Introduction to Forcing

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Journées GT Calculabilités
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Outline

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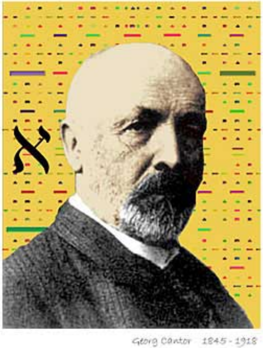
The work of Cantor



In the second half of the 19th century, german mathematician, **Georg Cantor** laid the foundations of **set theory**. He defined, ordinal and cardinal numbers, and developed their arithmetic.

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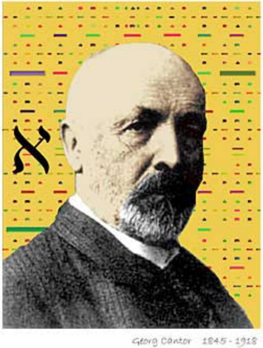

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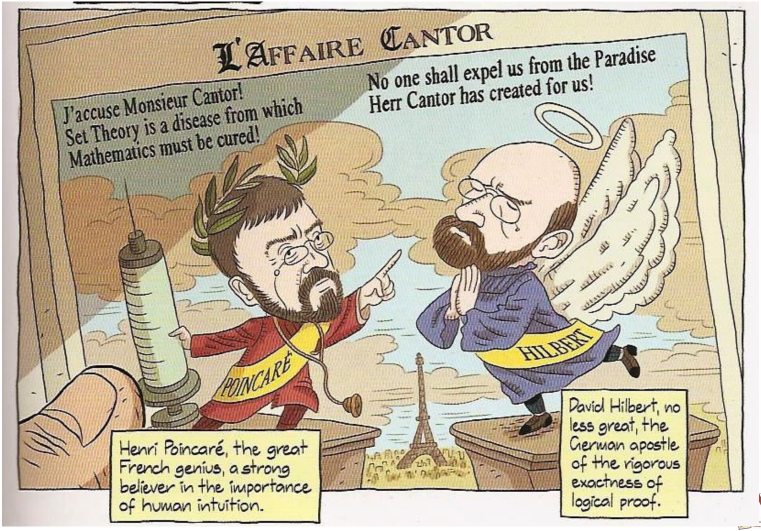
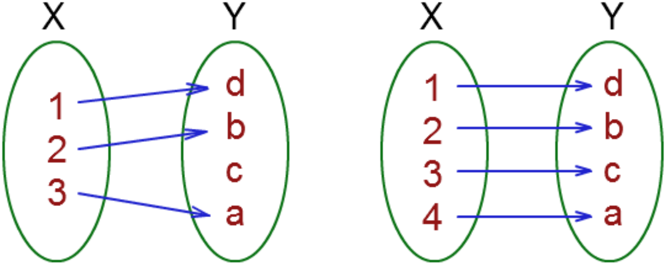
Cantor’s work provoked a lot of controversy.



Definition

Let X and Y be sets. We write $X \preceq Y$ if there is an injection from X to Y. We write $X \approx Y$ if there is a bijection between X et Y.



Theorem (Cantor - Bernstein)

Suppose that $X \preceq Y$ *and* $Y \preceq X$*. Then* $X \approx Y$*.*

Proposition

X *is infinite iff* $X \approx X \setminus \{x\}$*, for any* $x \in X$*.*

Definition

X *is* ***countable*** *if* $X \approx \mathbb{N}$*.*

Proposition (Cantor)

1. *If* A_n *is countable, for all* n*, then* $\bigcup_n A_n$ *is countable.*
2. $A^n \approx A$*, for any infinite set* A *and integer* $n \geq 1$*.*

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However, there are infinite sets that are not countable. By the famous **diagonal argument** we have.

Theorem (Cantor)

The set of reals $\mathbb{R}$ is uncountable.

Cantor spent the rest of his life trying to prove the following.

Continuum Hypothesis (CH)

Let X be an infinite set of reals. Then either $X \approx \mathbb{N}$ or $X \approx \mathbb{R}$.

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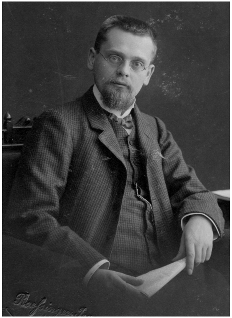
Zermelo-Fraenkel set theory



Following a tumultuous period in the Foundations of Mathematics, in the early 20th century, Ernst Zermelo and Abraham Fraenkel formulated set theory as a **first order theory** ZF whose only nonlogical symbol is $\in$. This was later augmented by adding the **Axiom of Choice**.

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ZFC axioms

The axioms of Zermelo-Fraenkel set theory with choice ZFC

In principle all of mathematics can be derived from these axioms

Extensionality	$\forall X\,\forall Y\,[\,X = Y \quad\Leftrightarrow\quad \forall z(z \in X \;\Leftrightarrow\; z \in Y)\,]$
Pairing	$\forall x\,\forall y\,\exists Z\,\forall z\,[\,z \in Z \quad\Leftrightarrow\quad z = x \text{ or } z = y\,]$
Union	$\forall X\,\exists Y\,\forall y\,[\,y \in Y \quad\Leftrightarrow\quad \exists Z(Z \in X \text{ and } y \in Z)\,]$
Empty set	$\exists X\,\forall y\,[\,y \notin X\,]$ (this set X is denoted by $\emptyset$)
Infinity	$\exists X\,[\,\emptyset \in X \text{ and } \forall x(x \in X \Rightarrow x \cup \{x\} \in X)\,]$
Power set	$\forall X\,\exists Y\,\forall Z\,[\,Z \in Y \quad\Leftrightarrow\quad \forall z(z \in Z \;\Rightarrow\; z \in X)\,]$
Replacement	$\forall x \in X\,\exists! y\, P(x,y) \quad\Rightarrow\quad [\,\exists Y\,\forall y\,(y \in Y \;\Leftrightarrow\; \exists x \in X\,(P(x,y)))\,]$
Regularity	$\forall X\,[\,X \neq \emptyset \quad\Rightarrow\quad \exists Y \in X\,(X \cap Y = \emptyset)\,]$
Axiom of choice	$\forall X\,[\,\emptyset \notin X \text{ and } \forall Y, Z \in X(Y \neq Z \;\Rightarrow Y \cap Z = \emptyset) \quad\Rightarrow\quad \exists Y\,\forall Z \in X\,\exists! z \in Z\,(z \in Y)\,]$

$\forall$ = for all $\quad$ $\exists!$ = there exists a unique $\quad$ P is any formula that does not contain Y

$$z \in X \cup Y \;\Leftrightarrow\; z \in X \text{ or } z \in Y \qquad z \in X \cap Y \;\Leftrightarrow\; z \in X \text{ and } z \in Y$$

Ordinals

In principle, all of mathematics can be carried out in ZFC. So it is important to understand its strengths and limitations. The basic concept is that of an **ordinal**, which is a generalization of an integer.

Definition

1. *A **well order** on a set X is a total order $<$ on X such that every nonempty subset of X has a minimal element.*
2. *An **ordinal** is a set α which is transitive (i.e. if $x \in y \in \alpha$ then $x \in \alpha$) and well ordered by $\in$.*

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We have:

$$
\begin{gathered}
0 := \varnothing, \\
1 := \{0\} = \{\varnothing\}, \\
2 := \{0, 1\} = \{\varnothing, \{\varnothing\}\}, \\
3 := \{0, 1, 2\} = \{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}\}, \\
\ldots \\
\omega := \{0, 1, 2, 3, \ldots\}, \\
\omega + 1 := \{0, 1, 2, 3, \ldots, \omega\}, \\
\ldots \\
\omega \cdot 2 := \omega + \omega = \{0, 1, 2, 3, \ldots, \omega, \omega + 1, \omega + 2, \omega + 3, \ldots\}, \\
\ldots \\
\omega^2 := \{0, 1, \ldots, \omega, \omega + 1, \ldots, \omega \cdot 2, \omega \cdot 2 + 1, \ldots, \omega \cdot n, \omega \cdot n + 1, \ldots\}, \\
\ldots
\end{gathered}
$$

Definition

1. *The successor of an ordinal α is the ordinal $\alpha + 1 = \alpha \cup \{\alpha\}$.*
2. *An ordinal α is limit if $\alpha > 0$ and α is not a successor. The least limit ordinal is ω.*

Definition

*A **cardinal** is an ordinal α such that $\alpha \not\approx \beta$, for all $\beta < \alpha$*

Remark

1. All integers are cardinals, as well as ω. The ordinals $\omega + 1, \omega + 2, \ldots, \omega \cdot 2, \ldots$, are not cardinals.
2. The first cardinal $> \omega$ is denoted by ω_1 or $\aleph_1$, the second ω_2 or $\aleph_2$, etc.

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Von Neumann's Cumulative Hierarchy

We define the cumulative hierarchy.

- $V_0 = \varnothing$,
- (Successor case) $V_{\alpha+1} = \mathcal{P}(V_\alpha)$, for all α, where $\mathcal{P}(X)$ is the powerset of X,
- (Limit case) $V_\alpha = \bigcup\{V_\xi : \xi < \alpha\}$, for all limit α,
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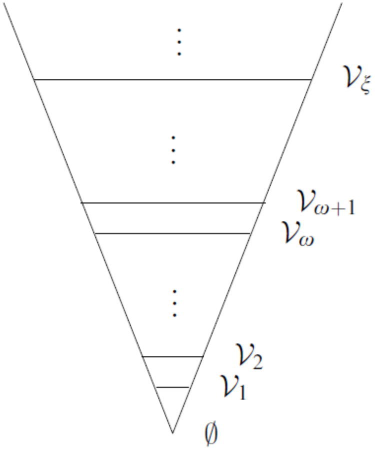
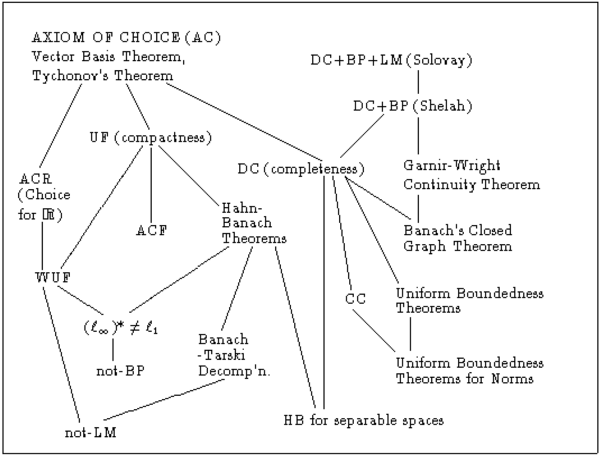


What about Choice?

And what about the **Axiom of Choice**? Well, it is necessary for some basic theorems in mathematics...



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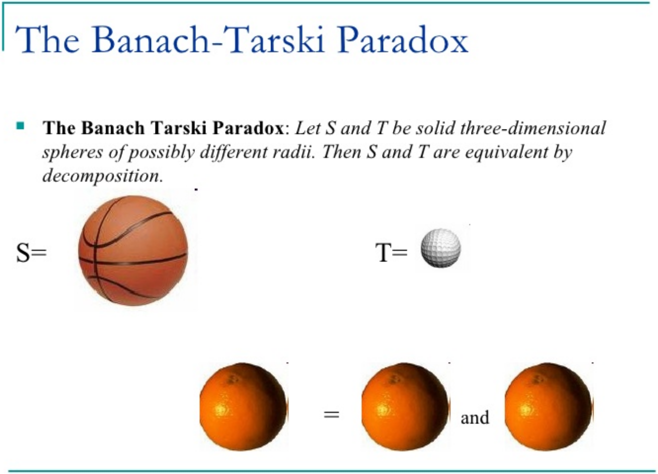
On the other hand it leads to some strange paradoxes...



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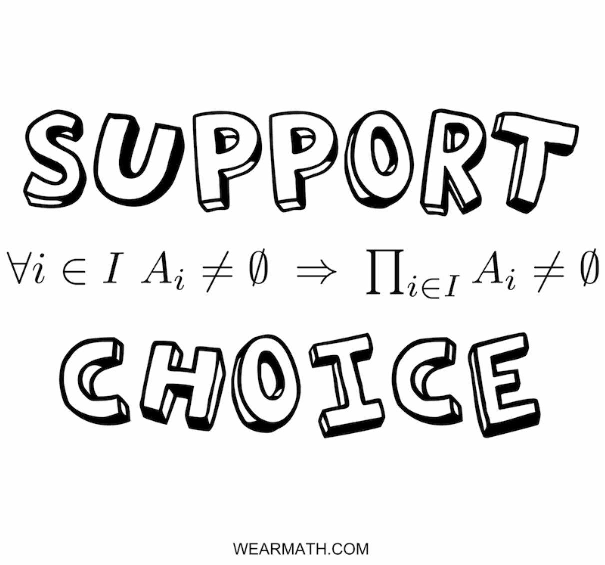
And in some countries it is still the topic of **hot** debate...



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Models of ZF

ZF is a first order theory, so we can consider models of ZF. A **model** of ZF is a **set** M with a binary relation E such that $(M, E) \models \mathrm{ZF}$. Note that E may not be the **true** membership relation $\in$.

ZF is recursive and contains arithmetic, hence by Gödel's **Incompleteness theorem**, if it is consistent then it is **incomplete**. In fact, ZF does not prove its own consistency.

But, wait! Isn't $\mathbb{V}$ a model of ZF?

Yes! But $\mathbb{V}$ is a **proper class** and the statement that $\mathbb{V}$ is a model of ZF cannot even be expressed as a first order statement by **Tarski's undefinability of truth**.

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Theorem (Kurt Gödel, 1940)

If the theory ZF is consistent, then so is ZFC + CH.

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Effective cumulative hierarchy: $\mathbb{L}$

The definable power set

For each set X, $\mathcal{P}_{\mathrm{Def}}(X)$ denotes the set of all $Y \subseteq X$ which are logically definable in the structure $(X, \in)$.

- (AC) $\mathcal{P}_{\mathrm{Def}}(X) = \mathcal{P}(X)$ if and only if X is finite.

Gödel's constructible universe, $\mathbb{L}$

Define L_α by induction on α as follows.

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$\mathbb{L}$ is a **proper class**, i.e. not a set. Formally, we prove the following metatheorem.

Theorem

1. $\mathrm{ZF} \vdash (\varphi)^{\mathbb{L}}$, *for every axiom* φ *of* ZF,
2. $\mathrm{ZF} \vdash (\mathbb{V} = \mathbb{L})^{\mathbb{L}}$,
3. $\mathrm{ZF} \vdash \mathbb{V} = \mathbb{L} \to \mathrm{CH}$, *and*
4. $\mathrm{ZF} \vdash \mathbb{V} = \mathbb{L} \to \mathrm{AC}$.

$\mathbb{L}$ is the smallest transitive class which is a model of ZF, hence with this method we **cannot** prove the independence of CH and AC.

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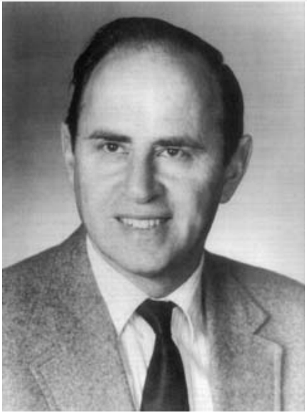
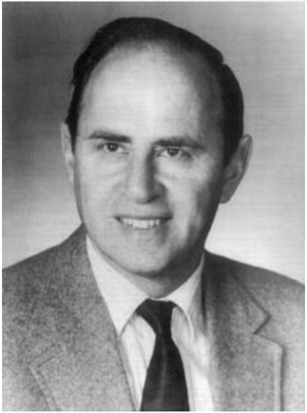
Independence of CH and AC



Theorem (Paul Cohen, 1963)

If the theory ZF *is consistent, then so are the theories* ZFC $+ \neg$CH *and* ZF $+ \neg$AC.

Independence of CH and AC



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Cohen's method of forcing

There are two equivalent ways of presenting forcing.

One is to work in V, but change the concept of **truth**. We fix a complete Boolean algebra $\mathbb{B}$ and define the $\mathbb{B}$-valued universe $\mathbb{V}^{\mathbb{B}}$. If $\varphi(x_1, \ldots, x_n)$ is a formula of set theory, and $\tau_1, \ldots, \tau_n \in \mathbb{V}^{\mathbb{B}}$, we can define $\|\varphi(\tau_1, \ldots, \tau_n)\|$, the $\mathbb{B}$-value of φ, which measure **how much** $\varphi(\tau_1, \ldots, \tau_n)$ is true in $\mathbb{V}^{\mathbb{B}}$. Then we show that $\|\varphi\| = \mathbf{1}_{\mathbb{B}}$, for every axiom φ of ZF. Moreover, if $\varphi_1, \ldots, \varphi_n \vdash \psi$ then

$$\|\varphi_1\| \wedge \ldots \wedge \|\varphi_n\| \leq \|\psi\|.$$

Then, by choosing carefully $\mathbb{B}$, we can make $\|\mathrm{CH}\|$ equal to $\mathbf{0}_{\mathbb{B}}$ or $\mathbf{1}_{\mathbb{B}}$.

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The second method is to assume that there is a countable, transitive set M such that $(M, \in)$ satisfies ZFC and work with actual models. Given a formula φ the truth value of φ may change when we change the model, so we must be careful.

Δ_0-formulas

A formula φ of set theory is Δ_0 if every quantifier φ is **bounded**, i.e. is of the form $\exists x \in y$ or $\forall x \in y$, for some variables x and y.

Absoluteness of Δ_0-formulas

If M is a transitive set, $\varphi(v)$ a Δ_0-formula and $a \in M$. Then $M \vDash \varphi(a)$ iff $\mathbb{V} \vDash \varphi(a)$.

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So, fix our ctm M and work for a while in M.

Forcing notions

A **forcing notion** is a partial order $(P, \leq)$ with the largest element 1_P.

Conditions

Elements of P are called **conditions**. If $p \leq q$ we say that p is **stronger** than q. If there is r such that $r \leq p, q$ we say that p and q are **compatible**. Otherwise, we say that they are **incompatible** and we write $p \perp q$. A set of incompatible conditions is called an **antichain**.

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Cohen's method of forcing

Dense sets

$D \subseteq P$ is called **dense** if for every $p \in P$ there is $q \in D$ with $q \leq p$.

Filters

A subset F of P is called a **filter** if:

1. if $p, q \in F$ then there is $r \leq p, q$ with $r \in F$,
2. if $p \in F$ and $p \leq q$ then $q \in F$.

Generic filters

A filter G is M-**generic** if $G \cap D \neq \varnothing$, for all dense $D \subseteq P$ with $D \in M$.

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Cohen's method of forcing

In nontrivial cases there are no M-generic filters in M, but it is easy to construct them in $\mathbb{V}$.

Baire category theorem

In $\mathbb{V}$, for every $p \in P$, there is an M-generic filter G such that $p \in G$.

Proof.

M is countable, so we can list all dense subsets of P which belong to M as $D_0, D_1, \ldots$. Then we build a sequence $p_0 \geq p_1 \geq \ldots$. Let $p_0 = p$. Given p_n, use the fact that D_n is dense to pick $p_{n+1} \in D_n$ such that $p_{n+1} \leq p_n$. Finally, let $G = \{q \in P : \exists n\, p_n \leq q\}$. $\square$

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Example of a forcing notion

Definition

Let P_0 consist of all finite partial functions from ω to $\{0,1\}$. The order is given by: $p \leq q$ iff $q \subseteq p$.

This definition is done in M, but it gives the same object in $\mathbb{V}$. What can we say about an M-generic filter G?

1. If $p, q \in G$ then $p \cup q$ is a function.
2. Let $g = \bigcup G$. Then g is a **total** function from ω to $\{0,1\}$.
3. Let $x_g = \{n : g(n) = 1\}$. Then x_g is infinite and co-infinite.
4. $x_g \notin M$.

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1. G is a filter, so if $p, q \in G$ there is $r \leq p, q$. Hence $p \cup q$ is a function.
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3. $E_n^0 = \{p \in P_0 : |p^{-1}(0)| \geq n\}$ and $E_n^1 = \{p \in P_0 : |p^{-1}(1)| \geq n\}$. Then E_n^0 and E_n^1 are dense, for all n, and hence intersect G.
4. Given a real $z \in M$ (think of $z : \omega \to \{0, 1\}$), let

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Fundamental theorem of forcing I

The fundamental theorem of forcing I

Let M be a ctm of ZFC, $(P, \leq) \in M$ a forcing notion and G an M-generic filter. Then there is a transitive set $M[G]$ such that:

1. $M \cup \{G\} \subseteq M[G]$,
2. $M[G] \cap \mathrm{ORD} = \mathrm{M} \cap \mathrm{ORD}$,
3. $M[G] \models \mathrm{ZFC}$,
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$M[G]$ is obtained by adding G to M and closing under simple set-theoretic operations.

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P-names

People living in M do not know G but they can still talk about $M[G]$. Every $t \in M[G]$ will have a **name** $\tau \in M$. In general, τ is not unique. One can interpret τ only once G is known. The following definition is done in M by $\in_*$-induction.

P-names

$\varnothing$ is a P-**name**. We say that τ is a P-**name** if every element of τ is of the form (q, σ), where $q \in P$ and σ is a P-name. Let M^P be the (class) of all P-names.

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Fundamental theorem of forcing II

Let G be an M-generic filter. We define $K_G(\tau)$ for every P-name τ.

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$M[G] = \{K_G(\tau) : \tau \in M \text{ and } \tau \text{ is a } P\text{-name}\}$.

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Canonical names

How do we show that $M \subseteq M[G]$ and $G \in M[G]$? First, we build a name for every element of M.

Canonical names

Let $\check{\varnothing} = \varnothing$. If $x \neq \varnothing$ let $\check{x} = \{(1_P, \check{y}) : y \in x\}$.

Since $1_P \in G$, it is easy to check that $K_G(\check{x}) = x$, for all $x \in M$.

Definition

Let $\Gamma = \{(p, \check{p}) : p \in P\}$.

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Language of forcing

The language $\mathcal{L}_f$ of forcing consists of symbols $\in$, $=$, a unary predicate S and a constant τ, for every P-name τ. We interpret $\mathcal{L}_f$ in $M[G]$. We let $\in$ and $=$ be as usual, τ is interpreted by $K_G(\tau)$, for every P-name τ. Finally, we let $S(x)$ iff $x \in M$.

Forcing relation

Let $p \in P$, φ a formula of $\mathcal{L}_f$, and $\tau_1, \ldots, \tau_n$ the P-names appearing in φ. We say that p **forces** φ and write $p \Vdash \varphi$ iff, for every M-generic filter G with $p \in G$, we have

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Fundamental theorem of forcing III and IV

The fundamental theorem of forcing III

Let φ be a closed formula of $\mathcal{L}_f$ and G an M-generic filter. Then

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The fundamental theorem of forcing IV - definability of the forcing relation

If $\varphi(x_1, \ldots, x_n)$ is a formula of $\mathcal{L}_{\mathrm{ZF}} \cup \{S\}$, then there is a formula $\theta(y, z, x_1, \ldots, x_n)$ such that, for every forcing notion $(P, \leq)$, $p \in P$, and P-names $\tau_1, \ldots, \tau_n$

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Forcing relation - atomic case

In M, we define $p \Vdash \tau_1 = \tau_2$, $p \Vdash \tau_1 \subseteq \tau_2$ and $p \Vdash \tau_1 \in \tau_2$, for $p \in P$ and P-names τ_1, τ_2 by induction on $(\mathrm{rank}(\tau_1), \mathrm{rank}(\tau_2))$.

Definition

1. *$p \Vdash \tau_1 = \tau_2$ iff $p \Vdash \tau_1 \subseteq \tau_2$ and $p \Vdash \tau_2 \subseteq \tau_1$.*
2. *$p \Vdash \tau_1 \subseteq \tau_2$ iff for every $(q, \sigma) \in \tau_1$ and $r \leq p, q$ there is $s \leq r$ such that $s \Vdash \sigma \in \tau_2$.*
3. *$p \Vdash \tau_1 \in \tau_2$ iff for every $q \leq p$ there is $(r, \sigma) \in \tau_2$ and $s \leq q, r$ such that $s \Vdash \tau_1 = \sigma$.*

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3. $p \Vdash \tau_1 \in \tau_2$ *iff for every* $q \leq p$ *there is* $(r, \sigma) \in \tau_2$ *and* $s \leq q, r$ *such that* $s \Vdash \tau_1 = \sigma$.

Forcing relation - connectives and quantifiers

Still in M, we continue to define $p \Vdash \varphi$, for non atomic φ.

Definition

1. $p \Vdash \varphi \wedge \psi$ *iff* $p \Vdash \varphi$ *and* $p \Vdash \psi$.
2. $p \Vdash \neg\varphi$ *iff* $q \nVdash \varphi$, *for all* $q \leq p$.
3. $p \Vdash \exists x\, \varphi(x)$ *iff for all* $q \leq p$ *there is* $r \leq q$ *and a* P*-name* τ *such that* $r \Vdash \varphi(\tau)$.

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3. *No* p *forces both* φ *and* $\neg\varphi$.

Forcing relation - connectives and quantifiers

Still in M, we continue to define $p \Vdash \varphi$, for non atomic φ.

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Proof of the Fundamental theorem of forcing

The proof of the Fundamental theorem of forcing is a straightforward, but tedious exercise. We prove:

$$M[G] \vDash \varphi(K_G(\tau_1), \ldots, K_G(\tau_n)) \text{ iff } \exists p \in G\, p \Vdash \varphi(\tau_1, \ldots, \tau_n).$$

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In applications, the hard part is designing the forcing notion that does what we want. We give a simple example.

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• **Collapsing cardinals** Let $\kappa > \omega$ be a cardinal in M. Force with $\mathrm{Fn}(\omega, \kappa)$. Then $\bigcup G$ is a total function from ω onto κ. So, κ is **not** a cardinal in M.

• **Adding many reals** Let $\kappa > \omega_1$ be a cardinal in M. Force with $\mathrm{Fn}(\kappa \times \omega, 2)$. Let G be generic. Then:

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Definition

P satisfies the ***countable antichain condition (c.a.c.)*** *if any antichain A in P is at most countable.*

Theorem

Suppose $P \in M$ and $M \vDash$ "P satisfies the c.a.c.". Then, for $\alpha \in M$

$$M[G] \vDash \alpha \text{ is a cardinal iff } M \vDash \alpha \text{ is a cardinal.}$$

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Given I and J, let $\mathrm{CPF}(I, J)$ set of **countable** partial functions from I to J. Let $p \leq q$ iff $q \subseteq p$.

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We need to check that $\omega_1^{M[G]} = \omega_1^M$ and that we did not add any reals.

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*P is **countably closed** if for any decreasing sequence $p_0 \geq p_1 \geq \dots$ there is q such that $q \leq p_n$, for all n.*

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If P is countably closed then $(2^\omega)^{M[G]} = (2^\omega)^M$ and $\omega_1^{M[G]} = \omega_1^M$.

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Negation of the Axiom of Choice

Question

If $M \models \mathrm{AC}$ then so does $M[G]$. So, how do we get a model of $\neg\mathrm{AC}$?

Sketch

Start with M, force with $\mathrm{Fn}(\omega, \omega)$ to get $M[G]$. Then define an intermediate model N, i.e. $M \subseteq N \subseteq M[G]$, such that $N \models \neg\mathrm{AC}$. N is a **symmetric model**, i.e. there is a group Σ in M acting on M^P and N is the set of all $K_G(\tau)$, for τ a P-name **invariant** under all $\sigma \in \Sigma$.

Details some other time....

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Thank You!