The algebra of full binary trees is affine complete

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To the memory of Kate Karagueuzian-Gibbons and Giliane Arnold

Abstract. A function on an algebra is congruence preserving if, for any congruence, it maps pairs of congruent elements onto pairs of congruent elements. We show that on the algebra of full binary trees whose leaves are labeled by letters of an alphabet containing at least three letters, a function is congruence preserving if and only if it is polynomial. This exhibits an example of a non commutative and non associative 1–affine complete algebra. As far as we know, it is the first example of such an algebra.

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1. Introduction

A function on an algebra is congruence preserving if, for any congruence, it maps pairs of congruent elements onto pairs of congruent elements. Such functions were introduced in Grätzer [4], where they are said to have the "substitution property".

A polynomial function on an algebra is a function defined by a term of the algebra using variables, constants and the operations of the algebra. Obviously, every polynomial function is congruence preserving. In most algebras this inclusion is strict. A very simple example where the inclusion is strict is the additive algebra of natural integers $\langle \mathbb{N}, + \rangle$, cf. [1]. Up to the example studied in [2], all affine complete algebras studied so far were commutative and associative, see [3, 7, 5, 9]. The example in [2] is the free monoid on an alphabet with at least three letters: its operation is non commutative but associative. The present paper is a follow-up of [2] though it does not depend on it. We here prove that the free algebra with one binary operation and at least three generators is 1–affine complete, i.e., every unary function preserving congruences is polynomial. This gives a nontrivial example of a non associative and non commutative 1–affine complete algebra.

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2. Preliminary definitions

For an algebra $\mathcal{A}$ with domain A, a congruence $\sim$ on $\mathcal{A}$ is an equivalence relation on A which is compatible with the operations of $\mathcal{A}$.

Lemma 2.1. *Let* $\mathcal{A} = \langle A\,,\,\star\rangle$, $\mathcal{B} = \langle B\,,\,*\rangle$ *be two algebras with binary operations* $\star$ *and* $*$, *and* $\theta\colon A \to B$ *a homomorphism. Then* $\sim_\theta$ *defined on* A *by* $x \sim_\theta y$ *iff* $\theta(x) = \theta(y)$ *is a congruence called the kernel* $\ker(\theta)$ *of* θ*.*

Definition 2.2. Let Σ be a nonempty alphabet whose elements are called letters. The free monoid generated by Σ is the algebra $\langle \Sigma^*, \cdot\rangle$. Its elements are the finite sequences (or words) of elements from Σ. It is endowed with the concatenation operation and the unit element is the empty word denoted by ε. The free monoid will be abbreviated as Σ^* in the sequel.

The length of a word $w \in \Sigma^*$ is the total number of occurrences of letters in w and it is denoted $|w|$.

As usual Σ^+ denotes the set $\Sigma^* \setminus \{\varepsilon\}$.

Definition 2.3. Let Γ be a subset of Σ. The projection π_Γ is the homomorphism $\Sigma^* \to \Gamma^*$ which erases all letters not in Γ and leaves those in Γ unchanged.

By Lemma 2.1 the relation $(x, y) \in \ker(\pi_\Gamma)$ is a congruence. We shall use the following homomorphisms on Σ^*.

Definition 2.4. Let $a \in \Sigma$ and $u \in \Sigma^*$. Then the substitution $\psi_{a\to u}$ is the homomorphism $\Sigma^* \to \Sigma^*$ which maps the letter a onto the word u and leaves other letters unchanged.

3. Full binary trees and their congruences

Let Σ be an alphabet, let $\Xi = \{\ \blacktriangleleft, \bullet\ , \blacktriangleright\ \}$ be an alphabet disjoint from Σ and let $\Theta = \Sigma \cup \Xi$. We shall represent the free groupoid with generators Σ as a set of words $\mathcal{T}(\Sigma)$ on the alphabet Θ together with the binary product operation $\star$.

Definition 3.1. The free binary algebra $\mathcal{B} = \langle \mathcal{T}(\Sigma), \star\rangle$ generated by Σ is defined as follows:

- Its carrier set $\mathcal{T}(\Sigma)$ is the least set of non empty words of Θ^+ also called "trees", inductively defined by
  - (1) each letter a in Σ is a tree a in $\mathcal{T}(\Sigma)$
  - (2) if t and t' are trees in $\mathcal{T}(\Sigma)$, then the word $\blacktriangleleft t \bullet t' \blacktriangleright$ is a tree in $\mathcal{T}(\Sigma)$
- The binary product operation $\star$ is defined by: $t \star t' = \blacktriangleleft t \bullet t' \blacktriangleright$

This product is neither commutative nor associative. The elements of $\mathcal{T}(\Sigma)$ can be viewed as full binary trees with leaves labeled by letters in the alphabet Σ. The trees of $\mathcal{T}(\Sigma)$ are said to be full because every node has either 0 or 2 children. See Figure 1.

Remark 3.2. Algebra $\mathcal{B}$ is most often called the free groupoid with generators Σ following [8], it is also called free magma with generators Σ [10]. We preferred the terminology "full binary trees" because it gives a better support for intuition. The operation $\star$ concatenates two trees as the left and right subtrees of a new root.

As $\mathcal{T}(\Sigma)$ is freely generated by Σ it enjoys the universal property stated below which will be of constant use.

Lemma 3.3. *Every mapping $\Sigma \to \mathcal{T}(\Sigma)$ can be uniquely extended to a homomorphism $\mathcal{T}(\Sigma) \to \mathcal{T}(\Sigma)$.*

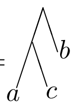
Lemma 3.4. *For $x, y, u, v \in \mathcal{T}(\Sigma)$, $x \star y = u \star v$ implies $x = y$ and $u = v$.*

Definition 3.5. Let $1 \notin \Sigma$. The set $\mathcal{S}$ of skeletons is the least set of words of $(\Xi \cup \{1\})^*$ inductively defined by

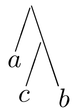
(i) 1 is a skeleton, and *(ii)* if s and s' are skeletons, then $\blacktriangleleft s \bullet s' \blacktriangleright$ is a skeleton.

– The skeleton of a tree t is the word $\sigma(t) = \pi_{\Xi \cup \{1\}}(t) \in (\Xi \cup \{1\})^*$ obtained by replacing all letters in Σ with 1.
– The foliage of a tree t in $\mathcal{T}(\Sigma)$ is the word $\varphi(t) = \pi_\Sigma(t) \in \Sigma^+$ obtained by erasing all letters not in Σ.

Note that the skeleton of a tree indeed belongs to $\mathcal{S}$.

$t =$ 

$t = \blacktriangleleft\blacktriangleleft a \bullet c \blacktriangleright \bullet b \blacktriangleright$
$\sigma(t) = \blacktriangleleft\blacktriangleleft 1 \bullet 1 \blacktriangleright \bullet 1 \blacktriangleright$
$\varphi(t) = acb$

$t' =$ 

$t' = \blacktriangleleft a \bullet \blacktriangleleft c \bullet b \blacktriangleright\blacktriangleright$
$\sigma(t) = \blacktriangleleft 1 \bullet \blacktriangleleft 1 \bullet 1 \blacktriangleright\blacktriangleright$
$\varphi(t) = acb$

FIGURE 1. A graphic representation of two trees

Proposition 3.6. *For all $t, t' \in \mathcal{T}(\Sigma)$,*
(1) $\sigma(t \star t') = \blacktriangleleft \sigma(t) \bullet \sigma(t') \blacktriangleright$,
(2) $\varphi(t \star t') = \varphi(t)\varphi(t')$,
(3) $|\sigma(t)| = 3|\varphi(t)| - 3$.

Proof. Point (3) is the variant (due to the extra symbols $\blacktriangleleft, \bullet, \blacktriangleright$) of the classical result that a full binary tree has one more leaf than it has nodes. □

Proposition 3.7. *(1) Let $u \in \Sigma^+$ and $s \in \mathcal{S}$ such that $|s| = 3|u| - 3$. Then there exists a unique tree $t = \tau(u, s)$ with foliage $\varphi(t) = u$ and skeleton $\sigma(t) = s$.*
(2) If t and t' are such that $\varphi(t) = \varphi(t')$ and $\sigma(t) = \sigma(t')$, then $t = t'$.

Proof. The proof is by induction on $|u|$. If $|u| = 1$ then $u = a$, $s = \varepsilon$ and $\tau(a, \varepsilon) = a$. If $|u| > 1$, there exists $u_1, u_2 \in \Sigma^+, s_1, s_2 \in \mathcal{S}$, such that $u = u_1 u_2$, $s = \blacktriangleleft s_1 \bullet s_2 \blacktriangleright$ and $|s_i| = 3|u_i| - 3$. Hence $\tau(u, s) = \tau(u_1, s_1) \star \tau(u_2, s_2)$.
(2) immediately follows from (1). □

Example 3.8. We give two congruences defined as kernels of homomorphisms.
(1) Equality of skeletons: $t \sim_\sigma t'$ iff $(t, t') \in \ker(\sigma)$.
(2) Equality of foliages: $t \sim_\varphi t'$ iff $(t, t') \in \ker(\varphi)$.

Other fundamental congruences are the kernels of the grafting homomorphisms defined below.

Definition 3.9. Let $a \in \Sigma$ and $\tau \in \mathcal{T}(\Sigma)$. Then $\gamma_{a\to\tau} : \mathcal{T}(\Sigma) \to \mathcal{T}(\Sigma)$ is the homomorphism on the free algebra of trees such that, for $b \in \Sigma$, the tree $\gamma_{a\to\tau}(b)$ is equal to τ if $b = a$, and to b otherwise.

The following Proposition and Lemma are easily proved by induction on t.

Proposition 3.10. *For all $\tau, t \in \mathcal{T}(\Sigma)$, $a \in \Sigma$, $\varphi(\gamma_{a\to\tau}(t)) = \psi_{a\to\varphi(\tau)}(\varphi(t)))$, i.e., the following diagram is commutative:*

$$\begin{array}{ccc} \mathcal{T}(\Sigma) & \xrightarrow{\gamma_{a\to\tau}} & \mathcal{T}(\Sigma) \\ \big\downarrow \varphi & & \big\downarrow \varphi \\ \Sigma^* & \xrightarrow{\psi_{a\to\varphi(\tau)}} & \Sigma^* \end{array}$$

Lemma 3.11. *A grafting $\gamma_{a\to\tau}$ is idempotent, i.e., $\gamma_{a\to\tau} \circ \gamma_{a\to\tau} = \gamma_{a\to\tau}$, if and only if the letter a does not appear in the foliage $\varphi(\tau)$.*

4. Congruence preserving functions on trees

We now study congruence preserving functions on the algebra $\langle \mathcal{T}(\Sigma), \star\rangle$. From now on, f, g will be congruence preserving functions on $\mathcal{T}(\Sigma)$.

Definition 4.1. A function $f\colon \mathcal{T}(\Sigma) \to \mathcal{T}(\Sigma)$ is congruence preserving (abbreviated into CP) if for all congruences $\sim$ on $\mathcal{T}(\Sigma)$, for all t, t' in $\mathcal{T}(\Sigma)$, $t \sim t' \implies f(t) \sim f(t')$.

We start with a very convenient result.

Proposition 4.2. *Let $\gamma_a = \gamma_{a\to\tau}$ and $\gamma_b = \gamma_{b\to\tau}$ be two graftings with $a \neq b$. For any t, t' if $\gamma_a(t) = \gamma_a(t')$ and $\gamma_b(t) = \gamma_b(t')$ then $t = t'$*

Proof. By induction on $\min(|t|, |t'|)$.

Basis If $\min(|t|, |t'|) = 1$ either **(i)** or **(ii)** holds.

(i) One of t, t' is a letter $c \in \{a, b\}$. Say $t = a$. Then $\gamma_b(t) = a$ hence $\gamma_b(t') = a$. This implies: either $t' = a$ hence $t = t'$, as wanted, or $t' = b$ and $\tau = a$ in which case $\gamma_a(t') = b$ contradicting $\gamma_a(t') = \gamma_a(t)$ since $\gamma_a(t) = \gamma_a(a) = \tau = a$.

(ii) One of t, t' is a letter $c \notin \{a, b\}$. Say $t = c$. Then $\gamma_a(c) = \gamma_b(c) = c$, hence $\gamma_a(t') = \gamma_b(t') = c$ implying $t' = c$ and $t = t'$ as wanted.

Induction Otherwise, we have $t = t_1 \star t_2$ and $t' = t'_1 \star t'_2$ with $\min(|t_i|, |t'_i|) < \min(|t|, |t'|)$, for $i = 1, 2$. By Lemma 3.4 $\gamma_a(t_1) \star \gamma_a(t_2) = \gamma_a(t'_1) \star \gamma_a(t'_2)$ implies $\gamma_a(t_i) = \gamma_a(t'_i)$ and $\gamma_b(t_i) = \gamma_b(t'_i)$. By the induction $t_i = t'_i$ hence $t = t'$. □

Proposition 4.3. *If f is CP then for every idempotent grafting $\gamma_{a\to t}$, we have $\gamma_{a\to t}(f(a)) = \gamma_{a\to t}(f(t))$.*

Proof. As $\gamma_{a\to t}$ is idempotent we have $\gamma_{a\to t}(a) = \gamma_{a\to t}(\gamma_{a\to t}(a))$. Now, $\gamma_{a\to t}(a) = t$ hence $\gamma_{a\to t}(a) = \gamma_{a\to t}(t)$. Since $\ker(\gamma_{a\to t})$ is a congruence and f is CP, we conclude $\gamma_{a\to t}(f(a)) = \gamma_{a\to t}(f(t))$. □

Corollary 4.4. *Let f, g be CP, if $f(a) = g(a)$, then for any idempotent grafting $\gamma_{a\to t}$, we have $\gamma_{a\to t}(f(t)) = \gamma_{a\to t}(g(t))$.*

Proof. By Proposition 4.3 we have, $\gamma_{a\to t}(f(t)) = \gamma_{a\to t}(f(a))$ and $\gamma_{a\to t}(g(a)) = \gamma_{a\to t}(g(t))$. As $f(a) = g(a)$, we infer $\gamma_{a\to t}(f(t)) = \gamma_{a\to t}(g(t))$. □

Proposition 4.3 and its Corollary tell us that the knowledge of $f(a)$ for all $a \in \Sigma$ gives a lot of information about the value of f on $\mathcal{T}(\Sigma)$. The following theorem shows that, in fact, f is completely determined by its value on Σ.

Theorem 4.5. *Suppose Σ has at least three letters, if f and g are CP functions on $\mathcal{T}(\Sigma)$ such that for all $a \in \Sigma$, $f(a) = g(a)$ then for all $t \in \mathcal{T}(\Sigma)$, $f(t) = g(t)$.*

Proof. Let $t \notin \Sigma$. The proof depends on the number $N(t)$ of letters of Σ which do not appear in the foliage $\varphi(t)$ of t.

1. Case $N(t) > 0$

Subcase $N(t) > 1$ Let a, b be two letters which do not occur in the foliage $\varphi(t)$ of t. Graftings $\gamma_{a\to t}$ and $\gamma_{b\to t}$ are idempotent. By Corollary 4.4 we have $\gamma_{c\to t}(f(t)) = \gamma_{c\to t}(g(t))$ for $c \in \{a, b\}$, and Proposition 4.2 yields $g(t) = f(t)$.

Subcase $N(t) = 1$ Let c be any letter and let t_c be the tree obtained by substituting c to all letters in t. Then $N(t) = |\Sigma| - 1 \geq 2$, and thus $g(t_c) = f(t_c)$. The trees t and t_c obviously have the same skeleton hence $t \sim_\sigma t_c$ (cf. Example 3.8 (1)). As f and g are congruence preserving, we thus have $f(t) \sim_\sigma f(t_c) = g(t_c) \sim_\sigma g(t)$ and $f(t)$ and $g(t)$ have the same skeleton.

Let c be the letter which does not appear in t. As $\gamma_{c\to t}$ is idempotent (cf. Lemma 3.11), we have by Corollary 4.4, $\gamma_{c\to t}(f(t)) = \gamma_{c\to t}(g(t))$. We prove the following Fact which, when applied to $u = f(t)$, $v = g(t)$, yields the result.

Fact *Let $u, v, t \in \mathcal{T}(\Sigma)$, $t \notin \Sigma$, if u and v have the same skeleton, and if $\gamma_{c\to t}(u) = \gamma_{c\to t}(v)$, then $u = v$.*

The proof is by induction on the common size of u and v, $|\sigma(u)| = |\sigma(v)|$.

Basis If $|\sigma(u)| = |\sigma(v)| = 1$, then $u = a$ and $v = b$. If $a = b$ the result is proved. Otherwise, we have $\gamma_{c\to t}(a) = \gamma_{c\to t}(b)$ which is possible only if one of the letters a, b (say a) is equal to c. But then we get $t = \gamma_{c\to t}(a) = \gamma_{c\to t}(b) = b$ contradicting the requirement that t is not a letter.

Induction Otherwise, we have $u = t_1 \star t_2$ and $v = t'_1 \star t'_2$ with with $\sigma(t_i) = \sigma(t'_i)$ and $\gamma_{c\to t}(u) = \gamma_{c\to t}(t_1) \star \gamma_{c\to t}(t_2)$, $\gamma_{c\to t}(v) = \gamma_{c\to t}(t'_1) \star \gamma_{c\to t}(t'_2)$ implying $\gamma_{c\to t}(t_i) = \gamma_{c\to t}(t'_i)$ by Lemma 3.4. By the induction $t_i = t'_i$ hence $u = v$.

2. Case $N(t) = 0$

Since $|\Sigma| \geq 3$ there exists a letter $c \notin \{a, b\}$. Then $\gamma_{c\to t}$ is idempotent. Let $t' = \gamma_{a\to c}(t)$. As $N(t') = 1$, we have $f(t') = g(t')$.

But t and t' are congruent for the congruence $\ker(\gamma_{a\to c})$, and as f is CP, we also have $\gamma_{a\to c}(f(t)) = \gamma_{a\to c}(f(t'))$. Similarly $\gamma_{a\to c}(g(t')) = \gamma_{a\to c}(g(t))$. As $f(t') = g(t')$, we infer $\gamma_{a\to c}(f(t)) = \gamma_{a\to c}(g(t))$. Similarly, $\gamma_{b\to c}(f(t)) = \gamma_{b\to c}(g(t))$. Thus, by Proposition 4.2, $f(t) = g(t)$. □

5. The algebra of full binary trees is 1–affine complete

Throughout this section, f is a fixed CP function on $\mathcal{T}(\Sigma)$. We first define polynomials on trees.

Definition 5.1. Let $x \notin \Sigma$ be a variable. A polynomial $T(x)$ is a tree on the alphabet $\Sigma \cup \{x\}$.

With every polynomial $T(x)$ we associate a polynomial function $T\colon \mathcal{T}(\Sigma) \to \mathcal{T}(\Sigma)$ defined by $T(t) = \gamma_{x\to t}(T(x))$. Obviously, every polynomial function is CP.

This section is devoted to proving the converse which amounts to saying that the algebra $\langle \mathcal{T}(\Sigma), *\rangle$ is 1–affine complete.

Theorem 5.2. *Every CP function is polynomial.*

By theorem 4.5, a CP function f is polynomial if there exists a polynomial $T_f(x)$ such that for all $a \in \Sigma$, $f(a) = T_f(a)$. Hence to prove Theorem 5.2, we will construct such a polynomial in the next subsection.

As $\sigma(a) = \varepsilon$ for all $a \in \Sigma$, if f is CP then all $f(a)$ have the same skeleton. For any pair $a, b \in \Sigma$ with $a \neq b$, we have $\gamma_{a\to b}(a) = \gamma_{a\to b}(b)$ and hence, by Lemma 2.1, $\gamma_{a\to b}(f(a)) = \gamma_{a\to b}(f(b))$. Thus, the next proposition can be applied to f.

Proposition 5.3. *Let $g : \Sigma \to \mathcal{T}(\Sigma)$ such that*

(1) all $g(a)$ have the same skeleton s and

(2) $\forall a \neq b$, $\gamma_{a\to b}(g(a)) = \gamma_{a\to b}(g(b))$. Then there exists a polynomial T_g such that $g(a) = T_g(a)$ for all $a \in \Sigma$.

Proof. The proof is by induction of the size of the common skeleton s.

Basis If $s = \varepsilon$ then each $g(a)$ is a letter in Σ. By assumption (2), we have $\gamma_{a\to b}(g(a)) = \gamma_{a\to b}(g(b))$. This last equality can happen when (i) $g(a) = g(b)$, or (ii) $\{g(a), g(b)\} = \{a, b\}$.

We first show that if there exists an a such that $g(a) = c \notin \{a, b\}$ then $\forall b \neq a$, $g(b) = c$. First for all $b \neq c$ we have either (i) $g(a) = g(b)$ or (ii) $\{a, b\} = \{g(a), g(b)\}$: (ii) is impossible since $g(a) = c \notin \{a, b\}$. Hence (i) holds and $g(b) = g(a) = c$. Next, for $b = c$, if $g(c) \neq c = g(a)$, we would infer from $\gamma_{a\to c}(g(a)) = \gamma_{a\to c}(g(c))$ that $\{a, c\} = \{c, g(c)\}$; similarly $\gamma_{d\to c}(g(d)) = \gamma_{d\to c}(g(c))$ implies that $\{d, c\} = \{c, g(c)\}$: hence $g(c) \in \{a, c\} \cap \{d, c\}$, and thus $g(c) = c$ holds also for c. Hence $T_g = c$.

Otherwise, $\forall a,\ g(a) = a$, hence $T_g = x$.

Induction Each $g(a)$ is equal to $g_1(a) \star g_2(a)$. It is easy to check that both g_i satisfy assumptions (1) and (2). Hence $T_g = T_{g_1} \star T_{g_2}$. □

6. Conclusion

We proved that, when Σ has at least three letters, the algebra $\mathcal{B}$ of full binary trees with leaves labeled by letters of Σ is a 1–affine complete algebra (non commutative and non associative). Our result extends to non commutative non associative algebras with unit by adding a unit element to $\mathcal{T}(\Sigma)$. By forgetting skeletons and replacing graftings $\gamma_{a\to\tau}$ with substitutions $\psi_{a\to u}$, the results in Sections 4 and 5 go through mutatis mutandis when $\mathcal{B}$ is replaced by the free monoid Σ^* on an alphabet Σ with at least three letters. This yields a simpler and shorter proof of the main result of [2], i.e., the 1–affine completeness of Σ^*. Surprisingly, the free monoid Σ^* when the alphabet Σ has just one letter is **not** 1–affine complete: Σ^* then reduces to the semigroup $\langle \mathbb{N}, +\rangle$ where "polynomial" functions are a strict subset of CP functions, e.g., $f(x) = \lfloor e^{1/a} a^x x! \rfloor$ for $a \in \mathbb{N} \setminus \{0,1\}$ is a non polynomial CP function, see [1].

We conjecture that

(1) $\mathcal{B}$ is affine complete, i.e., that CP functions of any arity on $\mathcal{T}(\Sigma)$ are also polynomial, and

(2) the algebra of binary trees (i.e., non full trees whose nodes might have 0,1 or 2 children) and whose leaves are labeled by letters of an alphabet Σ with at least three letters is also affine complete.

Extending the previous results when Σ has at most two letters yields open problems. The use of Σ was essential in the proof that $\mathcal{B}$ is 1–affine complete. Whether algebras of binary trees without labels would still be 1–affine complete is an open problem.

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