CHARACTERIZING CONGRUENCE PRESERVING FUNCTIONS
$\mathbb{Z}/n\mathbb{Z} \rightarrow \mathbb{Z}/m\mathbb{Z}$ VIA RATIONAL POLYNOMIALS

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Abstract
Using a simple basis of rational polynomial-like functions $P_0, \ldots, P_{n-1}$ for the free module of functions $\mathbb{Z}/n\mathbb{Z} \rightarrow \mathbb{Z}/m\mathbb{Z}$, we characterize the subfamily of congruence preserving functions as the set of linear combinations of the products $\text{lcm}(k)P_k$ where $\text{lcm}(k)$ is the least common multiple of $2, \ldots, k$ (viewed in $\mathbb{Z}/m\mathbb{Z}$). As a consequence, when $n \geq m$, the number of such functions is independent of $n$.

1. Introduction

The notion of a congruence preserving function on rings of residue classes was introduced in Chen [3] and studied in Bhargava [1].

Definition 1.1. Let $m,n \geq 1$. A function $f : \mathbb{Z}/n\mathbb{Z} \rightarrow \mathbb{Z}/m\mathbb{Z}$ is said to be congruence preserving if for all $d$ dividing $m$

\begin{equation}
\text{for all } a,b \in \{0,\ldots,n-1\} \quad a \equiv b \pmod{d} \text{ implies } f(a) \equiv f(b) \pmod{d}.
\end{equation}

Remark 1.2. 1. If $n \in \{1,2\}$ or $m = 1$ then every function $\mathbb{Z}/n\mathbb{Z} \rightarrow \mathbb{Z}/m\mathbb{Z}$ is trivially congruence preserving.

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2. Observe that since \( d \) is assumed to divide \( m \), equivalence modulo \( d \) is a congruence on \((\mathbb{Z}/m\mathbb{Z}, +, \times)\). However, since \( d \) is not supposed to divide \( n \), equivalence modulo \( d \) may not be a congruence on \((\mathbb{Z}/n\mathbb{Z}, +, \times)\).

**Example 1.3.** 1. For functions \( \mathbb{Z}/6\mathbb{Z} \to \mathbb{Z}/3\mathbb{Z} \), condition (1) reduces to the conditions \( f(3) \equiv f(0) \pmod{3} \), \( f(4) \equiv f(1) \pmod{3} \), \( f(5) \equiv f(2) \pmod{3} \).

2. For functions \( \mathbb{Z}/6\mathbb{Z} \to \mathbb{Z}/8\mathbb{Z} \), condition (1) reduces to \( f(2) \equiv f(0) \pmod{2} \), \( f(3) \equiv f(1) \pmod{2} \), \( f(4) \equiv f(0) \pmod{4} \), \( f(5) \equiv f(1) \pmod{4} \).

In this paper, we characterize congruence preserving functions \( \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z} \). We denote by \( \mathbb{Z} \) the set of integers and by \( \mathbb{N} \) the set of nonnegative integers (including zero).

**Definition 1.4.** The unary \( \text{lcm} \) function \( \mathbb{N} \to \mathbb{N} \) maps 0 to 1 and \( k \geq 1 \) to the least common multiple of \( 1, 2, \ldots, k \).

A natural way to associate with each map from \( \mathbb{N} \) to \( \mathbb{Z} \) a map from \( \mathbb{Z}/n\mathbb{Z} \) to \( \mathbb{Z}/m\mathbb{Z} \) is to restrict \( F \) to \( \{0, \ldots, n-1\} \) and take its values modulo \( m \).

**Definition 1.5.** With each map \( F : \mathbb{N} \to \mathbb{Z} \), we associate the map \( f : \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z} \) defined by \( f = \pi_m \circ F \circ \iota_n \), where \( \pi_m(x) = x \pmod{m} \), and \( \iota_n(z) \) is the unique element of \( \pi_n^{-1}(z) \cap \{0, \ldots, n-1\} \).

Definition 1.5 is best pictured by the commutativity of diagram (2).

\[
\begin{array}{ccc}
\mathbb{N} & \xrightarrow{F} & \mathbb{Z} \\
\downarrow{\iota_n} & & \downarrow{\pi_m} \\
\mathbb{Z}/n\mathbb{Z} & \xrightarrow{f} & \mathbb{Z}/m\mathbb{Z}
\end{array}
\] (2)

Applying Definition 1.5 to binomial coefficients, we obtain a basis of the \((\mathbb{Z}/m\mathbb{Z})\)-module of functions \( \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z} \).

**Proposition 1.6.** Let \( P_k : \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z} \) be associated with the \( \mathbb{N} \to \mathbb{N} \) binomial function \( x \mapsto \binom{x}{k} \). For every function \( f : \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z} \) there is a unique sequence \( (a_0, \ldots, a_{n-1}) \) of elements of \( \mathbb{Z}/m\mathbb{Z} \) such that

\[
f = \sum_{k=0}^{k=n-1} a_k P_k.
\]

In other words, the family \( \{P_0, \ldots, P_{n-1}\} \) is a basis of the \((\mathbb{Z}/m\mathbb{Z})\)-module of functions \( \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z} \).

Our main result can be stated as
Theorem 1.7. A function \( f : \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z} \) is congruence preserving if and only if, for each \( k = 0, \ldots, n - 1 \), in equation (3) the coefficient \( a_k \) is a multiple of the residue of \( \text{lcm}(k) \) in \( \mathbb{Z}/m\mathbb{Z} \).

The paper is organized as follows. Proposition 1.6 is proved in Section 2 where, after recalling Chen’s notion of a polynomial function \( \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z} \) (cf. [3]), we extend it to a notion of a rational polynomial function.

The proof of our main result, Theorem 1.7, is given in Section 3. We adapt the techniques of our paper [2], exploiting similarities between Definition 1.1 and the condition studied in [2] for functions \( f : \mathbb{N} \to \mathbb{Z} \) (namely, \( x - y \) divides \( f(x) - f(y) \) for all \( x, y \in \mathbb{N} \)). As a consequence of Theorem 1.7, the number of congruence preserving functions is independent of \( n \) for \( n \geq m \) and even for \( n \geq \text{gpp}(m) \) (the greatest prime power dividing \( m \)). Also, every congruence preserving function \( f : \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z} \) is a rational polynomial for a polynomial of degree strictly less than the minimum of \( n \) and \( \text{gpp}(m) \).

In Section 4 we use our main theorem to count the congruence preserving functions \( \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z} \). We thus get an expression equivalent to that obtained by Bhargava in [1] and which makes apparent the fact that, for \( n \geq \text{gpp}(m) \) (hence for \( n \geq m \)), this number depends only on \( m \) and is independent of \( n \).

2. Representing Functions \( \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z} \) by Rational Polynomials

In [3, 1], congruence preserving functions \( \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z} \) are introduced and studied together with an original notion of polynomial function \( \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z} \).

Definition 2.1 (Chen [3]). A function \( f : \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z} \) is polynomial if it is associated (in the sense of Definition 1.5) with a function \( F : \mathbb{N} \to \mathbb{Z} \) given by a polynomial in \( \mathbb{Z}[X] \).

Polynomial functions \( \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z} \) are obviously congruence preserving. Are all congruence preserving functions polynomial? Chen [3] observed that this is not the case for some values of \( n, m \), for instance \( n = 6, m = 8 \). He also proves that a stronger identity holds for infinitely many ordered pairs \( (n, m) \): every function \( \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z} \) is polynomial if and only if \( n \) is not greater than the first prime factor of \( m \) (in particular, this is the case when \( n = m \) and \( m \) is prime, cf. Kempner [4]).

Using counting arguments, Bhargava [1] characterizes the ordered pairs \( (n, m) \) such that every congruence preserving function \( f : \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z} \) is polynomial.

Some polynomials in \( \mathbb{Q}[X] \) (i.e., polynomials with rational coefficients) happen to map integers into integers.
Definition 2.2. For \( k \in \mathbb{N} \), let \( P_k \in \mathbb{Q}[X] \) be the following polynomial:

\[
P_k(x) = \binom{x}{k} = \frac{\prod_{i=0}^{k-1} (x - i)}{k!}.
\]

We will use the following examples later on:

\[
P_0(x) = 1, \quad P_1(x) = x, \quad P_2(x) = x(x - 1)/2, \quad P_3(x) = x(x - 1)(x - 2)/6, \quad P_4(x) = x(x - 1)(x - 2)(x - 3)/24, \quad P_5(x) = x(x - 1)(x - 2)(x - 3)(x - 4)/120.
\]

In [5], Pólya used the \( P_k \)'s to give the following very elegant and elementary characterization of polynomials in \( \mathbb{Q}[X] \) mapping integers to integers.

**Theorem 2.3** (Pólya). A polynomial in \( \mathbb{Q}[X] \) is integer-valued on \( \mathbb{Z} \) if and only if it can be written as a \( \mathbb{Z} \)-linear combination of the polynomials \( P_k, \ k = 0, 1, 2, \ldots \).

It turns out that the representation of functions \( \mathbb{N} \to \mathbb{Z} \) as \( \mathbb{Z} \)-linear combinations of the \( P_k \)'s used in [2] also fits in the case of functions \( \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z} \) : every such function is a \( (\mathbb{Z}/m\mathbb{Z}) \)-linear combination of the \( P_k \)'s.

**Definition 2.4.** 1. A function \( f : \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z} \) is rat-polynomial if is associated in the sense of Definition 1.5 with some polynomial in \( \mathbb{Q}[X] \).

2. The degree of a rat-polynomial function is the smallest degree of an associated polynomial in \( \mathbb{Q}[X] \).

3. We denote by \( P_k^{n,m} \) the rat-polynomial function \( \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z} \) associated with the polynomial \( P_k \) of Definition 2.2 in the sense of Definition 1.5. When there is no ambiguity, \( P_k^{n,m} \) will be denoted simply as \( P_k \).

**Remark 2.5.** In Definition 2.4, the polynomial crucially depends on the choice of representatives of elements of \( \mathbb{Z}/n\mathbb{Z} \): e.g., for \( n = m = 6, \ 0 \equiv 6 \pmod{6} \) but \( 0 = P_0(0) \neq P_2(6) = 3 \pmod{6} \). The chosen representatives for elements of \( \mathbb{Z}/n\mathbb{Z} \) will always be \( 0, 1, \ldots, n - 1 \).

We now prove the representation result by the \( P_k \)'s.

**Proof of Proposition 1.6.** Let us start with uniqueness. We have \( f(0) = a_0 \), and hence \( a_0 = f(0) \). We have \( f(1) = a_0 + a_1 \), and hence \( a_1 = f(1) - f(0) \). By induction, letting \( Q_k = \sum_{\ell=0}^{k} a_\ell P_\ell \), and noting that \( P_k(k) = 1 \), we have \( f(k) = Q_k(k) + a_k P_k(k) = Q_k(k) + a_k \), and hence \( a_k = f(k) - Q_k(k) \). We thus are able to determine \( a_k \) in \( \mathbb{Z}/m\mathbb{Z} \).

For existence, argue backwards to see that this sequence suits.

**Remark 2.6.** The evaluation of \( a_k P_k(x) \) in \( \mathbb{Z}/m\mathbb{Z} \) has to be done as follows: for \( x \) an element of \( \mathbb{Z}/n\mathbb{Z} \), we consider it as an element of \( \{0, \ldots, n-1\} \subseteq \mathbb{N} \) and we evaluate \( P_k(x) = \frac{1}{k!} \prod_{i=0}^{k-1} (x - i) \) as an element of \( \mathbb{Z} \), then we consider the remainder modulo \( m \), and finally we multiply the result by \( a_k \) in \( \mathbb{Z}/m\mathbb{Z} \). For instance, for
n = m = 8, we have $4 \cdot P_2(3) = 4 \times \frac{3 \times 2}{2} = 4 \times 3 = 4$, but we might be tempted to evaluate it as $4 \cdot P_2(3) = \frac{4 \times 3 \times 2}{2} = \frac{0}{2} = 0$, which does not correspond to our definition. However, dividing $a_k$ by a factor of the denominator is allowed.

**Corollary 2.7.** 1. Every function $f : \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$ is rat-polynomial with degree less than $n$.
2. The family of rat-polynomial functions $\{P_k \mid k = 0, 1, \ldots, n-1\}$ is a basis of the $(\mathbb{Z}/m\mathbb{Z})$-module of functions $\mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$.

**Example 2.8.** The function $f : \mathbb{Z}/6\mathbb{Z} \to \mathbb{Z}/6\mathbb{Z}$ such that $f(0) = 0$, $f(1) = 3$, $f(2) = 4$, $f(3) = 3$, $f(4) = 0$, $f(5) = 1$, is represented by the rational polynomial

$$P_f(x) = 3x + 4 \frac{x(x-1)}{2}$$

which can be simplified to $P_f(x) = 3x - x(x-1)$ on $\mathbb{Z}/6\mathbb{Z}$.

**Example 2.9.** The function $f : \mathbb{Z}/6\mathbb{Z} \to \mathbb{Z}/8\mathbb{Z}$ given by Chen [3] as a non-polynomial congruence preserving function, namely the function such that $f(0) = 0$, $f(1) = 3$, $f(2) = 4$, $f(3) = 1$, $f(4) = 4$, $f(5) = 7$, is represented by the rational polynomial with coefficients $a_0 = 0$, $a_1 = 3$, $a_2 = 6$, $a_3 = 2$, $a_4 = 4$, $a_5 = 4$. Thus,

$$P_f(x) = 3x + 6 \frac{x(x-1)}{2} + 2 \frac{x(x-1)(x-2)}{2} + 4 \frac{x(x-1)(x-2)(x-3)}{8}$$

$$+ 4 \frac{x(x-1)(x-2)(x-3)(x-4)}{8} = 3x + 3x(x-1) + x(x-1)(x-2) + \frac{x(x-1)(x-2)(x-3)}{2}$$

$$+ \frac{x(x-1)(x-2)(x-3)(x-4)}{2}.$$
The following is an immediate consequence of Lemma 3.2 (set \( a = b + n \)).

**Lemma 3.3.** If \( a \geq b \) and \( k \leq b \), then \( a - b \) divides \( \text{lcm}(k) \left( \binom{a}{k} - \binom{b}{k} \right) \) in \( \mathbb{N} \).

Besides these lemmata which deal with divisibility on integers, we shall use a classical result in \( \mathbb{Z}/m\mathbb{Z} \). For \( x, y \in \mathbb{Z} \) we say \( x \) divides \( y \) in \( \mathbb{Z}/m\mathbb{Z} \) if and only if the residue class of \( x \) divides the residue class of \( y \) in \( \mathbb{Z}/m\mathbb{Z} \).

**Lemma 3.4.** Let \( 1 \leq c_1, \ldots, c_k \leq m \) and let \( c \) be their least common multiple in \( \mathbb{N} \). If \( c_1, \ldots, c_k \) all divide \( a \) in \( \mathbb{Z}/m\mathbb{Z} \), then so does \( c \).

**Proof.** It suffices to consider the case \( k = 2 \) since the passage to any \( k \) is done via a straightforward induction. Let \( c = c_1b_1 = c_2b_2 \) with \( b_1, b_2 \) coprime. Let \( t, u \) be such that \( a = c_1 t = c_2 u \) in \( \mathbb{Z}/m\mathbb{Z} \). Then \( a = c_1 t \equiv c_2 u \) (mod \( m \)). Using Bézout’s identity, let \( \alpha, \beta \in \mathbb{Z} \) be such that \( \alpha b_1 + \beta b_2 = 1 \). Then \( c(t\alpha + u\beta) = c_1 b_1 t\alpha + c_2 b_2 u\beta \equiv a\alpha b_1 + a\beta b_2 \) (mod \( m \)), and hence \( c(t\alpha + u\beta) \equiv a \) (mod \( m \)), proving that \( c \) divides \( a \) in \( \mathbb{Z}/m\mathbb{Z} \).

**Proof of the “only if” part of Theorem 1.7.** Assume \( f : \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z} \) is congruence preserving and consider its decomposition \( f(x) = \sum_{k=0}^{n-1} a_k P_k(x) \) given by Proposition 1.6. We show that \( \text{lcm}(k) \) divides \( a_k \) in \( \mathbb{Z}/m\mathbb{Z} \) for all \( k < n \). The cases \( k = 0 \) and \( k = 1 \) are trivial since \( \text{lcm}(0) = \text{lcm}(1) = 1 \).

**Claim 1.** For all \( 2 \leq k < n \), \( k \) divides \( a_k \) in \( \mathbb{Z}/m\mathbb{Z} \).

**Proof.** Recall that \( f(k) = \sum_{i=0}^{n-1} a_i \binom{k}{i} = \sum_{i=0}^{k-1} a_i \binom{k}{i} \) since \( \binom{k}{i} = 0 \) for \( i > k \). We argue by induction on \( k \geq 2 \).

**Base case** \( k = 2 \). If \( 2 \) does not divide \( m \) then \( 2 \) and \( m \) are coprime, and hence \( 2 \) is invertible and divides \( a_2 \) in \( \mathbb{Z}/m\mathbb{Z} \). Assume \( 2 \) divides \( m \). As \( 2 \) divides \( 2 - 0 \) and \( f \) is congruence preserving, \( 2 \) also divides \( f(2) - f(0) = 2a_1 + a_2 \), and hence \( 2 \) divides \( a_2 \).

**Inductive step.** Let \( 2 < k < n - 1 \). The inductive hypothesis ensures that \( \ell \) divides \( a_\ell \) in \( \mathbb{Z}/m\mathbb{Z} \) for every \( \ell \leq k \). Let \( a_\ell \equiv \ell q_\ell \) (mod \( m \)) for \( 0 \leq \ell \leq k \). We prove that \( k + 1 \) divides \( a_{k+1} \) in \( \mathbb{Z}/m\mathbb{Z} \). First, observe that

\[
\begin{align*}
    f(k+1) - f(0) &= (k+1)a_1 + \left( \sum_{i=2}^{k} \binom{k+1}{i} a_i \right) + a_{k+1} \\
    &\equiv (k+1)a_1 + \left( \sum_{i=2}^{k} \binom{k+1}{i} iq_i \right) + a_{k+1} \pmod{m} \\
    f(k+1) - f(0) &= (k+1)a_1 + \left( \sum_{i=2}^{k} (k+1) \binom{k}{i-1} q_i \right) + am + a_{k+1}
\end{align*}
\]
for some $\alpha$. Let $d = gcd(k + 1, m)$. Since $d$ divides $m$ and $k + 1 - 0$ and $f$ is
congruence preserving, $d$ also divides $f(k + 1) - f(0)$. Using equality (4), we see
that $d$ divides the last term $a_{k+1}$ of the sum. Using Bézout’s identity, let $u, v$
be such that $u(k + 1) + vm = d$. Then $u(k + 1) \equiv d \pmod{m}$, and hence $k + 1$ divides
$d$ in $\mathbb{Z}/m\mathbb{Z}$. Since $d$ divides $a_{k+1}$, we conclude that $k + 1$ divides $a_{k+1}$ in $\mathbb{Z}/m\mathbb{Z}$. }

**Claim 2.** (i) For all $2 \leq p \leq k < n$, $p$ divides $a_k$ in $\mathbb{Z}/m\mathbb{Z}$.
(ii) For all $2 \leq k < n$, lcm($k$) divides $a_k$ in $\mathbb{Z}/m\mathbb{Z}$.

*Proof.* Assertion (ii) is a direct application of Lemma 3.4 and assertion (i). We
prove (i) by induction on $p \geq 2$. Both the base case and the inductive step of this
induction are proved by induction on $k$.

**Base case** $p = 2$. We have to prove that $2$ divides $a_k$ for all $k \geq 2$. If $2$ does not
divide $m$, then $2$ is invertible and divides all numbers in $\mathbb{Z}/m\mathbb{Z}$. Assume now that
$2$ divides $m$. We argue by induction on $k \geq 2$.

**Base case.** Apply Claim 1: $2$ divides $a_2$.

**Inductive step.** Let $k < n - 1$. Assuming that $2$ divides $a_i$ for all $2 \leq i \leq k$, we
prove that $2$ divides $a_{k+1}$. Two cases can occur.

**Subcase 1:** $k + 1$ is odd. Then $2$ divides $k$ and hence, by congruence preservation,$2$ divides $f(k + 1) - f(1)$. As $f(k + 1) - f(1) = ka_1 + \left(\sum_{i=2}^{k} a_i \binom{k+1}{i}\right) + a_{k+1}$,
and $2$ divides $k$ and also, by the induction hypothesis, $2$ divides $a_i$ for $2 \leq i \leq k$, we
see that $2$ divides $a_{k+1}$.

**Subcase 2:** $k + 1$ is even. By congruence preservation, $2$ divides $f(k + 1) - f(0) = (k+1)a_1 + \left(\sum_{i=2}^{k} a_i \binom{k+1}{i}\right) + a_{k+1}$. Since $2$ divides $k + 1$ and $a_i$ for $2 \leq i \leq k$
(induction hypothesis), we infer that $2$ divides $a_{k+1}$.

**Inductive step.** Let $2 \leq p < n - 1$ and assume that

$$
\text{for all } q \leq p \text{ and all } \ell \text{ such that } q \leq \ell < n, q \text{ divides } a_\ell \text{ in } \mathbb{Z}/m\mathbb{Z}. \quad (5)
$$

By induction on $k \geq p + 1$, we prove that $p + 1$ divides $a_k$ for all $k$ such that
$p + 1 \leq k < n$.

**Base case** $k = p + 1$. Apply Claim 1: $p + 1$ divides $a_{p+1}$.

**Inductive step.** Let $k < n - 1$. Assuming that $p + 1$ divides $a_i$ in $\mathbb{Z}/m\mathbb{Z}$ for all $i$
such that \( p + 1 \leq i \leq k \), we prove that \( p + 1 \) divides \( a_{k+1} \) in \( \mathbb{Z}/m\mathbb{Z} \). We have

\[
f(k + 1) - f(k - p) = \sum_{i=1}^{k-p} a_i \left( \binom{k+1}{i} - \binom{k-p}{i} \right) + \sum_{i=k+1-p}^{k} a_i \left( \binom{k+1}{i} - \binom{k-p}{i} \right) + a_{k+1} \quad (6)
\]

We first look at the terms of the first sum on the right side of (6) corresponding to \( 1 \leq i \leq p \). Applying (5) with \( \ell = i \), we see that \( q \) divides \( a_i \) in \( \mathbb{Z}/m\mathbb{Z} \) for all \( q \leq \min(p, i) = i \). Using Lemma 3.4, we conclude that \( \text{lcm}(i) \) divides \( a_i \) in \( \mathbb{Z}/m\mathbb{Z} \). Observing that \( (k+1) = (k-p) + (p+1) \), we can apply Lemma 3.2 (with \( k-p, p+1 \) and \( i \) in place of \( b, n \) and \( k \)) and conclude that \( p + 1 \) divides \( \text{lcm}(i) \left( \binom{k+1}{i} - \binom{k-p}{i} \right) \) in \( \mathbb{N} \). Thus, \( p + 1 \) divides \( a_i \left( \binom{k+1}{i} - \binom{k-p}{i} \right) \) in \( \mathbb{Z}/m\mathbb{Z} \).

We now turn to the terms of the first sum on the right side of (6) corresponding to \( p + 1 \leq i \leq k - p \) (if there are any). Each of these terms is divisible by \( p + 1 \) in \( \mathbb{Z}/m\mathbb{Z} \), because the induction hypothesis on \( k \) ensures that \( p + 1 \) divides \( a_i \) in \( \mathbb{Z}/m\mathbb{Z} \) whenever \( p + 1 \leq i \leq k \).

Consider next the terms of the second sum on the right side of (6). For those terms corresponding to values of \( i \) such that \( p + 1 \leq i \leq k \), divisibility by \( p + 1 \) in \( \mathbb{Z}/m\mathbb{Z} \) follows from the fact that, by the induction hypothesis on \( k \), \( p + 1 \) divides \( a_i \). It remains to look at the terms associated with the \( i \)'s such that \( k + 1 - p \leq i \leq p \) (there are such \( i \)'s in case \( k + 1 - p < i \leq p \)). For such \( i \)'s we have \( 0 \leq (k + 1) - i \leq (k + 1) - p < p + 1 \leq k + 1 \) and Lemma 3.1 (used with \( k+1, i \) and \( p+1 \) in place of \( n, k \) and \( p \)) implies that \( p + 1 \) divides \( \text{lcm}(i) \left( \binom{k+1}{i} - \binom{k-p}{i} \right) \). Now, for such \( i \)'s, the induction hypothesis (5) on \( p \) shows that \( \text{lcm}(i) \) divides \( a_i \) in \( \mathbb{Z}/m\mathbb{Z} \). A fortiori, \( p + 1 \) divides \( a_i \left( \binom{k+1}{i} - \binom{k-p}{i} \right) \) in \( \mathbb{Z}/m\mathbb{Z} \).

Let \( d = \gcd(p + 1, m) \). As \( p + 1 \) divides \( \mathbb{Z}/m\mathbb{Z} \) all terms of the two sums on the right side of (6) so does \( d \). Since \( d \) divides \( m \) and \( k + 1 - (k - p) = p + 1 \) and \( f \) is congruence preserving, \( d \) also divides \( f(k + 1) - f(k - p) \). Using equality (6), we conclude that \( d \) divides \( \mathbb{Z}/m\mathbb{Z} \) the last term \( a_{k+1} \). Using Bézout’s identity, let \( u, v \) be such that \( u(p + 1) + vm = d \). Then \( u(p + 1) \equiv d \pmod{m} \), and hence \( p + 1 \) divides \( d \) in \( \mathbb{Z}/m\mathbb{Z} \). As \( d \) divides \( a_{k+1} \) in \( \mathbb{Z}/m\mathbb{Z} \), we conclude that \( p + 1 \) divides \( a_{k+1} \) in \( \mathbb{Z}/m\mathbb{Z} \).

This ends the proof of the induction in the inductive step, and hence also the proof of Claim 2 and of the “only if” part of the Theorem.

**Proof of the “if” part of Theorem 1.7.** Assume \( f = \sum_{k=0}^{k=n-1} a_k P_k \) and that all of the \( a_k \)'s are divisible by \( \text{lcm}(k) \) in \( \mathbb{Z}/m\mathbb{Z} \). We can write \( f \) in the form \( f(n) = \sum_{k=0}^{n} c_k \text{lcm}(k) \binom{n}{k} \). We prove that \( f \) is congruence preserving, i.e., if \( 0 \leq i < a \leq n \)
n − 1 and d divides both m and a − b then d also divides f(a) − f(b). Observe that
\[
f(a) − f(b) = \left( \sum_{k=0}^{b} c_k \text{lcm}(k) \left( \binom{a}{k} - \binom{b}{k} \right) \right) + \sum_{k=b+1}^{a} c_k \text{lcm}(k) \left( \binom{a}{k} \right)
\]

By Lemma 3.3, a − b divides each term of the first sum. Consider the terms of the second sum. For \( 1 \leq k \leq a \), we have \( 0 \leq a - k < a - b \leq a \) and Lemma 3.1 (used with \( a, k \) and \( a - b \) in place of \( n, k \) and \( p \)) shows that \( a - b \) divides \( \text{lcm}(k) \left( \binom{a}{k} \right) \). Thus, \( a - b \) divides \( f(a) - f(b) \).

### 3.2. On a Family of Generators

We now sharpen the degree of the rat-polynomial representing a congruence preserving function \( \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z} \). We first state some properties of the \( \text{lcm} \) function in \( \mathbb{N} \).

**Lemma 3.5.** Let \( m \geq 1 \) be an integer with prime factorization \( m = p_1^{\alpha_1} \cdots p_\ell^{\alpha_\ell} \). Then \( \text{lcm}(k) = u \prod_{i=1}^{\ell} p_i^{\alpha_{i,k}} \), where \( u \) is coprime with \( m \) and \( \alpha_{i,k} = \max \{ \beta_i | p_i^{\beta_i} \leq k \} \).

**Definition 3.6.** Let \( m \geq 1 \) be an integer with prime factorization \( m = p_1^{\alpha_1} \cdots p_\ell^{\alpha_\ell} \). We let \( \text{gpp}(m) = \max \{ \alpha_i | i \in \{1, \ldots, \ell \} \} \) be the greatest power of prime dividing \( m \) in \( \mathbb{N} \).

**Lemma 3.7.** The number \( \text{gpp}(m) \) is the least integer \( k \) such that \( m \) divides \( \text{lcm}(k) \).

**Example 3.8.** We have \( \text{gpp}(8) = 8 \), \( \text{gpp}(12) = 4 \) and \( \text{gpp}(14) = 7 \). The successive values of the residues in \( \mathbb{Z}/m\mathbb{Z} \) of \( \text{lcm}(k) \) are

\[
\begin{array}{cccccccc}
  k & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
  \text{lcm}(k) \text{ in } \mathbb{Z}/8\mathbb{Z} & 1 & 2 & 2 & 4 & 4 & 4 & 4 & 0 \\
  \text{lcm}(k) \text{ in } \mathbb{Z}/12\mathbb{Z} & 1 & 2 & 6 & 0 & 0 & 0 & 0 & 0 \\
  \text{lcm}(k) \text{ in } \mathbb{Z}/14\mathbb{Z} & 1 & 2 & 6 & 12 & 4 & 4 & 0 & 0 \\
\end{array}
\]

For all \( \ell \geq \text{gpp}(m) \), \( \text{lcm}(\ell) \) is zero in \( \mathbb{Z}/m\mathbb{Z} \).

**Remark 3.9.** 1. Either \( \text{gpp}(m) = m \) or \( \text{gpp}(m) \leq m/2 \).
2. In general, \( \text{gpp}(m) \) is greater than \( \lambda(m) \), the least \( k \) such that \( m \) divides \( k! \) (a function considered in [3]): for \( m = 8 \), \( \text{gpp}(m) = 8 \) whereas \( \lambda(m) = 4 \).

Using Lemma 3.7, we can get a better version of Theorem 1.7.

**Theorem 3.10.** A function \( f: \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z} \) is congruence preserving if and only if it is associated in the sense of Definition 1.5 with a rational polynomial \( P = \sum_{k=0}^{d-1} a_k \binom{x}{k} \) where \( d = \min(n, \text{gpp}(m)) \) and such that \( \text{lcm}(k) \) divides \( a_k \) in \( \mathbb{Z}/m\mathbb{Z} \) for all \( k < d \).
Proof. For \( k \geq \text{gpp}(m) \), \( m \) divides \( \text{lcm}(k) \) hence the coefficient \( a_k \) is 0. \hfill \Box

**Theorem 3.11.** (i) Every congruence preserving function \( f : \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z} \) is rat-polynomial with degree less than \( \text{gpp}(m) \).
(ii) The family of rat-polynomial functions

\[
\mathcal{F} = \{ \text{lcm}(k)P_k \mid 0 \leq k < \text{min}(n, \text{gpp}(m)) \}
\]

generates the set of congruence preserving functions \( \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z} \).
(iii) \( \mathcal{F} \) is a basis of the set of congruence preserving functions if and only if \( m \) has no prime divisor \( p < \text{min}(n, m) \) (in case \( n \geq m \) this means that \( m \) is prime).

Proof. Assertions (i) and (ii) are restatements of Theorem 3.10. Let us prove (iii).

“Only If” part. Assume \( m \) has a prime divisor \( p < \text{min}(n, m) \) and let \( p \) be the least one. Then \( \text{lcm}(p) = pa \) with \( a \) coprime with \( m \), and hence \( \text{lcm}(p) \neq 0 \) in \( \mathbb{Z}/m\mathbb{Z} \). Since \( P_p(p) = 1 \) this shows that \( \text{lcm}(p)P_p \) is not the null function. However \( (m/p) \text{lcm}(p) = 0 \) in \( \mathbb{Z}/m\mathbb{Z} \) and hence \( (m/p) \text{lcm}(p)P_p \) is the null function. As \( (m/p) \neq 0 \) in \( \mathbb{Z}/m\mathbb{Z} \), this proves that \( \mathcal{F} \) cannot be a basis.

“If” part. Assume that \( m \) has no prime divisor \( p < \text{min}(n, m) \). We prove that \( \mathcal{F} \) is \((\mathbb{Z}/m\mathbb{Z})\)-linearly independent. Suppose that the \((\mathbb{Z}/m\mathbb{Z})\)-linear combination

\[
L = \sum_{k=0}^{\min(n, \text{gpp}(m)) - 1} a_k \text{lcm}(k)P_k
\]

is the null function \( \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z} \). By induction on \( k = 0, \ldots, \min(n, \text{gpp}(m)) - 1 \) we prove that \( a_k = 0 \).

- **Basic cases** \( k = 0, 1 \). From \( L(0) = a_0 \) and \( L(1) = a_0 + a_1 \) we deduce \( a_0 = a_1 = 0 \).

- **Induction step.** Assuming \( k \geq 2 \) and \( a_i = 0 \) for \( i = 0, \ldots, k - 1 \), we prove that \( a_k = 0 \). Observe that \( P_\ell(k) = \binom{k}{\ell} \) is 0 for \( k < \ell < n \). Since \( a_i = 0 \) for \( i = 0, \ldots, k - 1 \), and \( P_k(k) = 1 \) we get \( L(k) = a_k \text{lcm}(k) \). As \( k < \min(n, \text{gpp}(m)) \leq \min(n, m) \) and \( m \) has no prime divisor \( p < \min(n, m) \), the numbers \( \text{lcm}(k) \) and \( m \) are coprime. Thus, \( \text{lcm}(k) \) is invertible in \( \mathbb{Z}/m\mathbb{Z} \) and equality \( L(k) = a_k \text{lcm}(k) = 0 \) implies \( a_k = 0 \). \hfill \Box

### 4. Counting Congruence Preserving Functions

We now compute the number of congruence preserving functions \( \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z} \). As two different rational polynomials correspond to different functions by Proposition 1.6 (uniqueness of the representation by a rational polynomial), the number of congruence preserving functions \( \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z} \) is equal to the number of polynomials representing them.

**Proposition 4.1.** Let \( CP(n, m) \) be the number of congruence preserving functions \( \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z} \). Let \( m = p_1^{e_1}p_2^{e_2} \cdots p_\ell^{e_\ell} \) be the decomposition of \( m \) in powers of
primes. Let \( \mathcal{I} = \{ i \mid p_i^{e_i} < gpp(m) \} \) and \( \mathcal{J} = \{ i \mid p_i^{e_i} \geq gpp(m) \} \). Then
\[
CP(n, m) = \begin{cases} 
\prod_{i \in \mathcal{I}} p_i^{p_i + p_i^2 + \cdots + p_i^{e_i}} \times \cdots \times \prod_{i \in \mathcal{J}} p_i^{p_i + p_i^2 + \cdots + p_i^{\lcm(\log p, n)}} & \text{if } n \geq gpp(m), \\
\prod_{i \in \mathcal{J}} \prod_{k=1}^\ell p_i^{\nu_k} & \text{if } n < gpp(m).
\end{cases}
\]

Equivalently, writing \( E(p, \alpha) \) instead of \( p^\alpha \) for better readability, we have
\[
CP(n, m) = \begin{cases} 
\prod_{i \in \mathcal{I}} E(p_i, \sum_{k=1}^\ell p_i^k) \times \cdots \times \prod_{i \in \mathcal{J}} E(p_i, (\sum_{k=1}^\ell p_i^k) + n(\ell - \lcm(\log p, n))) & \text{if } n \geq gpp(m), \\
\prod_{i \in \mathcal{I}} \prod_{k=1}^\ell E(p_i, \nu_k) & \text{if } n < gpp(m).
\end{cases}
\]

**Corollary 4.2.** For \( n \geq gpp(m) \), \( CP(n, m) \) does not depend on \( n \).

**Proof of Proposition 4.1.** By Theorem 3.10, we must count the number of \( n \)-tuples of coefficients \((a_0, \ldots, a_{n-1})\), with, for \( k = 0, \ldots, n-1 \), \( a_k \) being a multiple of \( \lcm(k) \) in \( \mathbb{Z}/m\mathbb{Z} \).

**Claim 1.** For \( m = p_1^{e_1} p_2^{e_2} \cdots p_\ell^{e_\ell} \), for all \( n \), \( CP(n, m) = \prod_{i=1}^\ell CP(n, p_i^{e_i}) \).

**Proof of Claim 1.** Let \( E(r, k) \) be the set of multiples in \( \mathbb{Z}/r\mathbb{Z} \) of \( \lcm(k) \) and \( \lambda(r, k) \) the cardinal of \( E(r, k) \). The Chinese remainder theorem shows that the map \( \rho : z \mapsto (z \mod p_i^{e_i})_{i=1, \ldots, \ell} \) is an isomorphism and also that \( \rho \) maps the set \( E(m, k) \) onto the Cartesian product \( P = \prod_{i=1}^\ell E(p_i^{e_i}, k) \). Indeed, let \( (t_i)_{i=1, \ldots, \ell} \in P \). For each \( i = 1, \ldots, \ell \), there is \( 0 \leq q_i < p_i^{e_i} \) such that \( t_i \equiv q_i \lcm(k) \) (mod \( p_i^{e_i} \)). Applying the Chinese remainder theorem, there are \( 0 \leq t, q < m \) such that \( t \equiv t_i \) (mod \( p_i^{e_i} \)) and \( q \equiv q_i \) (mod \( p_i^{e_i} \)). Then \( t \equiv q \lcm(k) \) (mod \( m \)), and hence \( \rho(t) = (t_i)_{i=1, \ldots, \ell} \). This proves that \( \lambda(m, k) = \prod_{i=1}^\ell \lambda(p_i^{e_i}, k) \) for each \( k \). Thus, the number \( CP(n, m) \) of \( n \)-tuples \((a_0, \ldots, a_{n-1})\) such that \( \lcm(k) \) divides \( a_k \) is equal to
\[
CP(n, m) = \prod_{k<n} \lambda(m, k) = \prod_{k<n} \prod_{i=1}^\ell \lambda(p_i^{e_i}, k) = \prod_{i=1}^\ell \prod_{k<n} \lambda(p_i^{e_i}, k) = \prod_{i=1}^\ell CP(n, p_i^{e_i}).
\]

Claim 1 reduces the problem to that of counting the congruence preserving functions \( \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/p_i^{e_i}\mathbb{Z} \). We will use Theorem 3.10 to this end.

**Claim 2.** Letting \( \ell = \lcm(\log p, n) \) (and using the \( E(p, \alpha) \) notation for \( p^\alpha \)), we have
\[
CP(n, p^\ell) = \begin{cases} 
E(p, p + p^2 + \cdots + p^\ell) & \text{if } n \geq p^\ell, \\
E(p, p + p^2 + \cdots + p^\ell + (\ell - \ell)n) & \text{if } p^\ell \leq n < p^\ell.
\end{cases}
\]

**Proof of Claim 2.** By Theorem 3.10, as \( gpp(p^\ell) = p^\ell \), letting \( \nu = \inf(n, p^\ell) \), we have \( CP(n, p^\ell) = CP(\nu, p^\ell) = \prod_{k=0}^{\nu-1} \lambda(p^\ell, k) \). As we noted in the proof of Claim 1, for
Remark 4.3. 

1. $p^j \leq k < p^{j+1}$, the order $\lambda(p^e, k)$ of the subgroup generated by $lcm(k)$ in $\mathbb{Z}/p^e\mathbb{Z}$ is $p^{e-j}$, and there are $p^{j+1} - p^j$ such $k$'s. For $k = 0$, $lcm(0) = 1$ yields $\lambda(p^e, 0) = p^e$.

2. If $n \geq p^e$ then $CP(n, p^e) = CP(p^e, p^e) = p^e \prod_{j=0}^{e-1} p^{e-j} = p^M$ with

$$M = e + \sum_{j=0}^{e-1} (e - j)(p^{j+1} - p^j) = p + p^2 + \cdots + p^e$$

3. If $n < p^e$ then $p^f \leq n < p^e$ and

$$CP(n, p^e) = \prod_{k=0}^{n-1} \lambda(p^e, k) = p^e \prod_{j=0}^{n-1} p^{e-j} = p^M$$

$$M = e + \sum_{j=0}^{n-1} (e - j)(p^{j+1} - p^j) + \sum_{k=p^e}^{n-1} (e - k)$$

= $(e - k)p^f + (p + p^2 + \cdots + p^f) + (n - p^f)(e - l)$

= $(p + p^2 + \cdots + p^f) + n(e - l)$

This finishes the proof of Proposition 4.1.

\[\Box\]

Remark 4.3. In [1] the number of congruence preserving functions $\mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/p^e\mathbb{Z}$ is shown to be equal to $E(p, en - \sum_{k=1}^{n-1} \min\{e, \lfloor \log_p k \rfloor \})$. For $p^i \leq k < p^{i+1}$, we have $\lfloor \log_p k \rfloor = i$, and hence $\min\{e, \lfloor \log_p k \rfloor \} = \lfloor \log_p k \rfloor$ for $k \leq p^e$, and $\min\{e, \lfloor \log_p k \rfloor \} = e$ for $k \geq p^e$. Thus, we have

1. If $n \geq p^e$, then

$$\sum_{k=1}^{n-1} \min\{e, \lfloor \log_p k \rfloor \} = \sum_{k=1}^{p^e-1} \lfloor \log_p k \rfloor + \sum_{k=p^e}^{n-1} e = \sum_{j=0}^{e-1} j(p^{j+1} - p^j) + e(n - p^e)$$

= $-(p + \cdots + p^e) + ep^e + e(n - p^e)$, and hence $en - \sum_{k=1}^{n-1} \min\{e, \lfloor \log_p k \rfloor \} = p + \cdots + p^e$.

This coincides with our counting in Claim 2.

2. If $n < p^e$, and $l = \lfloor \log_p n \rfloor$, then, similarly,

$$\sum_{k=1}^{n-1} \lfloor \log_p k \rfloor = \sum_{k=1}^{l} \lfloor \log_p k \rfloor + \sum_{k=l+1}^{n-1} \lfloor \log_p k \rfloor = \sum_{j=0}^{l-1} j(p^{j+1} - p^j) + l(n - p^l)$$

= $-(p + \cdots + p^l) + nl$, and hence $en - \sum_{k=1}^{n-1} \lfloor \log_p k \rfloor = p + \cdots + p^l + (e - l)n$. Again, this coincides with our counting in Claim 2.

5. Conclusion

We proved that the rational polynomials $lcm(k) P_k$ generate the $\mathbb{Z}/m\mathbb{Z}$ submodule of congruence preserving functions $\mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$. When $n$ is larger than the greatest prime power dividing $m$, the number of functions in this submodule is independent of $n$. An open problem is the existence of a basis of this submodule.

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References


