In memoriam of Alan Robert WOODS: Some of the works inspired (especially in the French community of Logicians and Number Theorists) by his seminal thesis

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Abstract

We recall a first result of Alan WOODS: the undecidability of the first-order theory $Th(\mathbb{N}, S, \perp)$, where $\perp$ denotes the binary relation of coprimeness, with two different proofs. The question of the $(S, \perp)$-definability of the full first-order arithmetic leads to the famous Erdös-Woods Conjecture (EWC) (*There is some* $k \in \mathbb{N}$ *such that every natural number* x *is determined uniquely by the sequence* $S_0, S_1, \ldots, S_k$ *of sets of prime numbers defined by* $S_i = \{ p \mid p \text{ divides } x + i \}$). In case EWC is false, then a-b-c conjecture, Hall's conjecture, Hall-Schinzel's conjecture, and Lang-Waldschmidt's conjecture are also false (Michel LANGEVIN). After Alan WOODS, and except for EWC, the situations (mutual definability, decidability) of many (not all!) reasonable natural theories $Th(\mathbb{N}, R_1, R_2)$ – where R_1 is a subset of the relation of addition and R_2 a subset of the relation of divisibility – were investigated (Alexis BÈS, Patrick CÉGIELSKI, Françoise MAURIN, Denis RICHARD) with some surprising results. Concerning the *theory of addition and a predicate*, for primes essentially, nothing was known before Alan, in a joint work with C. G. JOCKUSH and P. T. BATEMAN, proved under Schinzel's hypothesis (a polynomial generalization of the twin-primes conjecture) that $Th(\mathbb{N}, +, \mathbb{P})$ is undecidable. Patrick CÉGIELSKI, Denis RICHARD, and Maxim VSEMIRNOV have shown an absolute result: the undecidability of $Th(\mathbb{N}, +, n \mapsto n \times f(n))$, where f is a good approximation of $n \mapsto p_n/n$ and p_n is the $(n+1)$-th prime.

Alan also proved that the class RUD of all rudimentary (i. e. Δ_0-definable) sets of positive integers is contained in the class consisting of the *spectra* $S_\Phi = \{|M| \text{ such that } M \text{ is finite and } M \models \Phi\}$ of those sentences Φ having only graphs as models. Malika MORE, Alex ESBELIN, and Frédéric OLIVE gave a toolbox for investigating rudimentary predicates and also extended results to second order languages.

The name of Alan is attached to another very known concept : the **Erdös-Woods numbers**, which are *the positive integers* k *such that there is a closed interval of integers* $[a, a+k]$ *with the property that every integer in the interval has a factor in common with either* a *or* $a + k$. Results on these numbers are due to M. BIENKOWSKI, Patrick CÉGIELSKI, D. L.

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Dowe, François Heroult, M. Korzeniowki, K. Lorys, N. Lygeros, Denis Richard, Maxim Vsemirnov, and others.

Although sure of not being at all exhaustive in speaking of the works of Alan Robert Woods, we add some results and interesting open questions about the p-destinies (Francis Nézondet) within $< \mathbb{N}, S, \perp >$.

1 Undecidability of successor and coprimality

In the *Open Days in Model Theory and Set Theory* held in Jadwisin in the Karpacz Mountains near Warsaw in September 1981, Jeff Paris gave a lecture in which he mentioned the thesis of Alan Woods, quoting one of Alan's results, the proof that $I\Delta_0 + \Delta_0 - PHP \vdash$ *Bertrand's theorem.* Denis Richard met Jeff for discussing about his lecture and two of his own results on questions appearing in the thesis of Julia Robinson, namely the $(\leq, \perp)$-definability of the whole arithmetic and the undecidability of $Th(\mathbb{N}, S, \perp)$, where S is the successor function. Jeff told him that one of his PhD students, namely Alan, had also proved those two results. Denis was glad that Jeff could know his arguments to prove the undecidability result just mentioned and compare them with those of Alan. He was at that time at the University Claude Bernard of Lyon, and organized a visit of Alan in order to give some lectures on his thesis. In fact, Alan visited Lyon in the following winter and they were able to compare their independent proofs of the undecidability of $Th(\mathbb{N}, S, \perp)$ and $Th(\mathbb{N}, =, S, \perp)$.

<u>Alan's Proof</u>.- Alan introduces infinitely many equivalence relations such as

- $x \approx y$ iff $\nu(x) = \nu(y)$, where $\nu(a)$ is the cardinal of the set of prime divisors of a
- $x \sim y$ iff x and y have the same set of prime divisors
- $x \sim_n y$ iff $(x \sim y) \wedge (x+1) \sim (y+1) \wedge \ldots \wedge (x+n) \sim (y+n)$

For $x \neq 0$, let us denote by $\overline{x}$, $\overline{x}^n$, and $\overline{\overline{x}}$, respectively, the $\sim$, $\sim_n$, and $\approx$ equivalence classes of which x is a representative.

Let $\overline{\overline{\mathbb{N}}} = \{\overline{\overline{x}} \mid x \in \mathbb{N}\backslash\{0\}\}$, $\overline{\mathbb{N}} = \{\overline{x} \mid x \in \mathbb{N}\backslash\{0\}\}$, $\overline{\mathbb{N}}^n = \{\overline{x}^n \mid x \in \mathbb{N}\backslash\{0\}\}$.

Alan proves $< \mathbb{N}, =, +, \times >$ is a model isomorph to $< \overline{\mathbb{N}}, =, +, \times >$ under the mapping ν^{-1}. Then he proves that, considered as predicates on $\mathbb{N}$, the expressions $x = y$, $x + y = z$, and $x \times y = z$ are $(S, \perp)$-definable and hence there is an effective way of transforming any sentence Φ of the language $L(=, +, \times)$ into an equivalent $L(S, \perp)$ sentence Φ^* asserting that

$$< \overline{\mathbb{N}}, =, +, \times > \models \Phi^*.$$

This result, of course, permits to deduce the undecidability of $Th(\mathbb{N}, S, \perp)$ and of $Th(\mathbb{N}, =, S, \perp)$ from the known undecidability of the full first order arithmetic. Also the existence of these formulas reduces the problem of the $(S, \perp)$-definability of multiplication to the question of whether ν^{-1} is $(S, \perp)$- definable.

Through some lemmas of definability using the Chinese Remainder Theorem, Alan proves the $(S, \perp)$-definability of relations $\overline{\overline{x}} \leq \overline{\overline{y}}$, $\overline{\overline{x}} + \overline{\overline{y}} = \overline{\overline{z}}$, and $\overline{\overline{x}} \times \overline{\overline{y}} = \overline{\overline{z}}$.

Richard's proof.- Instead of "Richard's proof", maybe it could be better to say "proof as a corollary of the following result, due to G.D BIRKHOFF and H.S. VANDIVER–ZSIGMONDY–CARMICHAEL".

Theorem.- (**ZBV-theorem**) *If x and y are coprime positive integers such that $x > y \geq 1$ then, for every $n > 0$, there exists at least a prime divisor of $x^n - y^n$ which divides none of the of $x^m - y^m$ for $0 < m < n$ (such divisors are called* primitive divisors *of $x^n - y^n$) except in the following cases*:

- $n = 1, x - y = 1$ (*since there is no prime divisor*)
- $n = 2, x + y = 2^\alpha$ (*for* $\alpha > 0$)
- $n = 6, x = 2, y = 1.$

Actually, it is not difficult to give a $(S, \perp)$-definition of $x = n$, for every n in $\mathbb{N}$, of $\mathbb{P}(x)$ (meaning x is prime), of the relation "*p is prime and x is a power of p different from 1*", of the relation "*x and y are powers (a priori different) primes and are equal*", of set-theoretic relations of union, intersection, and complementation on the supports (the *support* of an integer a is the set of the prime divisors of a), and of the relation "*$x \neq 1$ and x is a power of some prime and q is a primitive divisor of $x - 1$*".

Let p be some fixed prime. We denote by $p^{\mathbb{N}}$ the set of powers of the prime p, by $p^\alpha \oplus_p p^\beta = p^{\alpha+\beta}$ a binary function of addition and by $p^\alpha \otimes_p p^\beta = p^{\alpha\times\beta}$ a binary function of multiplication in $p^{\mathbb{N}}$. We also denote by $=_p$ the restriction to $p^{\mathbb{N}}$ of the equality relation. It is obvious that the structure $< p^{\mathbb{N}}, =_p, \oplus_p, \otimes_p >$ is isomorphic to the usual model $< \mathbb{N}, =, +, \times >$. Then we have proved that the structure $< p^{\mathbb{N}}, =_p, \oplus_p, \otimes_p >$ is $(S, \perp)$-definable in $\mathbb{N}$.

As a by-product, we get the fact that the $(S, \perp)$-definability of any map $n \mapsto p^n$, for any fixed prime p, would be sufficient for giving the $(S, \perp)$-definability of multiplication.

In fact, the two proofs have their advantages. On the one hand, Alan does not need any deep result in Number Theory (just the Chinese Remainder Theorem) and proves by using quotient structures the transfer of undecidability from full arithmetic to the weak first order $(S, \perp)$-arithmetic. On the other hand, the ZBV-theorem provides a powerful tool for investigating $Th(\mathbb{N}, S, \perp)$ and $Th(\mathbb{N}, =, S, \perp)$. The coding Denis RICHARD uses to transfer undecidability is explicit and, in case of a negative answer, we shall know in terms of Number Theory, exactly where the $(S, \perp)$-definability fails.

The previous result straightforwardly shows that :

(i) $z = x \times y$ is $(S, \perp)$-definable in $\mathbb{N}$

(ii) $z = x + y$ is $(S, \perp)$-definable in $\mathbb{N}$

(iii) $x \leq y$ is $(S, \perp)$-definable in $\mathbb{N}$

(iv) $x = y$ is $(S, \perp)$-definable in $\mathbb{N}$

(v) For any fixed prime p, the map $n \mapsto p^n$ is $(S, \perp)$-definable in $\mathbb{N}$.

Statements (i) to (iv) appear at first in the thesis of Alan [Woo-81], (v) in [Ric-82].

2 Erdös-Woods Conjecture (EWC)

(**Erdös-Woods Conjecture**) *There is some $k \in \mathbb{N}$ such that every natural number x is determined uniquely by the sequence $S_0, S_1, \ldots, S_k$ of sets of prime numbers defined by $S_i = \{ p \mid p \text{ divides } x+i\}$*

Let $Supp(a)$ denote the set of prime divisors of a. Consider the following conjectures of Number Theory.

Oesterlé-Masser's conjecture.- (Also called **a-b-c conjecture**) *For all $(a,b,c) \in (\mathbb{N}^*)^3$ such that $a+b=c$ and for every real r, there exists an effectively computable constant $K(r)$ such that*

$$\Pi_{p \in Supp(abc)}\ p > K(r)^{1-\epsilon(a)}$$

with $\epsilon(r)$ tends to 0 when r tends to infinity.

Hall's conjecture.- *If $x^3 \neq y^2$ then there exists an effectively computable constant C such that*

$$|x^3 - y^2| > [C.Max(x^3, y^2)]^{1/6}$$

Hall-Schinzel's conjecture. *If $x^m \neq y^n$ then there exists an effectively computable constant C such that*

$$|x^m - y^n| > [Max(x^m, y^n)]^C$$

Michel Langevin proved in 1988 [Lan-88] that the three previous conjectures are false in case EWC fails. Moreover every conjecture which is false if Hall's conjecture is false becomes false in turn if EWC is. This is the case for Lang-Waldschmidt's conjecture (which gives lower bounds of linear forms of logarithms) and for Vojta's conjecture about abelian varieties.

Obviously, a positive solution to one of the conjectures above would give a positive solution to EWC and Alan was right when he gave the title of his thesis [Woo-81]. Denis Richard [Ric-88] was able to prove, after some discussion with A. Schinzel, that if the a-b-c conjecture is true then the equality of supports (sets of prime divisors) leads to the equality of numbers

$$Supp[x^{2^n} - 1] = Supp[y^{2^n} - 1] \Rightarrow x = y.$$

Before leaving EWC, we must note that Maxim Vsemirnov was able to extend the EWC to a conjecture on polynomial rings. Parallel to the second proof using the so-called ZBV-theorem, Maxim Vsemirnov developed an investigation of Carmichaël numbers useful for coding in the logical approach of EWC [Vse-02].

3 Definability, Decidability, and Undecidability within theories of $\mathbb{N}$ in various first-order languages

After Alan's thesis, a lot of PhD students and experienced researchers investigate the questions first opened by Julia ROBINSON, and then revisited by Alan and members of the team of Denis RICHARD.

Here is an (non-exhaustive) abstract table giving some of the results on decidability found between 1980 and 2000. The theories in the first four lines are undecidable and decidable in the two last lines.

$$
\begin{array}{ccccc}
 & & <\mathbb{N},=,+,\times> & & \\
 & & \uparrow\downarrow & & \\
<\mathbb{N},\perp,<> & \rightarrow & <\mathbb{N},|,<> & \rightarrow & <\mathbb{N},=,\times,<> \\
\text{[Woo-81]} & \leftarrow & & & \\
\uparrow(\downarrow ?) & & \uparrow\downarrow & & \uparrow\downarrow \\
<\mathbb{N},\perp,S> & \rightarrow & <\mathbb{N},|,S> & \rightarrow & <\mathbb{N},=,\times,S> \\
\text{[Woo-81]} & \leftarrow ? & \text{[Rob-49]} & \leftarrow & \\
\text{[PB-81]} & & & & \\
\uparrow\neg\downarrow & & \uparrow\downarrow & & \uparrow\downarrow \\
<\mathbb{N},\perp,<_{\Pi}> & \rightarrow & <\mathbb{N},|,<_{\Pi}> & \rightarrow & <\mathbb{N},=,\times,<_{\Pi}> \\
\text{[BR-98]} & & \text{[BR-98]} & & \text{[BR-98]} \\
\uparrow\neg\downarrow & & \uparrow\neg\downarrow & & \uparrow\neg\downarrow \\
<\mathbb{N},\perp,<_{P_2}> & \rightarrow & <\mathbb{N},|,<_{P_2}> & \rightarrow & <\mathbb{N},=,\times,<_{P_2}> \\
\text{[BR-98]} & & \text{[BR-98]} & & \text{[BR-98]} \\
 & & & & \uparrow\neg\downarrow \\
 & & & & <\mathbb{N},=,\times,<_{\mathbb{P}}> \\
 & & & & \text{[Mau-97]} \\
 & & & & \uparrow\neg\downarrow \\
 & & & & <\mathbb{N},=,\times,(\overline{n})_{n\in\mathbb{N}}> \\
 & & & & \text{[Sko-30]}
\end{array}
$$

4 Results about addition and primes

Before 1981, almost nothing was known about the association of the operation of addition and the predicate of being a prime. Of course, there is the famous theorem of SCHNIRELMANN on the fact that every integer is a sum of less than 27 (nowadays 19) primes and the theorem of VINOGRADOV which reduces asymptotically this sum to three primes. Such results show, at least, the depth and the interest of $Th(\mathbb{N},+,\mathbb{P})$.

The first results on this area were conditional results under a conjecture of SCHINZEL. Let SH denote *Schinzel's hypothesis*, i.e. the following generalization of the twin primes conjectures:

(SH) *Let $f_1(x), f_2(x), \ldots, f_n(x)$ irreducible polynomials over $\mathbb{Z}$, each having a positive leading coefficient. If there is no prime p such that $p|f_1(x)f_2(x)\ldots f_n(x)$ then there are infinitely many integers x such that $[f_1(x)\in\mathbb{P}\wedge f_2(x)\in\mathbb{P}\wedge\ldots\wedge f_n(x)\in\mathbb{P}]$.*

Let **(DC)** denote *Dickson's conjecture*, which is (SH) restricted to linear polynomials. In 1981, Alan proved

(DC) $\Rightarrow Th_{\forall}(\mathbb{N}, +, \mathbb{P})$ *is decidable.*

In 1983, Belyakov and Martyanov extended this result of Alan by showing

(DC) $\Rightarrow Th_{\forall}(\mathbb{Z}, +, \mathbb{P}, 1)$ *is decidable.*

In 1993, Alan, T. Bateman, and C.G. Jockush [BJW-93] showed, using arguments from automata theory, a conditional result on a subtheory of $Th(\mathbb{N}, +, \mathbb{P})$, namely

(DC) $\Rightarrow Th_2(\mathbb{N}, S, \mathbb{P})$ *is decidable.*

So that multiplication is $(+, \mathbb{P})$-definable and hence

(SH) $\Rightarrow Th(\mathbb{N}, S, \mathbb{P})$ *is undecidable.*

Maurice Boffa briefly proves in the one-page paper [Bof-98] that

(SH) $\Rightarrow Th(\mathbb{N}, +, \{primes\ of\ the\ form\ an + b\ for\ a \perp b\})$ *is undecidable.*

Patrick Cégielski, Yuri Matiiassevich, Denis Richard, and Maxim Vsemirnov decided to investigate on unconditional results around the same theory $Th(\mathbb{N}, +, \mathbb{P})$. In 1996 [CMR-96], they proved that

$Th(\mathbb{N}, n \mapsto p^n, R)$ for $R \in \{\times, |, \perp\}$ *are undecidable.*

Now remember that

$$p_n = nlog(n) + n(log(log(n) - 1) + O(nlog(log(n))/log(n)).$$

In [CRV-07], they proved the following unconditional theorem

The theories $Th(\mathbb{N}, +, n \mapsto nlog(n) + n(log(log(n) - 1))$, $Th(\mathbb{N}, +, n \mapsto p_n, n \mapsto rem(p_n, n))$, *and* $Th(\mathbb{N}, +, n \mapsto nf(n))$ *are undecidable, for any function* f *"close" to the integer part (floor) of* p_n/n.

Below, PP denotes the set of products of two primes, X the set of primary numbers (powers of a prime number), and $<_A$ the restricted natural order of $\mathbb{N}$ to A for a part A of $\mathbb{N}$.

R. Villemaire and Véronique Bruyère showed in 1992 that if $V_k(x)$ is the valuation of an integer for the (k+1)-th prime then $Th(\mathbb{N}, +, V_k)$ is decidable but $Th(\mathbb{N}, +, V_k, V_h)$ is undecidable for k and h multiplicatively independent integers. Alexis Bès [Bes-96] completed these results by showing that V_h can be replaced by any set of n-tuples h-recognizable by a finite automaton.

To complete this part of the problems on addition and primes first opened by Alan, we must mention the result of Françoise Maurin [Mau-97] showing that $Th(\mathbb{N}, <_P, \times)$ is decidable so that $Th(\mathbb{N}, <_P, \mathbb{P})$ is decidable although $Th(\mathbb{N}, <_{PP}, \mathbb{P})$ and $Th(\mathbb{N}, <_X, \mathbb{P})$ are undecidable [BR-98].

5 Rudimentary predicates, Spectra, and Rudimentary languages of first and second order

Alan Woods, in chapter 2 of his thesis, considered the problem of defining addition and multiplication on the natural numbers by first order bounded quantifier formulas involving only the predicate of coprimeness ($x \perp y$) and the natural order predicate ($\leq$).

He also proved as an application that the class RUD of all rudimentary (i.e. Δ_0-definable) sets of positive integers is contained in the class consisting of the spectra $S_\Phi = \{|M| \text{ such that } M \text{ is finite and } M \models \Phi\}$ of those sentences Φ having only graphs as models.

The equality between these classes holds if and only if S_Φ is rudimentary for every first order sentence Φ.

An analogous result holds for partial orderings instead of graphs. Alan noted that if S_Φ is rudimentary for every first order sentence Φ then we have NP $\neq$ co-NP. It is worth to note that Alan was often concerned with key-questions.

This part of Alan's thesis inspired a lot of logicians and computer scientists. In the team of Denis Richard, Alex Esbelin and Malika More stubbornly worked to make some links with *Grzegorczyk small classes* (Esbelin), to investigate *binary spectra by explicit polynomial transformations of graphs* (More), to give tools for recognizing rudimentary predicates (Esbelin and More) and also to extend results to second order languages (More and Frédéric Olive).

In his thesis [Esb-94], Alex Esbelin studied the problem of equality of RUD with the first class $(E^0)^*$ containing the projections, the constant functions, the successor function and closed under composition and bounded recursion. He investigated two specific cases of $(E^0)^*$: 1) the over-rudimentary classes, the associated functions of which contain RUD (it is possible somehow counting in all these classes) and he has founded results of derecursivation; 2) the sub-rudimentary classes, the associated classes of relations of which are strictly contained in RUD and he solved the problem of derecursivation for these classes. Roughly speaking, derecursivation consisys to found the complete definability of the values of a function or of a relation from the variables and the parameters occurring in these without using the history of the computation, hence without recursive function.

The question of whether a given primitive recursive relation is rudimentary is in some cases difficult and is related to several well-known open questions in theoretical computer science. In [EM-98], Alex Esbelin and Malika More present systematic tools to study this question, with various applications. Namely, they give rudimentary definitions of various classical spectra. One of them gives a sufficient condition for the collapsing of the first classes of the Grzegorczyk's hierarchy. Also they prove there exist one-to-one functions with rudimentary graph that provide representations of various sets of spectra into RUD. Their results are obtained via arithmetical coding of finite structures by a transformation of the first-order sentences into rudimentary formulas describing the same semantic properties.

In [MO-97], Malika More and Frédéric Olive make conspicuous the equivalence between three notions respectively coming from theory of recursive functions and relations, from complexity of computing and, finally, from (finite) model theory. On one hand, we know rudimentary languages are characterized by the linear hierarchy. On the other hand, we can prove this complexity class corresponds to second-order monadic logic with addition. The authors enlighten the deep links there is between those far distant topics by providing a straightforward logical characterization of the rudimentary languages and also a result of representation of the second-order logic within those languages. (They use arithmetical tools for doing that).

6 Erdös-Woods numbers

About the works of Alan, Gordon ROYLE said in *SymOmega* on the 18th of december 2011: "*Mathematically, his name is attached to at least one concept theErdös-Woods numbers are the integer numbers k such that there is a closed interval of integers* $[a, a + k]$ *with the property that every integer in the interval has a factor in common with either a or* $a + k$".

As we have already seen, many interesting problems in number theory emerge from the thesis of Alan WOODS [Woo-81]. The most famous of them is now known as *Erdös-Woods conjecture*, after its publication in the book [Guy-81] of Richard GUY.

Erdös-Woods conjecture. *There exists an integer* k *such that integers* x *and* y *are equal if and only if, for* $i = 0, \ldots, k$, *integers* $x + i$ *and* $y + i$ *have same prime divisors.*

This problem is a source of an active domain of research.

In relation with this problem, Alan WOODS had conjectured ([Woo-81], p. 88) that for any ordered pair (a, d) of natural numbers, with $d \geq 3$, there exists a natural number c such that $a < c < a + d$ and c is coprime with a and with $a + d$. In other words

$$\forall a, \forall d > 2, \exists c\, [a < c < a + d \wedge a \perp c \wedge c \perp a + d]$$

Very quickly, he realized the conjecture was false, by finding the counterexample (2184, 16) (published in [Dow-89]). In 1989, David DOWE [Dow-89] proved that there exist infinitely many such numbers d. We call such numbers d *Erdös-Woods numbers.*

The main aim of the paper [CHR-03] is to prove that the set of Erdös-Woods numbers is recursive.

A second aim is to give the first values of Erdös-Woods numbers and to show there is a lot of natural open problems concerning these numbers.

<u>Notation</u>.- Let us denote by $NoCoprimeness(a, d)$ the property

$$\forall c\, [a < c < a + d \rightarrow \neg(a \perp c) \vee \neg(c \perp a + d)]$$

Let us begin with some remarks.

<u>Remarks</u>.- (1) The relation $NoCoprimeness(a, d)$ is recursive: it is easy to write a program to check whether an ordered pair belongs to it.

(2) The set $\{(a, d) \mid NoCoprimeness(a, d)\}$ is infinite: we know an element (2184, 16) of this set and it is easy to check that for every $k \neq 0$, the ordered pair $(2184, 2184 + k.2.3.5.7.11.13)$ is also an element of this set.

(3) The two unary relations, projections of $NoCoprimeness(a, d)$, defined by

$$ExtremNoCoprime(a) \text{ iff } \exists d\ NoCoprimeness(a, d)$$

$$AmplitudeNoCoprime(d) \text{ iff } \exists a\ NoCoprimeness(a, d)$$

are recursively enumerable: it is easy to write a program to list elements of these sets (but not in natural order, unfortunately).

First elements of *AmplitudeNoCoprime*.- Theoretical reasons allows us to implement an algorithm (in language C) to compute first elements of the set *AmplitudeNoCoprime*. The algorithm is not very efficient, but it allows to test quickly the first six hundred integers. We obtain the beginning of the set *AmplitudeNoCoprime*:

16; 22; 34; 36; 46; 56; 64; 66; 70; 76; 78; 86; 88; 92; 94; 96; 100; 106;112; 116;118; 120; 124; 130; 134; 142; 144; 146; 154; 160; 162; 186;190; 196; 204; 210; 216; 218; 220; 222; 232; 238; 246; 248; 250; 256;260; 262; 268; 276; 280; 286; 288; 292; 296; 298; 300; 302; 306; 310;316; 320; 324; 326; 328; 330; 336; 340; 342; 346; 356; 366; 372; 378.

On odd elements of *AmplitudeNoCoprime*.- Dowe has found [Dow-89] an infinite subset of *AmplitudeNoCoprime*, every element being even. He conjectures every element of *AmplitudeNoCoprime* is even. Marcin Bienkowski, Mirek Korzeniowski, and Krysztof Lorys, from Wroclaw University (Poland), have found the counterexamples $d = 903$ and 2545 by computation, then a general method to generate many other examples: 4533, 5067, 8759, 9071, 9269, 10353, 11035, 11625, 11865, 13629, 15395, ... Nik Lygeros, from Lyon 1 University (France), has independently found the counterexample $d = 903$, making precise the related extremity:

a = 9 522 262 666 954 293 438 213 814 248 428 848 908 865 242 615359 435 357 454 655 023 337 655 961 661 185 909 720 220 963 272 377170 658 485 583 462 437 556 704 487 000 825 482 523 721 777 298 113 684783 645 994 814 078 222 557 560 883 686 154 164 437 824 554 543412 509 895 747 350 810 845 757 048 244 101 596740 520097 753981 676 715 670 944 384 183 107 626 409 084 843 313 577 681 531 093 717028 660 116 797 728 892 253 375 798 305 738 503 033 846 246 769 704747 450 128 124 100 053 617.

Lygeros found other values ($d = 907$ and 909), proving that the previous sufficient condition is not necessary. Also he discovered that the solution $d = 903$ is an old result from Erdös and Seldfridge [ES-71].

On even squares of *AmplitudeNoCoprime*.- We see, in scanning the above list, that every even square except 4 appears at the beginning. However $676 = 26 \times 26$, $1156 = 34 \times 34$ and $1024 = 32 \times 32$ are not Erdös-Woods numbers.

On prime elements of *AmplitudeNoCoprime*.- An Erdös-Woods number may be a prime number as 15 493 and 18 637 show.

Open problems

The above list of first elements of $AmplitudeNoCoprime$ suggests a great number of open problems, curiously similar to problems for the set of primes.

We may implement a program to compute, for an Erdös-Woods number d, the smallest associated extremity a. Numerical experiments suggest that $2 \perp a+1$ whenever the amplitude d is even, hence 2 divides a. Is it a general property?

The solution $(a, 903)$, with the a found by Nik LYGEROS, shows it is not the case for d odd.

<u>Open problem 1</u>.- (**Even extremity for even amplitude**). *For an even d, is every element a such that $NoCoprimeness(a, d)$ even?*

We may note we have a great number of twin Erdös-Woods numbers among the first elements of $AmplitudeNoCoprime$: 34 and 36, 64 and 66, 76 and 78, 86 and 88, 92 and 94, ...

<u>Open problem 2</u>.- (**Infinity of twin Erdös-Woods numbers**) *There exists an infinity of integers d such that d, $d+2$ belong to $AmplitudeNoCoprime$.*

Indeed we also have a sequence of three consecutive even Erdös-Woods numbers (as 92, 94, 96), even four consecutive ones (as 216, 218, 220, 222).

<u>Open problem 3</u>.- (**Polignac's conjecture for Erdös-Woods numbers**) *For any integer k, there exists an even integer d such that $d, d+2, d+4, \ldots, d+2.k$ belong to $AmplitudeNoCoprime$.*

Nik LYGEROS has searched segments of consecutive natural numbers which are not Erdös-Woods numbers. He has found long such segments.

<u>Open problem 4</u>.- *There exist segments of any length without elements of $AmplitudeNoCoprime$:*

$$\forall k, \exists e\ [e, e+1, \ldots, e+k \notin AmplitudeNoCoprime]$$

Passing from patterns in $AmplitudeNoCoprime$ to complexity, we may remark that the algorithm we have given to decide whether an integer belongs to $AmplitudeNoCoprime$ is worst than exponential. It is interesting to improve it if it is possible.

<u>Open problem 5</u>.- (**Complexity of** $AmplitudeNoCoprime$) *To which complexity classes does $AmplitudeNoCoprime$ belong?*

<u>Open problem 6</u>.- *Find a lower bound for $AmplitudeNoCoprime$.*

Also we may ask questions à la Vallée-Poussin–Hadamard, passing from the complexity to the density of the set $AmplitudeNoCoprime$. Let us denote by $d(n)$ the cardinality of the set $\{d \leq n \mid AmplitudeNoCoprime(d)\}$.

Open problem 7.- *Find a (simple) function f such that $d(n) \sim f(n)$.*

Open problem 8.- *Is the density of $AmplitudeNoCoprime$ linear? More precisely*

$$d(n) = O(n)?$$

The last open problems we suggest concern Logic, more precisely Weak Arithmetics. Problems of definability and decidability are important: Presburger's proof of decidability for the elementary theory of $< \mathbb{N}, + >$ implies that a set $X \subseteq \mathbb{N}$ is definable in $< \mathbb{N}, + >$ iff X is ultimately periodic. The negative solution given by MATIYASEVICH to Hilbert's Tenth Problem relies on the fact that the exponential function is existentially definable in $< \mathbb{N}, +, \times >$. In the same way, the following problems deserve consideration.

Open problem 9.- *Is the theory $Th(\mathbb{N}, NoCoprimeness, R)$ decidable? where R is some relation or function to specify.* (Addition + is an interesting candidate).

At the opposite, we may search for undecidability.

Open problem 10.- *Is the theory $Th(N, +, AmplitudeNoCoprime)$ def-complete* (i.e. is multiplication definable in the underlying structure)?

7 Is Nézondet's method a beginning to an answer to EWC?

Via the whole theory of p-destinies, Francis NÉZONDET proved a surprising result which is too complicated to discuss here. Roughly speaking, NÉZONDET constructs in a functional language (not relational) a non-standard structure, the theory which is very like that of the standard model and in which EWC is false.

Last open problems

Using p-destinies, Marcel GUILLAUME, Jilei YIN and Denis RICHARD have tried to construct an algorithm for deciding $Th_3(\mathbb{N}, S, \perp)$. It turns out that two sets are the keys of this construction, namely

$$X = \{a \in \mathbb{N} \mid \exists b \; (Supp(a) \subseteq Supp(a) \wedge Supp(b) \neq Supp(a)$$

$$\wedge \neg(b - 1 \perp a) \wedge \neg(b + 1 \perp a)\}$$

$$Y = \{a \in \mathbb{N} \mid \exists b \; (Supp() \subseteq Supp(a) \wedge Supp(b) \neq Supp(a)$$

$$\wedge (b - 1 \perp a) \wedge \neg(b + 1 \perp a)\}$$

It can be proved that $a \in Y \cap (2\mathbb{N} + 1)$ iff $\exists p \in Supp(a), \exists q \in Supp(a)$ ($ord_p(q)$ *is even*), where $ord_p(q)$ is the order of the element q within the group $(\mathbb{Z}/q\mathbb{Z})*$.

The construction of the essential 3-destiny of $Th_3(N, S, \perp)$ depends on the answer (yes or no) to the following nine questions

- 1) $\exists a = (2^n - 1) \wedge |Supp(a)| = 2 \wedge a \notin Y$

Answer: YES with $a = 2^{227} - 1$ or $a = 2^{269} - 1$.

- 2) $\exists a = 2(2^n - 1) \wedge |Supp(a)| = 3 \wedge a \notin Y$

Answer: YES with $a = 2(2^{269} - 1)$.

- 3) $\exists a = (2^n - 1) \wedge |Supp(a)| \geq 3 \wedge a \notin Y$

Answer: OPEN

- 4) $\exists a = (2^n + 1) \wedge n \in 2\mathbb{N} \wedge |Supp(a)| = 2 \wedge a \notin Y$

Answer: OPEN

- 5) $\exists a = (2^n + 1) \wedge n \in 2\mathbb{N} \wedge |Supp(a)| \geq 3 \wedge a \notin Y$

Answer: OPEN

- 6) $\exists a = (2^n - 1) \wedge n \in 2\mathbb{N} + 1 \wedge |Supp(a)| \geq 3 \wedge a \in Y \backslash X$

Answer: YES with $2^{25} - 1 = 31.601.1801$

- 7) $\exists a = (2^n - 1) \wedge n \in 2\mathbb{N} \wedge |Supp(a)| \geq 3 \wedge a \notin X$

Answer: NO

- 8) $\exists a = (2^n + 1) \wedge n \in 2\mathbb{N} + 1 \wedge |Supp(a)| \geq 3 \wedge a \notin X$

Answer: OPEN

-9) $\exists a = (2^n + 1) \wedge n \in 2\mathbb{N} \wedge |Supp(a)| \geq 3 \wedge a \notin Y \backslash X$

Answer: OPEN

8 Conclusion

We believe neither in paradise nor in hell. But as SARTRE puts it in *Les mots*, authors write with the aim of some kind of survival. In the case of mathematicians, we cannot survive only by our theorems and works, but also, and above all, by our questions. Questions of Alan Robert WOODS are so pertinent, deep and difficult that we guess he will last very long.

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