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joint work with

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DEGREE SPECTRA OF Π^0_1 CLASSES

Cantor Space

Underlying set: $\{0,1\}^\omega$.

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Topology:

Basic open sets are of the form

$$[\sigma] =$$

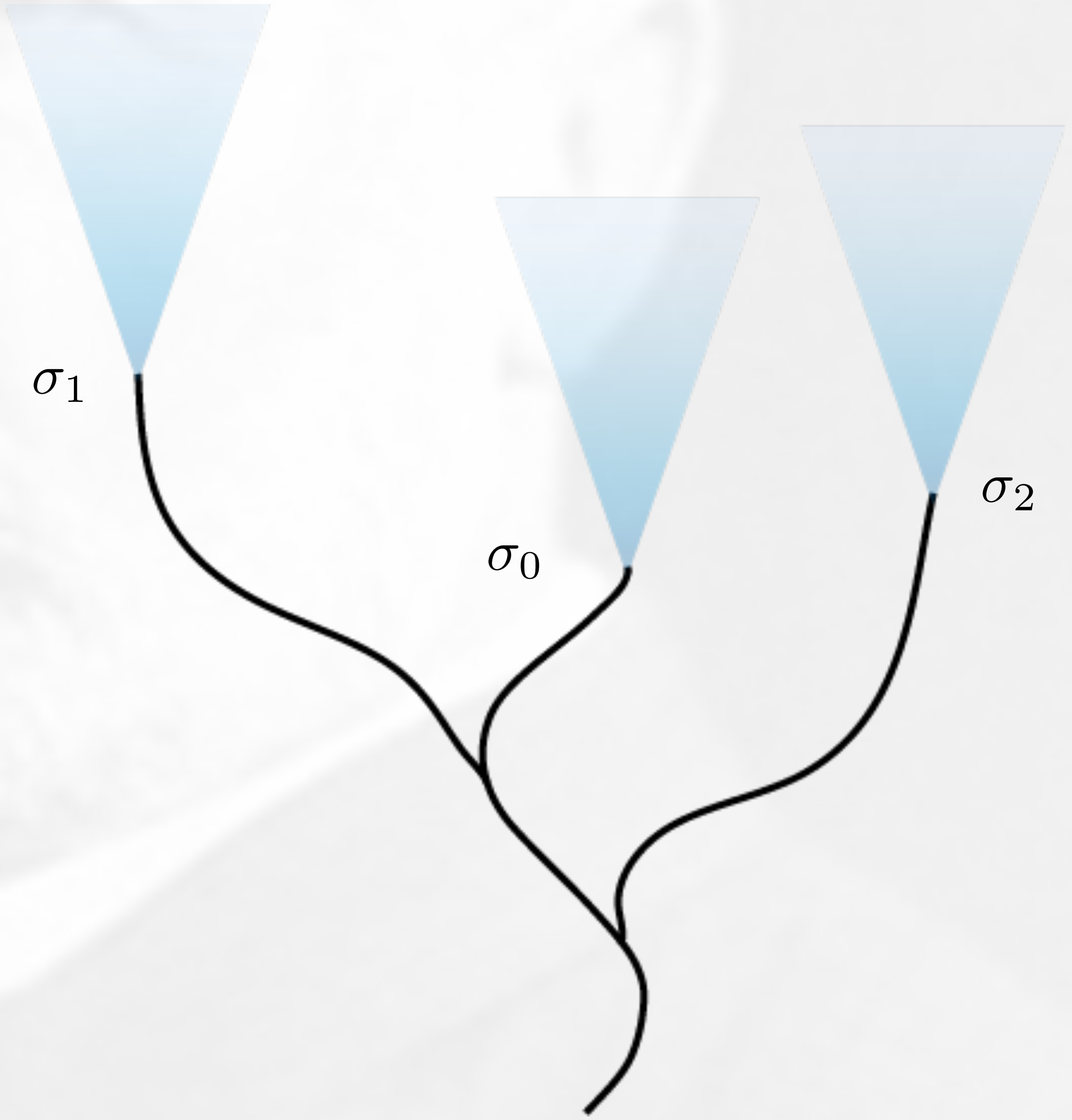
Cantor Space

Underlying set: $\{0,1\}^\omega$.

Topology:

Open sets are countable

unions of basic open sets



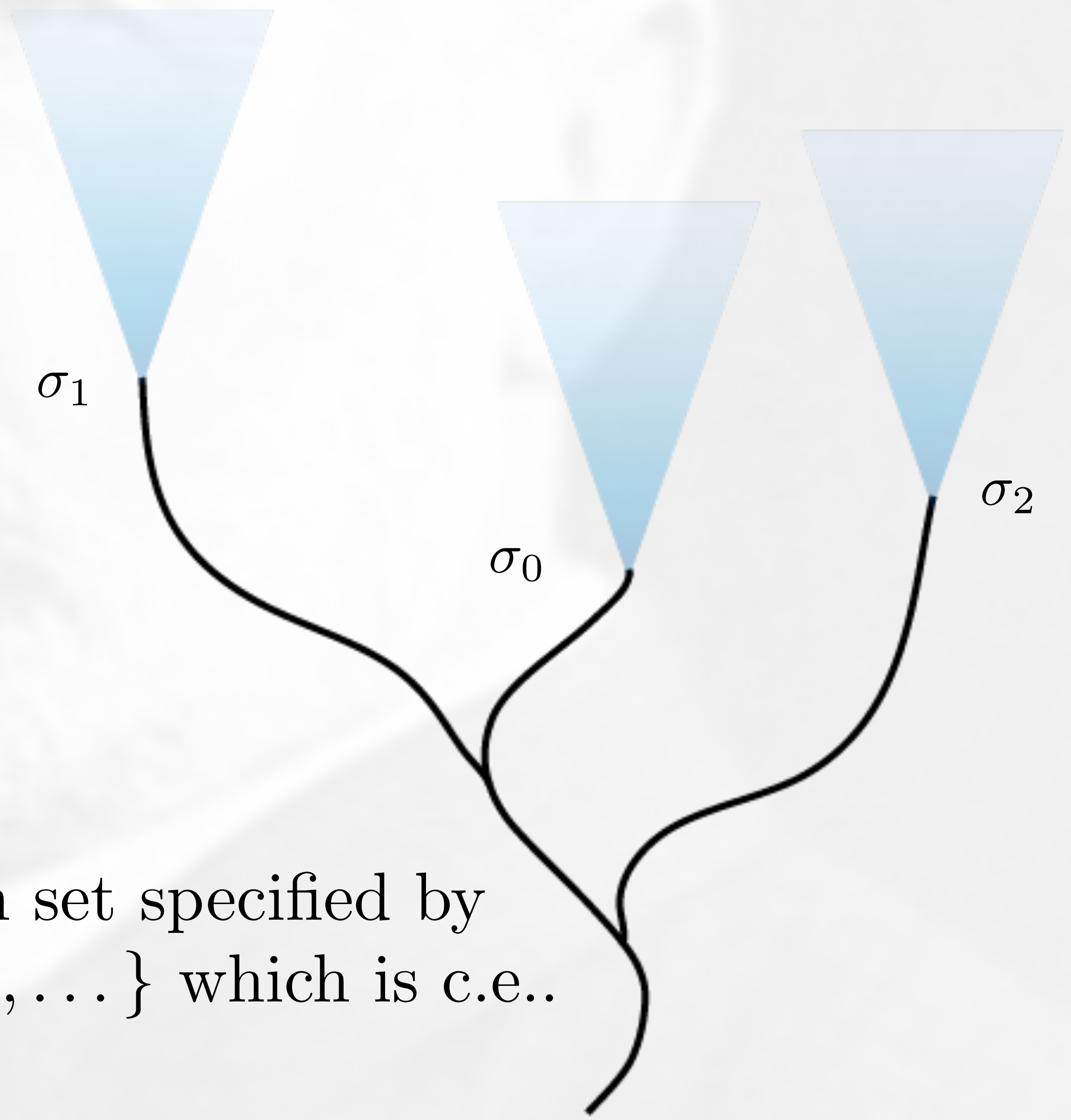
Cantor Space

Underlying set: $\{0,1\}^\omega$.

Topology:

A Σ^0_1 class (subset of Cantor space) is one defined by an existential formula:

$$X \in \mathcal{P} \leftrightarrow \exists n R[X, n].$$



Equivalently, a Σ^0_1 class is an open set specified by a set of defining strings $\{\sigma_0, \sigma_1, \sigma_2, \ldots\}$ which is c.e..

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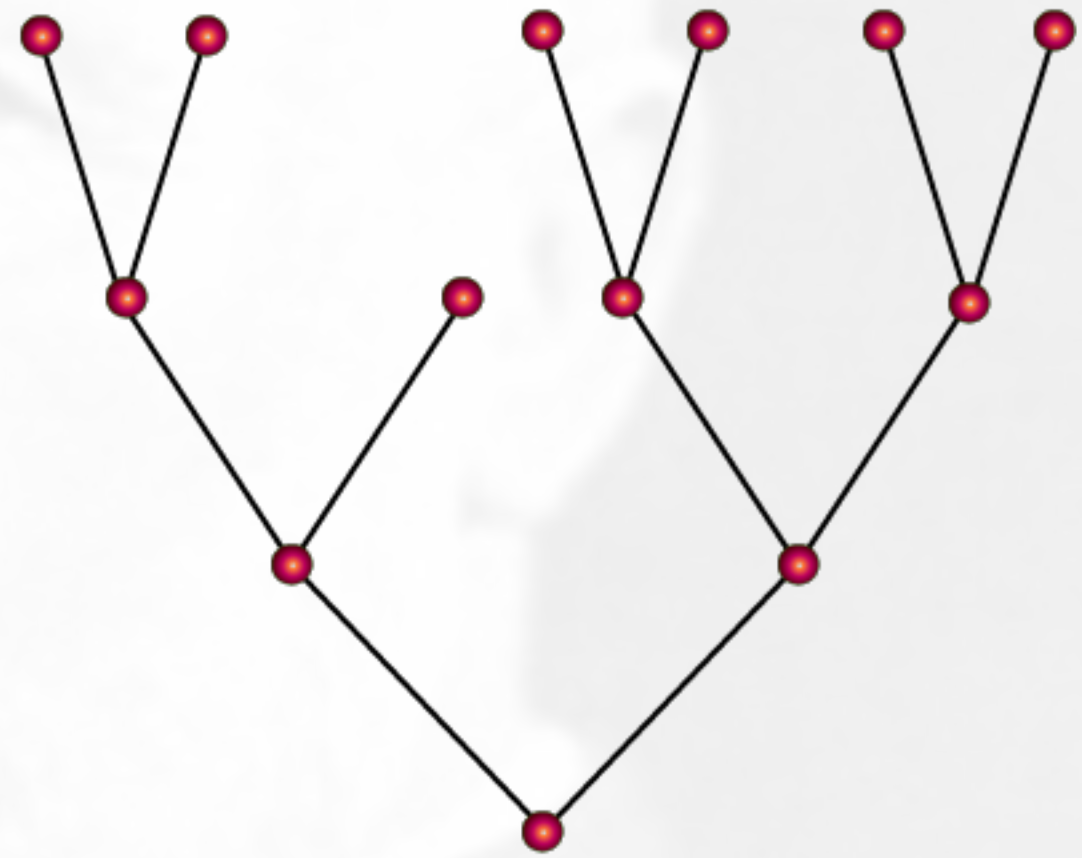

Topology:

A Π^0_1 class (subset of Cantor space) is one defined by a universal formula:

$$X \in \mathcal{P} \leftrightarrow \forall n R[X,n].$$

Equivalently, $\mathcal{P}$ is a Π^0_1 class iff it is the set of infinite paths through a downward closed and computable set of finite binary strings Λ.

5 BASIC FACTS

FACT 1 [Weak König's Lemma]. If $\Lambda \subseteq 2^{<\omega}$ is downward closed and infinite then there is an infinite path through Λ.

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FACT 4 [Low Basis Theorem]. We call a Turing degree $\boldsymbol{a}$ *low* if $\boldsymbol{a}' = \mathbf{0}'$. Every non-empty Π^0_1 class has a member of low degree.

Proving the Low Basis Theorem

We suppose $\mathcal{P}$ is non-empty and is the set of infinite paths through $\Lambda \subseteq 2^{<\omega}$ which is downward closed and computable.

We run a construction which makes use of an oracle for $\emptyset'$. At stage 0 we define $\Lambda_0 = \Lambda$, and at each stage $s+1$ we define $\Lambda_{s+1} \subseteq \Lambda_s$ such that $[\Lambda_{s+1}]$ (the set of infinite paths through Λ_{s+1}) is non-empty.

$$\Lambda_0 \supseteq \Lambda_1 \supseteq \Lambda_2 \supseteq \cdots$$

Ultimately we can then consider $\mathcal{Q} = \bigcap_s [\Lambda_s]$, which by compactness will be non-empty.

At stage $s+1$ we define Λ_{s+1} so as to decide whether $\Psi_s^A(s)\downarrow$ for all $A \in \mathcal{Q}$ once and for all. Since we compute A' for all $A \in \mathcal{Q}$ as the construction progresses (using only our oracle for $\emptyset'$), $\mathcal{Q}$ must have precisely one member, and this must be of low degree.

Proving the Low Basis Theorem

Convention: we shall suppose that $\Psi_s^\sigma(n)\downarrow$ only if this computation converges in at most $|\sigma|$ many steps.

Stage 0. Define $\Lambda_0 = \Lambda$.
Stage $s+1$. We are given Λ_s such that $[\Lambda_s]$ is non-empty. We ask:

"Does there exist an n such that $\Psi_s^\sigma(s)\downarrow$ for all $\sigma \in \Lambda_s$ of length n?"

If so: Then we can define $\Lambda_{s+1} = \Lambda_s$, and we already know that $s \in A'$ for all $A \in \mathcal{Q}$.
If not: Then we can define Λ_{s+1} to be all those strings σ in Λ_s for which $\Psi_s^\sigma(s)\uparrow$. Since there exist such σ of every length, it follows by WKL that $[\Lambda_{s+1}]$ is non-empty. In this case we have established that $s \notin A'$ for all $A \in \mathcal{Q}$.

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FACT 4 [Low Basis Theorem]. We call a Turing degree $\boldsymbol{a}$ *low* if $\boldsymbol{a}' = \boldsymbol{0}'$. Every non-empty Π^0_1 class has a member of low degree.

We say a degree $\boldsymbol{a}$ is *hyperimmune-free* if for every $f : \omega \to \omega$ computable in $\boldsymbol{a}$, there exists a computable function g which majorises f, i.e. such that $g(n) > f(n)$ for all n.

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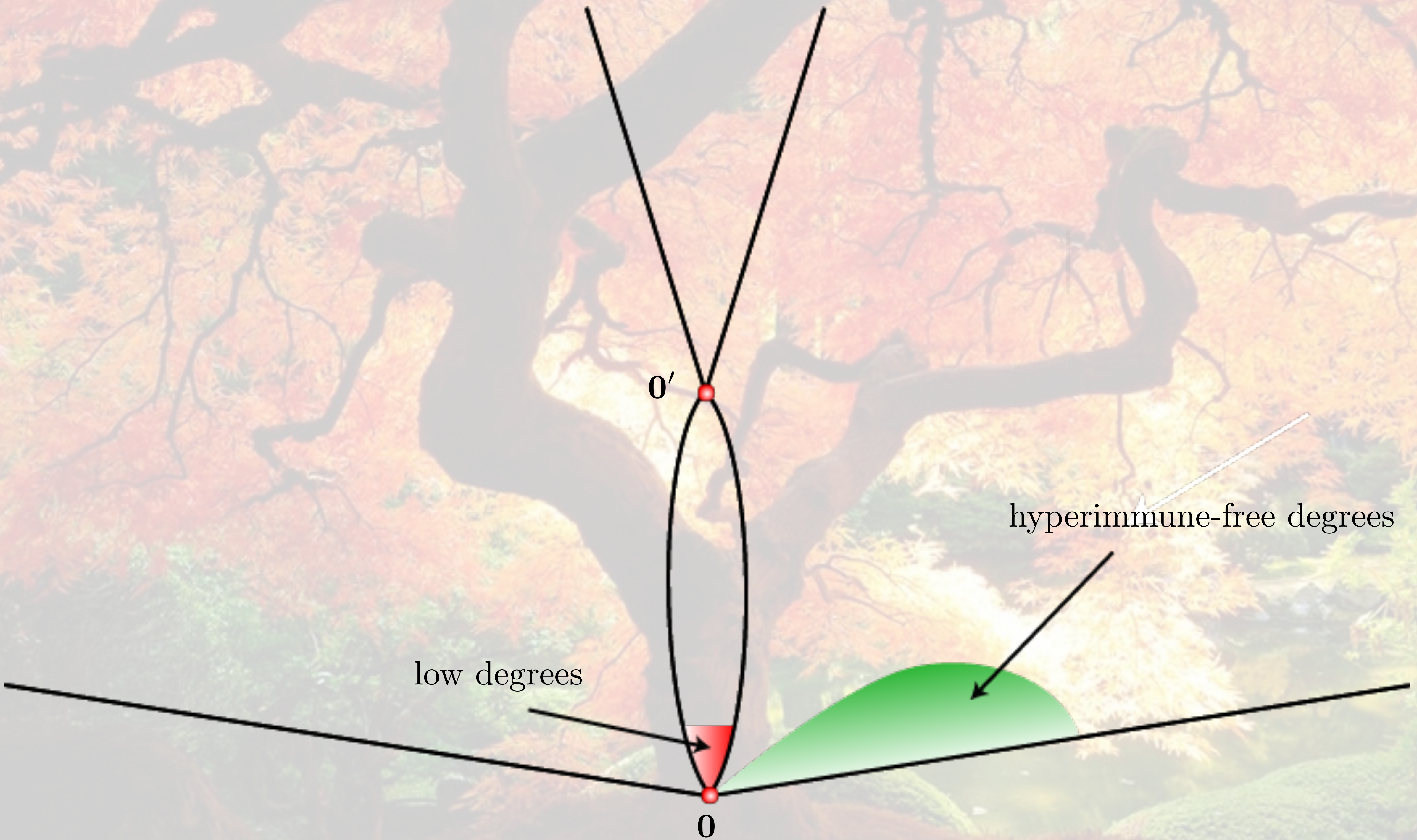
FACT 5 [Hyperimmune-free Basis Theorem]. Every non-empty Π^0_1 class has a member of hyperimmune-free degree.

The hyperimmune-free degrees



Defining the Structure

For $\mathcal{P} \subseteq 2^\omega$ we define $S(\mathcal{P})$, the *degree spectrum* of $\mathcal{P}$, to be the set of Turing degrees $\boldsymbol{a}$ such that there exists $A \in \mathcal{P}$ of degree $\boldsymbol{a}$.

We define $\mathfrak{P} = \{S(\mathcal{P}) : \mathcal{P} \text{ is a } \Pi^0_1 \text{ class}\}$ and we consider the elements of $\mathfrak{P}$ to be ordered by inclusion.

The following facts are easily derived:

(i) $(\mathfrak{P}, <)$ has a greatest element $\mathbf{1}_{\mathfrak{P}} = \mathfrak{D}$ and a least element $\mathbf{0}_{\mathfrak{P}} = \emptyset$.

(ii) $(\mathfrak{P}, <)$ is an uppersemilattice.

(iii) There is at least one minimal element $\{\mathbf{0}\}$.

Some easy examples of elements of the structure

The set of PA degrees is an element of the structure, as is the set of Martin-Löf random degrees.

Stephan has shown that the degrees which are PA and Martin-Löf random are precisely the degrees above $\mathbf{0}'$. This suffices to show that when $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$, their intersection $\boldsymbol{\alpha} \cap \boldsymbol{\beta}$ need not be a degree spectrum. This *does not* mean than $\boldsymbol{\alpha} \wedge \boldsymbol{\beta}$ doesn't exist though..

Question: Is $(\mathfrak{P}, <)$ a lattice?

The hf and low basis theorems combine to show that no non-trivial upper conner can be a degree spectrum. The question becomes more interesting when we add in $\mathbf{0}$: for which $\boldsymbol{a}$ is it the case that $\{\mathbf{0}\} \cup \{\boldsymbol{b} : \boldsymbol{b} \geq \boldsymbol{a}\}$ is a degree spectrum?

PROVING THAT FOR $\boldsymbol{a} \leq \mathbf{0}'$, $\{\mathbf{0}, \boldsymbol{a}\}$ IS A DEGREE SPECTRUM.

We construct $\mathcal{P} \subseteq \{0,1,2\}^\omega$. We construct Λ which is downward closed and computable – at stage s we decide which strings of length s are in Λ.

We suppose given an approximation $\{\sigma_s\}_{s\in\omega}$ to $A \in \boldsymbol{a}$ (and suppose σ_s is of length s). For any $\tau \in 3^{<\omega}$ we let $g(\tau)$ be the binary string obtained by removing all 2s. The idea is that we construct Λ so that all infinite paths B either end with infinitely many 2s, or else satisfy $g(B) = A$.

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such that $g(\tau) \subset \sigma_s$ is longest.

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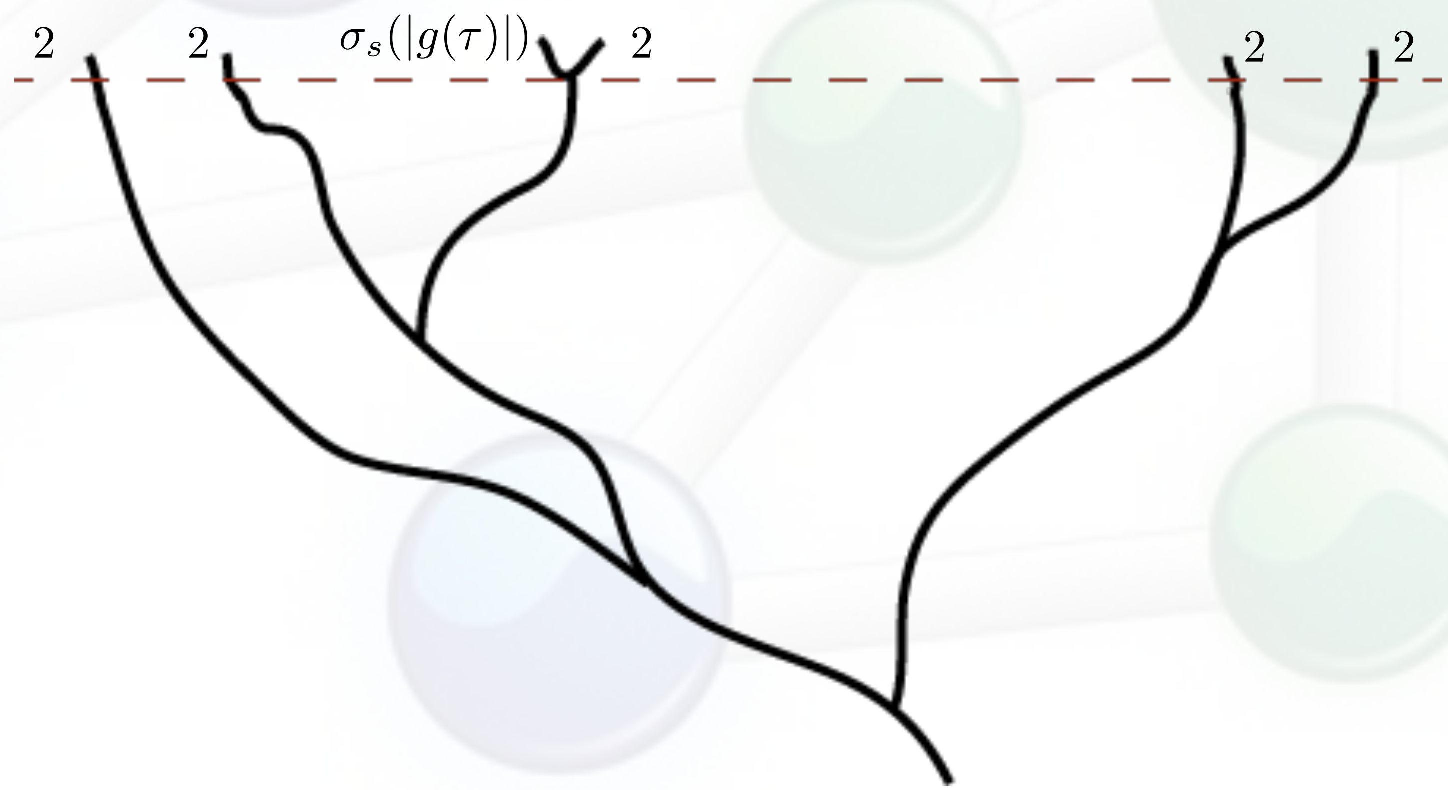

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Stage 0. Enumerate $\emptyset$ into Λ.
Stage $s > 0$. Choose that leaf τ such that amongst those leaves τ' with $g(\tau') \subset \sigma_s$, $g(\tau)$ is the longest. Enumerate $\tau * \sigma_s(g(|\tau|))$ and $\tau * 2$ into Λ. For every other leaf τ' enumerate $\tau' * 2$ into Λ.

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Here we used 2s as the 'filler', but for any Π^0_1 class $\mathcal{P}$, we could have used $\mathcal{P}$ as the filler instead, to give a new class with degree spectrum $S(\mathcal{P}) \cup \{\boldsymbol{a}\}$.

PROVING THAT FOR $\boldsymbol{a} \leq \mathbf{0}'$, $\{\mathbf{0}, \boldsymbol{a}\}$ IS A DEGREE SPECTRUM.

This proof is easily modified to show that for any $\boldsymbol{a} \leq \mathbf{0}'$, $\{\mathbf{0}\} \cup \{\boldsymbol{b} : \boldsymbol{b} \geq \boldsymbol{a}\}$ is a degree spectrum.

..rather than simply coding A into one path, we now have one coding path for each set of the form $A \oplus B$, where:

$$A \oplus B(2n) = A(n)$$

$$A \oplus B(2n+1) = B(n).$$

Some easy examples of elements of the structure

Recall that a degree is fixed point free iff it contains a DNC (diagonally non-computable) function, where f is DNC if for all n we have $f(n) \neq \Psi_n(n)$.

Simpson has shown that the fixed point free degrees are a degree spectrum.

Some easy examples of elements of the structure

The set of all c.e. degrees is the degree spectrum of a Π^0_1 class.

Proof. We let W_i denote the i^{th} c.e. set according to some fixed effective listing. Let f be a computable function such that, for any $i \in \omega$, $S([\Lambda_{f(i)}]) = \{\mathbf{0}, \boldsymbol{a}_i\}$ where $\boldsymbol{a}_i$ is the degree of W_i. Now let Λ be the computable set of strings which contains a copy of $\Lambda_{f(i)}$ above each string $0^i \star 1$, where 0^i is the sequence of i many zeros. $\square$

So here we have a degree spectrum $\boldsymbol{\alpha}$ which has non-empty intersection with every other non-empty degree spectrum $\boldsymbol{\beta}$. The analogue of this doesn't hold for the special degree spectra though..

Some easy examples of elements of the structure

For any special Π^0_1 class $\mathcal{P}_0$ there exists a special Π^0_1 class $\mathcal{P}_1$ such that no member of $\mathcal{P}_1$ computes any member of $\mathcal{P}_0$.

Proof. Suppose given downward closed and computable Λ such that $\mathcal{P}_0 = [\Lambda]$. We define an approximation to a 2-branching T such that $\mathcal{P}_1 = [T]$ satisfies the statement of the theorem. Those τ in T of level $2i+1$ will be defined so as to satisfy requirement:

Θ_i: If $A \in \mathcal{P}_1$ then $A \neq \Psi_i(\emptyset)$.

Those τ in T of level $2i+2$ will be defined so as to satisfy requirement:

Ξ_i: If $A \in \mathcal{P}_1$ and $\Psi_i(A)$ is total then $\Psi_i(A) \notin \mathcal{P}_0$.

Some easy examples of elements of the structure

Stage 0. Enumerate λ into T.
Stage $s > 0$. Consider all the strings in T to be ordered first according to their level in T and then lexicographically. Find the least string $\tau \in T$ (if any) such that either:

(1) τ is of level $2i+1$ and $\tau \subset \Psi_i(\emptyset)[s]$. In this case let τ_0 be the immediate predecessor of τ in T and let τ_1 be a leaf of T extending τ_0 and incompatible with τ. Remove all strings properly extending τ_0 from T and then enumerate in two incompatible extensions of τ_1.

(2) τ is of level $2i+2$, $\Psi_i(\tau)$ is compatible with some string in Λ of length s and there exists a leaf τ' of T extending τ such that $\Psi_i(\tau')$ properly extends $\Psi_i(\tau)$. In this case remove all strings extending (and including) τ from T, other than τ'.

Once these instructions are completed, choose two incompatible strings extending each leaf of T, and enumerate these strings into T.

The invisible degrees and cupping

Next we want to want to answer questions of the form, is it the case for every $\boldsymbol{\alpha} > \mathbf{0}_{\mathfrak{P}}$ there exists $\boldsymbol{\beta} < \mathbf{1}_{\mathfrak{P}}$ with $\boldsymbol{\alpha} \vee \boldsymbol{\beta} = \mathbf{1}_{\mathfrak{P}}$? Do there exist any $\boldsymbol{\alpha}, \boldsymbol{\beta} < \mathbf{1}_{\mathfrak{P}}$ with $\boldsymbol{\alpha} \vee \boldsymbol{\beta} = \mathbf{1}_{\mathfrak{P}}$?

We'll be able to answer this by establishing the existence of *invisible* degrees – degrees which don't belong to any spectrum other than $\mathbf{1}_{\mathfrak{P}}$.

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The principal theorem here is this:

For any countable sequence of F_σ sets, $\{\mathcal{Q}_k\}_{k\in\omega}$ say, there exists a degree $\mathbf{a}$ such that, for any $k \in \omega$, if $\boldsymbol{a} \in S(\mathcal{Q}_k)$ then $S(\mathcal{Q}_k) = \mathfrak{D}$. In fact $\boldsymbol{a}$ can be chosen to be hyperimmune-free and minimal.

For any countable sequence of F_σ sets, $\{\mathcal{Q}_k\}_{k\in\omega}$ say, there exists a degree **a** such that, for any $k \in \omega$, if $\boldsymbol{a} \in S(\mathcal{Q}_k)$ then $S(\mathcal{Q}_k) = \mathfrak{D}$. In fact $\boldsymbol{a}$ can be chosen to be hyperimmune-free and minimal.

Proof. We will in fact show that the conclusion holds for a countable sequence of closed sets. That this is sufficient to imply the full theorem is immediate since any F_σ set is the union of a countable sequence of closed sets. More specifically, let $\{\mathcal{Q}_k\}_{k\in\omega}$ be a sequence of F_σ sets, with $\mathcal{Q}_k = \bigcup_{j\in\omega} \mathcal{P}_{k,j}$ with $\mathcal{P}_{k,j}$ closed for each $k, j \in \omega$, and let **a** be such that if $\boldsymbol{a} \in S(\mathcal{P}_{k,j})$ then $S(\mathcal{P}_{k,j}) = \mathfrak{D}$. If $\boldsymbol{a} \in S(\mathcal{Q}_k)$ then there exists a j such that $\boldsymbol{a} \in S(\mathcal{P}_{k,j})$, giving $S(\mathcal{Q}_k) = \mathfrak{D}$ as required. So suppose we are given a sequence $\{\mathcal{P}_k\}_{k\in\omega}$ of closed sets.

For any countable sequence of F_σ sets, $\{\mathcal{Q}_k\}_{k\in\omega}$ say, there exists a degree **a** such that, for any $k \in \omega$, if $\boldsymbol{a} \in S(\mathcal{Q}_k)$ then $S(\mathcal{Q}_k) = \mathfrak{D}$. In fact $\boldsymbol{a}$ can be chosen to be hyperimmune-free and minimal.

Proof (continued) .. so now we have to construct A, such that no $\mathcal{P}_k$ has a member of degree $\boldsymbol{a} = \deg(A)$ unless it has a member of every degree. This means that sometimes we will have to be able to demonstrate that $\mathcal{P}_k$ does contain a member of every degree..

..it suffices to show that $[T] \subseteq \mathcal{P}_k$ for some 2-branching computable $T \subseteq 2^{<\omega}$.

For future reference, in order to show that $\mathcal{P}_k$ contains a member of every degree above $\boldsymbol{b}$ (for some $\boldsymbol{b}$) it also suffices to show that $[T] \subseteq \mathcal{P}_k$ for some 2-branching $\boldsymbol{b}$ computable pointed $T \subseteq 2^{<\omega}$.

For any countable sequence of F_σ sets, $\{\mathcal{Q}_k\}_{k\in\omega}$ say, there exists a degree **a** such that, for any $k \in \omega$, if $\boldsymbol{a} \in S(\mathcal{Q}_k)$ then $S(\mathcal{Q}_k) = \mathfrak{D}$. In fact $\boldsymbol{a}$ can be chosen to be hyperimmune-free and minimal.

Proof (continued) For every k, let $\mathcal{P}_k = [\Upsilon_k]$ for some downward closed $\Upsilon_k \subseteq 2^{<\omega}$. We shall use a simple forcing argument in order to construct $A = \bigcup_s \sigma_s$ which is of hyperimmune-free minimal degree. Initially we define $\sigma_0 = \lambda$ and we define T_0 to be the identity tree. At every stage s, given computable 2-branching T_s and σ_s which is the string of level 0 in T_s, we define $\sigma_{s+1} \supset \sigma_s$ and $T_{s+1} \subset T_s$.

For each $\langle i, j, k\rangle$ we must ensure:

If Ψ_i^A is total and belongs to $\mathcal{P}_k$ and if $\Psi_j(\Psi_i^A) = A$ then $\mathcal{P}_k$ contains a member of every degree.

For any countable sequence of F_σ sets, $\{\mathcal{Q}_k\}_{k\in\omega}$ say, there exists a degree **a** such that, for any $k \in \omega$, if $\boldsymbol{a} \in S(\mathcal{Q}_k)$ then $S(\mathcal{Q}_k) = \mathfrak{D}$. In fact $\boldsymbol{a}$ can be chosen to be hyperimmune-free and minimal.

Proof (continued) At stage s we act according to the first of the following situations which applies. Let $s = \langle i, j, k \rangle$.

(1) There exists $\sigma \in T_s$ such that no two strings extending σ in T_s are Ψ_i-splitting. In this case we define σ_{s+1} to be the first such σ properly extending σ_s and we define T_{s+1} to be the set of strings in T_s extending σ. In so doing we have ensured that $\Psi_i(A)$ is either computable or partial.

For any countable sequence of F_σ sets, $\{\mathcal{Q}_k\}_{k\in\omega}$ say, there exists a degree **a** such that, for any $k \in \omega$, if $\boldsymbol{a} \in S(\mathcal{Q}_k)$ then $S(\mathcal{Q}_k) = \mathfrak{D}$. In fact $\boldsymbol{a}$ can be chosen to be hyperimmune-free and minimal.

Proof (continued) At stage s we act according to the first of the following situations which applies. Let $s = \langle i, j, k \rangle$.

(2) Since the previous case does not apply we may let T_s^0 be a computable, 2-branching and Ψ_i-splitting subset of T_s containing σ_s (we shall eventually define T_{s+1} to be a subset of T_s^0). There exists $\sigma \in T_s^0$ such that either $\Psi_j(\Psi_i(\sigma))$ is incompatible with σ or else for no string $\tau \supset \sigma$ in T_s^0 is it the case that $\Psi_j(\Psi_i(\tau))$ properly extends $\Psi_j(\Psi_i(\sigma))$. In this case we define σ_{s+1} to be the first such σ properly extending σ_s and we define T_{s+1} to be the set of strings in T_s^0 extending σ. In so doing we have ensured it is not the case that $\Psi_j(\Psi_i(A)) = A$.

For any countable sequence of F_σ sets, $\{\mathcal{Q}_k\}_{k\in\omega}$ say, there exists a degree $\mathbf{a}$ such that, for any $k \in \omega$, if $\boldsymbol{a} \in S(\mathcal{Q}_k)$ then $S(\mathcal{Q}_k) = \mathfrak{D}$. In fact $\boldsymbol{a}$ can be chosen to be hyperimmune-free and minimal.

Proof (continued) At stage s we act according to the first of the following situations which applies. Let $s = \langle i, j, k \rangle$.

(3) Since the previous case does not apply we may let T_s^1 be a computable and 2-branching subset of T_s^0 containing σ_s such that, whenever $\tau, \tau' \in T_s^1$ and $\tau' \supset \tau$, we have that $\Psi_j(\Psi_i(\tau'))$ properly extends $\Psi_j(\Psi_i(\tau))$. There exists $\sigma \in T_s^1$ such that $\Psi_i(\sigma)$ is not in Υ_k. In this case we define σ_{s+1} to be the first such σ properly extending σ_s and we define T_{s+1} to be the set of strings in T_s^1 extending σ. In so doing we have ensured that $\Psi_i(A) \notin \mathcal{P}_k$.

For any countable sequence of F_σ sets, $\{\mathcal{Q}_k\}_{k\in\omega}$ say, there exists a degree **a** such that, for any $k \in \omega$, if $\boldsymbol{a} \in S(\mathcal{Q}_k)$ then $S(\mathcal{Q}_k) = \mathfrak{D}$. In fact $\boldsymbol{a}$ can be chosen to be hyperimmune-free and minimal.

Proof (continued) At stage s we act according to the first of the following situations which applies. Let $s = \langle i, j, k \rangle$.

(4) Since none of the previous cases apply we have that, for all $B \in [T_s^1]$, $\Psi_i(B) \in \mathcal{P}_k$ and $\Psi_j(\Psi_i(B)) = B$. Since T_s^1 is 2-branching and computable it follows that $\mathcal{P}_s$ contains a member of every Turing degree. In this case we define σ_{s+1} to be some proper extension of σ_s in T_s^1 and we define T_{s+1} to be the set of strings in T_s^1 which extend σ_{s+1}.

The invisible degrees and cupping

If we restrict ourselves to the arithmetical hierarchy, then we can do a little better:

There exists a hyperimmune-free minimal degree $\boldsymbol{a}$ below $\mathbf{0}''$, such that no Σ^0_3 class contains a member of degree $\boldsymbol{a}$ unless it contains a member of every degree.

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If we analyse these proofs a little further, then we reach some interesting conclusions:

An F_σ set $\mathcal{Q}$ contains a member of every degree above $\boldsymbol{b}$ iff there exists some $\boldsymbol{b}$-computable perfect and pointed T with $[T] \subseteq \mathcal{Q}$.

A Σ^0_3 class $\mathcal{P}$ contains a member of every degree above $\boldsymbol{b}$ iff there exists some $\boldsymbol{b}$-computable perfect and pointed T with $[T] \subseteq \mathcal{P}$.

The invisible degrees and cupping

An F_σ set $\mathcal{Q}$ contains a member of every degree above $\boldsymbol{b}$ iff there exists some $\boldsymbol{b}$-computable perfect and pointed T with $[T] \subseteq \mathcal{Q}$.

Proof. Let $\mathcal{Q} = \bigcup_k \mathcal{P}_k$ where each $\mathcal{P}_k$ is closed. Given B of degree $\boldsymbol{b}$, consider running the previous construction but beginning with T_0 as the set of strings of the form $\tau_0 \oplus \tau_1$ such that $\tau_0 \subset B$—so that all the T_s are now computable in B. Let the set A constructed be of degree $\boldsymbol{a}$. If $\mathcal{Q}$ contains a member of every degree above $\boldsymbol{b}$ then, in particular, some $\mathcal{P}_k$ must contain a member of degree $\boldsymbol{a}$. Therefore, for some $s = \langle i, j, k \rangle$ it must be that case (4) applies. Then $T = \{\Psi_i(\sigma) : \ \sigma \in T_{s+1}\}$ is B-computable perfect and pointed, and $[T] \subseteq \mathcal{Q}$. $\square$

The invisible degrees and cupping

We say that a degree $\mathbf{a}$ is *invisible* if any Π^0_1 class which contains a member of degree $\mathbf{a}$ contains a member of every degree.

The existence of invisible degrees immediately suffices to give the following corollary: There do not exist $\boldsymbol{\alpha} < \mathbf{1}_{\mathfrak{P}}$ and $\boldsymbol{\beta} < \mathbf{1}_{\mathfrak{P}}$ with $\boldsymbol{\alpha} \vee \boldsymbol{\beta} = \mathbf{1}_{\mathfrak{P}}$.

Proof. If neither of $\boldsymbol{\alpha}$ or $\boldsymbol{\beta}$ contain any invisible degrees then neither does their union. $\square$

The invisible degrees and cupping

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Proof. If neither of $\boldsymbol{\alpha}$ or $\boldsymbol{\beta}$ contain any invisible degrees then neither does their union. $\square$

All weakly 2-generics are invisible. In fact they are all strongly invisible:

We say $\boldsymbol{a}$ is *strongly invisible* if any Π^0_1 class which has a member of non-zero degree below $\boldsymbol{a}$ has a member of every degree.

The invisible degrees and cupping

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QUESTION: Are all invisible degrees strongly invisible? This holds iff the invisible degrees are downward closed avoiding $\mathbf{0}$.

The invisible degrees and cupping

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In order to answer this, we need to consider a third category of invisibility (and then things will work out nicely):

We say that a degree $\mathbf{a}$ is *weakly invisible* if any Π^0_1 class which contains a member of every degree above $\mathbf{a}$ contains a member of every degree.

The invisible degrees and cupping

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We say that a degree $\mathbf{a}$ is *weakly invisible* if any Π^0_1 class which contains a member of every degree above $\mathbf{a}$ contains a member of every degree.

There exists a non-zero weakly invisible degree which is not invisible.

The invisible degrees and cupping

A degree is weakly invisible iff it is bounded by an invisible degree.

Proof. Any degree bounded by an invisible degree is clearly weakly invisible, so it suffices to show that any weakly invisible degree is bounded by an invisible degree. In order to see this suppose given B of degree $\boldsymbol{b}$ which is weakly invisible, let $\mathcal{P}_k$ be the k^{th} Π^0_1 class and consider running the construction of an invisible degree but beginning with T_0 as the set of strings of the form $\tau_0 \oplus \tau_1$ such that $\tau_0 \subset B$. Now whenever case (4) applies we may deduce that $\mathcal{P}_k$ contains $[T]$ for some perfect pointed $\boldsymbol{b}$-computable T. Since $\boldsymbol{b}$ is weakly invisible, $\mathcal{P}_k$ therefore contains a member of every degree. $\square$

..so there exist invisible degrees which are not strongly invisible.

Minimal Elements

We say that $\boldsymbol{\alpha} \neq \mathbf{0}_{\mathfrak{P}}$ is subclass invariant if for any Π^0_1 class $\mathcal{P}$ with $S(\mathcal{P}) = \boldsymbol{\alpha}$ and any nonempty Π^0_1 class $\mathcal{P}' \subseteq \mathcal{P}$, $S(\mathcal{P}') = \boldsymbol{\alpha}$. We say that $\boldsymbol{\alpha} \neq \mathbf{0}_{\mathfrak{P}}$ is weakly subclass invariant if there exists a Π^0_1 class $\mathcal{P}$ with $S(\mathcal{P}) = \boldsymbol{\alpha}$ and for any nonempty Π^0_1 class $\mathcal{P}' \subseteq \mathcal{P}$, $S(\mathcal{P}') = \boldsymbol{\alpha}$.

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Clearly any $\boldsymbol{\alpha}$ which is minimal must be subclass invariant. Now suppose that $\boldsymbol{\alpha}$ is subclass invariant, let $\mathcal{P}$ be a Π^0_1 class with $S(\mathcal{P}) = \boldsymbol{\alpha}$ and suppose that $\mathcal{P}'$ is a nonempty Π^0_1 class with $S(\mathcal{P}') \subset \boldsymbol{\alpha}$. Then $\{0 \star A : A \in \mathcal{P}\} \cup \{1 \star A : A \in \mathcal{P}'\}$ is a Π^0_1 class with degree spectrum $\boldsymbol{\alpha}$ and witnesses the fact that $\boldsymbol{\alpha}$ is not subclass invariant, a contradiction. Thus being subclass invariant is equivalent to minimality.

We say that $\boldsymbol{\alpha} \neq \mathbf{0}_{\mathfrak{P}}$ is subclass invariant if for any Π^0_1 class $\mathcal{P}$ with $S(\mathcal{P}) = \boldsymbol{\alpha}$ and any nonempty Π^0_1 class $\mathcal{P}' \subseteq \mathcal{P}$, $S(\mathcal{P}') = \boldsymbol{\alpha}$. We say that $\boldsymbol{\alpha} \neq \mathbf{0}_{\mathfrak{P}}$ is weakly subclass invariant if there exists a Π^0_1 class $\mathcal{P}$ with $S(\mathcal{P}) = \boldsymbol{\alpha}$ and for any nonempty Π^0_1 class $\mathcal{P}' \subseteq \mathcal{P}$, $S(\mathcal{P}') = \boldsymbol{\alpha}$.

Suppose that $\boldsymbol{\alpha}$ is weakly subclass invariant. If a Π^0_1 class contains any member of any hyperimmune-free degree in $\boldsymbol{\alpha}$ then it contains a member of every degree in $\boldsymbol{\alpha}$.

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Suppose that $\boldsymbol{\alpha}$ is weakly subclass invariant. If a Π^0_1 class contains any member of any hyperimmune-free degree in $\boldsymbol{\alpha}$ then it contains a member of every degree in $\boldsymbol{\alpha}$.

Proof. Let $\mathcal{P}_0$ be a Π^0_1 class with $S(\mathcal{P}_0) = \boldsymbol{\alpha}$ and such that for any nonempty Π^0_1 class $\mathcal{P} \subseteq \mathcal{P}_0$, $S(\mathcal{P}) = \boldsymbol{\alpha}$. Let $\mathcal{P}_1$ be a Π^0_1 class which contains A of hyperimmune-free degree in $\boldsymbol{\alpha}$. Then there exists $B \equiv_{tt} A$ and which is in $\mathcal{P}_0$. Let i, j be such that the total functionals Φ_i and Φ_j satisfy $\Phi_i(A) = B$ and $\Phi_j(B) = A$. Now let $\mathcal{P}_2$ be the Π^0_1 class $\{C : C \in \mathcal{P}_0 \ \&\ (\exists D \in \mathcal{P}_1)[\Phi_i(D) = C \ \&\ \Phi_j(C) = D]\}$. Then $S(\mathcal{P}_2) \subseteq S(\mathcal{P}_1)$. Since $\mathcal{P}_2$ is nonempty and $\mathcal{P}_2 \subseteq \mathcal{P}_0$, $S(\mathcal{P}_2)$ and $S(\mathcal{P}_1)$ contain every degree in $\boldsymbol{\alpha}$. $\square$

We say that $\boldsymbol{\alpha} \neq \mathbf{0}_{\mathfrak{P}}$ is subclass invariant if for any Π^0_1 class $\mathcal{P}$ with $S(\mathcal{P}) = \boldsymbol{\alpha}$ and any nonempty Π^0_1 class $\mathcal{P}' \subseteq \mathcal{P}$, $S(\mathcal{P}') = \boldsymbol{\alpha}$. We say that $\boldsymbol{\alpha} \neq \mathbf{0}_{\mathfrak{P}}$ is weakly subclass invariant if there exists a Π^0_1 class $\mathcal{P}$ with $S(\mathcal{P}) = \boldsymbol{\alpha}$ and for any nonempty Π^0_1 class $\mathcal{P}' \subseteq \mathcal{P}$, $S(\mathcal{P}') = \boldsymbol{\alpha}$.

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Suppose that $\boldsymbol{\alpha}$ is weakly subclass invariant. If a Π^0_1 class contains any member of any hyperimmune-free degree in $\boldsymbol{\alpha}$ then it contains a member of every degree in $\boldsymbol{\alpha}$.

$\boldsymbol{\alpha}$ is minimal iff it is weakly subclass invariant.

Proof. If $\boldsymbol{\alpha}$ is weakly subclass invariant then, by the hyperimmune-free basis theorem, any nonempty Π^0_1 class which contains only members of degree in $\boldsymbol{\alpha}$ contains a member of hyperimmune-free degree in $\boldsymbol{\alpha}$ and therefore, by the previous result contains a member of every degree in $\boldsymbol{\alpha}$. $\square$

Minimal Elements

This immediately gives us nice examples of minimal elements..

The Martin-Löf random degrees are a minimal element of the structure.

Proof. Any nonempty Π^0_1 class containing only random sets is witness to the fact that this class is weakly subclass invariant since any Π^0_1 class containing a random set is of positive measure and any Π^0_1 class of positive measure contains a member of every random degree. $\square$

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..and so are the PA degrees.

Minimal Elements

Any non-empty Π^0_1 class containing only $\{0,1\}$-valued DNC functions contains a member of every PA degree.

Proof. If Λ is computable and downward closed then consider $\Psi_i(\emptyset)$ such that $\Psi_i(\emptyset; i) \downarrow= n$ iff there exists some $l > i$ such that $\tau(i) = n$ for all $\tau \in \Lambda$ of length l. By the uniformity of the recursion theorem it follows that there exists computable f such that, whenever $[\Lambda_j]$ is non-empty and contains only $\{0,1\}$-valued DNC functions, there exist $A, B \in [\Lambda_j]$ with $A(f(j)) = 0$ and $B(f(j)) = 1$.

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Now suppose given j_0 such that $[\Lambda_{j_0}]$ is non-empty and contains only $\{0,1\}$-valued DNC functions. Let A be a $\{0,1\}$-valued DNC function. We construct $B = \bigcup_s \sigma_s$ which is in $[\Lambda_{j_0}]$ and is of the same degree as A.

Minimal Elements

Stage 0. Define $\sigma_0 = \lambda$.

Stage $s > 0$. We have already decided j_{s-1} and σ_{s-1}. There exists $C \in [\Lambda_{j_{s-1}}]$ with $C(f(j_{s-1})) = A(s-1)$. Using the oracle for A we can therefore compute σ of length $f(j_{s-1})+1$ such that $\sigma(f(j_{s-1})) = A(s-1)$ and which is an initial segment of some $C \in [\Lambda_{j_{s-1}}]$ (this follows using the standard argument that any $\{0,1\}$-valued DNC function computes a member of any non-empty Π^0_1 class). Define $\sigma_s = \sigma$ and define j_s so that $[\Lambda_{j_s}]$ is the set of all $C \in [\Lambda_{j_{s-1}}]$ which extend σ.

That B computes A follows from the fact that an oracle for B allows us to retrace every step of the construction defining B.

$\square$

Minimal Elements

So now we have a number of minimal elements..but are there infinitely many? For every $\boldsymbol{\alpha}$, does there exist some $\boldsymbol{\beta}$ such that $\boldsymbol{\alpha} \wedge \boldsymbol{\beta} = \mathbf{0}_{\mathfrak{P}}$?

For any $\boldsymbol{\alpha} < \mathbf{1}_{\mathfrak{P}}$ there exists $\boldsymbol{\beta}$ which is minimal and such that $\boldsymbol{\beta} \not\leq \boldsymbol{\alpha}$.

Minimal Elements

So now we have a number of minimal elements..but are there infinitely many? For every $\boldsymbol{\alpha}$, does there exist some $\boldsymbol{\beta}$ such that $\boldsymbol{\alpha} \wedge \boldsymbol{\beta} = \mathbf{0}_{\mathfrak{P}}$?

For any $\boldsymbol{\alpha} < \mathbf{1}_{\mathfrak{P}}$ there exists $\boldsymbol{\beta}$ which is minimal and such that $\boldsymbol{\beta} \not\leq \boldsymbol{\alpha}$.

Therefore $(\mathfrak{P}, <)$ has an infinite number of minimal elements.

Proof. Suppose that $(\mathfrak{P}, <)$ has only a finite number of minimal elements, $\boldsymbol{\alpha}_0, \cdots, \boldsymbol{\alpha}_k$ say. Then $\boldsymbol{\alpha}_0 \cup \cdots \cup \boldsymbol{\alpha}_k$ is an element $\boldsymbol{\alpha}$ of $\mathfrak{P}$ and by the previous result there exists $\boldsymbol{\beta}$ which is minimal and such that $\boldsymbol{\beta} \not\leq \boldsymbol{\alpha}$, a contradiction. $\square$

For any $\boldsymbol{\alpha} < \mathbf{1}_{\mathfrak{P}}$ there exists $\boldsymbol{\beta}$ which is minimal and such that $\boldsymbol{\beta} \not\leq \boldsymbol{\alpha}$.

Some comments on the proof. First of all, how will we build $\mathcal{P}$ whose degree spectrum is minimal?

For any $\boldsymbol{\alpha} < \mathbf{1}_{\mathfrak{P}}$ there exists $\boldsymbol{\beta}$ which is minimal and such that $\boldsymbol{\beta} \not\leq \boldsymbol{\alpha}$.

Some comments on the proof. First of all, how will we build $\mathcal{P}$ whose degree spectrum is minimal?

We say that T is *homogenous* if all strings of the same level in T are of the same length and, whenever σ_0 and σ_1 are of the same level in T and $\sigma_0 \star \tau \in T$, $\sigma_1 \star \tau$ is also in T. Recall that a Π^0_1 class $\mathcal{P}$ is *thin* if for any Π^0_1 class $\mathcal{P}' \subset \mathcal{P}$ there exists a clopen set $\mathcal{Q}$ such that $\mathcal{P}' = \mathcal{P} \cap \mathcal{Q}$.

So in order to construct $\mathcal{P}$ whose degree spectrum is minimal, we ensure that $\mathcal{P}$ is thin and that $\mathcal{P} = [T]$ for some $T \subseteq 2^{<\omega}$ which is homogenous.

For any $\boldsymbol{\alpha} < \mathbf{1}_{\mathfrak{P}}$ there exists $\boldsymbol{\beta}$ which is minimal and such that $\boldsymbol{\beta} \not\leq \boldsymbol{\alpha}$.

Some comments on the proof. Then we also have to ensure that $\boldsymbol{\beta} \not\leq \boldsymbol{\alpha}$. We suppose we are given downward closed and computable Λ such that $[\Lambda]$ has degree spectrum $\boldsymbol{\alpha}$. We ensure $\mathcal{P}$ contains a non-computable A such that the following requirements are satisfied:

Ξ_i: If $\Psi_i(A)$ is total and non-computable then $\Psi_i(A) \notin [\Lambda]$.

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A difficulty with this is that it requires us to search for splittings inside the Π^0_1 class that we are building. It is quite possible to have a Π^0_1 class $\mathcal{Q}$ and $A \in \mathcal{Q}$ such that $\Psi_i(A)$ is total and non-computable, but such that no two initial segments of elements of $\mathcal{Q}$ are Ψ_i-splitting.

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The solution we adopt is to construct A which is of hyperimmune-free degree. Then if $\Psi_i(A) = B$, there exists a total Turing functional Φ such that $\Phi(A) = B$. The totality of Φ means that $\{\Phi(C) : C \in \mathcal{Q}\}$ is a Π^0_1 class, and so cannot consist of a single non-computable member.

Π^0_1 MATES

The existence of invisible degrees serves to highlight the fact that there exist degrees $\boldsymbol{a} \neq \boldsymbol{b}$ such that any Π^0_1 class which contains a member of degree $\boldsymbol{a}$ must also contain a member of degree $\boldsymbol{b}$. Let's investigate this idea further:

We define the Π^0_1 mates of $\boldsymbol{a}$ to be $\bigcap\{\boldsymbol{\alpha} \in \mathfrak{P} \ : \ \boldsymbol{a} \in \boldsymbol{\alpha}\}$.

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We define the Π^0_1 mates of $\boldsymbol{a}$ to be $\bigcap\{\boldsymbol{\alpha} \in \mathfrak{P} \ : \ \boldsymbol{a} \in \boldsymbol{\alpha}\}$.

What could the Π^0_1 mates of $\boldsymbol{a}$ look like?

$\mathfrak{D}$ $\qquad\qquad$ $\{\boldsymbol{a}\}$ $\qquad\qquad$ $\{\mathbf{0}, \boldsymbol{a}\}$

Π^0_1 MATES

The following theorem provides a wealth of further examples...

For any $\boldsymbol{\alpha} < \mathbf{1}_{\mathfrak{P}}$ there exists $\boldsymbol{a} \notin \boldsymbol{\alpha}$ such that the Π^0_1-mates of $\boldsymbol{a}$ are $\boldsymbol{\alpha} \cup \{\boldsymbol{a}\}$ and, moreover, such that $\boldsymbol{\alpha} \cup \{\boldsymbol{a}\} \in \mathfrak{P}$.

This tells us that every element of the structure has a strong minimal cover.

Sufficiency sets and anti-basis theorems

We say that $\boldsymbol{\alpha}$ is a *sufficiency set* for $\boldsymbol{a}$ if every Π^0_1 class that contains a member of every degree in $\boldsymbol{\alpha}$ also contains a member of degree $\boldsymbol{a}$.

What can we say about these sufficiency sets?

- If $\boldsymbol{\alpha}$ is a sufficiency set for $\boldsymbol{a}$ then there exists some countable $\boldsymbol{\beta} \subseteq \boldsymbol{\alpha}$ which is a sufficiency set for $\boldsymbol{a}$

(for each element of $\mathfrak{P}$ which does not contain every element of $\boldsymbol{\alpha}$ choose some element of $\boldsymbol{\alpha}$ not in this set).

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What can we say about these sufficiency sets?

- If $\boldsymbol{\alpha}$ is a sufficiency set for $\boldsymbol{a}$ then there exists some countable $\boldsymbol{\beta} \subseteq \boldsymbol{\alpha}$ which is a sufficiency set for $\boldsymbol{a}$
- For every countably infinite sufficiency set $\boldsymbol{\alpha}$ for $\boldsymbol{a}$ there exists some proper subset which is also a sufficiency set for $\boldsymbol{a}$.

In order to see this let $\boldsymbol{a}_0$ and $\boldsymbol{a}_1$ be distinct elements of $\boldsymbol{\alpha}$. If each $\boldsymbol{\alpha} - \{\boldsymbol{a}_i\}$ is not a sufficiency set for $\boldsymbol{a}$ then let each $\mathcal{P}_i$ be a Π^0_1 class which contains a member of every degree in $\boldsymbol{\alpha} - \{\boldsymbol{a}_i\}$ but does not contain a member of degree $\boldsymbol{a}$. Then $\mathcal{P}_0 \cup \mathcal{P}_1$ witnesses the fact that $\boldsymbol{\alpha}$ is not a sufficiency set for $\boldsymbol{a}$, a contradiction.

Sufficiency sets and anti-basis theorems

We say that $\boldsymbol{\alpha}$ is a *sufficiency set* for $\boldsymbol{a}$ if every Π^0_1 class that contains a member of every degree in $\boldsymbol{\alpha}$ also contains a member of degree $\boldsymbol{a}$.

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- If $\boldsymbol{\alpha}$ is a sufficiency set for $\boldsymbol{a}$ then there exists some countable $\boldsymbol{\beta} \subseteq \boldsymbol{\alpha}$ which is a sufficiency set for $\boldsymbol{a}$.
- For every countably infinite sufficiency set $\boldsymbol{\alpha}$ for $\boldsymbol{a}$ there exists some proper subset which is also a sufficiency set for $\boldsymbol{a}$.
- If $\boldsymbol{\alpha}$ is a finite sufficiency set for $\boldsymbol{a}$ then there exists some element $\boldsymbol{b}$ of $\boldsymbol{\alpha}$ such that $\{\boldsymbol{b}\}$ is a sufficiency set for $\boldsymbol{a}$.

Sufficiency sets and anti-basis theorems

The following theorem shows, however, that it is possible to find $\boldsymbol{a}$ and a countable $\boldsymbol{\alpha}$ which is a sufficiency set for $\boldsymbol{a}$, such that no finite subset of $\boldsymbol{\alpha}$ is a sufficiency set for $\boldsymbol{a}$.

[Low Anti-basis Theorem] Any Π^0_1 class that contains a member of every low degree contains a member of every degree.

Proof. Let Λ be downward closed and computable and such that $[\Lambda]$ does not contain a member of every degree. We define non-computable $A = \bigcup_i \sigma_i$ which is of low degree and such that for each i, if $\Psi_i(A)$ is total and non-computable then it is not an element of $[\Lambda]$. The fact that A is of low degree follows because we run the construction using an oracle for $\emptyset'$ and decide whether $\Psi_i(A; i)\downarrow$ at each stage $2i+2$.

Sufficiency sets and anti-basis theorems

In order to define the construction, we make use of a function σ, such that $\sigma(i, \tau)$ is defined as follows.

Given inputs i and τ, let T be the Ψ_i-splitting tree above τ. Let the strings in T be ordered according to their level and then from left to right. Since Λ does not contain a member of every degree, there exists a least string σ in T such that either σ is a leaf of T, or else $\Psi_i(\sigma) \notin \Lambda$. We define $\sigma(i, \tau)$ to be that string.

Sufficiency sets and anti-basis theorems

Stage 0. Define $\sigma_0 = \lambda$.

Stage $2i+1$. Define $\sigma_{2i+1} = \sigma(i, \sigma_{2i})$.

Stage $2i+2$. If there exists $\sigma \supset \sigma_{2i+1}$ such that $\Psi_i(\sigma; i) \downarrow$ then define σ_{2i+2} to be some extension of σ which is not an initial segment of $\Psi_i(\emptyset)$. Otherwise define σ_{2i+2} to be some extension of σ_{2i+1} which is not an initial segment of $\Psi_i(\emptyset)$.

..and finally..the structure is a lattice.

We noted previously that the intersection of the degree spectra of two Π^0_1 classes need not be the degree spectrum of a Π^0_1 class. The following result suffices to show, however, that $(\mathfrak{P}, <)$ is a lattice.

The intersection of the degree spectra of two Π^0_1 classes is the degree spectrum of a Π^0_1 class if it is the superset of the degree spectrum of a non-empty Π^0_1 class.

Proof. (Vague sketch) Let $\mathcal{P}_0$ and $\mathcal{P}_1$ be Π^0_1 classes with degree spectra $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ respectively, and suppose that $\mathcal{P}_2$ is a non-empty Π^0_1 class with degree spectrum $\boldsymbol{\gamma} \subseteq \boldsymbol{\alpha} \cap \boldsymbol{\beta}$.

..and finally..the structure is a lattice.

For each $i \leq 2$ let Λ_i be a downward closed and computable set of strings with $[\Lambda_i] = \mathcal{P}_i$. For each (i, j) we define:

$$\mathcal{Q}_{i,j} = \{A \in \mathcal{P}_0 : \ \Psi_i(A) \in \mathcal{P}_1 \text{ and } \Psi_j(\Psi_i(A)) = A\}.$$

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In order to define a downward closed and computable set of strings Λ such that $\mathcal{P} = [\Lambda]$ satisfies $S(\mathcal{P}) = \boldsymbol{\alpha} \cap \boldsymbol{\beta}$, we proceed as follows.

We begin by putting all strings in Λ_2 into Λ.

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In order to define a downward closed and computable set of strings Λ such that $\mathcal{P} = [\Lambda]$ satisfies $S(\mathcal{P}) = \boldsymbol{\alpha} \cap \boldsymbol{\beta}$, we proceed as follows.

We begin by putting all strings in Λ_2 into Λ.

Let us say $\sigma \notin \Lambda_2$ is terminal in Λ_2, if all proper initial segments of σ are in Λ_2. We consider strings to be ordered first by length and then from left to right. All strings which are terminal in Λ_2 we place in Λ and above the x^{th} terminal string we place a Π^0_1 class $\mathcal{P}_{i,j}$ with degree spectrum $S(\mathcal{Q}_{i,j}) \cup \boldsymbol{\gamma}$, where $x = \langle i,j \rangle$. This is done using Λ_2 as a 'filler'.


Thanks for listening!