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Computable Models of Theories with Few Models

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Abstract In this paper we investigate computable models of $\aleph_1$-categorical theories and Ehrenfeucht theories. For instance, we give an example of an $\aleph_1$-categorical but not $\aleph_0$-categorical theory T such that all the countable models of T except its prime model have computable presentations. We also show that there exists an $\aleph_1$-categorical but not $\aleph_0$-categorical theory T such that all the countable models of T except the saturated model, have computable presentations.

1 Introduction We begin by presenting some basic definitions from effective model theory. A *computable structure* is one with a computable domain and uniformly computable atomic relations. Without lost of generality, we can always suppose that the domain of every computable structure is the set of all natural numbers ω and that its language does not contain function symbols. If a structure $\mathcal{A}$ is isomorphic to a computable structure $\mathcal{B}$, then $\mathcal{A}$ is *computably presentable* and $\mathcal{B}$ is a *computable presentation of* $\mathcal{A}$. Let σ be an effective signature. Let $\sigma_0 \subset \sigma_1 \subset \sigma_2 \subset \cdots$ be an effective sequence of finite signatures such that $\sigma = \bigcup_t \sigma_t$. It is clear that a structure $\mathcal{A}$ of signature σ is computable if and only if there exists an effective sequence $\mathcal{A}_0 \subset \mathcal{A}_1 \subset \mathcal{A}_2 \subset \cdots$ of finite structures such that for each i the domain of $\mathcal{A}_i$ is $\{0, \ldots, t_i\}$, the function $i \longrightarrow t_i$ is computable, $\mathcal{A}_i$ is a structure of signature σ_i, $\mathcal{A}_{i+1}$ is an expansion and extension of $\mathcal{A}_i$, and the structure $\mathcal{A}$ is the union $\bigcup_i \mathcal{A}_i$. The domain of $\mathcal{A}$ is denoted by A. For a structure $\mathcal{A}$ of signature σ we write $P^{\mathcal{A}}$ to denote the interpretation of the predicate symbol $P \in \sigma$ in $\mathcal{A}$. When it does not cause confusion, we write P instead of $P^{\mathcal{A}}$. In this paper we only deal with finite or countable structures.

A basic question in computable model theory is whether a given first-order theory T has a computable model. A standard Henkin type construction shows that each

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decidable theory, that is, the theory whose set of theorems is computable, has a computable model. Moreover, the satisfaction predicate for this model is computable. Such computable models are called *decidable*. Constructing computable (decidable) presentations for specific models of T has been an intensive area of research in effective model theory (see Ershov [2], Goncharov [4], and Millar [9]). For example, the computableness of homogeneous models, in particular of prime and saturated models has been well studied. In [2] and [9] it is proved that the saturated model of T has a decidable presentation if and only if there exists a procedure which uniformly computes the set of all types of T. Goncharov [4] and Harrington [8] gave criteria for prime models to have decidable presentations. It is also known that the decidability of the saturated model of T implies the existence of a decidable presentation of the prime model of T ([2], Morley [10]). Thus, a general question arises as to how computable models of undecidable theories behave in comparison to computable models of decidable theories. In this paper we investigate computable models of complete theories with "few countable models" [10]. Examples of such theories are theories with countably many countable models such as $\aleph_1$-categorical theories and theories with finitely many countable models (Ehrenfeucht theories).

In [1], Baldwin and Lachlan developed the theory of $\aleph_1$-categoricity in terms of strongly minimal sets. They settled affirmatively Vaught's conjecture for $\aleph_1$-categorical complete theories by proving that each complete $\aleph_1$-categorical theory has either exactly one or $\aleph_0$ many countable models up to isomorphisms. Their paper also shows that all the countable models of any $\aleph_1$-categorical theory T can be listed in an $\omega + 1$ chain.

$$\text{chain}(T)\colon \quad \mathcal{A}_0 \prec \mathcal{A}_1 \prec \cdots \prec \mathcal{A}_n \prec \cdots \prec \mathcal{A}_\omega$$

of elementary embeddings with $\mathcal{A}_0$ and $\mathcal{A}_\omega$ being the prime and saturated models of T, respectively [1]. The results of Baldwin and Lachlan lead one to investigate the effective content of $\aleph_1$-categorical theories and their models. Based on the theory developed by Baldwin and Lachlan, Harrington and Khissamiev [6] proved that every countable model of each decidable $\aleph_1$-categorical theory T has a decidable presentation.

This result of Harrington and Khissamiev motivated the study of computable models of $\aleph_1$-categorical undecidable theories. In 1972, Goncharov [3] constructed an example of an $\aleph_1$-categorical but not $\aleph_0$-categorical theory T for which the only model with a computable presentation is the prime model, that is, the first element of chain(T). Later in 1980, Kudeiberganov [7] modified Goncharov's construction to provide an example of an $\aleph_1$-categorical but not $\aleph_0$-categorical theory T with exactly *n computable* models. These models are the first n elements of chain(T). These results lead to the following two questions which have remained open.

Question 1.1 (Goncharov [5]) If an $\aleph_1$-categorical but not $\aleph_0$-categorical theory T has a computable model, is the prime model of T computably presentable?

Question 1.2 If all models $\mathcal{A}_0, \mathcal{A}_1, \ldots, \mathcal{A}_i, \ldots, i \in \omega$, in chain($T$) of an $\aleph_1$-categorical but not $\aleph_0$-categorical theory T, have computable presentations, is the saturated model $\mathcal{A}_\omega$ of T computably presentable?

The above result of Harrington and Khissamiev also inspired Nerode to ask whether the hypothesis of $\aleph_1$-categoricity of T can be replaced by the hypothesis that T has only finitely many countable models, that is, whether every countable model of a decidable Ehrenfeucht theory has a decidable presentation. Morley noted that if the countable saturated model of such a theory is decidable, then the theory has at least three computable models [10]. Lachlan answered Nerode's question by giving an example of a decidable theory with exactly six models of which only the prime one has a computable presentation. Later, for each natural number $n > 3$, Peretyatkin constructed an example of decidable theory with exactly n models such that the prime model of the theory is computable and none of the other models of the theory has computable presentations [11]. In [7] Kudeiberganov constructed an example of a theory with exactly three models such that the theory has only one computable model and that model is prime. The saturated model of the theory cannot be decidable, since otherwise, all three models of the theory would have computable presentations. These results lead Morley to ask whether any countable model of a decidable Ehrenfeucht theory T with a decidable saturated model has a decidable presentation [10]. There is a natural analog of this question for computable models.

Question 1.3 If the saturated model of an Ehrenfeucht theory is computable, does there exist a nonsaturated, computable model of the theory?

In this paper we answer the above three questions by providing appropriate counterexamples. Our examples of models which answer the first two questions have infinite signatures. However these questions remain open for theories of finite signatures.

The general problem suggested by these results is to characterize *the spectrum of computable models of $\aleph_1$-categorical theories*: let T be an $\aleph_1$-categorical but not $\aleph_0$-categorical complete theory. Consider chain(T). *The spectrum of computable models of T*, denoted by **SRM(T)**, is the set

$$\{i \leq \omega \mid \text{ the model } \mathcal{A}_i \text{ in chain}(T) \text{ has a computable presentation.}\}$$

Problem 1.4 Describe all subsets of ω which are of the form **SRM(T)** for some $\aleph_1$-categorical theory T.

The result of Harrington and Khissamiev shows that if T is decidable, then **SRM(T)**$= \omega \bigcup \{\omega\}$. The results of Goncharov and Kudeiberganov show that the sets $\{1, \dots, n\}$, where $n \in \omega$, are spectra of computable models of $\aleph_1$-categorical theories. In this paper we show that the sets $\omega - \{0\} \bigcup \{\omega\}$ and ω are also spectra of computable models of $\aleph_1$-categorical theories.

2 Main results

The results of this paper are based on the idea of coding Σ^0_2 or Π^0_2 sets with certain recursion-theoretic properties into $\aleph_1$-categorical theories. Our first result is the following theorem which answers Question 1.1.

Theorem 2.1 *There exists an $\aleph_1$-categorical but not ω-categorical theory T such that all the countable models of T except its prime model have computable presentations (and so* $\mathbf{SRM}(T) = \omega - \{0\} \bigcup \{\omega\}$*).*

Before proving this theorem we would like to give the basic idea of our proof. For an infinite subset $S \subset \omega$ we construct a structure $\mathcal{A}_S$ of infinite signature $(P_0, P_1, P_2, \ldots)$ where each P_i is a binary predicate symbol. We will show that the theory T_S of the structure $\mathcal{A}_S$ is $\aleph_1$-categorical and $\mathcal{A}_S$ is the prime model of T_S. The countable models of T_S will have the following property: every nonprime model $\mathcal{A}$ of T_S has a computable presentation if and only if the set S is a Σ^0_2-set. The existence of a computable presentation of the prime model will imply that the set S has a certain recursion theoretic property. Our recursion theoretic lemma (Lemma 2.6) will show that there exists a Σ^0_2-set S which does not have this property.

2.1 The construction of cubes Let n be a nonzero natural number. Let $\sigma_n = (P_0, \ldots, P_{n-1})$ be a signature such that each P_i is a binary predicate symbol. For each nonzero natural number n we define a finite structure of signature σ_n, called an *n-cube*, as follows: a 1-cube C_1 is a structure $(\{a, b\}, P_0)$ such that $P_0(x, y)$ holds in C_1 if and only if $x = a$ and $y = b$ or $y = a$ and $x = b$.

Suppose that n-cubes have been defined. Let $\mathcal{A} = (A, P_0^{\mathcal{A}}, \ldots, P_{n-1}^{\mathcal{A}})$ and $\mathcal{B} = (B, P_0^{\mathcal{B}}, \ldots, P_{n-1}^{\mathcal{B}})$ be n-cubes such that $A \bigcap B = \varnothing$. These two n-cubes are isomorphic. Let f be an isomorphism from $\mathcal{A}$ to $\mathcal{B}$. Then an $n+1$-cube C_{n+1} is

$$(A \bigcup B, P_0^{\mathcal{A}} \bigcup P_0^{\mathcal{B}}, \ldots, P_{n-1}^{\mathcal{A}} \bigcup P_{n-1}^{\mathcal{B}}, P_n),$$

where $P_n(x, y)$ holds if and only if $f(x) = y$ or $f^{-1}(x) = y$. It follows that we can naturally define an *ω-cube* $C_\omega = \bigcup_{i \in \omega} C_i$ as an increasing union of n-cubes formed in this way.

An ω-cube C_ω is a structure of the infinite signature $\sigma = (P_0, P_2, \ldots)$. From these definitions of cubes we make the following claim.

Claim 2.2 *For each $n \leq \omega$ any two n-cubes are isomorphic.*

Each binary predicate P_i in any cube $\mathcal{A}$ is a partial function and sets up a one to one mapping from $\mathrm{dom}(P_i)$ onto $\mathrm{range}(P_i)$. Therefore we can also write $P_i(x) = y$ instead $P_i(x, y)$. Moreover, by the definition of P_i, $\mathrm{dom}(P_i) = \mathrm{range}(P_i)$.

2.2 Construction of $\mathcal{A}_S$ For each natural number $n \in \omega$ consider an n-cube denoted by $\mathcal{A}_n$. Assume that $A_n \bigcap A_t = \varnothing$ for all $n \neq t$. Let S be a subset of ω. Define a structure $\mathcal{A}_S$ by

$$\mathcal{A}_S = \bigcup_{n \in S} \mathcal{A}_n.$$

Thus the structure $\mathcal{A}_S$ is the disjoint union of all cubes $\mathcal{A}_n$, $n \in S$, with the natural interpretations of predicate symbols of signature σ. Let T_S be the theory of the structure $\mathcal{A}_S$.

Claim 2.3 *If S is an infinite set, then the theory T_S is $\aleph_1$-categorical but not $\aleph_0$-categorical.*

Proof: The model $\mathcal{A}_S$ satisfies the following list of statements. It is easy to see that this list of statements can be written as an (infinite) set of statements in the first-order logic.

1. $\forall x \exists y P_0(x, y)$ and for each n, P_n is a partial one to one function.

2. For all $n \neq m$ and for all x, $P_n(x) \neq P_m(x)$.
3. For each n and for all x if $P_n(x)$ is defined, then $P_0(x), P_1(x), \dots, P_{n-1}(x)$ are also defined.
4. For all n, m and for all x if $P_n(x)$ and $P_m(P_n(x))$ are defined, then $P_m(P_n(x)) = P_n(P_m(x))$.
5. For all $k, n > n_1 \geq n_2 \geq \cdots \geq n_{k-1} \geq n_k$, for all elements x, $P_{n_1}(\dots, (P_{n_k}(x), \dots) \neq P_n(x)$.
6. For each $n \in \omega$, $n \in S$ if and only if there exists exactly one n-cube that is not contained in an $n+1$-cube.

Let $\mathcal{M}$ be a model which satisfies all the above statements. Then for each $n \in S$, $\mathcal{M}$ must have an n-cube which is not contained in an $n+1$-cube. Moreover, if an $x \in M$ does not belong to any n-cube for $n \in S$, then x is in an ω-cube. Note that each ω-cube is countable. Using the previous claim it can be seen that any two models which satisfy the above list of axioms are isomorphic if and only if these two models have the same number of ω-cubes. Suppose that $\mathcal{M}_1$ and $\mathcal{M}_2$ are models of T_S and their cardinalities are $\aleph_1$. Since each cube is a countable set it follows that the number of ω-cubes in $\mathcal{M}_1$ and $\mathcal{M}_2$ is $\aleph_1$. Therefore the models $\mathcal{M}_1$ and $\mathcal{M}_2$ are isomorphic. Hence T_S is an $\aleph_1$-categorical but not $\aleph_0$-categorical theory. $\square$

Claim 2.4 *The set S is in Σ_2^0 if and only if every nonprime model of T_S possesses a computable presentation.*

Proof: Each ω-cube has a computable presentation. Therefore it suffices to prove that $S \in \Sigma_2^0$ if and only if the nonprime model $\mathcal{M}$ of T_S with exactly one ω-cube has a computable presentation. If $\mathcal{M}$ is computable, then $s \in S$ if and only if $\exists x \exists y \forall z (P_s(x, y) \,\&\, \neg P_{s+1}(x, z))$. Therefore $S \in \Sigma_2^0$.

Now suppose that $S \in \Sigma_2^0$. There exists a computable function f such that for every $n \in \omega$, $n \in S$ if and only if $W_{f(n)}$ is finite. We construct an effective sequence

$$\mathcal{M}_0 \subset \mathcal{M}_1 \subset \mathcal{M}_2 \subset \cdots$$

of finite structures by stages such that

1. the model $\mathcal{M}$ is isomorphic to $\bigcup_n \mathcal{M}_n$;
2. each $\mathcal{M}_t$ has exactly $t+1$ cubes and the function $t \longrightarrow \text{card}(\mathcal{M}_t)$ is computable;
3. each $\mathcal{M}_t$ is a structure of signature $(P_0, \dots, P_{n_i})$, where $i \longrightarrow n_i$ is a computable function.

Stage 0 Construct a 1-cube $\mathcal{M}_0$ and mark this structure with the symbol $\square_\omega$.

Stage s+1 Suppose that $\mathcal{M}_s$ has been constructed as the disjoint union

$$\mathcal{M}_{s,0} \bigcup \mathcal{M}_{s,1} \bigcup \cdots \bigcup \mathcal{M}_{s,s} \bigcup \mathcal{M}_{s,\omega},$$

where each $\mathcal{M}_{s,i}$, $i \leq s$ is an i-cube, and $\mathcal{M}_{s,\omega}$ is the cube marked with $\square_\omega$ at the previous stage. Compute $W_{f(0),s+1}, \dots, W_{f(s),n+1}, W_{f(s+1),s+1}$. For each $i \leq s+1$ define $\mathcal{M}_{i,s+1}$ and $\mathcal{M}_{s+1,\omega}$ as follows.

1. If $W_{f(i),s+1} = W_{f(i),s}$, then let $\mathcal{M}_{i,s+1} = \mathcal{M}_{i,s}$.
2. If $W_{f(i),s+1} \neq W_{f(i),s}$, then construct a new i-cube and let $\mathcal{M}_{i,s+1}$ be this new cube.
3. Extend the cube $\mathcal{M}_{s,\omega}$ to a finite cube denoted by $\mathcal{M}_{s+1,\omega}$ such that for each $i \leq s$ if $W_{f(i),s+1} \neq W_{f(i),s}$, then $\mathcal{M}_{s+1,\omega}$ contains $\mathcal{M}_{s,i}$.

Let $\mathcal{M}_{s+1}$ be

$$\mathcal{M}_{s+1,0} \bigcup \mathcal{M}_{s+1,1} \bigcup \cdots \bigcup \mathcal{M}_{s+1,s+1} \bigcup \mathcal{M}_{s+1,\omega}.$$

Define

$$\mathcal{M}_\omega = \bigcup_s \mathcal{M}_s.$$

By the construction, the structure $\mathcal{M}_\omega$ is computable. The construction of $\mathcal{M}_\omega$ guarantees that the structure $\mathcal{M}_\omega$ is isomorphic to the model $\mathcal{M}$. $\square$

Now we need the following definition and recursion theoretic lemma. We will prove the lemma in the next section.

Definition 2.5 A function f is *limitwise monotonic* if there exists a computable function $\varphi(x,t)$ such that $\varphi(x,t) \leq \varphi(x,t+1)$ for all $x, t \in \omega$, $\lim_t \varphi(x,t)$ exists for every $x \in \omega$ and $f(x) = \lim_t \varphi(x,t)$.

Lemma 2.6 (Recursion theoretic lemma) *There exists a Δ^0_2 set A which is not the range of any limitwise monotonic function.*

Proof of Theorem 2.1: We need the following lemma.

Lemma 2.7 *If the prime model $\mathcal{A}_S$ is computable, then the set S is the range of a limitwise monotonic function.*

Proof: Let $x \in \mathcal{A}_S$. Note that each cube in $\mathcal{A}_S$ is finite. Define $\varphi(x)$ to be an s such that x is in an s-cube and this cube is not contained in an $s+1$-cube. It is clear that φ witnesses that S is the range of a limitwise monotonic function. $\square$

By the recursion theoretic lemma there exists an $S \in \Delta^0_2$ which is not the range of any limitwise monotonic function. Consider the structure $\mathcal{A}_S$ and its theory T_S. The claims above and Lemma 2.7 show that T_S is the required theory and so prove Theorem 2.1. $\square$

Now we give an answer to Question 1.2. The idea of our proof is the following. We take a Π^0_2 but not Σ^0_2 set S and code this set into a theory T_S. The language of T_S will contain infinitely many unary predicates $P_0, P_1, \ldots$ and infinitely many predicates of arity n for each $n \in \omega$. We will prove that T_S is an $\aleph_1$-categorical but not $\aleph_0$-categorical theory. Our construction of T_S guarantees that all the countable models of T_S, except the saturated model, have computable presentations. The existence of a computable presentation for the saturated model will imply that the set S is a Σ^0_2 set. This will contradict with the choice of S.

Theorem 2.8 *There exists an $\aleph_1$-categorical but not $\aleph_0$-categorical theory T such that all the countable models of T except the saturated model, have computable presentations.*

Proof: We construct a structure of the infinite signature

$$(P_0, P_1, \ldots, R_{1,0}, R_{1,1}, R_{1,2}, \ldots, R_{k,0}, R_{k,1}, R_{k,2}, \ldots),$$

where each P_i is a unary predicate and each $R_{k,s}$ is a predicate of arity k.

Let S be a $(\Pi_2^0 \setminus \Sigma_2^0)$ set. There exists a computable predicate H such that $n \in S$ if and only if $\forall x \exists y H(x, y, n)$ holds. Below we present a step by step construction of a computable structure denoted by $\mathcal{A}_S$ and prove that the theory T_S of this structure satisfies the requirements of the theorem.

Stage 0 Let $\mathcal{A}_0 = (\{0\}, P_0)$, where $P_0(0)$ holds.

Stage t+1 The domain A_{t+1} of $\mathcal{A}_{t+1}$ is $\{0, \ldots, t+1\}$. The signature of $\mathcal{A}_{t+1}$ is

$$\sigma_{t+1} = (P_0, \ldots, P_{t+1}, R_{1,0}, \ldots, R_{1,t+1}, \ldots, R_{t+1,0}, \ldots, R_{t+1,t+1}).$$

For each $i \leq t+1$ let $P_i(x)$ hold if and only if $x \geq i$. For $k, s \leq t+1$, let $R_{k,s}(x_1, \ldots, x_k)$ hold if and only if $x_1, \ldots, x_k$ are pairwise different and for the maximal number $j \leq t+1$ such that all $P_j(x_1), \ldots, P_j(x_k)$ hold we have $\forall n \leq s \exists m \leq j H(n, m, k)$. We have defined the model $\mathcal{A}_{t+1}$.

Thus we have an effective sequence $\mathcal{A}_0, \mathcal{A}_1, \mathcal{A}_2, \ldots$ of finite structures such that each $\mathcal{A}_{i+1}$ is an extension and expansion of $\mathcal{A}_i$. Therefore we can define $\mathcal{A}_S$ by

$$\mathcal{A}_S = \bigcup_i \mathcal{A}_i.$$

It is clear that the model $\mathcal{A}_S$ is computable.

Claim 2.9 *The theory T_S of the model $\mathcal{A}_S$ is $\aleph_1$-categorical but not $\aleph_0$-categorical.*

Proof: The model $\mathcal{A}_S$ satisfies the following list of properties which can be written as an infinite set of statements in the language of the first-order logic.

1. For all x if $P_{i+1}(x)$ holds, then $P_i(x)$ also holds. Moreover, $\forall x P_0(x)$ is true.
2. For each $i \in \omega$ there exists a unique x such that $P_i(x) \,\&\, \neg P_{i+1}(x)$, $i \in \omega$.
3. For all $k, s \in \omega$, if $R_{k,s}(x_1, \ldots, x_k)$ holds, then $x_1, \ldots, x_k$ are pairwise distinct.
4. Let $k \in S$. For every $s \in \omega$ there exists a $j \in \omega$ such that $\forall n \leq s \exists m < j H(n, m, s)$. Let j_s be the minimal number which has this property. Then for all pairwise distinct $x_1, \ldots, x_k$ if $P_{j_s}(x_1) \,\&\, \cdots P_{j_s}(x_k)$ holds, then $R_{k,s}(x_1, \ldots, x_k)$ holds.
5. Let $k \notin S$. There exists an s_0 such that for all $s \geq s_0$ and for all $x_1, \ldots, x_k$, $R_{k,s}(x_1, \ldots, x_k)$ does not hold.

Let $\mathcal{A}$ be a model of T_S. Consider the set $\bigcap_i P_i^{\mathcal{A}}$. For any two elements $a, b \in \bigcap_i P_i^{\mathcal{A}}$ there exists an automorphism α of the model $\mathcal{A}$ such that $\alpha(a) = b$. Thus a proof of $\aleph_1$-categoricity can be based on the following observation. Two models $\mathcal{B}$ and $\mathcal{C}$ of the theory T_S are isomorphic if and only if the cardinalities of the sets $\bigcap_i P_i^{\mathcal{B}}$ and $\bigcap_i P_i^{\mathcal{C}}$ are equal. Hence if $\mathcal{B}$ and $\mathcal{C}$ are models of cardinality $\aleph_1$, then both $\bigcap_i P_i^{\mathcal{B}}$ and $\bigcap_i P_i^{\mathcal{C}}$ have exactly $\aleph_1$ elements. It follows that $\mathcal{B}$ and $\mathcal{C}$ are isomorphic. □

From the proof of Claim 2.9, it follows that if $\mathcal{B}$ is a countable unsaturated model of the theory T_S, then $\bigcap P_i^{\mathcal{B}}$ has a finite number of elements.

Claim 2.10 *If C is a countable and unsaturated model of T_S, then C has a computable presentation.*

Proof: Let C be a countable, unsaturated model of T_S. The set $\bigcap_i P_i^C$ has a finite number of elements, say n. We construct a computable presentation of C by stages.

Let $a_1, \ldots, a_n$ be new symbols. In our construction of a computable presentation $\mathcal{A}$ of C we put the elements $a_1, \ldots, a_n$ into $\bigcap_i P_i^{\mathcal{A}}$. Let $p_1, \ldots, p_n$ be the all elements of $S \bigcap \{0, 1, \ldots, n\}$.

Stage 0 Define $\mathcal{A}_0 = (\{0, a_1, \ldots, a_n\}, P_0)$, letting $P_0(0), P_0(a_1), \ldots, P_0(a_n)$ hold.

Stage t+1 The domain A_{t+1} of $\mathcal{A}_{t+1}$ is $\{0, \ldots, t+1, a_1, \ldots, a_n\}$. The signature of the $\mathcal{A}_{t+1}$ is

$$\sigma_{t+1} = (P_0, \ldots, P_{t+1}, R_{1,0}, \ldots, R_{1,t+1}, \ldots, R_{t+1,0}, \ldots, R_{t+1,t+1}).$$

For each $i \leq t+1$ let $P_i(x)$ hold if and only if $x \geq i$ or $x \in \{a_1, \ldots, a_n\}$. For $k, s \leq t+1$, let $R_{k,s}(x_1, \ldots, x_s)$ hold if and only if one of the followings holds:

1. $k \in \{p_1, \ldots, p_n\}$, $(x_1, \ldots, x_k) \in \{a_1, \ldots, a_n\}^n$, and $x_1, \ldots, x_k$ are pairwise distinct, or
2. $\{x_1, \ldots, x_k\} \setminus \{a_1, \ldots, a_n\} \neq \emptyset$, the elements $x_1, \ldots, x_k$ are pairwise different, and for the maximal number $j \leq t+1$ such that all $P_j(x_1), \ldots, P_j(x_k)$ hold we have $\forall n \leq s \exists m \leq j H(n, m, k)$.

Thus this stage defines the structure $\mathcal{A}_{t+1}$. For each $i \in \omega$, $\mathcal{A}_{i+1}$ is an extension and expansion of $\mathcal{A}_i$. Define $\mathcal{A}$ by $\mathcal{A} = \bigcup_i \mathcal{A}_i$. It is clear that the structure $\mathcal{A}$ is computable and isomorphic to the model C. □

Claim 2.11 *The countable saturated model $\mathcal{B}$ of T does not have a computable presentation.*

Proof: Suppose that $\mathcal{B}$ is computable. Since $\mathcal{B}$ is saturated the number of elements in $\bigcap_i P_i^{\mathcal{B}}$ is infinite. It can be checked that for each $k \in \omega$, $k \in S$ if and only if there exist different elements $y_1, \ldots, y_k$ from $\bigcap_i P_i^{\mathcal{B}}$ such that for all $s \geq 1$, $R_{k,s}(y_1, \ldots, y_k)$ holds. The set S would then be a Σ_2^0-set. This contradicts with our assumption that $S \in \Pi_2^0 \setminus \Sigma_2^0$. □

These claims prove Theorem 2.8. □

Thus the above theorems prove the following corollary about *spectra of computable models* (**SRM**) of $\aleph_1$-categorical theories.

Corollary 2.12

1. *There exists an $\aleph_1$-categorical but not ω-categorical theory T such that* $\mathbf{SRM}(T) = \omega - \{0\} \bigcup \{\omega\}$.

2. *There exists an $\aleph_1$-categorical but not $\aleph_0$-categorical theory T such that* $\mathbf{SRM}(T) = \omega$.

In the next theorem, which answers Question 1.3, we provide an example of a theory T_S with exactly three countable models of which only the saturated model is computably presentable. To prove that T_S has exactly three countable models, we use the known ideas which show that the theory of the model $(Q, \leq, c_0, c_1, \ldots)$, where $\leq$ is the linear ordering of rationals, and the constants are such that $c_0 > c_1 > c_2 > \cdots$, has exactly three countable models [12].

Theorem 2.13 *There exists a theory T with exactly three countable models such that the only model of T which has a computable presentation is the saturated model.*

Proof: Let Q be the set of all rational numbers. For each cardinal number $m \in \omega \bigcup \{\omega\}$ define a structure $Q_0(m)$ as follows. The domain of the structure is

$$\{q \in Q | 1 \leq q\} \bigcup \{c_{q,1}, \ldots, c_{q,m} | q \in Q\},$$

where $\{c_{q,i} | q \in Q, 1 \leq i \leq m\}$ is a set of new elements. The signature of the model is $(\leq, f)$, where $\leq$ is a binary predicate and f is a unary function symbol. The predicate $\leq$ and the function f are defined as follows. For all x, y we have $x \leq y$ if and only if $x, y \in Q$ and x is less than or equal to y as rational numbers. For all z, y define $f(z) = y$ if and only if for some rational number q, $y = q$ and $z \in \{c_{q,1}, \ldots, c_{q,m}\}$ or $y = z = q$. Let $Q(m)$ be the structure obtained from $Q_0(m)$ by removing the elements $1, c_{1,1}, \ldots, c_{1,m}$ from the domain of $Q_0(m)$.

If $\mathcal{A}$ and $\mathcal{B}$ are isomorphic copies of the structures $Q_0(n)$ and $Q_0(m)$, respectively, and $A \bigcap B = \emptyset$, then one can naturally define the isomorphism type of the structure $Q_0(n) + Q_0(m)$ as follows. The domain of the new structure is $A \bigcup B$. The predicate $\leq$ in the new structure is the least partial ordering which contains the partial ordering of $\mathcal{A}$, the partial ordering of $\mathcal{B}$, and the relation $\{(x, y) | x \in A \,\&\, f^{\mathcal{A}}(x) = x \,\&\, y \in B \,\&\, f^{\mathcal{B}}(y) = y\}$. The unary function f in the new structure is the union of the unary operations of the first and the second structures.

If $n_0, n_1, n_2, \ldots, n_i, \ldots, i < \omega$ is a sequence of natural numbers, then as above we can define the structure

$$Q_0(n_0) + Q_0(n_1) + Q_0(n_2) + \cdots.$$

Let S be a set in Δ^0_2 which is not the range of a limitwise monotonic function. There exists a computable function g such that, for all $n h(n) = \lim_s g(n, s)$ exists and $\mathrm{range}(h) = S$. Consider the model $Q_0(S)$ defined by

$$Q_0(h(0)) + Q_0(h(1)) + Q_0(h(2)) + \cdots.$$

Define the theory T_S to be the theory of the structue $Q_0(S)$.

Claim 2.14 *The theory T_S has exactly three countable models.*

Proof: The first model of T_S is $Q_0(S)$. This model is the prime model of the theory T_S. The second model of T_S is

$$Q'(S) = Q_0(h(0)) + Q_0(h(1)) + Q_0(h(2)) + \cdots + Q_0(\omega).$$

The third model $\mathcal{M}$ of T_S is

$$Q(h(0)) + Q_0(h(1)) + Q_0(h(2)) + \cdots + Q(\omega).$$

These structures are indeed models of T_S. To see this, note that $Q_0(S)$ is a submodel of $Q'(S)$, and $Q'(S)$ is a submodel of $\mathcal{M}$. It can be checked that for any formula $\exists x\varphi(x, a_1, \ldots, a_n)$ and all $a_1, \ldots, a_n \in Q_0(S)(a_1, \ldots, a_n \in Q'(S))$ if the formula $\exists x\varphi(x, a_1, \ldots, a_n)$ is true in $Q'(S)$—in $\mathcal{M}$—then there exists a $b \in Q_0(S)(b \in Q'(S))$ such that $\varphi(b, a_1, \ldots, a_n)$ is true in $Q_0(S)$—in $Q'(S)$. Therefore the embeddings are elementary.

We have to prove that any countable model of T_S is isomorphic to one of the three models described above. Let $\mathcal{A}$ be a model of T_S. For each $i \in \omega$ we define by induction an element $a_i \in A$ as follows. The element a_0 is the minimal element with respect to the partial ordering in $\mathcal{A}$. Note that the set $\{b|b \neq a_0 \,\&\, f(b) = a_0\}$ has exactly $h(0)$ elements. Also put $k_0 = 0$.

Suppose that the elements $a_0, \ldots, a_{i-1} \in A$ and the numbers $k_0, \ldots, k_{i-1}$ have been defined. Let k_i be the least element such that $h(k_i) \neq h(k_j)$ for $j = 1, \ldots, i-1$. The element a_i is the one such that the following properties hold:

1. the set $\{b|b \neq a_i \,\&\, f(b) = a_i\}$ has exactly $h(k_i)$ elements;
2. for each $x < a_i$ the cardinality of the set $\{b|b \neq x \,\&\, f(b) = x\}$ is in $\{h(k_0), \ldots, h(k_{i-1}\}$.

Consider the sequence $a_0, a_1, a_2, \ldots$. Clearly $a_0 < a_1 < a_2 < \cdots$. Thus we have three cases.

Case 1: $\lim_i a_i$ does not exist and for any $x \in A$ such that $f(x) = x$ there exists an i such that $a_i \geq x$,

Case 2: $\lim_i a_i$ exists,

Case 3: $\lim_i a_i$ does not exist and there exists an x such that $f(x) = x$ and $x \geq a_i$ for all a_i.

In the first case $\mathcal{A}$ is isomorphic to $Q_0(S)$. In the second case $\mathcal{A}$ is ismorphic to $Q'(S)$. In the third case $\mathcal{A}$ is isomorphic to $\mathcal{M}$. Note that $Q_0(S)$ is the prime model. The model $Q'(S)$ is not saturated since it does not realize the type containing $\{x > a_i \,\&c > x|i \in \omega\}$, where $c = \lim_i a_i$. Hence $\mathcal{M}$ is the saturated model of T_S. □

Claim 2.15 *The unsaturated models of the theory T_S do not have computable presentations.*

Proof: Consider the prime model $Q_0(S)$. Suppose $Q_0(S)$ is a computable model. Then it can be easily checked that the set S is the range of a limitwise monotonic function. This contradicts the assumption on S. If the other unsaturated model

$$Q'(S) = Q_0(h(0)) + Q_0(h(1)) + Q_0(h(2)) + \cdots + Q_0(\omega)$$

were computable, then $Q_0(S)$ would be a computably enumerable submodel of the model $Q'(S)$. Hence $Q_0(S)$ would have a computable presentation. This is again a contradiction. □

Claim 2.16 *The saturated model $\mathcal{M}$ of the theory T has a computable presentation.*

Proof: We present a constuction of the saturated model $\mathcal{M}$ by stages. The construction will clearly show that the saturated model has a computable presentation.

Stage 0 Consider the structure $Q_0(g(0,0))+Q(\omega)$. Denote this model by $\mathcal{A}_0$.

Stage n+1 Suppose that $\mathcal{A}_n$ has been defined and is isomorphic to

$$Q_0(g(0,n)) + \cdots + Q_0(g(n,n)) + Q(\omega).$$

Compute

$$g(0,n+1), \ldots, g(n+1,n+1).$$

Let $i \le n$ be the minimal number such that $g(i,n) \neq g(i,n+1)$.$\mathcal{A}_n$ can be extended to a structure $\mathcal{A}_{n+1}$ isomorphic to

$$\begin{aligned} &Q_0(g(0,n+1)) + \cdots + Q_0(g(i-1,n+1)) \\ &\qquad + Q_0(g(i,n+1)) + \cdots + Q_0(g(n+1,n+1)) + Q(\omega). \end{aligned}$$

To see this, take the substructure

$$Q_0(g(i,n)) + \cdots + Q_0(g(n,n)) + Q(\omega)$$

of $\mathcal{A}_n$; extend this substructure to $Q(\omega)$; insert the new structure

$$Q_0(g(i,n+1)) + \cdots + Q_0(g(n+1,n+1))$$

between the structures

$$Q_0(g(0,n+1)) + \cdots + Q_0(g(i-1,n+1))$$

and the extended structure $Q(\omega)$. The structure obtained in this way is $\mathcal{A}_{n+1}$.

Thus we have the sequence

$$\mathcal{A}_0 \subset \mathcal{A}_1 \subset \mathcal{A}_2 \subset \cdots.$$

Define

$$\mathcal{A}_\omega = \bigcup_i \mathcal{A}_n.$$

It is easy to see that the model $\mathcal{A}_\omega$ is ismorphic to

$$Q_0(h(0)) + Q_0(h(1)) + \cdots + Q_0(h(n)) + \cdots + Q(\omega).$$

Now it is clear that the above description can be effectivized. □

These claims prove the theorem. □

Finally we have to prove the promised recursion theoretic lemma.

3 Proof of the recursion theoretic lemma Let $\varphi_e(x, t)$, $e \in \omega$, be a uniform enumeration of all partial computable functions φ such that for all $t' \geq t$ if $\varphi(x, t')$ is defined, then $\varphi(x, t)$ is defined and $\varphi(x, t) \leq \varphi(x, t')$. At stage s of our construction we define a finite set A_s in such a way that $A(y) = \lim_s A_s(y)$ exists for all y. We satisfy the requirement R_e asserting that, if $f_e(x) = \lim_t \varphi_e(x, t) < \omega$ for all x, then $\text{range}(f_e) \neq A$.

The strategy for a single R_e is as follows: at stage s pick a witness m_e, enumerate m_e into A (i.e., $A_s(m_e) = 1$). Now R_e is satisfied (since m_e remains in A) unless at some later stage t_0 we find an x such that $\varphi_e(x, t_0) = m_e$. If so, R_e ensures that $A(\varphi_e(x, t)) = 0$ for all $t \geq t_0$. Thus, either $f_e(x) \uparrow$ or $f_e(x) \downarrow$ and $f_e(x) \notin A$.

Keeping $\varphi_e(x, t)$ out of A for all $t \geq t_0$ can conflict with a lower priority $(i > e)$ requirement R_i since it may be the case that $m_i = \varphi_e(x, t')$ for some $t' > t_0$. However, if $f_e(x) \downarrow$, then this holds permanently for just one number, and if $f_e(x) \uparrow$, then the restriction is transitory for each number. So each lower priority R_i will be able to choose a stable witness at some stage.

3.1 Construction At stage s we try to determine the values of parameters m_e, x_e, and $n_e = \varphi_e(x_e, s)$ for R_e. Each parameter may remain undefined. Moreover, we define the approximation A_s to A at stage s.

Stage 0 Let $A_0 = \varnothing$, and declare all parameters to be undefined.

Stage s For each $e = 0, \dots, s-1$ in turn go through substage e by performing the following actions.

1. If m_e is undefined, let m_e be the least number in $\omega^{[e]}$ greater than all $m_i (i < e)$ which is not equal to any n_i. Let $A_s(m_e) = 1$ and proceed to the next subtage, or to stage $s+1$ if $e = s-1$.
2. If x_e is undefined and $\varphi_e(x, s) = m_e$ for some x, let $x_e = x, n_e = m_e$, and $A_s(n_e) = 0$, and proceed to the next stage $s+1$ if $e = s-1$.
3. Let $n_e = \varphi_e(x_e, s)$ and $A_s(n_e) = 0$. If $n_e = m_i$ for some $i > e$, declare all the parameters of the R_j, $j \geq i$, to be undefined.

For each y, if $A_s(y)$ is not determined by the end of stage s, then assign to $A_s(y)$ its previous value $A_{s-1}(y)$. The stage is now completed. Now we will verify that the construction succeeds.

Claim 3.1 *Each m_e is defined and is constant from some stage on.*

Proof: Suppose inductively that the claim holds for each $i < e$. Let s_0 be a stage such that each m_i has reached its limit for $i < e$, and if x_i ever becomes defined after s_0, and $\lim_s n_{i,s} < \infty$, then the limit has been reached at s_0. Moreover, let $k \geq e$ be the least number which does not equal any of these limits and is greater than all m_i for $i < e$. Also suppose that $n_{i,s_0} > k$ if $\lim_s n_{j,s} = \infty$, $(j < e)$. If m_e is cancelled after stage s_0, then $m_e = k$ is permanent from the next stage on. $\square$

Claim 3.2 *For each y, $\lim_s A_s(y)$ exists. Therefore the set $A = \lim_s A_s$ is a Δ^0_2-set.*

Proof: Suppose that $y \in \omega^{[e]}$, and let s_0 be a stage at which m_e has reached its limit. Since y can only be enumerated into A if $y = m_e$, after stage s_0, $A(y)$ can change at most once. This proves the claim. $\square$

Claim 3.3 *Suppose* $f_e(x) = \lim_t \varphi_e(x,t)$ *exists for each* x. *Then* $A \neq \mathrm{range}(f_e)$.

Proof: Suppose that $A = \mathrm{range}(f_e)$. Let s_0 be the stage at which m_e reaches its limit. Then at some stage $s > s_0$ we must reach the second instruction of the construction, otherwise $A(m_e) = 1$ but $m_e \notin \mathrm{range}(f_e)$. Suppose that $\varphi_e(x,s) = m_e$ for the minimal $s \geq s_0$ at which we reach the second instruction of the construction. It follows that for $t \geq s$, $n_e = \varphi_e(x,t)$ and $A_t(n_e) = 0$. So $A(f_e(x)) = 0$. This contradiction proves the claim and hence the lemma. $\square$

Remark 3.4 It is possible to make A d.r.e., that is, $A = B - C$ for some r.e. sets B, C. To do so, we have to set aside an interval I_e, roughly of size 2^e, for R_e, $I_0 < I_1 < \cdots$. As a first choice for m_e, we take the maximal element of I_e, and then we proceed downward. The point is that, if R_e is injured by R_i, $i < e$, via $n_i = m_e$, then all further values of n_i are above the next values of m_e (unless R_i injured itself later). Obviously A can be neither r.e. nor co-r.e.

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