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The MacDowell-Specker theorem and miniaturizations

1

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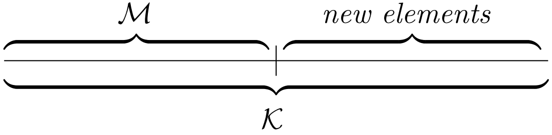
$\mathcal{LA}$=$\{<,+,\cdot,0,1\}$ `PA` Peano Arithmetic $\mathbb{N}$ standard model

For $\mathcal{M},\mathcal{K}$ structures for $\mathcal{L}A$, $\mathcal{K}$ is an elementary extension of $\mathcal{M}$, denoted by $\mathcal{M} < \mathcal{K}$, if $\mathcal{M} \subseteq \mathcal{K}$ and for all $\vec{a}\in\mathcal{M}$ and for all formulas $\varphi(\vec{x})$, $\mathcal{M}\models\varphi(\vec{a})$ iff $\mathcal{K}\models\varphi(\vec{a})$.

For any $n\in\mathbb{N}$, Σ_n-elementary extensions are defined analogously and denoted by $<_n$.

For $\mathcal{M},\mathcal{K}$ structures for $\mathcal{L}A$, $\mathcal{K}$ is a proper end extension of $\mathcal{M}$, denoted by $\mathcal{M} \subset_e \mathcal{K}$, if $\mathcal{M} \subset \mathcal{K}$ and

$$\text{for all } a\in\mathcal{M} \text{ and for all } b\in\mathcal{K}, \text{ if } b <^{\mathcal{K}} a, \text{ then } b\in\mathcal{M}$$



Fact. There exists a (countable) $\mathcal{M}$ such that $\mathbb{N} <_e \mathcal{M}$
(Let $\mathcal{M}$ be any (countable) non-standard model of $\mathtt{Th}(\mathbb{N})$)

Problem. Does this hold if we replace $\mathbb{N}$ with a non-standard model of $\mathtt{PA}$?

R. Mac Dowell & E. Specker. Modelle der Arithmetik. (German) 1961 Infinitistic Methods (Proc. Sympos. Foundations of Math., Warsaw, 1959) pp. 257–263 Pergamon, Oxford; Państwowe Wydawnictwo Naukowe, Warsaw.

Zu jedem Modell M gibt es eine echte arithmetische Erweiterung M' von gleicher Mächtigkeit, so daß die Elemente von $M' - M$ größer sind als die Elemente von M.

Theorem. For every model $\mathcal{M}$ of $\mathtt{PA}$ there exists $\mathcal{K}$ (of the same cardinality) such that $\mathcal{M} <_e \mathcal{K}$.

H. Gaifman. Models and types of Peano's arithmetic. Ann. Math. Logic 9 (1976), no. 3, 223–306.

Theorem. For every model $\mathcal{M}$ of PA there exists $\mathcal{K}$ (of the same cardinality) such that $\mathcal{K}$ is a conservative extension of $\mathcal{M}$.

Reminder. For $\mathcal{M}, \mathcal{K}$ structures for $\mathcal{L}A$, we say that $\mathcal{K}$ is a conservative extension of $\mathcal{M}$, if for any $\vec{b} \in \mathcal{K}$ and any formula $\theta(u, \vec{v})$ there exist $\vec{a} \in \mathcal{M}$ and a formula $\psi(u, \vec{w})$ such that

$$\{u : \mathcal{K} \models \theta(u, \vec{b})\} \cap \mathcal{M} = \{u : \mathcal{M} \models \psi(u, \vec{a})\}.$$

Fact. If $\mathcal{K}$ is a conservative extension of $\mathcal{M}$, then $\mathcal{K} \subseteq_e \mathcal{M}$.

Aim. Miniaturize the MacDowell-Specker theorem, i.e., prove versions when PA is replaced by one of its fragments.

For $n\in\mathbb{N}$,

- $I\Sigma_n$: induction for Σ_n formulas (plus base theory)
- $B\Sigma_n$: $I\Sigma_0$ plus the collection schema

$$\forall t \forall \vec{z}\, [\forall x{<}t \exists y\, \varphi(x, y, \vec{z}) \rightarrow \exists s \forall x{<}t \exists y{<}s\, \varphi(x, y, \vec{z})]$$

  for any Σ_n formula $\varphi(x, y, \vec{z})$.

J. B. Paris & L. A. S. Kirby. Σ_n-collection schemas in arithmetic. Logic Colloquium '77 (Proc. Conf., Wrocław, 1977), pp. 199–209, Studies in Logic and the Foundations of Mathematics, 96, North-Holland, Amsterdam-New York, 1978.

Theorem. For any countable $\mathcal{M}$ and $n\geq 2$, if $\mathcal{M}\models B\Sigma_n$, then there exists $\mathcal{K}\models I\Delta_0$ such that $\mathcal{M} <_{n,e} \mathcal{K}$.

Remarks. (a) The result does not hold for $n=1$.
(b) It is unknown what holds for $n=0$ (more on this later).

Problem. Does the Paris & Kirby result hold for **any** model?

P. G. Clote. A note on the MacDowell-Specker theorem. Fund. Math. 127 (1987), no. 2, 163–170.

Theorem. For any $\mathcal{M}$ and $n \geq 2$, if $\mathcal{M} \models B\Sigma_n$, then there exists $\mathcal{K} \models I\Delta_0$ such that $\mathcal{M} <_{n,e} \mathcal{K}$.

P. Clote. Addendum to: "A note on the MacDowell-Specker theorem" [Fund. Math. 127 (1987), no. 2, 163–170]. Fund. Math. 158 (1998), no. 3, 301–302.

Theorem. For any $\mathcal{M}$ and $n \geq 2$, if $\mathcal{M} \models I\Sigma_n$, then there exists $\mathcal{K} \models I\Delta_0$ such that $\mathcal{M} <_{n,e} \mathcal{K}$.

Reminder. For any $n \in \mathbb{N}$, $I\Sigma_n$ is strictly stronger than $B\Sigma_n$.

P. Clote & J. Krajíček. Open problems. Arithmetic, Proof Theory and Computational Complexity (Prague, 1991), 1–19, vol. 23 of Oxford Logic Guides, Oxford University Press, 1993.

Fundamental problem F. (Paris, Wilkie) Does every countable model $\mathcal{M}$ of $I\Delta_0 + B\Sigma_1$ have an end extension to a model of $I\Delta_0$ [59]? This is the most popular problem in the area but appears to be difficult and several problems below are related to it (and hopefully more tractable).

Problem*. Does every countable model $\mathcal{M}$ of $B\Sigma_1$ have a proper Σ_0-elementary end extension to a model of $I\Delta_0$?

Fact. For any structures $\mathcal{M}, \mathcal{K}$, if $\mathcal{M} \subset_e \mathcal{K}$, then $\mathcal{M} <_0 \mathcal{K}$.

([59]) A. Wilkie & J. Paris. On the existence of end extensions of models of bounded induction. Logic, methodology and philosophy of science, VIII (Moscow, 1987), 143–161, Stud. Logic Found. Math., 126, North-Holland, Amsterdam, 1989.

Theorem. For any countable model $\mathcal{M}$ of $B\Sigma_1$, if $\mathcal{M}$ is $I\Delta_0$-full, then there exists $\mathcal{K} \models I\Delta_0$ such that $\mathcal{M} \subset_e \mathcal{K}$.

Theorem. For any countable $\mathcal{M} \models B\Sigma_1$, each of the following conditions implies that $\mathcal{M}$ is $I\Delta_0$-full:

(i) $\mathcal{M}$ is short Π_1–recursively saturated

(ii) $\mathcal{M} \models exp$ (*exp* denotes "exponentiation is total")

(iii) $I\Delta_0 \vdash \neg\Delta_0 H$ and there exists $\gamma \in \mathcal{M} - \mathbb{N}$ s.t. $\mathcal{M} \models \forall x \exists y (y = x^\gamma)$

(iv) $I\Delta_0 \vdash \neg\Delta_0 H$ and there exists $a \in \mathcal{M}$ s.t. $\mathcal{M} = a^{\mathbb{N}}$.

$I\Delta_0 \vdash \neg\Delta_0 H$: the Δ_0 hierarchy provably collapses in $I\Delta_0$

Remark ([59]). A direct proof that any countable model of $I\Delta_0 + B\Sigma_1$ which is closed under exponentiation has a proper end extension to a model of $I\Delta_0$ may be obtained by mimicking the proof of Theorem 4 but with "Semantic Tableau Consistency of Γ" in place of Γ-fullness and adding a new constant symbol $\pi > \mathcal{M}$ to ensure that the end extension is proper.

C. Dimitracopoulos & V. S. Paschalis. End extensions of models of weak arithmetic theories. Notre Dame J. Form. Log. 57 (2016), no. 2, 181–193.

Theorem. For any countable model $\mathcal{M}$ of $B\Sigma_1$, if $\mathcal{M}$ satisfies any of the Wilkie-Paris conditions (i), (iii) and (iv), then there exists $\mathcal{K} \models I\Delta_0$ such that $\mathcal{M} \subset_e \mathcal{K}$.

Idea. Prove variants of the Arithmetized Completeness Theorem, thus avoiding the use of $I\Delta_0$-fullness.

C. Dimitracopoulos & V. Paschalis. End extensions of models of fragments of PA. Arch. Math. Logic 59 (2020), no. 7-8, 817–833.

Theorem. (a) Clote's result. (b) For any $\mathcal{M} \models I\Sigma_1$, there exists $\mathcal{K} \models I\Delta_0$ s. t. $\mathcal{M} \subset_e \mathcal{K}$. (recall $I\Sigma_1$ strictly stronger than $B\Sigma_1$)

Remark. Result (b) follows from earlier work of A. Enayat and T. L. Wong - see

T. L. Wong. Interpreting weak König's lemma using the arithmetized completeness theorem. Proc. Amer. Math. Soc. 144 (2016), no. 9, 4021–4024.

A. Enayat & T. L. Wong. Unifying the model theory of first-order and second-order arithmetic via WKL_0^*. Ann. Pure Appl. Logic 168 (2017), no. 6, 1247–1283.

For $n \in \mathbb{N}$,

- $I\Delta_n$: induction for provably Δ_n formulas (plus base theory) $\forall x \forall \vec{y}(\varphi(x,\vec{y}) \leftrightarrow \psi(x,\vec{y})) \to I\varphi$, where $\varphi \in \Sigma_n$, $\psi \in \Pi_n$ and $I\varphi$ denotes the induction axiom for φ
- $L\Delta_n$: least number principle for provably Δ_n formulas $\forall x \forall \vec{y}(\varphi(x,\vec{y}) \leftrightarrow \psi(x,\vec{y})) \to L\varphi$, where $\varphi \in \Sigma_n$, $\psi \in \Pi_n$ and $L\varphi$ denotes the least number axiom for φ

Theorem. For any $n \geq 1$, $B\Sigma_n \Leftrightarrow L\Delta_n \Rightarrow I\Delta_n$

Problem. (Technical problem no. 34 in Clote & Krajíček)

For $n \geq 1$, is $I\Delta_n$ equivalent to $B\Sigma_n$?

T. Slaman. Σ_n-bounding and Δ_n-induction. *Proc. Amer. Math. Soc.* 132 (2004), 2449–2456.

Theorem. (i) For $n \geq 2$, $I\Delta_n \Leftrightarrow B\Sigma_n$.
(ii) $I\Delta_1 + exp \Rightarrow B\Sigma_1$ (hence $I\Delta_1 + exp \Leftrightarrow B\Sigma_1 + exp$).

Problem 1. Does every model of $B\Sigma_1{+}exp$ have a proper end extension satisfying $I\Delta_0$?

Problem 2. Does every model of $I\Delta_1{+}exp$ have a proper end extension satisfying $I\Delta_0$?

Remarks. Without assuming Slaman's result,
(i) If "yes" to Problem 2, then "yes" to Problem 1.
(ii) If "yes" to Problem 2, then $I\Delta_1{+}exp{\Rightarrow}B\Sigma_1$, i.e., (b) of Slaman's result follows. This is based on the fact that for any structures $\mathcal{M}, \mathcal{K}$, if $\mathcal{M} \subset_e \mathcal{K}$, then $\mathcal{M} \models I\Delta_0$.

C. Dimitracopoulos & V. Paschalis. End extensions of models of $I\Delta_1$. In preparation.

Theorem. Every model of $I\Delta_1{+}exp$ has a proper end extension to a model of $IOpen$ (= induction for open formulas).

N. Thapen. A note on Δ_1 induction and Σ_1 collection. *Fund. Math.* 186 (2005), 79–84.

e_p is the axiom $\forall x \exists y\,(x<p(y) \wedge$ "x^y exists"$)$,
where p is any primitive recursive function

Theorem. $I\Delta_1 + e_p \Rightarrow B\Sigma_1$.

Remarks. (i) Thapen's result implies part (b) of Slaman's result, since *exp* is (equivalent to) e_p for the specific primitive recursive function $p(y)=y+1$.
(ii) Another instance of e_p is Ω_1, i.e., the axiom $\forall x \exists y\,(y=x^{|x|})$, where $|x|$ denotes the length of x, since Ω_1 is (equivalent to) e_p for the specific primitive recursive function $p(y) = 2^{y+1}$.

Problem. Does every model of $I\Delta_1 + e_p$ have a proper end extension satisfying $I\Delta_0$?

Theorem. Every model of $I\Delta_1 + e_p$ has a proper end extension to a model of *IOpen*.

S. Boughattas. L'arithmétique ouverte et ses modèles non-standards. *J. Symbolic Logic* 56 (1991), 700–714.

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ΠΑΤΡΙΚ, ΤΥΧΕΙΝ ΑΡΙΣΤΟΝ!
PATRICK, ALL THE BEST TO YOU!