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# DESTINIES AND DECIDABILITY

Abstract. Francis Nézondet has introduced a new tool in Logic, called *destiny*, used to study open problems in Number Theory. We give a sufficient condition of computation of destinies of a structure.

## Introduction

In his thesis (Nézondet, 1997), Francis Nézondet has introduced the notion of *destinies* as a general tool of Logic, linked to back and forth of Roland Fraïssé. For the moment, the main application of this tool is as a way to attack some open problems in Number Theory.

Because the theory of destinies is not well known, we have to recall part of it in section 1, with a presentation different from the original one. In section 2, we give a sufficient condition of computation of destinies of a structure.

## 1. Theory of destinies

Here our aim is not to present the theory of destinies in the more general setting. We use a restricted formulation suited to problems of decidability in Number Theory. Also we change the vocabulary used by Francis Nézondet but not concepts.

### 1.1. Application of destinies.

Let $\langle \mathbb{N}, R_1, R_2, ..., R_k \rangle$ a relational structure whose universe is the set $\mathbb{N}$ of natural integers and whose one of these relations is equality.

For a given integer $p$, let denote by

$$Th_p(\mathbb{N}, R_1, R_2, ..., R_k)$$

the set of first order sentences of signature $(R_1, R_2, ..., R_k)$ in prenex form, with less than $p$ quantifiers realised in the structure $\langle \mathbb{N}, R_1, R_2, ..., R_k \rangle$. We call it the **p-theory** of $\langle \mathbb{N}, R_1, R_2, ..., R_k \rangle$.

For a given structure and a given $p$, the p-theory is a finite set of sentences, hence classically it is decidable. In other words, for a given structure and a given $p$, there exists a recursive function from the set of first order sentences of signature $(R_1, R_2, ..., R_k)$ with less than $p$ quantifiers in prenex form into the set $\mathbb{B}$ of booleans whose image is true if the sentence is realized in the structure.

It is not the case from an intuitionist point of view. The problem to give explicitly such a function is not obvious.

Francis Nézondet has introduced the notion of p-destinies. For a given structure and a given $p$, there exists a finite set of p-destinies, as we will see. But explicitly to exhibit its set is a very difficult problem. However, the exhibition of this finite set gives the function we are searching.

### 1.2. Destinies in a leisurely presentation.

The main idea is to visually present the various possible configurations with $p$ objects of the structure. To simplify our presentation, we consider that relations $R_1$, $R_2$, ... , $R_k$ are unary or binary. To illustrate our presentation, we choose the traditional example of the structure $< \mathbb{N}, <, P >$, where $<$ is the natural order relation and $P$ the unary predicate "is a prime". Let us note that equality relation $=$ is not explicitly mentioned but we use it.

#### 1.2.1. Primitive forest of height p.

**Definition 1.** *The* **primitive forest of height** $p$ *is constituted of the infinite set of (infinite) k-labelled trees of height p verifying the following conditions:*

- *each node is a natural integer;*
- *for every unary relation $R_i$, a node $x$ has coloration $c_i$ if we have $R_i(x)$;*
- *for every binary relation $R_j$, an edge $(x, y)$ has coloration $c_j$ if we have $R_j(x, y)$.*

*We call an element of this primitive forest a* **primitive tree of height** $p$.

**Example.** It is difficult to represent an infinite set of infinite trees. Figure 1 represent the beginning of the first primitive tree of height 2 for structure $\langle \mathbb{N}, <, P \rangle$. For instance node 2 has coloration P and edge (0,1) has coloration $<$.

**Remark.** There is only a tree whose root is a given natural integer.

#### 1.2.2. Pruned trees.

To have each possible situation, it is not necessary to keep every branches of the above trees: we have to keep only one instance of equivalent branches. Hence the notion of *pruning* is interesting.

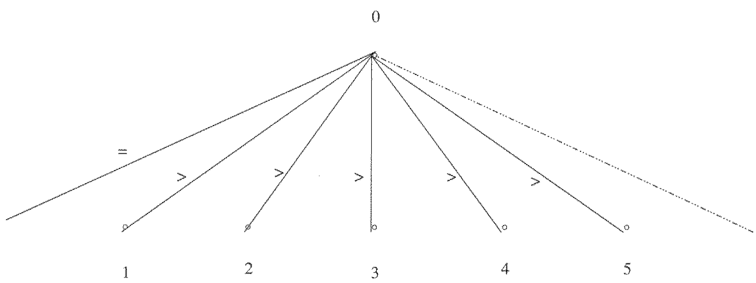


Fig. 1. Primitive tree.

**Definition 2.** *A* **branch** $b_1 = (x_{1,1}, ..., x_{1,p})$ *is* **equivalent** *to a branch* $b_2 = (x_{2,1}, ..., x_{2,p})$ *iff the node* $x_{1,i}$ *has the same set of colorations than the node* $x_{2,i}$ *and if the edge* $(x_{1,i}, x_{1,i+1})$ *has the same set of colorations than the edge* $(x_{2,i}, x_{2,i+1})$*, for every* $i$.

**Definition 3.** *The* **natural order** *between branches of a tree whose nodes are natural numbers is defined by:*

*a branch* $b_1 = (x_{1,1}, ..., x_{1,p})$ *is* **less than** *a branch* $b_2 = (x_{2,1}, ..., x_{2,p})$ *if the word* $ax_{1,1}a...ax_{1,p}$ *is less than the word* $ax_{2,1}a...ax_{2,p}$ *in lexicographical order*[1].

**Definition 4.** *The* **pruned tree** *associated to a primitive tree* $T$ *is the tree where a branch* $b$ *is pruned (i.e. removed) if it is equivalent to a lesser branch*

**Remark.** A pruned tree is a finite tree.

**Example.** Figure 2 represent the pruned tree associated to the tree of the first figure.

### 1.2.3. Essential forest.

To have an instance of each possible situation, it is not necessary to keep every pruned tree because many pruned trees are equivalent (i.e. represent the same situation). Hence the interest of the notion of essential forest.

[1]We need a new symbol, the marker 'a', different from digits '0', '1', ... ,'9' if we use the decimal expansion of natural integers to avoid to have to use the infinite alphabet of all natural numbers.

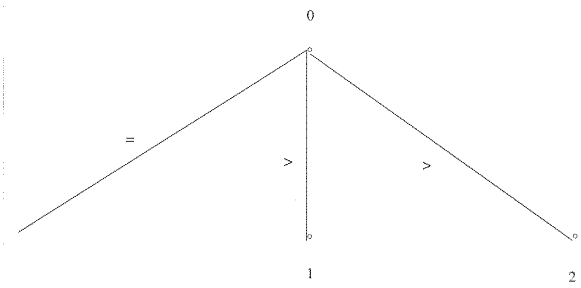


Fig. 2. Pruned tree.

**Definition 5.** *The* **pruned trees** *T1 and T2 are* **equivalent** *if they have the same number of branches [hence we have* $T1 = (b_{1,1}, ..., b_{1,q})$ *and* $T2 = (b_{2,1}, ..., b_{2,q})$*] and if branch* $b_{1,i}$ *is equivalent to branch* $b_{2,i}$ *for every* $i$*.*

Trees are naturally ordered by values of roots.

**Definition 6.** *A pruned tree belongs to the* **essential forest** *if it is not equivalent to a lesser pruned tree.*

*An element of the essential forest is called a* **destiny**, *more precisely a* **p-destiny** *if he has heigth* $p$*.*

*We denote by*

$$Dest_p(\mathbb{N}, R_1, R_2, ..., R_k)$$

*the essential forest of height* $p$ *of the structure* $\langle \mathbb{N}, R_1, R_2, ..., R_k \rangle$*.*

**Example.** Figure 3 represents the essential forest of the structure $\langle \mathbb{N}, <, P \rangle$ of height two.

**Remark.** The essential forest of height $p$ is finite.

**1.3. Problems concerning computation of essential forests.**

For the structure $\langle \mathbb{N}, <, P \rangle$ which illustrates our purpose, it is easy to compute the essential forests. For other structures, it is not so easy. The difficulties are of two orders:

- when we try to compute the pruned tree of a given tree, we begin to remove a certain number of branches but how to know if there exists, in the infinite set of other branches, a branch not equivalent to the (not yet removed) branches?

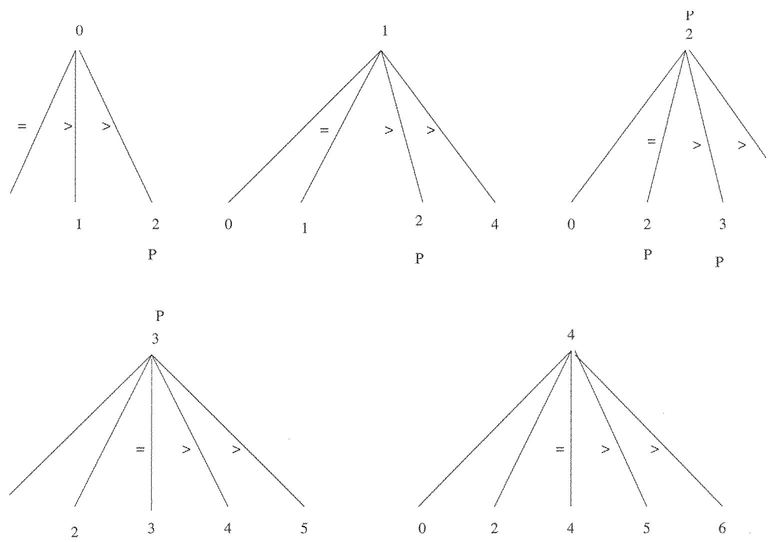


Fig. 3. Essential forest.

- when we try to compute the essential forest, we begin to remove a certain number of pruned trees but how to know if there exists, in the infinite set of other pruned trees, a pruned tree not equivalent to the (not yet removed) pruned trees?

### 1.4. Forest of configurations.

Intuitively, an essential tree represents a configuration. But some configurations are not realised in a given structure, i.e. are not equivalent to a pruned tree of this structure. Hence we have to consider the new notion of pruned tree associated to a signature and not to the structure itself.

**Definition 7.** *Let $(r_1, r_2, ..., r_k)$ be a relational signature with $r_1 = 2$. The relation $R_1$ will be interpreted by the equality relation. The* **forest of configurations** *of height $p$ is the forest of non equivalent pruned trees of height $p$, for any structure of this signature.*

The forest of configurations of a given height is finite.

## 2. Decidability and essential forest

Classically, theories $Th_p(\mathbb{N}, R_1, R_2, ..., R_k)$ are decidable. Howewer the map which associates $Th_p(\mathbb{N}, R_1, R_2, ..., R_k)$ to $p$ may be recursive or not recursive. In the same manner, the map which associates $Dest_p(\mathbb{N}, R_1, R_2, ..., R_k)$ to $p$ may be recursive or not recursive.

**Definition 7.** *The* **flattening map** *of a structure* $\langle \mathbb{N}, R_1, R_2, ..., R_k \rangle$ *is the map*

$$p \mapsto Th_p(\mathbb{N}, R_1, R_2, ..., R_k).$$

*The* **destiny map** *of a structure* $\langle \mathbb{N}, R_1, R_2, ..., R_k \rangle$ *is the map*

$$p \mapsto Dest_p(\mathbb{N}, R_1, R_2, ..., R_k).$$

**Remark.** Obviously the flattening map of a structure is recursive iff the theory $Th(\mathbb{N}, R_1, R_2, ..., R_k)$ is decidable. We do not have such a nice characterization for the destiny map.

**Theorem 1.** *If the relational structure* $\langle \mathbb{N}, R_1, R_2, ..., R_k \rangle$ *satisfies*
- $Th(\mathbb{N}, R_1, R_2, ..., R_k, (i)_{i \in \mathbb{N}})$ *is decidable, where each element of* $\mathbb{N}$ *is considered as a constant;*
- *every relation* $R_i$ *is recursive*

*then its destiny map is recursive.*

**Proof.** Let us begin with two facts.

**Fact 1** *For each primitive tree, we may compute the associated pruned tree.*

Because every relation $R_i$ is recursive, it is easy to know if a branch is equivalent to a lesser branch. Hence we keep the first branch, we compare the second branch to the first one to know if we keep it. And so on, we compare a branch to the lesser (not removed) branches to know if it is a new branch to keep it or not.

The problem is to know when we have finished, i.e. when we are sure there is no new branch (no equivalent to kept branches) to keep. For a given signature $(r_1, r_2, ..., r_k)$ and a given $p$, there is a finite number of non equivalent branches, in other words of **types** of branches. A type of branches, with root equal to a natural integer $a$, appears iff a first order sentence is true in $Th(\mathbb{N}, R_1, R_2, ..., R_k, (i)_{i \in \mathbb{N}})$. Because this last theory is decidable, we may decide if we have to search for such a branch or not.

**Fact 2** *For each configuration, we may decide if there is a pruned tree equivalent to it.*

It is the same idea: there exists a pruned tree equivalent to a given configuration iff a first order sentence is true in $Th(\mathbb{N}, R_1, R_2, ..., R_k)$; because this last theory is decidable, we may decide whether it is the case or not.

The proof of the theorem follows from these two facts. We compute the first pruned tree. If an other configuration appears in the essential forest, we compute pruned trees until we have this configuration. We continue until there is no other configuration. The procedure finishes because the number of configurations is finite. CQFD

**Corollary.** *Destiny map is recursive for structures* $\langle \mathbb{N}, + \rangle$, $\langle \mathbb{N}, \times \rangle$,...

**Problem.** *What is the link between complexities of relations* $R_i$ *and complexity of* $Th(\mathbb{N}, R_1, R_2, ..., R_k)$ *and complexity of destiny map?*

**Remark.** We don't know if destiny map is recursive for $\langle \mathbb{N}, S, P \rangle$ but, classically, the restriction of destiny map to a finite number of integers $p$ is recursive.

## References

1. F. Nézondet, *p-destinées et applications à la théorie du successeur et de la co-primarité sur les entiers*, Ph. D. thesis, 1997, Université de Clermont-Ferrand 1 (France).

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