satisfy a given universal sentence.

For a set $\mathcal{C}$ of groups, we denote by $\mathcal{S}(\mathcal{C})$ the set of its subgroups.

Proposition 8.5. *For any set $\mathcal{C}$ of groups, we have $T_\forall(\mathcal{C}) = T_\forall(\mathcal{S}(\mathcal{C})) = T_\forall(\overline{\mathcal{S}(\mathcal{C})})$.*

Proof. This follows directly from the facts that:

- Satisfying a universal sentence is a closed property in $\mathcal{G}$;
- If $H \leq G$, then $H \models T_\forall(G)$. □

The main link between the topology of the space of marked groups and universal theories is given by the following lemma.

Lemma 8.6. *For any set $\mathcal{C}$ of groups and basic clopen set $\Omega^k_{R,S}$, we have the equivalence:*

$$\mathcal{S}(\mathcal{C}) \cap \Omega^k_{R,S} = \emptyset \iff \left(\forall \underline{x},\ \bigwedge R(\underline{x}) = 1 \implies \bigvee S(\underline{x}) = 1\right) \in T_\forall(\mathcal{C}).$$

Proof. The implication $\impliedby$ follows directly from the fact that each element of $\mathcal{S}(\mathcal{C})$ satisfies all sentences of $T_\forall(\mathcal{C})$.

For the converse implication, suppose that $(\forall \underline{x},\ \bigwedge R(\underline{x}) = 1 \implies \bigvee S(\underline{x}) = 1) \notin T_\forall(\mathcal{C})$. Then there is a group in $\mathcal{C}$ with elements that violate the sentence $\forall \underline{x},\ \bigwedge R(\underline{x}) = 1 \implies \bigvee S(\underline{x}) = 1$. These elements generate a group in $\mathcal{S}(\mathcal{C})$ which is in $\Omega^k_{R,S}$. □

Proposition 8.7. *Let $\mathcal{C}$ be a set of groups and G a finitely generated group. We have the equivalence*

$$G \models T_\forall(\mathcal{C}) \iff G \in \overline{\mathcal{S}(\mathcal{C})}.$$

Proof. The implication $\impliedby$ is a direct consequence of Proposition 8.5 together with the fact that satisfying a universal sentence is a closed property.

Suppose now that $G \models T_\forall(\mathcal{C})$. Fix a k-marking of G, and let $\Omega^k_{R,S}$ be a basic open set that contains it. Thus the sentence $\forall \underline{x},\ \bigwedge R(\underline{x}) = 1 \implies \bigvee S(\underline{x}) = 1$ is not satisfied by G. By Lemma 8.6, this implies that $\mathcal{S}(\mathcal{C}) \cap \Omega^k_{R,S} \neq \emptyset$. Since this holds for any basic set containing G, we obtain $G \in \overline{\mathcal{S}(\mathcal{C})}$. □

We get the immediate corollary:

Corollary 8.8. *The universal theory $T_\forall(\mathcal{C})$ of $\mathcal{C}$ is semi-decidable if and only if*

$$\mathcal{S}(\mathcal{C}) \cap \Omega^k_{R,S} \neq \emptyset$$

is a semi-decidable property of k, R and S.

Proof. Any sentence of $T_\forall(\mathcal{C})$, put in conjunctive normal form, is a conjunction of sentences of the form $\forall \underline{x},\ \bigwedge R(\underline{x}) = 1 \implies \bigvee S(\underline{x}) = 1$.

But, for any theory $\mathcal{T}$, being able to semi-decide "finite conjunctions of statements in $\mathcal{T}$" is equivalent to being able to semi-decide statements in $\mathcal{T}$.

Thus we can conclude by Lemma 8.6. □

Recall that a subset $A \subseteq \mathcal{G}_k$ is called *computably compact* if there is an algorithm which, given a cover of A by basic open sets $(\Omega_n^k)_{n\in\mathbb{N}}$, outputs some N such that $(\Omega_n^k)_{n\leq N}$ also covers A (in particular, such N should always exist, and A is compact). The following is a well-known fact in computable topology:

Proposition 8.9 ([BP03], Theorem 4.9)**.** *A subset $A \subseteq \mathcal{G}_k$ is computably compact if and only if it is compact and there is an algorithm which, given a finite sequence $(\Omega_n^k)_{n\leq N}$ of basic open sets, stops if and only if $A \subseteq (\Omega_n^k)_{n\leq N}$.*

See also [Pau16] for a more recent approach to computable compactness.

Lemma 8.10. *A compact subset $A \subseteq \mathcal{G}_k$ is computably compact if and only if*

$$\mathcal{C} \cap \Omega_{R,S}^k = \emptyset$$

is a semi-decidable property of R and S.

Proof. We have the equivalence

$$\begin{aligned} \mathcal{C} \cap \Omega_{R,S} = \emptyset &\iff \mathcal{C} \subseteq \mathcal{G}_k \setminus \Omega_{R,S}, \\ &\iff \mathcal{C} \subseteq \bigcup_{r\in R} \Omega_{r\neq 1} \cup \bigcup_{s\in S} \Omega_{s=1}. \end{aligned}$$

Thus by Proposition 8.9, if $A \subseteq \mathcal{G}_k$ is computably compact, $\mathcal{C} \cap \Omega_{R,S} = \emptyset$ is semi-decidable.

Conversely, consider a finite union $\bigcup_{n\leq N} \Omega_n$. Then, $\mathcal{C} \subseteq \bigcup_{n\leq N} \Omega_n$ if and only if $\mathcal{C}$ does not meet the complement $\left(\bigcup_{n\leq N} \Omega_n\right)^c$ of $\bigcup_{n\leq N} \Omega_n$. But we have

$$\begin{aligned} \left(\bigcup_{n\leq N} \Omega_n\right)^c &= \bigcap_{n\leq N} \Omega_n^c \\ &= \bigcap_{n\leq N} \left(\bigcup_{i\leq A_n} \Omega_{n,i}\right) \end{aligned}$$

where each Ω_n^c was written as a union $\bigcup_{i\leq A_n} \Omega_{n,i}$ of basic sets.

We use distributivity of intersection over unions to rewrite the above as:

$$\bigcap_{n\leq N} \left(\bigcup_{i\leq A_n} \Omega_{n,i}\right) = \bigcup_{\sigma} \bigcap_{n\leq N} \Omega_{n,\sigma(n)},$$

where σ ranges over all choice functions $\sigma : \{1, .., N\} \to \mathbb{N}$ with $\sigma(n) \leq A_n$ for each n. Note that each intersection $\bigcap_{n\leq N} \Omega_{n,\sigma(n)}$ can be expressed as a single basic open set Ω_σ.

And, furthermore,

$$\mathcal{C} \cap \bigcup_{\sigma} \Omega_\sigma = \emptyset \iff \bigwedge_{\sigma} (\mathcal{C} \cap \Omega_\sigma = \emptyset),$$

which indeed shows that if $\mathcal{C} \cap \Omega_\sigma = \emptyset$ is a semi-decidable property of σ, then $\mathcal{C} \subseteq \bigcup_{n\leq N} \Omega_n$ is a semi-decidable property of the finite tuple $(\Omega_n)_{n\leq N}$. □

Corollary 8.11. *Let A be a subset of $\mathcal{G}$ and put $A_k = A \cap \mathcal{G}_k$. The sets A_k are uniformly computably compact if and only if*

$$\mathcal{C} \cap \Omega_{R,S}^k = \emptyset$$

is a semi-decidable property of k, R and S.

The following well-known fact is an effective version of the statement that if X is compact, then its compact and closed subsets agree.

Proposition 8.12 ([BP03], Corollary 4.14)**.** *Let X be computably compact. Then a subset $A \subseteq X$ is computably closed if and only if it is computably compact. And this statement is uniform in A.*

Here again, [Pau16] provides a more modern treatment of this fact.
A first corollary we obtain that each $\mathcal{G}_k$ is computably compact.

Corollary 8.13. *For each k, $\mathcal{G}_k$ is computably compact, and this is uniform in k.*

Proof. This follows from the fact that $\mathcal{G}_k$ is a computably closed subset of the Cantor space. □

Rewriting Proposition 8.12 using σ-compactness of $\mathcal{G} = \bigcup \mathcal{G}_k$, we immediately obtain:

Corollary 8.14. *Let A be a subset of $\mathcal{G}$. Then A is computably closed if and only if the sets $A_k = A \cap \mathcal{G}_k$ are uniformly computably compact.*

Theorem 8.15. *Let $\mathcal{C}$ be a set of groups. We have the equivalence*

$$T_\forall(\mathcal{C}) \text{ is not c.e.} \iff \overline{\mathcal{S}(\mathcal{C})} \text{ is not computably closed.}$$

Proof. We have the equivalences:

$T_\forall(\mathcal{C})$ is semi-decidable		
$\iff$	$\mathcal{C} \cap \Omega^k_{R,S} = \emptyset$ is a semi-decidable property of R, S and k	Corollary 8.8
$\iff$	the sets $\overline{\mathcal{S}(\mathcal{C})}_k = \overline{\mathcal{S}(\mathcal{C})} \cap \mathcal{G}_k$ are uniformly computably compact	Corollary 8.11
$\iff$	$\overline{\mathcal{S}(\mathcal{C})}$ is computably closed.	Corollary 8.14

□

8.3. **Applications.** Recall that the set of LEF groups is the closure of the set of finite groups in $\mathcal{G}$, and that the set of limit groups is the closure of the set of free groups in $\mathcal{G}$.

Theorem 8.16. *The following sets are computably closed:*

- *The set of limit groups,*
- *The closure of $\mathcal{S}(H)$, where H is any hyperbolic group.*

The following sets are closed, but not computably so:

- *The closure of the set $\mathcal{G}^+$ of groups with solvable word-problem,*
- *The set of LEF groups,*
- *More generally the closure of any set of finite groups that contains all finite three solvable groups,*
- *The closure of the set of finite nilpotent groups,*
- *The closure of the set of hyperbolic groups.*

Proof. Each statement is a consequence of Theorem 8.15. We use it together with:

- Makanin's Theorem [Mak85] for limit groups, and its extension to hyperbolic groups by Dahmani and Guirardel [DG11] (the torsion free case being due to Sela [Sel09]);
- Theorem B for the closure of the set $\mathcal{G}^+$;
- Slobodskoi's Theorem [Slo81] which shows that the universal theory of finite groups is not c.e., and two variations: [Kha83] for finite nilpotent groups, and [KMS17] for sets of finite groups that contain all finite three solvable groups.
- The fact the the universal theory of hyperbolic groups is not semi-decidable is due to Osin [Osi09]. □

Problem 8.17. Are the following sets computably closed?

- The closure of the set of torsion groups;
- The closure of the set of groups with a finite exponent;
- The set of sofic groups.

8.4. **Isolated Groups.** It does not seem possible to prove that a word problem algorithm belongs to an isolated group (even with a partial algorithm). From this, we conjecture that the set of isolated groups, while open, is not Λ_{WP}-semi-decidable. The same conjecture goes for finitely presented simple groups, which form a subset of the set of isolated groups.

Conjecture 8.18. *The sets of isolated group and of finitely presented simple groups are not Λ_{WP}-semi-decidable.*

Remark that the impossibility of partially recognizing isolated groups is also an open problem for groups described by finite presentations.

Conjecture 8.19. *The set of isolated group is not Λ_{FP}-semi-decidable.*

While the Adian-Rabin theorem implies that no algorithm stops exactly on finite presentations of non-isolated groups (since being isolated is a Markov property, as any group with unsolvable word problem provides a negative witness for it), it fails to prove that no algorithm stops exactly on finite presentations of isolated groups. The problem of proving that the set of finite presentations of simple groups is not c.e. is, to the best of our knowledge, still open. (The question appears for instance in [Mos73].)

This is related to the Higman-Boone conjecture, which asks whether any group with solvable word problem embeds in a finitely presented simple group. Indeed, a proof of the Higman-Boone conjecture would imply that there exists simple groups of arbitrarily difficult (while solvable) word problem, in terms of time complexity. By a simple diagonal argument, we have:

Lemma 8.20 ([Rau21], Proposition 26)**.** *Let A be a set of finitely presented groups with uniformly solvable word problem. If A is Λ_{FP}-c.e., then there is a universal computable upper bound to the time complexity for the word problem for groups in A.*

And thus:

Proposition 8.21. *If the Higman-Boone conjecture holds, the set of finitely presented simple groups is not Λ_{FP}-semi-decidable.*

It is also unknown (the question appears in [dCGP07]) whether isolated groups are dense in $\mathcal{G}^+$. If they were, it would follow from Proposition 6.3 that no sequence of word problem algorithms which contains each isolated group can be enumerated. We could still arrive to that conclusion if we knew that the word problem is not uniform on isolated groups, that is to say, since all isolated groups are finitely presented, if we knew that a solution to the word problem of an isolated group cannot be retrieved from a finite presentation for this group. For instance, it is well known that the word problem is uniform on the set of simple groups [Kuz58], however, Kuznetsov's argument fails if we add the trivial group to the set of simple groups.

It would be very interesting to prove that the trivial group is unrecognizable from simple groups, from the finite presentation description. This would prove both that the word problem is not uniform on all isolated groups, and that the set of finite presentations of simple groups is not c.e.. However, too few finitely presented infinite simple groups are known as of now to obtain such results.

Jeandel has further investigated in [Jea17] the link between Kuznetsov's Theorem and the space of marked groups, and given a general framework where "Kuznetsov type arguments" do apply.

9. Subgroups of finitely presented groups with solvable word problem

9.1. **Higman-Clapham-Valiev Theorem for groups with solvable word problem.** After Higman's proof of his famous Embedding Theorem [Hig61], several theorems that resemble it were obtained.

In particular, it was remarked that the theorem is effective, meaning that it provides an algorithm that takes as input a recursive presentation for a group G, and outputs a finite presentation for a group H, together with a finite family of elements of H that generate G. Note that in terms of numbering types, this implies that the numbering type $\Lambda_{r.p.}$, associated to recursive presentation, is equivalent to the numbering type associated to the idea "a marked group (G, S) is described by a finite presentation of an overgroup of G together with words that define the elements of S in that overgroup". We leave out the details.

We will be interested here in the version of Higman's Theorem that preserves solvability of the word problem [Val75, Cla67]. This theorem is known as the Higman-Clapham-Valiev Theorem.

Historical remarks about these results can be found in [OS04]. The following formulation of the Higman-Clapham-Valiev Theorem can also be found in [McC70].

Theorem 9.1 (Higman-Clapham-Valiev, I)**.** *There exists a procedure that, given a recursive presentation for a marked group (G, S), produces a finite presentation for a group H, together with an embedding $G \hookrightarrow H$ described by the images of the generators of G, and such that if the word problem is solvable in G, then it is also solvable in H.*

One can also check that if one has access to a word problem algorithm for the group given as input to this procedure, one can obtain a word problem algorithm for the constructed finitely presented group. This yields:

Theorem 9.2 (Higman-Clapham-Valiev, II)**.** *There exists a procedure that, given a word problem algorithm for a finitely generated group, produces a finite presentation of a group in which it embeds, together with a word problem algorithm for this new group, and a set of elements that generate the first group.*

This proves that, in general, the description of a group by its word problem algorithm, or by a finite generating family inside a finitely presented group with solvable word problem, are equivalent (we leave it to the reader to render this statement precise: define a numbering of $\mathcal{G}$ associated to the idea "a group is given as a subgroup of a group described by a finite presentation together with a word problem algorithm", thus using the numbering $\nu_{WP} \wedge \nu_{FP}$ to describe the overgroup, the Higman-Clapham-Valiev Theorem implies that this numbering is equivalent to ν_{WP}).

Thus the study of algorithmic problems that can be solved from the word problem description is identical to the study of decision problems about subgroups of finitely presented groups with solvable word problem. We now prove Theorem A.

Theorem 9.3. *Let P be a marked group property. The following are equivalent:*

- *There exists a finitely presentable marked group (G, S) with solvable word problem where the problem "given elements $(w_1, ..., w_k)$ in G, decide whether the marked group they generate has P" is not semi-decidable;*
- *The property P is not Λ_{WP}-Banach-Mazur semi-decidable.*

Proof. Suppose that P is Λ_{WP}-Banach-Mazur semi-decidable. Fix a marked group (G, S) with solvable word problem. Consider an effective enumeration $\{T_1, T_2, ...\}$ of all finite sets of words over $(S \cup S^{-1})^*$. The sequence of marked groups $((\langle T_i \rangle, T_i))_{n\in\mathbb{N}}$ is a computable sequence. Thus if P is Banach-Mazur semi-decidable the set $\{n \in \mathbb{N}, P((\langle T_i \rangle, T_i))\}$ is a c.e. subset of $\mathbb{N}$. And thus it is possible to semi-decide P on subgroups of G.

Suppose that P is not Λ_{WP}-Banach-Mazur semi-decidable. There must exist a Λ_{WP}-computable sequence of marked groups $((G_n, S_n))_{n\in\mathbb{N}}$ such that $\{n \in \mathbb{N}, P((G_n, S_n))\}$ is not a c.e. subset of $\mathbb{N}$. Embedding the restricted direct product of the (G_n, S_n) in a finitely generated group with solvable word problem (see for instance [Dar15]) then in a finitely presented group with solvable word problem via Higman's embedding theorem yields the desired finitely presented group. □

The following theorem is a direct consequence of a joint application of the the above result with Markov's Lemma:

Theorem 9.4. *Suppose that a Λ_{WP}-computable sequence $(G_n)_{n\in\mathbb{N}}$ of k-marked groups effectively converges to a k-marked group H, and suppose that $H \notin \{G_n, n \in \mathbb{N}\}$. Then there exists a finitely presented group Γ, with solvable word problem, in which no algorithm can, given a tuple of elements of Γ that defines a marked group of $\{G_n, n \in \mathbb{N}\} \cup \{H\}$, stop if and only if this tuple defines H.*

Note that when the conjugacy problem is uniformly solvable for the groups in $(G_k)_{k\in\mathbb{N}}$, we may want to apply the version of Higman's Theorem due to Alexander Olshanskii and Mark Sapir ([OS04], and [OS05] for non-finitely generated groups) that preserves its solvability.

9.2. **Some examples.** We now give some examples of possible applications of Theorem 9.4. The *order problem* in a marked group asks for an algorithm that, given an element of the group, determines what is its order. In a group with solvable word problem, this is equivalent to being able to decide whether or not an element has infinite order. The *power problem* asks for an algorithm which, given two elements of a marked group, decides whether or not the first one is a power of the second one. (It is thus the subgroup membership problem restricted to cyclic subgroups.)

The following two propositions were proven by McCool in [McC70].

Proposition 9.5. *There exists a finitely presented group with solvable word problem, but unsolvable order problem.*

Proof. Apply Theorem 9.4 to a sequence of finite cyclic groups that converges to $\mathbb{Z}$. This yields a finitely presented group with solvable word problem in which one cannot decide whether a given element generates a subgroup isomorphic to $\mathbb{Z}$ or to a finite cyclic group. This is precisely a finitely presented group with solvable word problem, but unsolvable order problem. □

Proposition 9.6. *There exists a finitely presented group with solvable word problem, but unsolvable power problem.*

Proof. Apply Theorem 9.4 to the sequence of 2-markings of $\mathbb{Z}$ defined by the generating families $(1, k)$, $k \in \mathbb{N}^*$, which converges to (the only 2-marking of) $\mathbb{Z}^2$ when k goes to infinity (see [CG05]). This yields a finitely presented group with solvable word problem where, given a pair of commuting elements, one cannot decide whether they generate $\mathbb{Z}^2$, or if one of these elements is a power of the other: this is a group with unsolvable power problem. □

We can also use this theorem to strengthen a result that was recently obtained in [DI22].

Theorem 9.7. *There is a finitely presented group with solvable word problem in which the problem of deciding whether a given subgroup is amenable is neither semi-decidable nor co-semi-decidable.*

Proof. This is proven by using both a sequence of marked amenable groups which converges to a non-amenable group and a sequence of non-amenable marked groups that converges to an amenable marked group. Such examples were given in Section 7.1. □

The "not semi-decidable" half of this result is Theorem 6 in [DI22].

Theorem 9.4 can be applied to all the properties that appeared in Section 7.1 to produce results similar to this one. The three results given above were well known, but explaining them in terms of convergence in the space of marked groups unifies several existing constructions. What's more, for the rest of the effectively non-closed/non-open group properties presented in Section 7.1, the result obtained by applying Theorem 9.4 are new.

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