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Large Properties at Small Cardinals

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Grandes Propriétés pour Petits Cardinaux

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Don't settle for the horizon,
pursue infinity.
Jim Morrison

Dedicato a Giulio, papà e Federica,
la mia famiglia piccola, ma bella.

Contents

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Laura Fontanella
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Résumé

Les grands cardinaux jouent un rôle central dans la théorie des ensembles contemporaine, car un grand nombre de problèmes qui sont indépendants de la théorie classique ZFC peuvent être résolues sous l'hypothèses que des grands cardinaux existent. D'autre part, l'existence d'un grand cardinal constitue une hypothèse très forte. En effet même la notion la plus faible de grand cardinal, à savoir l'inaccessibilité, entraine l'existence d'un modèle de ZFC, donc d'après le Second Théorème de Gödel, l'existence de tels cardinaux ne peut pas se montrer dans ZFC. Pour cette raison la communauté scientifique ne concorde pas sur leur statut: elle se divise en ceux qui croient que les grands cardinaux existent, ceux qui doutent de leur existence et ceux qui pensent que ça n'a pas de sens d'affirmer ni qu'ils existent, ni qu'ils n'existent pas. Quel que soit notre point de vu, ces considérations mettent en évidence la nécessité de déterminer dans quels contextes ses axiomes sont nécessaires et quand, au contraire, il est possible de les remplacer par des hypothèses plus faibles.

Certaines propriétés de grands cardinaux peuvent s'exprimer en utilisant des notions de combinatoire infinie, comme dans le théorème suivant:

Théorème: Soit κ un cardinal inaccessible, alors

- κ est faiblement compact si et seulement s'il satisfait la propriété d'arbres (Erdös et Tarski [3]);
- κ est fortement compact si et seulement s'il satisfait la propriété d'arbres forte (Di Prisco - Zwicker [18], Donder - Weiss [22] et Jech [8]);
- κ est supercompact si et seulement s'il satisfait la propriété d'arbres super (Donder - Weiss [22], Jech [8] et Magidor [13]).

La propriété d'arbres (tree property) pour un cardinal régulier κ établie que tout κ-arbre (un arbre de hauteur κ et dont les niveaux ont tailles strictement inférieure à κ) a une branche de longueur κ. La propriété d'arbres forte (strong tree property) et la propriété d'arbres super (super tree property) concernent des objet particuliers qu'on appelle (κ, λ)-*arbres.* On peut dire, de façon informelle, qu'un (κ, λ)-arbre est un "arbre sur $[\lambda]^{<\kappa}$" dont les "niveaux" ont taille inférieure à κ. La propriété d'arbres super implique la propriété d'arbres forte, qui entraine la propriété d'arbres usuelle. L'intérêt pour ces propriétés est lié au fait que d'un côté elles

characterizent des grands cardinaux, de l'autre elle peuvent être satisfaite egalement par des petits cardinaux. Le théorème ci-dessu nous dit que d'un point de vu combinatoire, un cardinal qui satisfait l'une de ces propriétés "se comporte comme un grand cardinal".

Les caractérisations de strong compacité et supercompacité en term de propriétés combinatoires remontent aux années 70, mais un étude systématique de ces propriétés a été achevé seulement récemment par Weiss[1] [22]. En travaillant sur la propriété d'arbres super pour $\aleph_2$, Viale et Weiss (voir [21] et [20]) trouvèrent des résultats intéressants sur la force de consistance de l'axiome de forcing propres, PFA. Ils montrèrent que si on force un modèle de PFA avec un forcing qui change κ en ω_2 et satisfait la propriété de κ-recouvrement (κ-covering) et la propriété de κ-approximation (κ-approximation), alors κ est fortement compact; si de plus le forcing est propre, alors κ est supercompact. Comme tout forcing connu qui produit un modèle de PFA en changeant κ en ω_2 satisfait ces conditions, on peut dire que la force de consistence de PFA est un cardinal supercompact.

Il est plutôt naturel de se poser les questions suivantes.

- Quels cardinaux peuvent satisfaire la propriété d'arbres forte ou la propriété d'arbres super?
- Dans quel contextes peut-on remplacer l'hypothèse que des cardinaux fortement compacts ou supercompacts existent par l'assomption plus faible que des cardinaux avec ces propriétés existent?
- Est-t-il possible de caractériser de façon similaire (en terme de propriétés combinatoires) d'autres grands cardinaux?

Les résultats présentés dans cette thèse fournissent une réponse partielle à la première des ces trois questions. Nous citons quelque résultat classique concernant la propriété d'arbres usuelle.

- (König's Lemma) $\aleph_0$ a la propriété d'arbres;
- (Aronszajn) $\aleph_1$ n'a pas la propriété d'arbres;

Pour les cardinaux réguliers plus grands que $\aleph_1$, on ne peut pas montrer dans ZFC ni que la propriété d'arbres est satisfaite, ni qu'elle ne l'est pas.

[1] Les objets qu'on appelle ici (κ, λ)-arbres, sont appelés (κ, λ)-*thin lists* dans la thèse de Weiss [22]. La propriété d'arbres forte pour un cardinal régulier κ coïncide avec la propriété que Weiss appelle "(κ, λ)-TP pour tout $\lambda \geq \kappa$", et la propriété d'arbres super correspond à "(κ, λ)-ITP pour tout $\lambda \geq \kappa$".

- (Specker) Si $\tau^{<\tau} = \tau$, alors τ^+ n'a pas la propriété d'arbres.
- (Mitchell [15]) S'il y a un modèle de ZFC avec un cardinal faiblement compact, alors pour tout cardinal régulier τ tel que $\tau^{<\tau} = \tau$, il y a un modèle de ZFC dans lequel τ^{++} satisfait la propriété d'arbres.

De nombreux mathématiciens se sont dédiés à la recherche de modèles de la théorie des ensembles dans lesquels plusieurs cardinaux réguliers satisfont simultanéement la propriété d'arbres usuelle. Dans cette thèse nous montrerons que certains de ces résultats peuvent se généraliser aux propriété d'arbres forte ou super. Nous citons quelques résultats de ce type.

- (Abraham [1] 1983) S'il y a un modèle de ZFC avec un supercompact et un faiblement compact au dessus du supercompact, alors il y a un modèle de ZFC dans lequel la propriété d'arbres est satisfaite à la fois par $\aleph_2$ et par $\aleph_3$.
- (Cummings and Foreman [2] 1998) Si un modèle de ZFC contenant une infinité de cardinaux supercompacts existe, alors il y a un modèle de ZFC dans lequel tous les $\aleph_n$ (où n est un entier ≥ 2) satisfont la propriété d'arbres.
- (Magidor and Shelah [14] 1996) Suppose l'existence d'un modèle de ZFC avec une suite croissante $\langle\lambda\rangle_{n<\omega}$ telle que
  - (i) si $\lambda = \sup_{n\geq 0} \lambda_n$, alors chaque λ_n est λ^+-supercompact, pour tout $n > 0$;
  - (ii) λ_0 est le point critique d'un plongement élémentaire $j : V \to M$ où $j(\lambda_0) = \lambda_1$ et ${}^{\lambda^+}M \subseteq M$.

  Alors il y a un modèle de ZFC dans lequel $\aleph_{\omega+1}$ a la propriété d'arbres.
- (Sinapova [19] 2012) Assume l'existence d'une infinité de supercompacts, alors il y a un modèle de ZFC dans lequel $\aleph_{\omega+1}$ a la propriété d'arbres.
- (Neeman [17] 2012) Assume l'existence d'une infinité de supercompacts, alors il y a un modèle de ZFC dans lequel la propriété d'arbres est satisfaite par tous les $\aleph_n$ (avec $2 \leq n < \omega$) et par $\aleph_{\omega+1}$.
- (Friedman and Halilovic [7] 2011) Assume l'existence d'une infinité de supercompacts, alors il y a un modèl de ZFC dans lequel $\aleph_{\omega+2}$ a la propriété d'arbres.

Tous ces résultats étaient orientés vers la recherche d'un modèle de ZFC dans lequel tous les cardinaux réguliers satisfont simultanément la propriété d'arbres — l'existence d'un tel modèle reste une question ouverte. Nous

nous intéressons au même probléme pour les propriétés d'arbres forte et super. Weiss montra que pour tout $n \geq 2$, si on force avec le forcing de Mitchell sur un supercompact, on obtient un modèle de la théorie des ensembles dans lequel $\aleph_n$ satisfait même la propriété d'arbres super.

Théorème: (Weiss [22] 2010) Si un modèle de ZFC avec un supercompact existe, alors pour tout entier $n \geq 2$ il y a un modèle de ZFC dans lequel $\aleph_n$ a la propriété d'arbres super.

En utilisant une variation d'un forcing de Abraham (le même qui produit un modèle de la propriété d'arbres pour $\aleph_2$ et $\aleph_3$), on peut obtenir un modèle dans lequel la propriété d'arbres super est satisfaite à la fois par $\aleph_2$ et $\aleph_3$.

Théorème: (Fontanella [5] 2012) Si un modèle de ZFC avec deux supercompacts existe, alors il y a un modèle de ZFC dans lequel la propriété d'arbres super est satisfaite à la fois par $\aleph_2$ et $\aleph_3$.

Ce dernier résultat se généralise à tous les $\aleph_n$ (avec $2 \leq n < \omega$).

Théorème 1 : (Fontanella [4] 2012) Si un modèle de ZFC avec une infinité de supercompacts existe, alors il y a un modèle de ZFC dans lequel tous les $\aleph_n$ (avec $2 \leq n < \omega$) ont la propriété d'arbres super.

Un tel modèle s'obtient en utilisant une itération de forcing qui est due à Cummigs et Foreman (la même qu'ils avaient introduit pour obtenir un modèle de la propriété d'arbres usuelle pour tous les $\aleph_n$). Enfin, on peut montrer que même $\aleph_{\omega+1}$ peut satisfaire la propriété d'arbres forte.

Théorème 2 : (Fontanella [6] 2012) Suppose l'existence d'une infinité de supercompacts, alors il y a un modèle de ZFC dans lequel $\aleph_{\omega+1}$ a la propriété d'arbres forte.

Nous ne savons pas si ce théorème peut se généraliser à la propriété d'arbres super.

Dans cette dissertation nous montrons les Théorèmes 1 et 2 mentionnés ci-dessus. La thèse est structurée comme suit. Dans le Chapitre 1 nous verrons des résultats classiques concernants la propriété d'arbres usuelle et nous définirons les propriété d'arbres forte et super. Au Chapitre 2 nous illustrerons le forcing de Mitchell et nous montrerons que ceci produit un modèle de la propriété d'arbres super au successeur d'un cardinal régulier (c'est à dire nous montrerons le théorème de Weiss ci-dessus). Le

Chapitre 3 est dédié à l'itération de Cummings et Foreman pour un modèle de la propriété d'arbres pour tous les $\aleph_n$. Nous verrons au Chapitre 4 que dans ce modèle les $\aleph_n$ satisfont également la propriété d'arbres super. La thèse se termine avec le Chapitre 5 qui concerne la consistance de la propriété d'arbres super pour $\aleph_{\omega+1}$.

Introduction

Since its origins, set theory has strived for a fundamental target, namely to provide mathematics with its foundation. Large cardinals play a central role in this foundational programme, since many problems which are independent from classical set theory, ZFC, can be solved under large cardinal hypotheses. Nevertheless, the existence of a large cardinal corresponds to a very strong assumption. Indeed, even the weakest large cardinal notion, inaccessibility, yields a model of ZFC, hence by Gödel Second Incompleteness Theorem the existence of such cardinals cannot be proven within set theory. The scientific community is therefore divided on their epistemological status. Some set theorists believe large cardinal axioms are "true", others think they are "illegitimate" or even "false", and formalists say it does not make sense to claim they are either true or false. Aside from any philosophical position, those considerations suggest the importance of undertaking a systematic analysis of these cardinals that would clarify in what context they are really necessary and when, on the contrary, they can be replaced by weaker assumptions.

Some large cardinal notions can be characterized in terms of combinatorial properties (e.g. partition properties or infinite trees properties) like in the following theorem.

Theorem: Assume κ is an inaccessible cardinal, then

- κ is weakly compact if and only if it satisfies the tree property (Erdös and Tarski [3]);
- κ is strongly compact if and only if it satisfies the strong tree property (Di Prisco - Zwicker [18], Donder - Weiss [22] and Jech [8]);
- κ is supercompact if and only if it satisfies the super tree property (Donder - Weiss [22], Jech [8] and Magidor [13]).

Given a regular cardinal κ, we say that κ has the tree property when every κ-tree (i.e. every tree of height κ with levels of size less than κ) has a branch of length κ. The strong and super tree properties concern special objects known as (κ, λ)-*trees*. Roughly speaking, a (κ, λ)-tree is a "tree over $[\lambda]^{<\kappa}$" whose "levels" have size less than κ (this notion will be defined in Chapter 1). The super tree property implies the strong tree property, that entails the usual tree property in its turn. Our interest for these properties is motivated by the fact that they can be satisfied even by small cardinals. It follows from the theorem above, that any cardinal satisfying one of the previous properties is "large" from a combinatorial point of view.

While the previous characterizations date back to the early 1970s, a systematic study of the strong and the super tree properties has only recently been undertaken by Weiss[2] [22]. By working on the super tree property at $\aleph_2$, Viale and Weiss (see [21] and [20]) obtained new results about the consistency strength of the Proper Forcing Axiom (PFA). They proved that if one forces a model of PFA using a forcing that makes κ become ω_2 and satisfies the κ-covering and the κ-approximation properties, then κ has to be strongly compact; if the forcing is also proper, then κ is supercompact. Since every known forcing producing a model of PFA by collapsing κ to ω_2 satisfies those conditions, we can say that the consistency strength of PFA is, reasonably, a supercompact cardinal. Several natural questions arise:

- What cardinals can satisfy the strong or the super tree properties?
- How can we use the "strong compactness" or "supercompactness" of small cardinals satisfying the strong or the super tree properties?
- Is it possible to find analogous combinatorial characterizations of other large cardinals?

The results presented in this thesis partially answer the first of the above questions. We list a few classical results concerning the usual tree property.

- (König's Lemma) $\aleph_0$ has the tree property;
- (Aronszajn) $\aleph_1$ does not have the tree property.

For larger regular cardinals, we cannot prove within ZFC that the tree property holds or fails.

- (Specker) If $\tau^{<\tau} = \tau$, then the tree property fails at τ^+;
- (Mitchell [15]) If there is a model of ZFC with a weakly compact cardinal, then for every regular τ such that $\tau^{<\tau} = \tau$ there is a model of ZFC where τ^{++} has the tree property.

Many mathematicians worked on the construction of models of set theory in which distinct regular cardinals simultaneously satisfy the usual tree property. In this thesis, we prove that some of these results can be generalized to the strong or the super tree property. We list a few classical results of this sort.

[2]What we call (κ, λ)-tree is called (κ, λ)-*thin list* in Weiss' Phd dissertation [22]. The strong tree property at a regular cardinal κ is the property (κ, λ)-TP for all $\lambda \geq \kappa$, while the super tree property corresponds to (κ, λ)-ITP for all $\lambda \geq \kappa$.

- (Abraham [1] 1983) Assume there is a model of ZFC with a supercompact cardinal and a weakly compact cardinal above it, then there is a model of ZFC where both $\aleph_2$ and $\aleph_3$ have the tree property.

- (Cummings and Foreman [2] 1998) Assume there is a model of ZFC with infinitely many supercompact cardinals, then there is a model of ZFC where every cardinal of the form $\aleph_n$ with $2 \leq n < \omega$ has the tree property.

- (Magidor and Shelah [14] 1996) Assume there is a model of ZFC with an increasing sequence $\langle \lambda_n \rangle_{n<\omega}$ such that

  (i) if $\lambda = \sup_{n \geq 0} \lambda_n$, then λ_n is λ^+-supercompact, for all $n > 0$;

  (ii) λ_0 is the critical point of an embedding $j : V \to M$ where $j(\lambda_0) = \lambda_1$ and ${}^{\lambda^+}M \subseteq M$.

  Then there is a model of ZFC where $\aleph_{\omega+1}$ has the tree property.

- (Sinapova [19]) Assume there is a model of ZFC with infinitely many supercompact cardinals, then there is a model of ZFC where $\aleph_{\omega+1}$ has the tree property.

- (Neeman [17]) Assume there is a model of ZFC with infinitely many supercompact cardinals, then there is a model of ZFC where the tree property holds at every $\aleph_n$ with $n \geq 2$ and at $\aleph_{\omega+1}$.

- (Friedman and Halilovic [7]) Assume there is a model of ZFC with infinitely many supercompact cardinals, then there is a model of ZFC where the tree property holds at $\aleph_{\omega+2}$.

All these results were oriented toward the construction of a model where the tree property holds simultaneously at every regular cardinal — whether such a model can be found is still an open question. We want to investigate the same problem for the strong and the super tree properties. Weiss proved that for every integer $n \geq 2$, if we force with Mitchell's forcing over a supercompact cardinal, we get a model of set theory where even the super tree property holds at $\aleph_n$.

Theorem: (Weiss [22]) Assume there is a model of ZFC with a supercompact cardinal, then for every integer $n \geq 2$ there is a model of ZFC where $\aleph_n$ has the super tree property.

By considering a variation of Abraham's forcing (the one that produces a model of the usual tree property at $\aleph_2$ and $\aleph_3$) we can obtain a model where both $\aleph_2$ and $\aleph_3$ have the super tree property.

Theorem: (Fontanella [5]) Assume there is a model of ZFC with two supercompact cardinals, then there is a model of ZFC where both $\aleph_2$ and $\aleph_3$ have the super tree property.

This result can be generalized to all the $\aleph_n$ with $2 \leq n < \omega$. Indeed, one can prove that in the Cummings-Foreman's model the $\aleph_n$'s satisfy even the super tree property.

Theorem 1 : (Fontanella [4]) Assume there is a model of ZFC with infinitely many supercompact cardinals, then there is a model of ZFC where every cardinal $\aleph_n$ with $2 \leq n < \omega$ has the super tree property.

We can go further with our analysis and prove that even $\aleph_{\omega+1}$ can consistently satisfy the strong tree property.

Theorem 2 : (Fontanella [6]) Assume there is a model of ZFC with infinitely many supercompact cardinals, then there is a model of ZFC where $\aleph_{\omega+1}$ has the strong tree property.

Whether Theorem 2 can be generalized to the super tree property is still an open question.

In this dissertation we prove Theorem 1 and Theorem 2 above. The thesis is organized as follows. In Chapter 1 we discuss some general facts about the tree property and we give the definition of the strong and super tree properties at a regular cardinal. In Chapter 2 we analyze Mitchell's forcing construction and we show that it produces a model of the super tree property at the double successor of a regular cardinal (this is Weiss' theorem above). Chapter 3 is devoted to Cummings-Foreman's forcing iteration for a model of the tree property at every $\aleph_n$. We prove in Chapter 4 that in such a model even the super tree property holds at every $\aleph_n$ (Theorem 1). Finally, in Chapter 5, we prove the consistency of the strong tree property at $\aleph_{\omega+1}$ (Theorem 2).

The reader who is familiar with the tree property can skip the first two chapters except Section 1.3 that contains the definition of the strong and super tree properties. The reader who is interested only in the main results of [4] can read just Section 1.3, Section 2.2, Chapter 3 and Chapter 4. The reader who is interested only in the main results of [6] can read just Section 1.3 and Chapter 5.

Preliminaries

It may be useful to recall some terminology. The main reference for basic set theory is [8], while we will refer to [10] for large cardinals notions and to [11] for the forcing technique. The notation is standard and it is summarized at the end of this thesis.

Generalities

We recall the definition of closed unbounded subset of $[A]^{<\kappa}$ (club).

Definition I.0.1. *Assume κ is a cardinal, A is a set of size $\geq \kappa$ and $C \subseteq [A]^{<\kappa}$.*

(i) C is unbounded *if for every $x \in [A]^{<\kappa}$ there exists $y \in C$ such that $x \subseteq y$.*

(ii) C is closed *if for any $\subseteq$-increasing chain $\langle x_\gamma \rangle_{\gamma<\alpha}$ of sets in C, the union $\bigcup_{\gamma<\alpha} x_\gamma \in C$.*

(iii) C is a club *if it is closed and unbounded.*

(iv) C is stationary *if S has non-empty intersection with every club of $[A]^{<\kappa}$.*

The following lemma is known as the Δ-*system Lemma.*

Lemma I.0.2. *Assume that λ is a regular cardinal and $\kappa < \lambda$ is such that $\alpha^{<\kappa} < \lambda$, for every $\alpha < \lambda$. Let $\mathscr{F}$ be a family of sets of cardinality less than κ such that $|\mathscr{F}| = \lambda$. There exists a family $\mathscr{F}' \subseteq \mathscr{F}$ of size λ and a set R such that $X \cap Y = R$, for any two distinct $X, Y \in \mathscr{F}'$.*

We say that $\mathscr{F}'$ forms a Δ-*system* of *root* R For a proof of that lemma the reader can consult Jech's book [9, Theorem 9.19].

Lemma I.0.3. *(Pressing Down Lemma) If f is a regressive function on a stationary set $S \subseteq [A]^{<\kappa}$ (i.e. $f(x) \in x$, for every non empty $x \in S$), then there exists a stationary set $T \subseteq S$ such that f is constant on T.*

For a proof of that lemma see [9, Theorem 8.24].

Definition I.0.4. *Let η be a regular cardinal and $\theta > \eta$. Given $M \prec H_\theta$ of size η, we say that M is* internally approachable of length η *if it can be written as the union of an increasing continuous chain $\langle M_\xi : \xi < \eta \rangle$ of elementary submodels of H_θ of size less than η, such that for every $\eta' < \eta$, we have $\langle M_\xi : \xi < \eta' \rangle \in M_{\eta'+1}$.*

Note that the set $\{M \prec H_\theta;\ M \text{ is internally approachable of length } \eta\}$ is a stationary subset of $[H_\theta]^\eta$.

Sometime we will work with models of set theory having the following property.

Definition I.0.5. *If $V \subseteq W$ are two models of set theory with the same ordinals and η is a cardinal in W, we say that (V, W) has the* η-covering property *if and only if every set $X \subseteq V$ in W of cardinality less than η in W, is contained in a set $Y \in V$ of cardinality less than η in V.*

We also say that a forcing $\mathbb{P}$ has the η-covering property if every generic extension W by $\mathbb{P}$ is such that (V, W) has the η-covering property.

Forcing

Given a forcing $\mathbb{P}$ and conditions $p, q \in \mathbb{P}$, we use $p \leq q$ in the sense that p is stronger than q; we write $p||q$ when p and q are two compatible conditions (i.e. there is a condition $r \in \mathbb{P}$ such that $r \leq p$ and $r \leq q$).

Definition I.0.6. *A poset $\mathbb{P}$ is* separative *if whenever $q \not\leq p$, then some extension of q in $\mathbb{P}$ is incompatible with p.*

Every partial order can be turned into a separative poset. Indeed, one can define $p \prec q$ iff all extensions of p are compatible with q, and the resulting equivalence relation is given by $p \sim q$ iff $p \prec q$ and $q \prec p$, provides a separative poset. Then the set of all equivalence classes of $\mathbb{P}$ is separative. Assume that $\mathbb{P}$ is a forcing notion in a model V, we will use $V^{\mathbb{P}}$ to denote the class of $\mathbb{P}$-names. If $G \subseteq \mathbb{P}$ is a generic filter over V, then $V[G]$ denotes the generic extension of V determined by G. If $a \in V^{\mathbb{P}}$ and $G \subseteq \mathbb{P}$ is generic over V, then a^G denotes the interpretation of a in $V[G]$. Every element x of the ground model V is represented in a canonical way by a name $\check{x}$. However, to simplify the notation, we will use just x instead of $\check{x}$ in forcing formulas. The set $\dot{G} := \{(\check{p}, p);\ p \in \mathbb{P}\} \in V^{\mathbb{P}}$ is called the *canonical name for a generic filter for* $\mathbb{P}$, thus for every filter $G \subseteq \mathbb{P}$ generic over V, the interpretation of $\dot{G}$ in V^G is precisely G.

Definition I.0.7. *Given a forcing $\mathbb{P}$, we say that*

(i) $\mathbb{P}$ *is* κ-closed *if and only if every decreasing sequence of conditions of $\mathbb{P}$ of size less than κ has an infimum;*

(ii) $\mathbb{P}$ *is* κ-directed closed *if and only if for every set of less than κ pairwise compatible conditions of $\mathbb{P}$ has an infimum;*

(iii) $\mathbb{P}$ *is* κ-distributive *if and only if no sequence of ordinals of length less than κ is added by $\mathbb{P}$.*

(iv) $\mathbb{P}$ *is* κ-c.c. *when every antichain of* $\mathbb{P}$ *has size less than* κ;

(v) $\mathbb{P}$ *is* κ-Knaster *if and only if for all sequence of conditions* $\langle p_\alpha;\ \alpha < \kappa\rangle$, *there is* $X \subseteq \kappa$ *cofinal such that the conditions of the sequence* $\langle p_\alpha;\ \alpha \in X\rangle$ *are pairwise compatible.*

Given two forcings $\mathbb{P}$ and $\mathbb{Q}$, we will write $\mathbb{P} \equiv \mathbb{Q}$ when $\mathbb{P}$ and $\mathbb{Q}$ are equivalent, namely:

(i) for every filter $G_{\mathbb{P}} \subseteq \mathbb{P}$ which is generic over V, there exists a filter $G_{\mathbb{Q}} \subseteq \mathbb{Q}$ which is generic over V, and $V[G_{\mathbb{P}}] = V[G_{\mathbb{Q}}]$;

(ii) for every filter $G_{\mathbb{Q}} \subseteq \mathbb{Q}$ which is generic over V, there exists a filter $G_{\mathbb{P}} \subseteq \mathbb{P}$ which is generic over V, and $V[G_{\mathbb{P}}] = V[G_{\mathbb{Q}}]$.

If $\mathbb{P}$ is any forcing and $\dot{\mathbb{Q}}$ is a $\mathbb{P}$-name for a forcing, consider the class of all $(p, q) \in \mathbb{P} \times V^{\mathbb{P}}$ such that $p \Vdash q \in \dot{\mathbb{Q}}$. We define an ordering on the elements of this class by setting $(p, q) \leq (p', q')$ if and only if $p \leq p'$ and $p \Vdash q \leq q'$. Then $\mathbb{P} * \dot{\mathbb{Q}}$ denotes the set of all equivalence classes (corresponding to this ordering) of minimal rank.

Theorem I.0.8. *(Product Lemma) Assume* $\mathbb{P}$ *and* $\mathbb{Q}$ *are two forcing notions in* V. *For every* $G_P \subseteq \mathbb{P}$ *and* $G_Q \subseteq \mathbb{Q}$, *the following are equivalent:*

(i) $G_P \times G_Q$ *is generic for* $\mathbb{P} \times \mathbb{Q}$ *over* V;

(ii) G_P *is generic for* $\mathbb{P}$ *over* V *and* G_Q *is generic for* $\mathbb{Q}$ *over* $V[G_P]$;

(iii) G_Q *is generic for* $\mathbb{Q}$ *over* V *and* G_P *is generic for* $\mathbb{P}$ *over* $V[G_Q]$.

Furthermore, if (1)–(3) *holds, then* $V[G_P \times G_Q] = V[G_P][G_Q] = V[G_Q][G_P]$ *and we say that* G_P *and* G_Q *are* mutually generic.

For a proof of the previous theorem see for example [11, Theorem 1.4., Ch. VIII].

Lemma I.0.9. *(Easton's Lemma) Let* κ *be regular. If* $\mathbb{P}$ *has the* κ*-chain condition and* $\mathbb{Q}$ *is* κ*-closed, then*

(i) $\Vdash_{\mathbb{Q}} \mathbb{P}$ *has the* κ*-chain condition*;

(ii) $\Vdash_{\mathbb{P}} \mathbb{Q}$ *is a* κ*-distributive*;

(iii) If G *is* $\mathbb{P}$*-generic over* V *and* H *is* $\mathbb{Q}$*-generic over* V, *then* $(V, V[G][H])$ *has the* κ*-covering property;*

(iv) If $\mathbb{R}$ *is* κ*-closed, then* $\Vdash_{\mathbb{P}\times\mathbb{Q}}\ \mathbb{R}$ *is* κ*-distributive.*

For a proof of that lemma see [2, Lemma 2.11].

Classical Forcing Constructions

We will use the following forcing notions.

Definition I.0.10. *(Cohen) Let $\tau < \kappa$ be two regular cardinals.*

(i) $\mathrm{Add}(\tau, \kappa)$ *is the set of all $p : \kappa \to 2$ of size $< \tau$, partially ordered by reverse inclusion.*

(ii) $\mathrm{Add}(\kappa)$ *denotes* $\mathrm{Add}(\kappa, \kappa)$.

If $\mathbb{P} := \mathrm{Add}(\kappa, \lambda)$ and $\eta < \lambda$ we will often use $\mathbb{P} \restriction \eta$ for $\mathrm{Add}(\kappa, \eta)$. The poset $\mathrm{Add}(\kappa, \lambda)$ is κ-directed closed and it is $(2^{<\kappa})^+$-Knaster.

Definition I.0.11. *(The Lévy Collapse) Let $\kappa < \lambda$ be two cardinals with κ regular,*

(i) we denote by $\mathrm{Coll}(\kappa, \lambda)$ *the set* $\{p : \kappa \to \lambda;\ |\mathrm{dom}(p)| < \kappa\}$ *ordered by reverse inclusion;*

(ii) if λ is inaccessible, then $\mathrm{Coll}(\kappa, < \lambda) := \Pi_{\alpha<\lambda}\mathrm{Coll}(\kappa, \alpha)$.

Lemma I.0.12. *(Lévy) Let $\kappa < \lambda$ be two cardinals with κ regular, then* $\mathrm{Coll}(\kappa, \lambda)$ *collapses λ onto κ, i.e. λ has cardinality κ in the generic extension. Moreover,*

(i) every cardinal $\alpha \leq \kappa$ in V remains a cardinal in $V[G]$;

(ii) if $\lambda^{<\kappa} = \lambda$, then every cardinal $\alpha > \lambda$ remains a cardinal in the extension.

For a proof of that lemma see for example [8, Lemma 15.21].

Lemma I.0.13. *(Lévy) If κ is regular and $\lambda > \kappa$ is inaccessible. Then for every $G \subseteq$* $\mathrm{Coll}(\kappa, < \lambda)$ *generic over V,*

(i) every α such that $\kappa \leq \alpha < \lambda$ has cardinality κ in $V[G]$;

(ii) every cardinal $\leq \kappa$ and every cardinal $\geq \lambda$ remains a cardinal in $V[G]$.

Hence $V[G] \models \lambda = \kappa^+$.

For a proof of that lemma see for example [8, Theorem 15.22].

Projections

Definition I.0.14. *If $\mathbb{P}$ and $\mathbb{Q}$ are two posets, a* projection $\pi : \mathbb{P} \to \mathbb{Q}$ *is a function such that:*

(i) for all $p, p' \in \mathbb{P}$ if $p \leq p'$, then $\pi(p) \leq \pi(p')$;

(ii) $\pi(1_{\mathbb{P}}) = 1_{\mathbb{Q}}$;

(iii) for all $p \in \mathbb{P}$ if $q \leq \pi(p)$, then there is $p' \leq p$ such that $\pi(p') \leq q$.

We say that $\mathbb{P}$ is a projection of $\mathbb{Q}$ *when there is a projection* $\pi : \mathbb{Q} \to \mathbb{P}$.

Lemma I.0.15. *If $\pi : \mathbb{P} \to \mathbb{Q}$ is a projection, the following hold.*

(i) If $G_{\mathbb{P}} \subseteq \mathbb{P}$ is a generic filter over V, then $H := \{q \in \mathbb{Q};\ \exists p \in G\ (\pi(p) \leq q)\}$ is generic over V.

(ii) If $H \subseteq \mathbb{Q}$ is a generic filter over V, we define $\bar{\mathbb{P}} := \{p;\ \pi(p) \in H\}$ ordered as a suborder of $\mathbb{P}$. Then $\mathbb{P}$ is non-empty, and if G is generic for $\bar{\mathbb{P}}$ over $V[H]$, then G is generic for $\mathbb{P}$ over V. Moreover, $\pi[G]$ generates H.

(iii) Let $G \subseteq \mathbb{P}$ be generic and define H as in 1 and $\bar{\mathbb{P}}$ as in 2. Then $G \subseteq \bar{\mathbb{P}}$ and it is generic over $V[H]$. In other words, we can factor forcing with $\mathbb{P}$ as forcing with $\mathbb{Q}$ followed by forcing with $\bar{\mathbb{P}}$ over $V[G_{\mathbb{Q}}]$ (where $G_{\mathbb{Q}} \subseteq \mathbb{Q}$ is a generic filter over V).

Some of our projections $\pi : \mathbb{Q} \to \mathbb{P}$ will also have the following stronger property.

Definition I.0.16. *Let $\mathbb{P}$ and $\mathbb{Q}$ be two forcings. We say that $\pi : \mathbb{P} \to \mathbb{Q}$ is a* good projection *if and only if*

(i) $p \leq p'$ implies $\pi(p) \leq \pi(p')$;

(ii) $\pi(1_{\mathbb{P}}) = 1_{\mathbb{Q}}$;

(iii) for all $p \in \mathbb{P}$ and $q \leq \pi(p)$ there is $p' \leq p$ such that

(a) $\pi(p') = q$;

(b) for all $r \leq p$ if $\pi(r) \leq q$, then $r \leq p'$.

Condition p' in part 3 of the previous definition is essentially unique, in fact if p'' also has these properties then $p' \leq p'' \leq p'$. We denote by $\mathrm{Ext}(p, q)$ some extension of p with these properties.

Lemma I.0.17. *Let $\pi : \mathbb{P} \to \mathbb{Q}$ be a good projection and suppose that $H \subseteq \mathbb{Q}$ is a generic filter over V. We let*

$$\bar{\mathbb{P}} := \{p \in \mathbb{P};\ \pi(p) \in H\}$$

and define an ordering on $\leq^*$ on $\bar{\mathbb{P}}$ by letting

$$p \leq^* q \iff \exists r \leq \pi(p)\ (r \in H \wedge \mathrm{Ext}(p, r) \leq q).$$

Then forcing over $V[H]$ with $\bar{\mathbb{P}}$ ordered as a subset of $\mathbb{P}$ is equivalent to forcing over $V[H]$ with $\bar{\mathbb{P}}$ ordered by $\leq^*$.

For example if $\mathbb{P}$ is an iteration, $\mathbb{Q} = \mathbb{P} \upharpoonright \beta$ is an initial segment of $\mathbb{P}$, and π is the map $p \mapsto p \upharpoonright \beta$, then $\mathrm{Ext}(p, q) = q \frown (p \upharpoonright \mathrm{dom}(p) - \beta)$.

Suppose that $\mathbb{P}$ and $\mathbb{Q}$ are two forcings with the same underlying set but different orderings, and suppose that the identity function projects $\mathbb{P}$ into $\mathbb{Q}$. If $G \subseteq \mathbb{P}$ is a generic filter, G is directed as a subset of $\mathbb{Q}$ and generates a generic filter H for $\mathbb{Q}$. If $H \subseteq \mathbb{Q}$ is a generic filter, then forcing with H considered as a suborder of $\mathbb{P}$ produces a generic filter $G \subseteq \mathbb{P}$ such that $G \subseteq H$ and G generates H.

Elementary Embeddings

We will use the following lemma repeatedly and without comments.

Lemma I.0.18. *(Silver) Let $j : M \to N$ be an elementary embedding between inner models of* ZFC*. Let $\mathbb{P} \in M$ be a forcing and suppose that G is $\mathbb{P}$-generic over M, H is $j(\mathbb{P})$-generic over N, and $j[G] \subseteq H$. Then there is a unique $j^* : M[G] \to N[H]$ such that $j^* \upharpoonright M = j$ and $j^*(G) = H$.*

Proof. If $j[G] \subseteq H$, then the map $j^*(\dot{x}^G) = j(\dot{x})^H$ is well defined and satisfies the required properties. □

The following theorem by Laver [12] will be deeply used in the following chapters.

Theorem I.0.19. *(Laver) Let κ be a supercompact cardinal. There exists a* Laver function*, namely a function $L : \kappa \to V_\kappa$ satisfying the following property for every cardinal $\lambda \geq \kappa$. Given a set x with $|\mathrm{t.c.}(x)| \leq \lambda$, there exists a λ-supercompact embedding $j : V \to M$ with critical point κ such that $j(L)(\kappa) = x$.*

For a proof of this theorem see for example [9, Theorem 20.21, Ch. 20].

1

Tree Properties

In this chapter we discuss the tree property, the strong tree property and the super tree property as well as their connection with large cardinals.

1.1 Compactness

There are several ways to define weakly compact, strongly compact and supercompact cardinals. We focus on their characterizations in terms of properties of filters. We recall the definition of a filter on a given set.

Definition 1.1.1. *A* filter *on a set S is a collection F of subsets of S such that*

(i) $S \in F$ *and* $\emptyset \notin F$,

(ii) *if* $X, Y \in F$, *then* $X \cap Y \in F$,

(iii) *if* $X \in F$, *then for every* $Y \supseteq X$ *in* $\mathscr{P}(S)$, *we have* $Y \in F$.

More generally we can define the notion of S-filter for a given family of sets S.

Definition 1.1.2. *Let S be a non-empty family of sets and let $F \subseteq S$, we say that F is an S-*filter *if it satisfies*

(i) $\emptyset \notin F$,

(ii) *if* $X, Y \in F$, *then* $X \cap Y \in F$,

(iii) *for every* $X, Y \in S$ *if* $X \in F$ *and* $X \subseteq Y$, *then* $Y \in F$.

In order to define weak compactness, strong compactness and supercompactness we need to talk about κ-complete filters and normal filters.

Definition 1.1.3. *Let κ be a regular uncountable cardinal and let $F \subset \mathscr{P}(\kappa)$, we say that F is κ-*complete *if and only if it is closed under intersection of less than κ-many sets, i.e. for every $\gamma < \kappa$ and for every family $\{X_\delta;\ \delta < \alpha\}$ of sets in F, the intersection $\bigcap_{\delta<\alpha} X_\delta$ is in F.*

Definition 1.1.4. *Let κ be a regular uncountable cardinal and let F be a filter over $[S]^{<\kappa}$. We say that F is* normal *if and only if*

(i) F it is κ-complete and for every $s \in S$, the set $\{x \in [S]^{<\kappa};\ s \in x\} \in F$;

(ii) F is closed under diagonal intersections $\Delta_{s\in S} X_s := \{x \in [S]^{<\kappa};\ x \in \bigcap_{s\in x} X_s\}$.

Definition 1.1.5. *Let κ be a regular uncountable cardinal.*

(i) κ is weakly compact *if and only if for every κ-complete family $S \subseteq \mathscr{P}(\kappa)$ of size κ, every κ-complete S-filter F can be extended to a κ-complete S-filter that decides S, i.e. for every $x \in S$ with $\kappa - x \in S$, either x or $\kappa - x$ is in the filter;*

(ii) κ is strongly compact *if and only if every κ-complete filter on a set S can be extended to a κ-complete ultrafilter on S;*

(iii) κ is supercompact *if and only if for every set S of size at least κ, there exists a normal ultrafilter on $[S]^{<\kappa}$.*

It should be clear from the previous definition that supercompactness implies strong compactness, which implies weak compactness. Moreover weakly compact cardinals are (strongly) inaccessible. For supercompact cardinals we will mainly use the following characterization in terms of elementary embeddings.

Theorem 1.1.6. *A cardinal κ is* supercompact *if and only if for every cardinal $\lambda \geq \kappa$, there exists an elementary embedding $j : V \to M$ with critical point κ such that*

(i) $j(\kappa) > \lambda$,

(ii) ${}^{\lambda}M \subseteq M$, i.e. M is closed by subsets of size λ.

Such an embedding is called a λ-supercompact embedding.

For more details on elementary embeddings of the universe, the reader can consult [10, §4, Ch. 1].

Lemma 1.1.7. *If κ is a supercompact cardinal and $j : V \to M$ is a λ-supercompact embedding with critical point κ, then the corresponding ultrafilter U on $[\lambda]^{<\kappa}$ contains every club of $[\lambda]^{<\kappa}$.*

Proof. U is defined by $X \in U \iff j[\lambda] \in j(X)$ and it is a normal ultrafilter on $[\lambda]^{<\kappa}$. Let $C \subseteq [\lambda]^{<\kappa}$ be a club, we want to show that $j[\lambda] \in j(C)$. Consider the set $D := j(C) \cap [\ j[\lambda]\ ]^{<\kappa}$, then $D \in M$. Since C is unbounded, $\bigcup D = j[\lambda]$ and D satisfies the property that for every $X, Y \in D$ there is $Z \in D$ such that $X \cup Y \subseteq Z$. So there exists in M a $\subseteq$-increasing chain $\langle X_\alpha \rangle_{\alpha<\lambda}$ of elements of D such that $\bigcup_{\alpha<\lambda} X_\alpha = j[\lambda]$. Now $j(C)$ is a club (by elementarity) and $\lambda < j(\kappa)$ so $j[\lambda] = \bigcup_{\alpha<\lambda} X_\alpha \in j(C)$. □

1.2 The Tree Property

In this section we discuss some well-known results concerning the tree property.

Definition 1.2.1. *A* tree *is a partial order* $\langle T, < \rangle$ *such that for every* $t \in T$ *the set* $\{s \in T;\ s < t\}$ *of* predecessors *of* t, *or* $\text{pred}_T(t)$, *is well-ordered by* $<$. *The elements of* T *are called* nodes.

Definition 1.2.2. *Given a tree* $\langle T, < \rangle$ *and a subset* $T \subseteq T^*$, *we say that* $\langle T^*, < \rangle$ *is a* sub-tree *of* $\langle T, < \rangle$ *if* T^* *is closed by* $<$, *namely for every* $s, t \in T$ *if* $t \in T^*$ *and* $s < t$, *then* $s \in T^*$.

We shall abuse notation and refer to T when we mean $\langle T, < \rangle$.

Definition 1.2.3. *Let* T *be a tree.*

(a) *The* α-th level *of* T, *or* $\text{Lev}_\alpha(T)$, *is the set* $\{t \in T;\ \text{o.t.}(\text{pred}_T(t)) = \alpha\}$.

(b) *The* height *of* T, $\text{ht}(T)$, *is the least* α *such that* $\text{Lev}_\alpha(T) = \emptyset$.

(c) *A* branch *of* T *is a maximal linearly ordered subset of* T.

(d) *A* cofinal branch *of* T *is a branch* b *such that* $b \cap \text{Lev}_\alpha(T) \neq \emptyset$, *for every* $\alpha < \text{ht}(T)$.

When there is no ambiguity, we will simply write Lev_α instead of $\text{Lev}_\alpha(T)$. Given $\alpha < \beta$ and $t \in \text{Lev}_\beta$, we denote by $t \upharpoonright \alpha$ the unique predecessor of t in Lev_α. Given two nodes $s, t \in T$, we say that s and t are *comparable* when there is $u \in T$ such that $s < u$ and $t < u$ — by definition of tree, this is equivalent to $s < t$ or $t < s$.

Definition 1.2.4. *Let* κ *be a regular cardinal.*

(i) *A* κ-tree *is a tree* T *of height* κ *such that every level of* T *has size less than* κ.

(ii) *We say that* κ *has the* tree property *if, and only if, every* κ*-tree has a cofinal branch.*

The tree property provides a nice characterization of weakly compact cardinals.

Theorem 1.2.5. *(Erdös and Tarski [3]) An inaccessible cardinal* κ *is weakly compact if and only if it satisfies the tree property.*

Proof. First we prove the forward direction. Let T be a κ-tree and for every $\alpha < \kappa$ let $\gamma_\alpha < \kappa$ be the size of Lev_α. We fix an enumeration $\text{Lev}_\alpha = \{t_i^\alpha; i < \gamma_\alpha\}$ of every level of the tree and we choose nodes $s_\alpha \in \text{Lev}_\alpha$, for every $\alpha < \kappa$. Let S be a family of subset of κ containing all the sets of the form

$$R_{\alpha,I,a} := \{\beta > \alpha;\ \exists i \in I\ (s_\beta \restriction \alpha = t_i^\alpha)\} \cup a,$$

where $\alpha < \kappa$, $I \subseteq \gamma_\alpha$ and $a \subseteq \alpha$. In other words $R_{\alpha,I,a}$ is the set of all β's above α such that $s_\beta \restriction \alpha$ has index in I, plus a subset of $a \subseteq \alpha$. As κ is inaccessible, S has size κ — for every $\alpha < \kappa$, the are 2^{γ_α} many sets $I \subseteq \gamma_\alpha$ and 2^α many subets $a \subseteq \alpha$. If we close S by intersection of less than κ sets, we obtain a κ-complete family of size κ; we rename it S. Let F be the S-filter generated by the final segments of κ (i.e. $F := \{a \in S;\ \exists\alpha < \kappa\ (a \supseteq \kappa - \alpha)\}$). Note that F is a (non-empty) κ-complete S-filter. Since κ is weakly compact we can extend F to a κ-complete S-filter U that decides S. For every $\alpha < \kappa$ and $i < \gamma_\alpha$ consider $D_i^\alpha := \{\beta > \alpha;\ s_\beta \restriction \alpha = t_i^\alpha\}$ (both D_i^α and $\kappa - D_i^\alpha$ are in S). We claim that there exists $i_\alpha < \gamma_\alpha$ such that $D_{i_\alpha}^\alpha \in U$. Assume towards a contradiction that for every $i < \gamma$ we have $D_i^\alpha \notin U$, then $\kappa - D_i^\alpha \in U$. As $F \subseteq U$, we have $E_i^\alpha := (\kappa - D_i^\alpha) \cap (\kappa - \alpha) \in U$, hence $\bigcap_{i<\gamma_\alpha} E_i^\alpha \in U$ by κ-completeness of U. Since $E_i^\alpha = \{\beta > \alpha;\ s_\beta \restriction \alpha \neq t_i^\alpha\} \in U$ we have $\bigcap_{i<\gamma_\alpha} E_i^\alpha = \emptyset$, that contradicts $\emptyset \notin U$. We prove that the set $b := \{t_{i_\alpha}^\alpha\}_{\alpha<\kappa}$ is a linearly ordered subset of T. Indeed for every $\alpha, \beta < \kappa$, we have $D_{i_\alpha}^\alpha \cap D_{i_\beta}^\beta \in U$ so we can pick $\gamma \in D_{i_\alpha}^\alpha \cap D_{i_\beta}^\beta$. By definition $s_\gamma \restriction \alpha = t_{i_\alpha}^\alpha$ and $s_\gamma \restriction \beta = t_{i_\beta}^\beta$ hence $t_{i_\alpha}^\alpha$ and $t_{i_\beta}^\beta$ are comparable nodes. From b we can define a cofinal branch $b' := \{t \restriction \alpha;\ t \in b \text{ and } \alpha < \mathrm{ht}(T)\}$.

For the converse, assume κ is inaccessible and satisfies the tree property, we prove that κ is weakly compact. Let $S \subseteq \mathscr{P}(\kappa)$ be a set of size κ and let F be a κ-complete S-filter. Assume $S := \{s_\alpha;\ \alpha < \kappa\}$, observe that for every $\alpha < \kappa$ if the set $F_\alpha := \{s_\beta \in F;\ \beta < \alpha\}$ is non-empty, then $\bigcap F_\alpha$ is non empty because F is κ-complete. We define a tree T^* as follows. Let T be the set of all functions $t : \alpha \to 2$ such that (F_α is non-empty) and for some $\gamma \in \bigcap F_\alpha$ we have $t(\beta) = 1$ if and only if $\gamma \in s_\beta$. We let

$$T^* := \{t \restriction \alpha;\ t \in T \text{ and } \alpha < \mathrm{dom}(t)\}$$

ordered by extension. Since κ is inaccessible, T^* is a κ-tree. By hypothesis there exists a cofinal branch $\{b_\alpha\}_{\alpha<\kappa}$ such that $b_\alpha \in \mathrm{Lev}_\alpha$ for every $\alpha < \kappa$. Let $b : \kappa \to 2$ be $\bigcup_\alpha b_\alpha$, we define

$$U := \{s_\alpha \in S;\ b(\alpha) = 1\}.$$

It is easy to see that U is an S-filter, we check that U is κ-complete, extends F and decide S. Given $s_\alpha \in F$ we consider $b \restriction (\alpha + 1)$ which is $b_{\alpha+1}$. By definition of T and T^*, there is an ordinal $\gamma \in \bigcap F_{\alpha+1}$ such that $b_{\alpha+1}(\beta) = 1$ if and only if $\gamma \in s_\beta$. As $s_\alpha \in F_{\alpha+1}$, we have $\gamma \in s_\alpha$ and $b(\alpha) = b_{\alpha+1}(\alpha) = 1$. So $s_\alpha \in U$. Assume that both $x = s_\alpha$ and $\kappa - x = s_\beta$ are in S, and suppose

$x \notin U$. Let $\delta > \alpha, \beta$ be the minimum ordinal such that $F_\delta \neq \emptyset$. There exists $\gamma \in \bigcap F_\delta$ such that for every $\epsilon < \delta$ we have $b(\epsilon) = 1$ if and only if $\gamma \in s_\epsilon$. Since $s_\alpha \notin U$ we must have $b(\alpha) = 0$, hence $\gamma \notin s_\alpha$. Then $\gamma \in \kappa - s_\alpha = s_\beta$ and $b(\beta) = 1$. Therefore $s_\beta \in U$. To prove that U is κ-complete, consider a family $\{s_{\alpha_i}\}_{i<\delta}$ of sets in U with $\delta < \kappa$ and let $s_\beta := \bigcap_{i<\delta} s_{\alpha_i}$. Let $\alpha < \kappa$ be the minimum ordinal greater than $\max(\beta, \lim_{i<\delta} \alpha_i)$ and such that F_α is non-empty. Consider $b \restriction \alpha$, there exists $\gamma \in \bigcap F_\alpha$ such that $b(\beta) = 1$ if and only if $\gamma \in s_\beta$. Since $b(\alpha_i) = 1$ for every i, we have $\gamma \in \bigcap_{i<\delta} s_{\alpha_i} = s_\beta$. Hence $b(\beta) = 1$ and $s_\beta \in U$. □

Although the tree property characterizes large cardinals, even small cardinals can satisfy it. We now show that $\aleph_0$ satisfies the tree property.

Proposition 1.2.6. *(König's Lemma) Every $\aleph_0$-tree has a cofinal branch.*

Proof. The 0-the level of T is finite, while T is infinite. Since every element of T has a predecessor at Lev_0, this implies that there exists x_0 in Lev_0 such that $\{t \in T;\ t \geq x_0\}$ is infinite. Similarly, we may inductively pick nodes $x_n \in \mathrm{Lev}_n$, for every $n < \omega$, such that $x_n < x_{n+1}$ and $\{t \in T;\ t \geq x_n\}$ is infinite. The set $\{x_n;\ n < \omega\}$ is a cofinal branch of T. □

On the contrary the tree property fails at $\aleph_1$.

Theorem 1.2.7. *(Aronszajn) There exists an $\aleph_1$-tree with no cofinal branches.*

Proof. Consider the tree $\langle T, \sqsubset \rangle$ of all bounded injective sequences $s : \aleph_1 \to \aleph_0$ ordered by extension. T has height $\aleph_1$, since for every $\alpha < \aleph_1$, there exists an injective function from α to $\aleph_0$. Moreover, every branch of T is countable, indeed if b were an uncountable branch, then b would be an injective function from $\aleph_1$ into $\aleph_0$. Therefore T has no cofinal branches. Unfortunately T is not an $\aleph_1$-tree, since the levels are not all countable. We are going to define a sub-tree T^* of T with countable levels so that T^* will be an $\aleph_1$-tree.

Given $s, t \in T$, we write $s \sim t$ when $s(\zeta) = t(\zeta)$ for all but finitely many ζ's. First we define by induction for every $\alpha < \aleph_1$ a sequence $s_\alpha \in T$ such that

(i) $s_\alpha : \alpha \to \omega$;

(ii) for every $\beta < \alpha$, we have $s_\alpha \restriction \beta \sim s_\beta$;

(iii) $\omega - \mathrm{Im}(s_\alpha)$ is infinite.

Given s_α, use condition (iii) to pick $n \in \omega - \mathrm{Im}(s_\alpha)$ and define $s_{\alpha+1}$ as the sequence $s_\alpha \frown n$. For α limit, define s_α as follows. By inductive hypothesis $\langle s_\beta;\ \beta < \alpha \rangle$ is defined. Fix $\langle \alpha_n \rangle_{n<\omega}$ such that $\alpha = \lim \alpha_n$. First we define by induction a sequence $\langle t_n;\ n < \omega \rangle$ so that $t_n : \alpha_n \to \omega$

is injective, $t_n \sim s_{\alpha_n}$ and $t_{n+1} \upharpoonright \alpha_n = t_n$. Then we consider the function $t = \bigcup_n t_n$, it is injective and has domain α. We define s_α as follows.

$$s_\alpha(x) = \begin{cases} t(\alpha_{2n}) & \text{if } x = \alpha_n \\ t(x) & \text{if } x \notin \{\alpha_n;\ n < \omega\} \end{cases}$$

Then condition (iii) above holds for s_α, since every $t(\alpha_{2n+1}) \notin \mathrm{Im}(s_\alpha)$. This completes the definition of the s_α's.

We let T^* be the set of all sequences $t \in T$ such that $t : \alpha \to \omega$ and $t \sim s_\alpha$ for some $\alpha < \aleph_1$. Condition (i) ensures that $s_\alpha \in T^* \cap \mathrm{Lev}_\alpha(T)$, for every $\alpha < \aleph_1$, hence the height of T^* is $\aleph_1$. Moreover, for every α, we have just countably many $t : \alpha \to \omega$ such that $t \sim s_\alpha$, so T^* has countable levels and is therefore an $\aleph_1$-tree. As T^* is a sub-tree of T, it does not have cofinal branches. □

$\aleph_1$-trees with no cofinal branches were named after the author of the previous proposition, namely *Aronszajn trees.*

Definition 1.2.8. *For a regular cardinal κ, a κ-*Aronszajn tree *is a κ-tree with no cofinal branches.*

For regular cardinals larger than $\aleph_1$, we cannot prove within ZFC that the tree property holds or fails as it is shown by the following two theorems.

Theorem 1.2.9. *(Specker) If $\tau^{<\tau}$, then there exists a (special) τ^+-Aronszajn tree with no cofinal branches.*

Theorem 1.2.10. *(Mitchell [15]) If there is a model of ZFC with a weakly compact cardinal, then for every regular cardinal τ such that $\tau^{<\tau} = \tau$, there is a model of ZFC where τ^{++} has the tree property.*

Mitchell's forcing will be presented in Chapter 2.

1.3 The Strong and Super Tree Properties

The strong and the super tree property concern special objects that generalize the notion of κ-tree, for a regular cardinal κ.

Definition 1.3.1. *Given a regular cardinal $\kappa \geq \omega_2$ and an ordinal $\lambda \geq \kappa$, a (κ, λ)-*tree *is a set F satisfying the following properties:*

(i) for every $f \in F$, $f : X \to 2$, for some $X \in [\lambda]^{<\kappa}$

(ii) for all $f \in F$, if $X \subseteq \mathrm{dom}(f)$, then $f \upharpoonright X \in F$;

(iii) the set $\mathrm{Lev}_X(F) := \{f \in F;\ \mathrm{dom}(f) = X\}$ is non empty, for all $X \in [\lambda]^{<\kappa}$;

(iv) $|\mathrm{Lev}_X(F)| < \kappa$, for all $X \in [\lambda]^{<\kappa}$.

As usual, when there is no ambiguity, we will simply write Lev_X instead of $\mathrm{Lev}_X(F)$. In a (κ, λ)-tree, levels are not indexed by ordinals, but by *sets of ordinals.* So the predecessors of a node in a (κ, λ)-tree are not (necessarily) well ordered and a (κ, λ)-tree is not a tree. In Weiss Phd-thesis [22] (κ, λ)-trees were called $\mathscr{P}_\kappa\lambda$*-thin lists.*

Definition 1.3.2. *Given a regular cardinal* $\kappa \geq \omega_2$*, an ordinal* $\lambda \geq \kappa$ *and a* (κ, λ)*-tree* F,

(i) *a* cofinal branch *for* F *is a function* $b : \lambda \to 2$ *such that* $b \upharpoonright X \in \mathrm{Lev}_X(F)$, *for all* $X \in [\lambda]^{<\kappa}$;

(ii) *an* F-level sequence *is a function* $D : [\lambda]^{<\kappa} \to F$ *such that for every* $X \in [\lambda]^{<\kappa}$, $D(X) \in \mathrm{Lev}_X(F)$;

(iii) *given an* F*-level sequence* D, *an* ineffable branch *for* D *is a cofinal branch* $b : \lambda \to 2$ *such that* $\{X \in [\lambda]^{<\kappa};\ b \upharpoonright X = D(X)\}$ *is stationary.*

Definition 1.3.3. *Given a regular cardinal* $\kappa \geq \omega_2$ *and an ordinal* $\lambda \geq \kappa$,

(i) (κ, λ)-TP *holds if every* (κ, λ)*-tree has a cofinal branch;*

(ii) (κ, λ)-ITP *holds if for every* (κ, λ)*-tree* F *and for every* F*-level sequence* D, *there is an an ineffable branch for* D;

(iii) *we say that* κ *satisfies the* strong tree property *if* (κ, μ)-TP *holds, for all* $\mu \geq \kappa$;

(iv) *we say that* κ *satisfies the* super tree property *if* (κ, μ)-ITP *holds, for all* $\mu \geq \kappa$;

The strong tree property captures the combinatorial essence of strongly compact cardinals.

Theorem 1.3.4. *(Di Prisco - Zwicker [18], Donder - Weiss [22] and Jech [8]) If* κ *is an inaccessible cardinal, then* κ *is strongly compact if and only if* κ *satisfies the strong tree property.*

Proof. For the forward direction, suppose T is a (κ, λ)-tree T and pick for every $X \in [\lambda]^{<\kappa}$ an element $d_X \in \mathrm{Lev}_X$ (any element). Consider the filter F on $[\lambda]^{<\kappa}$ generated by the sets $\mathrm{Cone}(X) := \{Y \in [\lambda]^{<\kappa};\ Y \supseteq X\}$, namely for every $A \subseteq [\lambda]^{<\kappa}$ we have $A \in F$ if and only if there exists $X \in [\lambda]^{<\kappa}$ such that $A \supseteq \mathrm{Cone}(X)$. Then, F is κ-complete and since κ is strongly compact, it can be extended to a κ-complete ultrafilter U. For every $X \in [\lambda]^{<\kappa}$ we fix an enumeration $\mathrm{Lev}_X := \{f_i^X;\ i < \gamma_X\}$ (i.e. $\gamma_X < \kappa$ is the size of Lev_X) and we define sets

$$D_i^X := \{Y \in \mathrm{Cone}(X);\ d_Y \upharpoonright X = f_i^X\},$$

where $i < \gamma_X$. We show that $D_{i_X}^X \in U$ for some $i_X < \gamma_X$. If not, then every set $E_i^X := \{Y \in \mathrm{Cone}(X);\ d_Y \upharpoonright X \neq f_i^X\}$ is in U. By the κ-completeness of

U the set $E := \bigcap_{i<\gamma_X} E_i^X$ is in U, but E is empty and that contradicts the fact that U is a filter. We show that $b := \bigcup_{X\in[\lambda]^{<\kappa}} f_{i_X}^X$ is a function. Given $X, Y \in [\lambda]^{<\kappa}$, we have $D_{i_X}^X \cap D_{i_Y}^Y$ is in U, hence we can pick $Z \in D_{i_X}^X \cap D_{i_Y}^Y$ such that $d_Z \restriction X = f_{i_X}^X$ and $d_Z \restriction Y = f_{i_Y}^Y$. So b is a function and it is a cofinal branch for T.

For the the converse, let F be a κ-complete filter on a set S of size $\lambda \geq \kappa$. Fix an enumeration $\mathscr{P}(S) = \{x_\alpha;\ \alpha < \lambda\}$. Observe that for every $A \in [\lambda]^{<\kappa}$, if the set $F_A := \{x_\alpha \in F;\ \alpha \in A\}$ is non empty, then $\bigcap F_A$ is non empty (because F is κ-complete). We define T as the set of all functions $f : A \to 2$ with $A \in [\lambda]^{<\kappa}$ such that (F_A is non empty and) for some $c_f \in \bigcap F_A$ the function f satisfies $f(\alpha) = 1$ if and only if $c_f \in X_\alpha$. We let $T' := \{f \restriction A;\ f \in T \text{ and } A \in [\lambda]^{<\kappa}\}$ ordered by extension. Since κ is inaccessible, T' is a (κ, λ)-tree, and by hypothesis there exists a cofinal branch $b : \lambda \to 2$ for T'. We let U be defined by $x_\alpha \in U$ if and only if $b(\alpha) = 1$. It is easy to see that U is a filter, we claim that U is a κ-complete ultrafilter that extends F. Let $x_\alpha \in F$, then $b \restriction \{\alpha\} = f \in T$ and for some $c_f \in \bigcap F_{\{\alpha\}}$ we have $f(\beta) = 1$ if and only if $c_f \in X_\beta$. As $c_f \in x_\alpha$, we have $b(\alpha) = f(\alpha) = 1$ and $x_\alpha \in U$. Suppose $x_\alpha \notin U$ and let β be such that $x_\beta = S - x_\alpha$. Consider $A \in [\lambda]^{<\kappa}$ such that $\alpha, \beta \in A$ and F_A is non-empty. We have $b \restriction A = f \in T$ and $f(\gamma) = 1$ if and only if $c_f \in x_\gamma$. Since $x_\alpha \notin U$, we have $f(\alpha) = b(\alpha) = 0$, thus $c_f \notin x_\alpha$. It follows that $c_f \in S - x_\alpha = x_\beta$, hence $b(\beta) = f(\beta) = 1$ and $x_\beta \in U$. Finally we prove that U is κ-complete. Let $\{x_{\alpha_i}\}_{i<\gamma}$ be a family of sets in U and let β be such that $\bigcap_{i<\gamma} x_{\alpha_i} = x_\beta$. Pick $A \in [\lambda]^{<\kappa}$ such that $\beta \in A$, $\alpha_i \in A$ for all i, and F_A is non-empty. If $f = b \restriction A$, then there is $c_f \in \bigcap F_A$ such that $f(\delta) = 1$ if and only if $c_f \in x_\delta$. Since $b(\alpha_i) = 1$ for every i, we have $c_f \in \bigcap_{i<\gamma} x_i = x_\beta$. Therefore $b(\beta) = f(\beta) = 1$ and $x_\beta \in U$. □

We conclude this chapter by discussing a theorem by Magidor establishing that an inaccessible cardinal is supercompact if and only if it satisfies the super tree property. The forward direction is due to Donder, Weiss and Jech, we give a proof of his lemma by using elementary embeddings.

Lemma 1.3.5. *(Donder - Weiss [22], Jech [8]) If κ is supercompact, then κ has the super tree property.*

Proof. Let F be a (κ, λ)-tree for some $\lambda \geq \kappa$, and let D be an F-level sequence. Fix a $|\lambda|^{<\kappa}$-supercompact embedding $j : V \to M$ with critical point κ. By elementarity, $j(F)$ is a $(j(\kappa), j(\lambda))$-tree and $j(D)$ is a $j(F)$-level sequence. The set $j[\lambda]$ is in M and belongs to $[j(\lambda)]^{<j(\kappa)}$. Consider the function $f := j(D)(j[\lambda])$ and define $b : \lambda \to 2$ by $b(\alpha) := f(j(\alpha))$. We conclude the proof by showing that b is ineffable. Let $S := \{X \in [\lambda]^{<\kappa};\ b \restriction X = D(X)\}$, then $j(S) = \{X \in [j(\lambda)]^{<j(\kappa)};\ j(b) \restriction X = j(D)(X)\}$. As $j(b) \restriction j[\lambda] = f = j(D)(j[\lambda])$, we have $j[\lambda] \in j(S)$. So S is in the ultrafilter determined by j and it is stationary by Lemma 1.1.7. □

Similar arguments will be used in the following chapters to prove the main theorems of this thesis.

Theorem 1.3.6. *(Donder - Weiss [22], Jech [8] and Magidor [13]) If κ is an inaccessible cardinal, then κ is supercompact if and only if κ satisfies the super tree property*

Proof. Lemma 1.3.5 proves that supercompact cardinals have the super tree property. For the converse see [13, §2]. □

2

Mitchell's Forcing

In [15] Mitchell proved that starting from a weakly compact cardinal κ above a regular cardinal τ, one can define a forcing iteration that produces a model of set theory where τ^{++} has the tree property. Weiss improved Mitchell's result by proving that if κ is also supercompact, then this forcing construction produces a model where τ^{++} satisfies even the super tree property. In this chapter we present Mitchell's forcing and Weiss' result. We will also introduce tools from [5] and [4] that will be used in the rest of this thesis.

2.1 Mitchell's Forcing

Definition 2.1.1. *Let $\tau < \kappa$ be two regular cardinals.*

(i) We let $\mathbb{A}(\tau, \kappa)$ be the set of all $p \in \mathrm{Add}(\tau, \kappa)$ such that every $\alpha \in \mathrm{dom}(p)$ is a successor ordinal. As usual $\mathbb{A}(\tau, \kappa)$ is ordered by reverse inclusion.

(ii) For every set $E \subseteq \kappa$, we let $\mathbb{A}(\tau, \kappa) \upharpoonright E =: \{p \in \mathbb{A}(\tau, \kappa);\ \mathrm{dom}(p) \subseteq E\}$, ordered by reverse inclusion.

Definition 2.1.2. *(Mitchell [15]) Let $\tau < \kappa$ be a regular cardinals, Mitchell's forcing $\mathbb{M}(\tau, \kappa)$ is defined as follows. A pair (p, q) is a condition of $\mathbb{M}(\tau, \kappa)$ if and only if*

(i) $p \in \mathbb{A}(\tau, \kappa)$,

(ii) $q :]\tau, \kappa[\to V$ of size $\leq \tau$ such that every $\alpha \in]\tau, \kappa[$ is a successor cardinal and $\Vdash_{\mathbb{A}(\tau,\kappa)\upharpoonright\alpha} q(\alpha) \in \mathrm{Add}(\tau^+)$.

$\mathbb{M}(\tau, \kappa)$ is partially ordered by $(p, q) \leq (p', q')$ if and only if

(a) $p \leq p'$,

(b) $\mathrm{dom}(q') \subseteq \mathrm{dom}(q)$,

(c) for every $\alpha \in \mathrm{dom}(q')$, $p \upharpoonright \alpha \Vdash q(\alpha) \leq q'(\alpha)$.

(Mitchell's forcing is actually the set of all equivalence classes).

From now on $\tau < \kappa$ are two regular cardinals such that $\tau^{<\tau} = \tau$ and κ is inaccessible. Observe that the function $(p, q) \mapsto p$ is a projection of $\mathbb{M}(\tau, \kappa)$ into $\mathbb{A}(\tau, \kappa)$. We define

$$\mathbb{T}(\tau, \kappa) := \{(\emptyset, q);\ (\emptyset, q) \in \mathbb{M}(\tau, \kappa)\}$$

partially ordered as a subset of $\mathbb{M}(\tau, \kappa)$.

Lemma 2.1.3. *$\mathbb{M}(\tau, \kappa)$ is a projection of $\mathbb{A}(\tau, \kappa) \times \mathbb{T}(\tau, \kappa)$.*

Proof. The function π defined by $(p, (\emptyset, q)) \mapsto (p, q)$ is a projection of $\mathbb{A}(\tau, \kappa) \times \mathbb{T}(\tau, \kappa)$ into $\mathbb{M}(\tau, \kappa)$. It clearly preserves the ordering and $\pi(\emptyset, (\emptyset, \emptyset))$ equals $(\emptyset, \emptyset)$. We check that condition (iii) of Definition I.0.14 is satisfied. Assume $(p', q') \leq \pi(p, (\emptyset, q)) = (p, q)$, we are going to define a function q^* such that $(\emptyset, q^*) \leq (\emptyset, q)$ and $\pi(p', (\emptyset, q^*)) = (p', q')$. We let $\mathrm{dom}(q^*) = \mathrm{dom}(q')$ and for all $\alpha \in \mathrm{dom}(q') - \mathrm{dom}(q)$, we let $q^*(\alpha) := q'(\alpha)$. On the contrary, if $\alpha \in \mathrm{dom}(q)$, then $p' \upharpoonright \alpha \Vdash q'(\alpha) \leq q(\alpha)$. In that case we define $q^*(\alpha) \in V^{\mathbb{A}(\tau,\kappa)\upharpoonright\alpha}$ in such a way that $p' \upharpoonright \alpha \Vdash q^*(\alpha) = q'(\alpha)$ and if r is a condition of $\mathbb{A}(\tau, \kappa) \upharpoonright \alpha$ incompatible with $p' \upharpoonright \alpha$, then $r \Vdash q^*(\alpha) = q(\alpha)$. So we have $\Vdash_{\mathbb{A}(\tau,\kappa)\upharpoonright\alpha} q^*(\alpha) \leq q(\alpha)$, hence $(p', (\emptyset, q^*)) \leq (p, (\emptyset, q))$ and $\pi(p', (\emptyset, q^*)) = (p', q^*) = (p', q')$. □

Lemma 2.1.4. *$\mathbb{T}(\tau, \kappa)$ is τ^+-closed.*

Proof. Let $\langle(\emptyset, q_i)\rangle_{i<\gamma}$ be a decreasing sequence of conditions in $\mathbb{T}(\tau, \kappa)$ with $\gamma < \tau^+$. We define a function q such that $(\emptyset, q) \leq (\emptyset, q_i)$ for every $i < \gamma$. We let $\mathrm{dom}(q) := \bigcup_{i<\gamma} q_i$ and for all $\alpha \in \mathrm{dom}(q)$, we let i_α be the minimum index such that $\alpha \in q_{i_\alpha}$. Then we have

$\Vdash_{\mathbb{A}(\tau,\kappa)\upharpoonright\alpha}$ $\langle q_i(\alpha)\rangle_{i_\alpha \leq i<\gamma}$ is a decreasing sequence of conditions in $\mathrm{Add}(\tau^+)$.

There exists $q(\alpha) \in V^{\mathbb{A}(\tau,\kappa)\upharpoonright\alpha}$ such that $\Vdash_{\mathbb{A}(\tau,\kappa)\upharpoonright\alpha} \forall i \geq i_\alpha\ (q(\alpha) \leq q_i(\alpha))$. Thus $(\emptyset, q)$ is a lower bound for $\langle(\emptyset, q_i)\rangle_{i<\gamma}$. □

We now show that $\mathbb{M}(\tau, \kappa)$ makes $2^\tau = \tau^{++} = \kappa$.

Lemma 2.1.5. *$\mathbb{M}(\tau, \kappa)$ is κ-c.c, preserves τ^+ and makes $\kappa = \tau^{++} = 2^\tau$.*

Proof. Let $\mathbb{M} := \mathbb{M}(\tau, \kappa)$, we prove that $\mathbb{M}$ is even κ-Knaster, this is a standard application of the Δ-system Lemma. Assume $\langle(p_\alpha, q_\alpha)\rangle_{\alpha<\kappa}$ is a sequence of conditions in $\mathbb{M}$. Then the p_α's and the q_α's are functions of size at most τ. As κ is inaccessible, it satisfies $\gamma^{<\tau^+} < \kappa$ for all $\gamma < \kappa$. By the Δ-system Lemma there exists $I \subseteq \kappa$ of size κ such that $\{p_\alpha\}_{\alpha\in I}$ and $\{q_\alpha\}_{\alpha\in I}$ are two Δ-systems with roots p and q. It follows that for every $\alpha, \beta \in I$, we have $p_\alpha \cap p_\beta = p$ and $q_\alpha \cap q_\beta = q$, so $p' := p_\alpha \cup p_\beta$ and $q' := q_\alpha \cup q_\beta$ are functions and (p', q') is a condition of $\mathbb{M}$ stronger than both (p_α, q_α) and (p_β, q_β).

The proof that τ^+ is preserved is an easy consequence of Easton Lemma. Let $G \subseteq \mathbb{M}$ be generic over V and assume that $X \in V[G]$ is a set of ordinals of size τ in $V[G]$. By Lemma 2.1.3 there exists $g \times h \subseteq \mathbb{A}(\tau,\kappa) \times \mathbb{T}(\tau,\kappa)$ generic over V and projecting on G. By the Product Lemma and the Easton Lemma, we have $V[G] \subseteq V[g \times h] = V[g][h]$ and $(V, V[g][h])$ has the τ^+-covering property. In particular $X \in V[g][h]$ and it is covered by a set $Y \in V$ of size $\leq \tau$ in V.

Finally note that in $V^{\mathbb{A}(\tau,\kappa)\restriction\alpha}$ we have $2^\tau \geq \alpha$, hence introducing a subset of τ^+ over $V^{\mathbb{A}(\tau,\kappa)\restriction\alpha}$ collapses α to τ^+. It follows that $\mathbb{M}(\tau,\kappa)$ makes $\kappa = \tau^+ = 2^\tau$, see [15] for more details. □

Lemma 2.1.6. *$\mathbb{M}(\tau,\kappa)$ is τ-closed.*

Proof. Let $\langle (p_i, q_i)_{i<\mu} \rangle_{i<\mu}$ be a decreasing sequence of conditions in $\mathbb{M}(\tau,\kappa)$ with $\mu < \tau$. Define $p := \bigcup_{i<\mu} p_i$, then $p \in \mathbb{A}(\tau,\kappa)$ because τ is regular and. For every $\alpha \in \bigcup_{i<\mu} \mathrm{dom}(q_i)$ if i_α is the first index such that $\alpha \in q_{i_\alpha}$ we have

$$p \restriction \alpha \Vdash \ \langle q_i(\alpha) \rangle_{i_\alpha \leq i<\mu} \text{ is a decreasing sequence of conditions in } \mathbb{A}(\tau^+).$$

So there exists $q(\alpha) \in V^{\mathbb{A}(\tau,\kappa)\restriction\alpha}$ such that $p \restriction \alpha \Vdash q(\alpha) = \bigcup_{i_\alpha \leq i<\mu} q_i(\alpha)$. The condition (p, q) is a lower bound for the sequence $\langle (p_i, q_i)_{i<\mu} \rangle_{i<\mu}$. □

Note that the statement of previous lemma is true even if κ is not inaccessible.

Lemma 2.1.7. *Let $\lambda > \tau$ be inaccessible in V and assume W is a forcing extension of V where τ and λ are still cardinals. If (V, W) has the λ-covering property, then $\mathbb{A}(\tau,\eta)^V$ is λ-Knaster in W for every ordinal η.*

Proof. Let $\langle p_\alpha \rangle_{\alpha<\lambda}$ be a sequence of conditions of $\mathbb{A}(\tau,\eta)^V$ in W. We mimic the usual Δ-system argument. In W consider the set $A := \bigcup_{\alpha<\lambda} \mathrm{dom}(p_\alpha)$. Then A has size λ, hence there is a bijective function $h : A \to \lambda$. For every $\alpha < \lambda$ of cofinality $> \tau$, define $H(\alpha) := \sup(h[\mathrm{dom}(p_\alpha)] \cap \alpha)$. The function H so defined is regressive, so it has a fixed value γ on a stationary subset S of λ. Now $M := h^{-1}(\gamma)$ has size $< \lambda$ so we can use the λ-covering property to find $M' \in V$ of size $< \lambda$ such that $M \subseteq M'$. By the inaccessibility of λ in V we have that $[M']^{<\tau}$ has size less than λ in V. As λ remains a cardinal in W the same holds in W, so there exists q such that $p_\alpha \restriction M' = q$ for every α in a stationary subset $S' \subseteq S$. It follows that $\langle p_\alpha;\ \alpha \in S' \rangle$ is a Δ-system with root q. So for every $\alpha, \beta \in S'$, we have $p_\alpha \cap p_\beta = p$ hence $p' := p_\alpha \cup p_\beta$ is a condition of $\mathbb{A}(\tau,\kappa)^V$ which is stronger than both p_α and p_β. □

Let $\alpha \in]\tau,\kappa[$ be an inaccessible cardinal, then the function $(p, q) \mapsto (p \restriction \alpha, q \restriction \alpha)$ is a projection of $\mathbb{M}(\tau,\kappa)$ into $\mathbb{M}(\tau,\alpha)$.

Definition 2.1.8. *Assume $G_\alpha \subseteq \mathbb{M}(\tau,\alpha)$ is generic over V, we define in $V[G_\alpha]$ the poset $\mathbb{M}(\tau,\kappa-\alpha,G_\alpha)$ whose conditions are couples (p,q) such that*

(i) $p \in \mathbb{A}(\tau,\kappa)^V \restriction (\kappa-\alpha)$;

(ii) $q \in V[G_\alpha]$ *is a function of size* $\leq \tau$ *such that every* $\beta \in \mathrm{dom}(q)$ *is a successor cardinal in the interval* $\kappa-\alpha$ *and*

$$\Vdash^{V[G_\alpha]}_{\mathbb{A}(\tau,\kappa)^V\restriction(\beta-\alpha)} q(\alpha) \in \mathrm{Add}(\tau^+).$$

The conditions of $\mathbb{M}(\tau,\kappa-\alpha,G_\alpha)$ are ordered as in Definition 2.1.2.

Remark 2.1.9. *Similar arguments as for the proof of Lemma 2.1.3 and Lemma 2.1.4 show that $\mathbb{M}(\tau,\kappa-\alpha,G_\alpha)$ is a projection of $\mathbb{A}(\tau,\kappa)^V \restriction (\kappa-\alpha)\times\mathbb{T}(\tau,\kappa-\alpha,G_\alpha)$ where $\mathbb{T}(\tau,\kappa-\alpha,G_\alpha)$ is a τ^+-closed poset.*

The following lemma shows that $\mathbb{M}(\tau,\kappa)$ is equivalent to $\mathbb{M}(\tau,\alpha)*\mathbb{M}(\tau,\kappa-\alpha,\dot{G}_\alpha)$ where $\dot{G}_\alpha$ is the canonical name for a generic filter for $\mathbb{M}(\tau,\alpha)$.

Lemma 2.1.10. *$\mathbb{M}(\tau,\alpha)*\mathbb{M}(\tau,\kappa-\alpha,\dot{G}_\alpha)$ contains a dense set isomorphic to $\mathbb{M}(\tau,\kappa)$.*

Proof. Let $i:\mathbb{M}(\tau,\kappa)\to\mathbb{M}(\tau,\alpha)*\mathbb{M}(\tau,\kappa-\alpha,\dot{G}_\alpha)$ be the map defined by $i(p,q)=((p\restriction\alpha,q\restriction\alpha),(p\restriction\kappa-\alpha,\bar{q}))$ where $\bar{q}$ is the unique function with domain $\mathrm{dom}(q)-\alpha$ such that for every $\beta\in\mathrm{dom}(q)\restriction\alpha$, we have $\bar{q}(\beta)\in V[G_\alpha]^{\mathbb{A}(\tau,\kappa)\restriction\beta}$ and $\bar{q}(\beta)=q(\beta)$. It is easy to see that i is order preserving. We prove that i is a dense embedding (i.e. the image of i is dense). Assume $((r,s),\dot{a})\in\mathbb{M}(\tau,\alpha)*\mathbb{M}(\tau,\kappa-\alpha,\dot{G}_\alpha)$, then

$$(r,s)\Vdash_{\mathbb{M}(\tau,\alpha)} \dot{a}=(\dot{p},\dot{q})\in\mathbb{M}(\tau,\kappa-\alpha,\dot{G}_\alpha).$$

So there is a condition $(r',s')\leq(r,s)$ and a function $p\in\mathbb{A}(\tau,\kappa)\restriction(\kappa-\alpha)$ such that $(r',s')\Vdash\dot{p}=p$. Work in $V[G_\alpha]$. Let q be the interpretation of $\dot{q}$ in $V[G_\alpha]$. Then every $q(\beta)$ is a $\mathbb{A}(\tau,\kappa)\restriction(\beta-\alpha)$-name for a bounded function on τ^+. We can assume that $q(\beta)$ consists of pairs $(t,(\gamma,i))$ with $t\in\mathbb{A}(\tau,\kappa)\restriction(\beta-\alpha)$ and $(\gamma,i)\in\tau^+\times 2$ such that $(t,(\gamma,i)),(t^*,(\gamma,i))\in q(\beta)$ implies t and t^* are incompatible. Lemma 2.1.7 implies that $\mathbb{A}(\tau,\kappa)$ is τ^+-c.c. so we get that $q(\beta)$ has cardinality $<\tau^+$. As in Lemma 2.1.5 we can prove that every set of ordinals of cardinality $<\tau^+$ in $V[G_\alpha]$ is already in $V[g_\alpha]$ where g_α is the projection of G_α generic for $\mathbb{A}(\tau,\alpha)$ over V. It follows that $q\in V[g_\alpha]$, that is $(r',s')\Vdash_{\mathbb{M}(\tau,\alpha)} \dot{q}\in V[\dot{g}_\alpha]$. So there are $(r'',s'')\leq(r',s')$ and $\dot{q}'\in V^{\mathbb{A}(\tau,\alpha)}$ such that $(r'',s'')\Vdash_{\mathbb{M}(\tau,\alpha)} \dot{q}'=\dot{q}$. Since $\mathbb{A}(\tau,\alpha)$ is τ^+-c.c. we can assume that $\mathrm{dom}(\dot{q}')=D\in V$ and using $\mathbb{A}(\tau,\alpha)\times\mathbb{A}(\beta-\alpha)$, we define a function q^* with domain D by letting $q^*(\beta)$ be the interpretation of $\dot{q}'(\beta)$ in the generic extension determined by $\mathbb{A}(\tau,\alpha)\times\mathbb{A}(\beta-\alpha)$. Finally if $c:=(r''\frown p,s''\frown q^*)$, then $i(c)\leq((r,s),\dot{a})$ as required. □

2.2 Preserving Branches

It will be important, in what follows, that certain forcings cannot add ineffable branches. The following lemma generalizes a result by Silver (see [11, Lemma 3.4, Ch. VIII] or [22, Proposition 2.1.12]).

Lemma 2.2.1. *(First Preservation Lemma) Let θ be a regular cardinal and $\mu \geq \theta$ be any ordinal. Assume that F is a (θ, μ)-tree and $\mathbb{Q}$ is an η^+-closed forcing with $\eta < \theta \leq 2^\eta$. For every filter $G_\mathbb{Q} \subseteq \mathbb{Q}$ generic over V, every cofinal branch for F in $V[G_\mathbb{Q}]$ is already in V.*

Proof. We can assume, without loss of generality, that η is minimal such that $2^\eta \geq \theta$. Assume towards a contradiction that $\mathbb{Q}$ adds a cofinal branch to F, let $\dot{b}$ be a $\mathbb{Q}$-name for such a function. For all $\alpha \leq \eta$ and all $s \in {}^\alpha 2$, we are going to define by induction three objects $a_\alpha \in [\mu]^{<\theta}$, $f_s \in \mathrm{Lev}_{a_\alpha}$ and $p_s \in \mathbb{Q}$ such that:

(i) $p_s \Vdash \dot{b} \upharpoonright a_\alpha = f_s$;

(ii) $f_{s \frown 0}(\beta) \neq f_{s \frown 1}(\beta)$, for some $\beta < \mu$;

(iii) if $s \subseteq t$, then $p_t \leq p_s$;

(iv) if $\alpha < \beta$, then $a_\alpha \subset a_\beta$.

Let $\alpha < \eta$, assume that a_α, f_s and p_s have been defined for all $s \in {}^\alpha 2$. We define $a_{\alpha+1}$, f_s, and p_s, for all $s \in {}^{\alpha+1}2$. Let t be in ${}^\alpha 2$, we can find an ordinal $\beta_t \in \mu$ and two conditions $p_{t\frown 0}, p_{t \frown 1} \leq p_t$ such that $p_{t\frown 0} \Vdash \dot{b}(\beta_t) = 0$ and $p_{t\frown 1} \Vdash \dot{b}(\beta_t) = 1$. (otherwise, $\dot{b}$ would be a name for a cofinal branch which is already in V). Let $a_{\alpha+1} := a_\alpha \cup \{\beta_t;\ t \in {}^\alpha 2\}$, then $|a_{\alpha+1}| < \theta$, because $2^\alpha < \theta$. We just defined, for every $s \in {}^{\alpha+1}2$, a condition p_s. Now, by strengthening p_s if necessary, we can find $f_s \in \mathrm{Lev}_{a_{\alpha+1}}$ such that

$$p_s \Vdash \dot{b} \upharpoonright a_{\alpha+1} = f_s.$$

Finally, $f_{t\frown 0}(\beta_t) \neq f_{t\frown 1}(\beta_t)$, for all $t \in {}^\alpha 2$: because $p_{t\frown 0} \Vdash f_{t\frown 0}(\beta_t) = \dot{b}(\beta_t) = 0$, while $p_{t\frown 1} \Vdash f_{t\frown 1}(\beta_t) = \dot{b}(\beta_t) = 1$.

If α is a limit ordinal $\leq \eta$, let t be any function in ${}^\alpha 2$. Since $\mathbb{Q}$ is η^+-closed, there is a condition p_t such that $p_t \leq p_{t\upharpoonright\beta}$, for all $\beta < \alpha$. Define $a_\alpha := \bigcup_{\beta<\alpha} a_\beta$. By strengthening p_t if necessary, we can find $f_t \in \mathrm{Lev}_{a_\alpha}$ such that $p_t \Vdash \dot{b} \upharpoonright a_\alpha = f_t$. That completes the construction.

We show that $|\mathrm{Lev}_{a_\eta}| \geq {}^\eta 2 \geq \theta$, thus a contradiction is obtained. Let $s \neq t$ be two functions in ${}^\eta 2$, we are going to prove that $f_s \neq f_t$. Let α be the minimum ordinal less than η such that $s(\alpha) \neq t(\alpha)$, without loss of generality $r \frown 0 \sqsubset s$ and $r \frown 1 \sqsubset t$, for some $r \in {}^\alpha 2$. By construction,

$$p_s \leq p_{r\frown 0} \Vdash \dot{b} \upharpoonright a_{\alpha+1} = f_{r\frown 0} \text{ and } p_t \leq p_{r\frown 1} \Vdash \dot{b} \upharpoonright a_{\alpha+1} = f_{r\frown 1},$$

where $f_{r^\frown 0}(\beta) \neq f_{r^\frown 1}(\beta)$, for some β. Moreover, $p_s \Vdash \dot{b} \restriction a_\eta = f_s$ and $p_t \Vdash \dot{b} \restriction a_\eta = f_t$, hence $f_s \restriction a_{\alpha+1}(\beta) = f_{r^\frown 0}(\beta) \neq f_{r^\frown 1}(\beta) = f_t \restriction a_{\alpha+1}(\beta)$, thus $f_s \neq f_t$. That completes the proof. □

The following definition is due to Veličković.

Definition 2.2.2. *Let θ be a regular cardinal and let $\mathbb{P} \subseteq \mathrm{Add}(\tau, \eta)$ (where $\eta \geq \tau$ is any ordinal). We say that $\mathbb{P}$ has the θ-*sunflower property *if for every club $C \subseteq [H_\chi]^{<\theta}$ where $\chi \geq \theta$ and for every sequence of conditions $\langle p_X;\ X \in C\rangle$ there exists a cofinal $S \subseteq C$ and $q \in \mathbb{P}$ such that $p_X \restriction X = q$ for every $X \in S$. We say that $\langle p_X;\ X \in S\rangle$ forms a* sunflower *with* root *q.*

Now we prove that a forcing that has the θ-sunflower property cannot add cofinal branches to a given (θ, μ)-tree.

Lemma 2.2.3. *(Second Preservation Lemma) Let $\mu \geq \theta$ be any ordinal and assume that F is a (θ, μ)-tree and $\mathbb{P}$ is a forcing that has the θ-sunflower property. For every filter $G_\mathbb{P} \subseteq \mathbb{P}$ generic over V, every cofinal branch for F in $V[G_\mathbb{P}]$ is already in V.*

Proof. Fix a condition $p \in G_\mathbb{P}$ and a $\mathbb{P}$-name $\dot{b}$ for a cofinal branch b of F in $V[G_\mathbb{P}]$. Let χ be large enough for the following argument and consider the set C of all the elementary substructures of H_θ of size less than θ. For every $X \in C$, there exists a condition $p_X \leq p$ and a function $f_X \in F$ such that $p_X \Vdash \dot{b} \restriction (X \cap \mu) = f_X$. Since $\mathbb{P}$ has the θ-sunflower property there is a stationary set $S \subseteq C$ such that the sequence $\langle p_X;\ X \in S\rangle$ forms a sunflower with root $q \in \mathbb{P}$. We show that $B = \bigcup_{X \in S} f_X$ is a function. For every $X, Y \in S$, take some $Z \in S$ such that $X \cup Y, \mathrm{dom}(p_X), \mathrm{dom}(p_Y) \subseteq Z$, then

$$p_X \cap p_Z = p_X \cap p_Z \restriction Z = p_X \cap q = q$$

and similarly $p_Y \cap p_Z = q$. So $p_X || p_Z$ and $p_Z || p_Y$. Let $r \leq p_X, p_Z$ and $s \leq p_Y, p_Z$, then $r \Vdash f_X \restriction (X \cap Y) = \dot{b} \restriction (X \cap Y) = f_Z \restriction (X \cap Y)$ and $s \Vdash f_Y \restriction (X \cap Y) = \dot{b} \restriction (X \cap Y) = f_Z \restriction (X \cap Y)$, therefore

$$f_X \restriction (X \cap Y) = f_Z \restriction (X \cap Y) = f_Y \restriction (X \cap Y)$$

hence $f_X \cup f_Y$ is a function. This would be enough if we had to prove just that a cofinal branch exists in V, but we want to prove that $B = b$. We show that for every $\alpha \in \mu$, the set $D_\alpha = \{r;\ r \Vdash \dot{b}(\alpha) = B(\alpha)\}$ is predense below q (that completes the proof since p and q are compatible). Let $r \leq q$, there exists $X \in S$ such that $\alpha, \mathrm{dom}(r) \subseteq X$. We have

$$p_X \cap r = p_X \restriction X \cap r = q \cap r = q.$$

So r and p_X are compatible and $p_X \Vdash \dot{b}(\alpha) = f_X(\alpha) = B(\alpha)$, thus $p_X \in D_\alpha$. □

Lemma 2.2.4. *Assume $\gamma^{<\tau} < \theta$ for every $\gamma < \tau$, then $\mathrm{Add}(\tau, \kappa)$ has the θ-sunflower property.*

Proof. Let $C \subseteq [H_\chi]^{<\theta}$ be club where $\chi \geq \theta$ and let $\langle p_X;\ X \in C\rangle$ be a sequence of conditions of $\mathrm{Add}(\tau, \kappa)$. We consider the set S of all the structures $X \prec H_\chi$ such that X is internally approachable of length τ. Since S is stationary, the set $= C \cap S$ is also stationary. For every $X \in C \cap S$, the condition p_X has length less than τ so there exists $M_X \in X$ of size less than τ such that $p_X \restriction X \subseteq M_X$. By the Pressing Down Lemma the function $X \mapsto M_X$ is constant on a stationary subset $S' \subseteq C \cap S$. We let M be such that $M = M_X$ for every $X \in S'$, then M has size less than τ as well as $A := \bigcup_{X \in S'} p_X \restriction M$. If $\gamma = |A|$, then the size of $[A]^{<\tau}$ is $\gamma^{<\tau}$ which is less than θ by hypothesis. It follows that there are less than θ possible values for $p_X \restriction M$, thus for some $q \in \mathrm{Add}(\tau, \kappa)$ and for some $S^* \subseteq S'$ cofinal, we have $q = p_X \restriction X$ for every $X \in S^*$. □

Observe that the previous lemma works for any subposet $\mathbb{B}$ of $\mathrm{Add}(\tau, \kappa)$ such that $\mathbb{B}$ is closed under restriction of its functions (i.e. if $f \in \mathbb{B}$ and $X \subseteq \mathrm{dom}(f)$, then $f \restriction X \in \mathbb{B}$). In particular $\mathrm{Add}(\tau, \kappa)$ can be replaced by $\mathbb{A}(\tau, \kappa)$ in the statement of the previous lemma.

The First and Second PreservationLemmas will be deeply used in Chapter 4.

2.3 The Super Tree Property at the double successor of a regular cardinal

Now we are ready to prove Weiss' theorem. The general idea is the following. Assume we want to obtain a model of the super tree property at $\aleph_2$ for example. We start with a supercompact cardinal κ; by Lemma 1.3.5 κ is inaccessible and it satisfies the super tree property. By forcing with $\mathbb{M}(\aleph_0, \kappa)$, we obtain a model in which κ is $\aleph_2$ and still satisfies the the super tree property. So in the generic extension, $\aleph_2$ has the super tree property.

Theorem 2.3.1. *(Weiss [22]) Let τ be a regular cardinal such that $\tau^{<\tau} = \tau$ (in particular it is the case if GCH holds in V) and let κ a supercompact cardinal above it, then forcing with $\mathbb{M}(\tau, \kappa)$ produces a model where τ^{++} has the super tree property.*

Proof. We fix a generic filter $G \subseteq \mathbb{M}(\tau, \kappa)$. We know that $\kappa = \tau^{++}$ in $V[G]$, so we want to prove that κ has the super tree property in that model. For $\lambda \geq \kappa$, we fix in $V[G]$ a (κ, λ)-tree F and an F-level sequence D. Let $\sigma := |\lambda|^{<\kappa}$, by the supercompactness of κ, there exists a σ-supercompact embedding $j : V \to N$ with critical point κ.

Let $j(\mathbb{M}) \restriction \kappa$ denote the set of all couples $(p \restriction \kappa, q \restriction \kappa)$ such that $(p, q) \in j(\mathbb{M})$ (ordered as a subposet of $j(\mathbb{M})$). Then

$$j(\mathbb{M}) \restriction \kappa = \mathbb{M}(\tau, \kappa)^N = \mathbb{M}(\tau, \kappa)^V.$$

Force over V to get a $j(\mathbb{M})$-generic filter H such that $H \restriction \kappa = G$. As $\mathbb{M}(\tau, \kappa)$ is κ-c.c. in V, we have that $j \restriction \mathbb{M}$ is a complete embedding from $\mathbb{M}$ into $j(\mathbb{M})$, hence we can lift j to an elementary embedding

$$j^* : V[G] \to N[H].$$

Rename j^* by j. In $V[H]$, the set $j(F)$ is a $(j(\kappa), j(\lambda))$-tree and $j(D)$ is a $j(F)$-level sequence. By the closure of N, the tree F and the F-level sequence D are in $N[G]$. We claim that there exists in $N[H]$ an ineffable branch b for D. Let $a := j[\lambda]$, clearly $a \in [j(\lambda)]^{<j(\kappa)}$. Consider $f := j(D)(a)$ and let $b : \lambda \to 2$ be the function defined by $b(\alpha) := f(j(\alpha))$. Then b is an ineffable branch for D, because a is in the image of the set $S := \{X;\ b \restriction X = D(X)\}$ as $j(b) \restriction a = f$, hence S is stationary.

To conclude the proof it is enough to show that b is already in $N[G]$, hence in $V[G]$. Indeed, if $b \in N[G]$, then b is ineffable because $\{X \in [\lambda]^{<|\kappa|} \cap N[G];\ b \restriction X = D(X)\}$ is stationary in $N[H]$, hence it is stationary in $N[G]$. We assume towards a contradiction that $b \notin N[G]$. Step by step, we want to prove that $b \notin N[H]$, that will lead us to a contradiction. We will use repeatedly and without comments the resemblance between V and N. The forcing $\mathbb{M}(\tau, j(\kappa) - \kappa, G)^N$ is a projection of

$$\mathbb{A}(\tau, j(\kappa) - \kappa)^N \times \mathbb{T}(\tau, j(\kappa) - \kappa, G)$$

where $\mathbb{T}(\tau, j(\kappa) - \kappa, G)$ is a τ^+-closed poset in $N[G]$. Let $h_A \times h_T \subseteq \mathbb{A}(\tau, j(\kappa) - \kappa)^N \times \mathbb{T}(\tau, j(\kappa) - \kappa, G)$ be generic over $N[G]$ such that $G * (h_A \times h_T)$ projects on H. In $N[G]$ we have $\kappa = \tau^{++} = 2^\tau$ and F is an (τ^{++}, λ)-tree, so we can apply the First Preservation Lemma (Lemma 2.2.1). It follows that $b \notin N[G][h_T]$, where $G * h_T$ is the projection of H to $\mathbb{M}(\tau, \kappa) * \mathbb{T}(\tau, j(\kappa) - \kappa, G)$. It remains to prove that forcing with $\mathbb{A}(\tau, j(\kappa) - \kappa)^N$ over $N[G][h_T]$ could not add the branch b. The filter h_T collapsed κ to have size τ^+.

We now prove the following claim.

Claim 1. *$\mathbb{A}(\tau, j(\kappa) - \kappa)^N$ has the τ^+ sunflower property in $N[G][h_T]$.*

Proof. We mimic the proof of Lemma 2.2.4. Assume that for some club $C \subseteq H_\chi^{<\tau^+}$ we have conditions $\langle p_X;\ X \in C\rangle$ of $\mathbb{A}(\tau, j(\kappa) - \kappa)^N$ in $N[G][h_T]$. We consider the set S of all the substructures $X \prec H_\chi$ internally approachable of length τ which are in C. The set S is stationary. For every $X \in S$ there is $M_X \in X$ of size less than τ such that $p_X \restriction X \subseteq M_X$. By the pressing down lemma there is M and $S' \subseteq S$ stationary such that $M_X = M$ for every $X \in S'$. So the set $A := \bigcup_{X \in S} p_X \restriction M$ has size $< \tau$ in $N[G][h_T]$.

Since $\mathbb{T}(\tau, j(\kappa) - \kappa, G)$ is τ^+-closed and $\mathbb{M}(\tau, \kappa)$ is τ-closed, A was already in N and has size $\gamma < \tau$ in N. By hypothesis $\gamma^{<\tau} \leq \tau < \tau^+$ in N and τ^+ is preserved in $N[G][h_T]$ so A has size less than τ^+ in $N[G][h_T]$ as well. It follows that there are less than τ^+ possible values for $p_X \restriction X$ in $N[G][h_T]$. Thus for some $q \in \mathbb{A}(\tau, j(\kappa) - \kappa)^N$ and for a cofinal subset $S^* \subseteq S'$ we have $p_X \restriction X = q$ for every $X \in S^*$. $\square$

We would like to apply the Second Preservation Theorem, but F is not exactly a (τ^+, λ)-tree. However, the argument is the same. Suppose that b is in $N[G][h_T][h_A]$ and let $\dot{b}$ be an $\mathbb{A}(\tau, j(\kappa) - \kappa)^N$-name for b. Work in $N[G][h_T]$. Take χ large enough for the argument that follows and consider the set C of all the elementary substructures $X \prec H_\chi$ of size τ. Observe that for every $X \in C$, we have $X \cap \lambda \in [\lambda]^\tau$ both in the sense of $N[G][h_T]$ and $N[G]$ So we can fix for every $X \in C$ a condition $p_X \in \mathbb{A}(\tau, j(\kappa) - \kappa)^N$ and a function f_X such that

$$p_X \Vdash \dot{b} \restriction (X \cap \lambda) = f_X.$$

By the τ^+-sunflower property of $\mathbb{A}(\tau, j(\kappa) - \kappa)^N$, the sequence $\langle p_X;\ X \in C\rangle$ can be refined into a sunflower $\langle p_X;\ X \in S\rangle$ with root q, where $S \subseteq C$ is stationary. If we let $B := \bigcup_{X \in S} f_X$, then B is a function (the argument is the same as for the Second Preservation Lemma). We show with a density argument that $q \Vdash B = \dot{b}$. Let $\alpha \in \lambda$ be any ordinal, we prove that $D_\alpha := \{r;\ r \Vdash \dot{b}(\alpha) = B(\alpha)\}$ is predense below q. Let $r \leq q$, then there is $X \in S$ such that $\{\alpha\}, \mathrm{dom}(r) \subseteq X$. It follows that $p_X \cap r = p_X \restriction X \cap r = q$, so p_X and r are compatible and $p_X \in D_\alpha$. So b is in $N[G][h_T]$, while we proved that it was not in $N[G][h_T]$, a contradiction. $\square$

3

Cummings-Foreman Iteration

In this chapter we discuss a forcing iteration by Cummings and Foreman [2]. Such construction was introduced to prove the consistency of the usual tree property at every cardinal of the form $\aleph_n$ with $n \geq 2$. In Chapter 4 we will prove that Cummings- Foreman's iteration produces a model where every $\aleph_n$ with $n \geq 2$ satisfy even the super tree property. None of the results of this Chapter is due to the author.

3.1 The Building Block of the Iteration

The following definition is due to Cummings and Foreman and generalizes Mitchell construction. A few considerations will help the reader to understand the definition of this iteration. In Chapter 2 we have seen that to produce a model of the super tree property for a cardinal $\aleph_{n+2}$ (where $n < \omega$) we can start with a supercompact cardinal κ and use Mitchell's forcing $\mathbb{M}(\aleph_n, \kappa)$ to turn κ into $\aleph_{n+2}$ while preserving the super tree property at κ. A naive attempt to construct a model where the super tree property holds simultaneously for two cardinals $\aleph_{n+2}$ and $\aleph_{n+3}$, would be to start with two supercompact cardinals $\kappa < \lambda$, and force with $\mathbb{M}(\aleph_n, \kappa)$ first, and then with $\mathbb{M}(\aleph_{n+1}, \lambda)$. The problem with that approach is that, at the second step of this iteration, we could lose the super tree property at κ, that is at $\aleph_{n+2}$. For this reason, the first step of the iteration must be reformulated so that, not only it will turn κ into $\aleph_{n+2}$ and preserve the super tree property at κ, but it will also "anticipate a fragment" of $\mathbb{M}(\aleph_{n+1}, \lambda)$. We are going to define a forcing $\mathbb{R}(\tau, \kappa, V, W, L)$ that will constitute the main brick of Cummings and Foreman's iteration (Definition 3.3.1). If κ is supercompact cardinal in the model V, then $\mathbb{R}(\tau, \kappa, V, W, L)$ turns κ into τ^{++} and it makes τ^{++} satisfy the super tree property in a larger model W. The parameter L refers to a function that is basically a Laver function for κ, such function will be used to "guess the tail" of the iteration.

Definition 3.1.1. *Let $V \subseteq W$ be two models of set theory and suppose that for some τ, κ, we have $W \models$ ($\tau < \kappa$ is regular and κ is inaccessible). Let*

$\mathbb{P} := \mathrm{Add}(\tau,\kappa)^V$ and suppose that $W \models \mathbb{P}$ is τ^+-c.c. and τ-distributive. Let $L \in W$ be a function with $L : \kappa \to (V_\kappa)^W$. Define in W a forcing

$$\mathbb{R} := \mathbb{R}(\tau,\kappa,V,W,L)$$

as follows. The definition is by induction; for each $\beta \le \kappa$ we will define a forcing $\mathbb{R} \restriction \beta$ and we will finally set $\mathbb{R} := \mathbb{R} \restriction \kappa$. $\mathbb{R} \restriction 0$ is the trivial forcing. (p,q,f) is a condition in $\mathbb{R} \restriction \beta$ if and only if

(i) $p \in \mathbb{P} \restriction \beta := \mathrm{Add}(\tau,\beta)^V$;

(ii) *q is a partial function on β, $|q| \le \tau$, $\mathrm{dom}(q)$ consists of successor ordinals, and if $\alpha \in \mathrm{dom}(q)$, then $q(\alpha) \in W^{\mathbb{P}\restriction\alpha}$ and $\Vdash^W_{\mathbb{P}\restriction\alpha} q(\alpha) \in \mathrm{Add}(\tau^+)$*

(iii) *f is a partial function on β, $|f| \le \tau$, $\mathrm{dom}(f)$ consists of limit ordinals and $\mathrm{dom}(f)$ is a subset of*

$$\{\alpha;\ \Vdash^W_{\mathbb{R}\restriction\alpha} L(\alpha) \text{ is a } \tau^+\text{-directed closed forcing }\}$$

(iv) *If $\alpha \in \mathrm{dom}(f)$, then $f(\alpha) \in W^{\mathbb{R}\restriction\alpha}$ and $\Vdash^W_{\mathbb{R}\restriction\alpha} f(\alpha) \in L(\alpha)$.*

The conditions in $\mathbb{R} \restriction \beta$ are ordered by $(p',q',f') \le (p,q,f)$ if and only if

(i) $p' \le p$;

(ii) *for all $\alpha \in \mathrm{dom}(q)$, $p' \restriction \alpha \Vdash^W_{\mathbb{P}\restriction\alpha} q'(\alpha) \le q(\alpha)$;*

(iii) *for all $\alpha \in \mathrm{dom}(f)$, $(p',q',f') \restriction \alpha \Vdash^W_{\mathbb{R}\restriction\alpha} f'(\alpha) \le f(\alpha)$.*

Let us discuss some easy properties of that forcing.

Lemma 3.1.2. *$\mathbb{R}$ has size κ and it is κ-Knaster.*

Proof. To prove that $\mathbb{R}$ has size κ it is enough to observe that at each α there are fewer than κ possibilities for $q(\alpha)$ or $f(\alpha)$. The proof that $\mathbb{R}$ is κ-Knaster is a Δ-system argument like in Lemma 2.1.5. □

Lemma 3.1.3. *In the situation of Definition 3.1.1, $\mathbb{R}$ can be projected to $\mathbb{P}$, $\mathbb{R} \restriction \alpha * L(\alpha)$, and $\mathbb{P} \restriction \alpha * \dot{\mathbb{A}}$ where $\dot{\mathbb{A}}$ is a $\mathbb{P} \restriction \alpha$-name for Add($\tau^+$).*

Proof. The projection maps are defined as follows:

(i) $\pi_0 : (p,q,f) \mapsto p$ is the projection to $\mathbb{P}$;

(ii) $\pi_1 : (p,q,f) \mapsto ((p,q,f) \restriction \alpha, f(\alpha))$ is the projection to $\mathbb{R} \restriction \alpha * L(\alpha)$;

(iii) $\pi_2 : (p,q,f) \mapsto (p \restriction \alpha, q(\alpha))$ is the projection to $\mathbb{P} \restriction \alpha * \dot{\mathbb{A}}$.

See also [2, Lemma 3.3]. □

Lemma 3.1.4. *$\mathbb{R}$ adds at least κ subsets to τ.*

Proof. $\mathbb{R}$ projects to $\mathbb{P}$ and $\mathbb{P}$ adds at least κ subsets to τ. $\square$

Lemma 3.1.5. *$\mathbb{R}$ collapses every cardinal between τ^+ and κ to τ^+.*

Proof. Let α be a cardinal between τ^+ and κ. $\mathbb{R}$ projects to a forcing which makes $2^\tau \geq \alpha$ and then adds a Cohen subset of τ^+. This forcing collapses α to τ^+ $\square$

Lemma 3.1.6. *Assume $g \subseteq \mathbb{P}$ is a generic filter and $\mathbb{P}$ is τ-distributive in W, then $\mathbb{R}/g$ is τ-directed closed in $W[g]$. In particular if $\mathbb{P}$ is τ-closed, then $\mathbb{R}$ is τ-closed.*

Proof. In $W[g]$, let $\langle (p_i, q_i, f_i);\ i < \gamma \rangle$ be a sequence of less than τ pairwise compatible conditions of $\mathbb{R}/g$. Since $\mathbb{P}$ is τ-distributive, the sequence belongs to W. By definition of $\mathbb{R}/g$, we have $p_i \in g$ for every g, so we can fix a condition p such that $p \leq p_i$ for every $i < \gamma$ (as $\mathbb{P}$ is separative, we can take for example $p \in g$ such that $p \Vdash p_i \in \dot{g}$ for all i, where $\dot{g}$ is the canonical name for a generic filter for $\mathbb{P}$). We define a function q with domain $\bigcup\limits_{i<\gamma} \mathrm{dom} q_i$ as follows. For every $\alpha \in \mathrm{dom}(q)$, let $I_\alpha \subseteq \gamma$ such that $\alpha \in \mathrm{dom}(q_i)$ for every $i \in I_\alpha$. Then we have

$$p \restriction \alpha \Vdash \langle q_i(\alpha);\ i \in I_\alpha \rangle \text{ are pairwise compatible conditions in } \mathrm{Add}(\tau^+).$$

Therefore there is $q(\alpha) \in W^{\mathbb{P}\restriction\alpha}$ such that $p \restriction \alpha \Vdash q(\alpha) \leq q_i(\alpha)$ for every $i \in I_\alpha$. Now we define a function f with domain $\bigcup\limits_{i<\gamma} \mathrm{dom}(f_i)$. By induction on α, we define $f(\alpha)$ so that $(p, q, f) \restriction \alpha$ is a lower bound for the sequence $\langle (p_i, q_i, f_i) \restriction \alpha;\ i < \gamma \rangle$. Assume that $f(\beta)$ has been defined for every $\beta < \alpha$, and let $J_\alpha \subseteq \gamma$ such that $\alpha \in \mathrm{dom}(f_i)$ for every $i \in J_\alpha$, then

$$(p, q, f) \restriction \alpha \Vdash \langle f_i(\alpha);\ i \in J_\alpha \rangle \text{ are pairwise compatible conditions in } L(\alpha).$$

By definition we have $\Vdash^W_{\mathbb{R}\restriction\alpha} L(\alpha)$ is τ^+-directed closed, so there is $f(\alpha) \in W^{\mathbb{R}\restriction\alpha}$ such that $(p, q, f) \restriction \alpha \Vdash f(\alpha) \leq f_i(\alpha)$, for every $i \in J_\alpha$. That completes the definition of f. Finally the condition (p, q, f) is a lower bound for the sequence $\langle (p_i, q_i, f_i);\ i < \gamma \rangle$. $\square$

3.2 Factoring $\mathbb{R}$

Definition 3.2.1. *We define the poset*

$$\mathbb{U} := \mathbb{U}(\tau, \kappa, V, W, L) := \{(0, q, f);\ (0, q, f) \in \mathbb{R}\}$$

ordered as a subset of $\mathbb{R}$.

Lemma 3.2.2. *$\mathbb{U}$ is κ-c.c.*

Proof. It follows from Lemma 3.1.2. □

Lemma 3.2.3. *In W, the poset $\mathbb{U}$ is τ^+-directed closed.*

Proof. We write out explicitly the definition of the ordering on $\mathbb{U}$. $(0, q', f') \leq (0, q, f)$ if and only if

(i) $\mathrm{dom}(q) \subseteq \mathrm{dom}(q')$ and $\Vdash^W_{\mathbb{P}\restriction\alpha}\ q'(\alpha) \leq q(\alpha)$ for all $\alpha \in \mathrm{dom}(q_0)$;

(ii) $\mathrm{dom}(f) \subseteq \mathrm{dom}(f')$ and $(0, q', f') \restriction \alpha \Vdash^W_{\mathbb{R}\restriction\alpha} f'(\alpha) \leq f(\alpha)$ for all $\alpha \in \mathrm{dom}(f)$.

Let $\{(0, q_\eta, f_\eta)\}_{\eta<\tau}$ be a directed set of conditions. We define

$$A_1 := \bigcup_{\eta<\tau} \mathrm{dom}(q_\eta)$$

and observe that A_1 has size at most τ. We will define a function q with domain A_1. For $\alpha \in A_1$, consider $\{q_\eta(\alpha);\ \eta < \tau\}$. If $\eta, \zeta < \tau$, then for some $\mu < \tau$ we have that $(0, q_\mu, f_\mu)$ is a common refinement of $(0, q_\eta, f_\eta)$ and $(0, q_\zeta, f_\zeta)$. In particular

$$\Vdash\ q_\mu(\alpha) \leq q_\eta(\alpha), q_\zeta(\alpha).$$

So we can look at $\{q_\eta(\alpha);\ \eta < \tau\}$ as a name in $W^{\mathbb{P}\restriction\alpha}$ for a directed set of size τ in $\mathrm{Add}(\tau^+)^{W[g_\alpha]}$ where g_α is a generic filter for $\mathbb{P} \restriction \alpha$ over W. Then we can find $r(\alpha)$ which is forced to be the greatest lower bound. In particular, $\Vdash r(\alpha) \leq q_\eta(\alpha)$ for all $\eta < \tau$.

Let $A_2 := \bigcup_{\eta<\tau} \mathrm{dom}(f_\eta)$ and observe that A_2 has size at most τ. We will define by induction on α a function g with domain A_2 such that

$$(0, r, g) \restriction \alpha \Vdash\ g(\alpha) \leq f_\eta(\alpha) \text{ for all } \alpha, \eta.$$

Fix α, as we remarked already if $\eta, \zeta < \tau$, then for some $\mu < \tau$ we have that $(0, q_\mu, f_\mu)$ is a common refinement of $(0, q_\eta, f_\eta)$ and $(0, q_\zeta, f_\zeta)$. In particular

$$(0, q_\mu, f_\mu) \restriction \alpha \Vdash\ f_\mu(\alpha) \leq f_\eta(\alpha), f_\zeta(\alpha).$$

By induction $(0, r, g) \restriction \alpha \leq (0, q_\mu, f_\mu) \restriction \alpha$ so

$$(0, r, g) \restriction \alpha \Vdash\ f_\mu(\alpha) \leq f_\eta(\alpha), f_\zeta(\alpha).$$

Now $(0, r, g) \restriction \alpha$ forces that $\{f_\eta(\alpha);\ \eta < \tau\}$ is directed. We define $g(\alpha)$ to be a name which denotes the greatest lower bound of $\{f_\eta(\alpha);\ \eta < \tau\}$ is that set is directed, and the trivial condition otherwise. In particular

$$(0, r, g) \restriction \alpha \Vdash\ g(\alpha) \leq f_\eta(\alpha) \text{ for all } \eta.$$

At the end we have constructed a condition $(0, r, g)$ which is a lower bound for the directed set $\{(0, q_\eta, f_\eta);\ \eta < \tau\}$. It remains to show that $(0, r, g)$ is an infimum. Let $(0, s, h)$ be any conditions such that for all η, we have $(0, s, h) \leq (0, q_\eta, f_\eta)$. Clearly $A_1 \subseteq \mathrm{dom}(s)$ and $A_2 \subseteq \mathrm{dom}(h)$. For each $\alpha \in \mathrm{dom}(s)$ we have $\Vdash s(\alpha) \leq f_\eta(\alpha)$ for all η, and since $r(\alpha)$ is forced to be a greater lower bound $\Vdash s(\alpha) \leq r(\alpha)$. We attempt to show by induction that $(0, s, h) \restriction \alpha \Vdash h(\alpha) \leq g(\alpha)$. If it is true below α, then $(0, s, h) \restriction \alpha \leq (0, r, g) \restriction \alpha$ so that $(0, s, h) \restriction \alpha$ forces that $\{f_\eta(\alpha);\ \eta < \tau\}$ is directed. It also follows from the hypothesis that $(0, s, h) \restriction \alpha \Vdash h(\alpha) \leq f_\eta(\alpha)$ for all η, so that by our choice of $g(\alpha)$ we have $(0, s, h) \restriction \alpha \Vdash h(\alpha) \leq g(\alpha)$. So the induction goes through and at the end we have shown that $(0, s, h) \leq (0, r, g)$. Hence $(0, r, g)$ is the greatest lower bound. □

Lemma 3.2.4. *In $\mathbb{R}$, the condition (p, q, f) is the greatest lower bound for $(p, 0, 0)$ and $(0, q, f)$.*

Proof. Clearly it is a lower bound. Suppose that (p', q', f') is also a lower bound, then by definition $p' \leq p$, $p \restriction \alpha \Vdash\ q'(\alpha) \leq q(\alpha)$ and

$$(p', q', f') \restriction \alpha \Vdash f'(\alpha) \leq f(\alpha).$$

That is to say $(p', q', f') \leq (p, q, f)$. □

Lemma 3.2.5. *If $\mathbb{P}$ and $\mathbb{U}$ are as above, then*

(i) $\mathbb{P} \times \mathbb{U}$ *is κ-c.c.;*

(ii) If $g \times u \subseteq \mathbb{P} \times \mathbb{U}$ is generic over W, then all τ-sequences of ordinals in $W[g \times u]$ are in $W[g]$.

Proof. By Easton's Lemma, $\mathbb{P}$ is τ^+-c.c. in $W[u]$. Since $\mathbb{U}$ is κ-c.c. in W, the product $\mathbb{P} \times \mathbb{U}$ is κ-c.c. Easton's Lemma also implies that all τ-sequences of ordinals from $W[g \times u]$ are in $W[g]$. □

Lemma 3.2.6. *Let $\pi : \mathbb{P} \times \mathbb{U} \to \mathbb{R}$ be the function given by $\pi(p, (0, q, f)) = (p, q, f)$. Then π is a projection.*

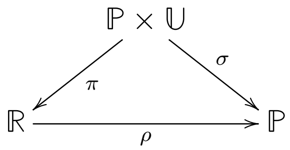
Proof. It is clear that π preserves the identity and respects the ordering relation. Let $(p', q', f') \leq (p, q, f)$ be in $\mathbb{R}$. Observe that for all α, we have $p' \restriction \alpha \Vdash\ q'(\alpha) \leq q(\alpha)$. Define $\bar{q}(\alpha)$ as a name with the following property: for every G generic for $\mathbb{P} \restriction \alpha$, $\bar{q}(\alpha)$ interprets as $q'(\alpha)^G$ if $p' \restriction \alpha \in G$, and interprets as $q(\alpha)^G$ otherwise. By construction, we have $\Vdash\ \bar{q}(\alpha) \leq q(\alpha)$ and $p' \restriction \alpha \Vdash\ \bar{q}(\alpha) = q'(\alpha)$. Now we attempt to define by induction a term $\bar{f}(\alpha)$ such that $(p', \bar{q}, \bar{f}) \restriction \alpha \Vdash \bar{f}(\alpha) = f'(\alpha)$ and $(0, \bar{q}, \bar{f}) \Vdash\ \bar{f}(\alpha) \leq f(\alpha)$. If we have done this for stages below α, then the conditions $(p', \bar{q}, \bar{f}) \restriction \alpha$ and $(p', q', f') \restriction \alpha$ are equivalent in $\mathbb{R} \restriction \alpha$. By hypothesis,

$$(p', q', f') \restriction \alpha \Vdash\ f'(\alpha) \leq f(\alpha).$$

Define $\bar{f}(\alpha)$ as follows: for any generic filter G for $\mathbb{R} \restriction \alpha$, the interpretation of $\bar{f}(\alpha)$ is $f'(\alpha)^G$ if $(p', \bar{q}, \bar{f}) \restriction \alpha \in G$ and $f(\alpha)^G$ otherwise. Now $(p', \bar{q}, \bar{f}) \restriction \alpha \Vdash \bar{f}(\alpha) = f'(\alpha)$ and $\Vdash \bar{f}(\alpha) \leq f(\alpha)$, so we are done.

At the end of this construction we have shown that the conditions $(p', \bar{q}, \bar{f})$ and (p', q', f') are equivalent in $\mathbb{R}$ and $(0, \bar{q}, \bar{f}) \leq (0, q, f)$, which is what is needed. □

Recall that we also have projections $\rho : \mathbb{R} \to \mathbb{P}$ and $\sigma : \mathbb{P} \times \mathbb{U} \to \mathbb{P}$ given by $\rho(p, q, f) = p$ and $\sigma(p, (0, q, f)) = p$. The following diagram is commutative:



Until the end of this section G is generic for $\mathbb{R}$ over W and g is a projection of G generic for $\mathbb{P}$ over W.

Lemma 3.2.7. *If $X \in W[G]$ is a set of ordinals of size τ, then $X \in W[g]$.*

Proof. We have seen that $W[G]$ can be embedded in a larger extension $W[g][H]$, where H is generic for $\mathbb{U}$ over $W[g]$. As $\mathbb{P}$ is τ^+-c.c. and $\mathbb{U}$ is τ^+-closed, Easton's lemma implies that $X \in W[g]$. □

Lemma 3.2.8. *$\mathbb{R}$ preserves τ^+ and forces that $2^\tau = \kappa = \tau^{++}$.*

Proof. If $f \in W[G]$ and $f : \tau \to (\tau^+)^W$ then by Lemma 3.2.7 $f \in W[g]$ and g is generic for a τ^+-c.c. forcing so that f is bounded. Similarly $\mathscr{P}(\tau) \cap W[G] = \mathscr{P}(\tau) \cap W[g]$ and $2^\tau = \kappa$ in $W[g]$. By Lemma 3.1.5 $\mathbb{R}$ collapses every cardinal in the interval $[\tau^+, \kappa[$ to τ^+. □

Lemma 3.2.9. *$\mathbb{R}$ is τ-distributive in W, so preserves cardinals $\leq \tau$.*

Proof. If $X \in W[G]$ is a set of ordinals of size less than τ, then $X \in W[g]$ by Lemma 3.2.7. Moreover $\mathbb{P}$ is τ-distributive in W so $X \in W$. □

Lemma 3.2.10. *If $H \subseteq \mathbb{U}$ is generic over W, then in $W[H]$ we have $\tau^{++} = \kappa$.*

Proof. $\mathbb{U}$ must collapse κ to τ^{++} because

$$W[H] \models \mathbb{P} \text{ is } \tau^+\text{-c.c. and } \tau\text{-distributive}$$

and κ is collapsed to τ^{++} in $W[H \times g]$. □

Lemma 3.2.11. *If $X \in W[G]$ is a set of ordinals of size τ, then X is covered by a set of size τ in W (i.e. $(W, W[G])$ has the τ-covering property).*

Proof. It follows from the fact that $X \in W[g]$ and $\mathbb{P}$ is τ^+-c.c. in W. □

Now we want to explore some properties of the forcing that takes us from $W[G]$ to $W[g \times H]$ where $H \subseteq \mathbb{U}$ is generic over W.

Definition 3.2.12. *We define in* $W[G]$ *the forcing*

$$\mathbb{S} := \mathbb{S}(\tau, \kappa, V, W, L, G) := \{(p, (0, q, f));\ (p, q, f) \in G\}$$

ordered as a suborder of $\mathbb{P} \times \mathbb{U}$.

Lemma 3.2.13. $W[G] \models \mathbb{S}$ *is* τ^+*-distributive,* τ*-closed and* κ*-c.c.. In particular forcing with* $\mathbb{S}$ *over* $W[G]$ *does not collapse cardinals.*

Proof. The κ-chain condition and the $< \tau^+$-distributivity follow from our earlier remakrs about $\mathbb{P} \times \mathbb{U}$. It remains to show that $\mathbb{S}$ is τ-closed. Let $\langle (p_\zeta, (0, q_\zeta, f_\zeta)) \rangle_{\zeta < \mu}$ be a decreasing sequence of conditions from $\mathbb{S}$ for some $\mu < \tau$. By the $< \tau$-distributivity of $\mathbb{R}$, this sequence is in W. Let $p := \bigcup_\zeta p_\zeta$, then $p \in g$. Since the sequence $\langle (0, q_\zeta, f_\zeta) \rangle_{\zeta < \mu}$ is decreasing in $\mathbb{U}$ we may perform the construction of Lemma 3.2.3 to get an infimum $(0, \bar{q}, \bar{f})$ for this sequence. We claim that $(0, \bar{q}, \bar{f}) \in G$. We already know that $p \in g$. Fix a successor $\alpha < \kappa$, then by definition of $\bar{q}(\alpha)$ we see that $\bar{q}^{g \restriction \alpha}$ is the infimum of the sequence $\langle q_\zeta(\alpha)^{g \restriction \alpha};\ \zeta < \mu \rangle$ in the forcing $\mathrm{Add}(\tau^+)^{W[g \restriction \alpha]}$. If we let G^0_α be the $\mathrm{Add}(\tau^+)^{W[g \restriction \alpha]}$ generic filter added by G, then we know that $q_\zeta(\alpha)^{g \restriction \alpha} \in G^0_\alpha$ for all ζ and so $\bar{q}(\alpha)^{g \restriction \alpha} \in G^0_\alpha$. For each relevant limit $\alpha < \kappa$ let G^1_α denote the generic filter for $L(\alpha)^{G \restriction \alpha}$ added by G. We will prove by induction on α, that $\bar{f}(\alpha)^{G \restriction \alpha} \in G^1_\alpha$. Suppose that we have done this up to stage α, so that in particular $(0, \bar{q}, \bar{f}) \restriction \alpha \in G \restriction \alpha$. Since $(0, \bar{q}, \bar{f})$ is a lower bound for $\langle (0, q_\zeta, f_\zeta) \rangle_{\zeta < \mu}$ the condition $(0, \bar{q}, \bar{f}) \restriction \alpha$ forces that $\langle f_\zeta(\alpha);\ \zeta < \mu \rangle$ is decreasing, so that $\langle f_\zeta(\alpha)^{G \restriction \alpha} \rangle$ is a decreasing sequence of members of G^1_α. Moreover, $\bar{q}(\alpha)^{G \restriction \alpha}$ is the infimum of this sequence, so that $\bar{q}(\alpha)^{G \restriction \alpha} \in G^1_\alpha$ and we are done. □

We conclude the analysis of $\mathbb{R}$ by looking at the forcing obtained when we factor $\mathbb{R}$ over one of its initial segments. Fix $\beta < \kappa$, the projection $\pi : \mathbb{R} \to \mathbb{R} \restriction \beta$ given by restriction is a good projection, in particular if X_β is generic for $\mathbb{R} \restriction \beta$ then we may consider the forcing to prolong X_β to a generic filter for $\mathbb{R}$ as given by the following definition.

Definition 3.2.14. *Let* $\beta < \kappa$ *and* X_β *be generic for* $\mathbb{R} \restriction \beta$ *over* W, *we define*

$$\mathbb{R}^* := \mathbb{R}^*(\tau, \kappa, V, W, L, X_\beta) = \{r \in \mathbb{R};\ r \restriction \beta \in X_\beta\}$$

with the ordering given by $r' \leq r \iff \exists s \in X_\beta\ \mathrm{Ext}(r', s) \leq r$. *Note that here* $\mathrm{Ext}(r', s)$ *is just the extension of* r' *in which* $r' \restriction \beta$ *is replaced by* s.

Definition 3.2.15. *Given β and X_β as above, define*

$$\mathbb{U}^* := \mathbb{U}^*(\tau, \kappa, V, W, L, X_\beta) := \{(0, q, f);\ (0, q, f) \in \mathbb{R}^*\}$$

ordered as a suborder of $\mathbb{R}^*$. We also define

$$\mathbb{P}^* := \{p \in \mathbb{P};\ (p, 0, 0) \in \mathbb{R}^*\}$$

ordered as a suborder of $\mathbb{P}$.

$\mathbb{P}^*$ is essentially $\mathbb{P} \restriction \kappa - \beta = \mathrm{Add}(\tau, \kappa - \beta)^V$.

Lemma 3.2.16. *The following hold:*

(1) *the function $\pi : \mathbb{P}^* \times \mathbb{U}^* \to \mathbb{R}^*$ defined by $\pi(p, (0, q, f)) \mapsto (p, q, f)$ is a projection;*

(2) *$\mathbb{U}^*$ is τ^+-closed in $W[X_\beta]$.*

Proof. Claim (1) follows from Lemma 3.2.4. We prove Claim (2). Let $\dot{\tau}$ name a descending τ-sequence in $\mathbb{U}^*$. Let x_β be a generic filter added by X_β. We may assume that $\dot{\tau} \in W^{\mathbb{P}\restriction\beta}$ because all τ-sequences in $W[X_\beta]$ come from $W[x_\beta]$. We denote by $\dot{\tau}_\eta$ the canonical term for entry η in the sequence named by $\dot{\tau}$. We adopt the convention that

$$\mathrm{Left}(0, q, f) = q \text{ and } \mathrm{Right}(0, q, f) = f$$

Let X_γ be generic for $\mathbb{R} \restriction \gamma$ and x_γ be the corresponding generic filter for $\mathbb{P} \restriction \gamma$. Then $\dot{\tau}_\eta^{x_\gamma\restriction\beta}$ is a condition in $\mathbb{U}^*$ so that $[\mathrm{Right}(\dot{\tau}_\eta^{x_\gamma\restriction\beta})]^{X_\gamma} \in L(\gamma)^{X_\gamma}$. Similarly $[\mathrm{Left}(\dot{\tau}_\eta^{x_\gamma\restriction\beta})]^{x_\gamma} \in \mathrm{Add}(\tau^+)^{W[x_\gamma]}$.

We will define in W a condition $(0, q^*, f^*) \in \mathbb{R}$ such that

(i) $\mathrm{dom}(q^*) \subseteq \kappa - \beta$, $\mathrm{dom}(f^*) \subseteq \kappa - \beta$;

(ii) $\mathrm{dom}(q^*)$ is the set of $\gamma \geq \beta$ such that for some $\eta < \tau$, γ is a potential member of the domain of $\mathrm{Left}(\dot{\tau}_\eta)$;

(iii) $\mathrm{dom}(f^*)$ is the set of $\gamma \geq \beta$ such that for some $\eta < \tau$, γ is a potential member of the domain of $\mathrm{Right}(\dot{\tau}_\eta)$;

(iv) for all $\gamma \geq \beta$ if X_γ is generic for $\mathbb{R} \restriction \gamma$ and $(0, q^*, f^*) \restriction X_\gamma$, then for all $\eta < \tau$ we have $\dot{\tau}_\eta^{x_\gamma\restriction\beta} \in X_\gamma$.

As $\mathbb{P}$ is τ^+-c.c. in W, the domains are not too big. We will start by setting $(0, q^*, f^*) \restriction \beta = (0, 0, 0)$. Suppose we have defined $(0, q^*, f^*) \restriction \gamma$ successfully. We will now define $f^*(\gamma)$. Let X_γ be generic for $\mathbb{R} \restriction \gamma$ and assume that $(0, q^*, f^*) \restriction \gamma \in X_\gamma$. By our induction hypothesis, we have for every $\eta < \tau$ $\dot{\tau}_\eta^{x_\gamma\restriction\beta} \restriction \gamma \in X_\gamma$. We will work in $W[X_\gamma]$.

Claim 2. *Define a τ-sequence of conditions in $L(\gamma)^{X_\gamma}$ by*

$$p(\eta) = [\mathrm{Right}(\dot{\tau}_\eta^{x_\gamma\restriction\beta})].$$

Then this is a decreasing sequence.

Proof. Let $\zeta < \eta < \tau$ and suppose that

$$\dot{\tau}_{\zeta}^{x_\gamma \restriction \beta} = (0, q, f),$$

$$\dot{\tau}_{\eta}^{x_\gamma \restriction \beta} = (0, \bar{q}, \bar{f}).$$

Notice that $p(\gamma) = f(\gamma)^{X_\gamma}$ and $p(\eta) = \bar{f}(\gamma)^{X_\gamma}$. Moreover, $(0, q, f) \restriction \gamma$ and $(0, \bar{q}, \bar{f}) \restriction \gamma$ are in X_γ. We may choose $s \in X_\gamma \restriction \beta$ such that $s \Vdash \dot{\tau}_\zeta = (0, q, f)$, $s \Vdash \dot{\tau}_\eta = (0, \bar{q}, \bar{f})$, and $\mathrm{Ext}(s, (0, \bar{q}, \bar{f})) \leq (0, q, f)$ in $\mathbb{R}$. Now $\mathrm{Ext}(s, (0, \bar{q}, \bar{f})) \restriction \gamma \in X_\gamma$ so that by definition of extension in $\mathbb{R}$ we have $p(\eta) = \bar{f}(\gamma)^{X_\gamma} \leq f(\gamma)^{X_\gamma} = p(\zeta)$. This completes the proof of the claim. □

Now we choose $f^*(\gamma)$ to be a name, forced by $(0, q^*, f^*) \restriction \gamma$ to be a lower bound for that sequence $\vec{p}$. We observe that if we assume $(0, q^*, f^*) \restriction \gamma \in X_\gamma$, then $f^*(\gamma)^{X_\gamma} \leq [\mathrm{Right}(\dot{\tau}_\eta^{x_\gamma \restriction \beta})]^{X_\gamma}$. We choose q^* similarly. Let X_γ and x_γ as usual with $(0, q^*, f^*) \restriction \gamma \in X_\gamma$. Working in $W[x_\gamma]$ define a sequence q in $\mathrm{Add}(\tau^+)$ by $q(\eta) = [\mathrm{Left}(\dot{\tau}_\eta^{x_\gamma \restriction \beta})]^{x_\gamma}$. Working as before we can show that q is decreasing. Now choose $q^*(\gamma)$ to be a $\mathbb{P} \restriction \gamma$-name for a lower bound. We check that the induction hypothesis goes through. Suppose that $(0, q^*, f^*) \restriction \gamma + 1 \in X_{\gamma+1}$ and let $\eta < \tau$. Suppose that $\dot{\tau}_\eta^{x_{\gamma+1} \restriction \beta} = t_\eta = (0, q_\eta, f_\eta)$. Then $t_\eta \restriction \gamma \in X_\gamma$ and by construction we know that $q^*(\gamma)^{x_\gamma} \leq q_\eta(\gamma)^{x_\gamma}$ and $f^*(\gamma)^{X_\gamma} \leq f_\eta(\gamma)^{X_\gamma}$. So that $t_\eta \restriction \gamma + 1 \in X_{\gamma+1}$. Limits do not present a problem, so that the construction of $(0, q^*, f^*)$ can proceed. We finish by showing that we have constructed a lower bound.

Claim 3. *Let X_β be generic for $\mathbb{R} \restriction \beta$. Then $(0, q^*, f^*)$ is a lower bound in $\mathbb{U}^*$ for the sequence $\dot{\tau}^{x_\beta}$.*

Proof. Let $\zeta < \tau$ and suppose $\dot{\tau}_\zeta^{x_\beta} = (0, q, f)$. Choose $r \in X_\beta$ such that $r \leq (0, q, f) \restriction \beta$ and $r \Vdash \dot{\tau}_\zeta = (0, q, f)$. By construction

$$r \restriction \gamma \Vdash q^*(\gamma) \leq q(\gamma)$$

$$\mathrm{Ext}((0, q^*, f^*) \restriction \gamma, r) \Vdash f^*(\gamma) \leq f(\gamma)$$

for each γ so that $\mathrm{Ext}((0, q^*, f^*), r) \leq (0, q, f)$. □

That completes the proof of the lemma. □

3.3 Cummings-Foreman's Iteration

Definition 3.3.1. *(Cummings-Foreman's Model) We consider $\langle \kappa_n;\ n < \omega \rangle$ an increasing sequence of supercompact cardinals. For every $n < \omega$, let $L_n : \kappa_n \to V_{\kappa_n}$ be the Laver function for κ_n. We define a forcing iteration $\mathbb{R}_\omega$ of length ω as follows.*

(i) The first stage of the iteration $\mathbb{R}_1$ is the poset $\mathbb{Q}_0 := \mathbb{R}(\aleph_0, \kappa_0, V, V, L_0)$.

(ii) *Assume* $\dot{L}_1$ *is an* $\mathbb{R}_1$*-name for a function so that* $\Vdash_{\mathbb{R}_1} \dot{L}_1(\alpha) = L_1(\alpha)$*, if* $L_1(\alpha)$ *is a* $\mathbb{R}_1$*-name, and* $\Vdash_{\mathbb{R}_1} \dot{L}_1(\alpha) = 0$ *otherwise. We let* $\dot{\mathbb{Q}}_1$ *be an* $\mathbb{R}_1$*-name for* $\mathbb{R}(\aleph_1^V, \kappa_1, V, V[\dot{K}_1], \dot{L}_1)$*, where* $\dot{K}_1$ *denotes the canonical name for a generic filter for* $\mathbb{R}_1$*. We define* $\mathbb{R}_2 := \mathbb{Q}_0 * \dot{\mathbb{Q}}_1$.

(iii) *Suppose* $\mathbb{R}_n := \mathbb{Q}_0 * ... * \dot{\mathbb{Q}}_{n-1}$ *has been defined and assume* $\dot{L}_n$ *is an* $\mathbb{R}_n$*-name for a function so that* $\Vdash_{\mathbb{R}_n} \dot{L}_n(\alpha) = L_n(\alpha)$*, if* $L_n(\alpha)$ *is a* $\mathbb{R}_n$*-name, and* $\Vdash_{\mathbb{R}_n} \dot{L}_n(\alpha) = 0$*, otherwise. We let* $\dot{\mathbb{Q}}_n$ *be an* $\mathbb{R}_n$*-name for the poset* $\mathbb{R}(\kappa_{n-2}, \kappa_n, V[\dot{K}_{n-1}], V[\dot{K}_n], \dot{L}_n)$*, where* $\dot{K}_n$ *is the canonical name for a generic filter for* $\mathbb{R}_n$*. Finally, we define* $\mathbb{R}_{n+1} := \mathbb{Q}_0 * ... * \dot{\mathbb{Q}}_n$.

(iv) $\mathbb{R}_\omega$ *is the inverse limit of* $\langle \mathbb{R}_n;\ n < \omega \rangle$.

We also fix a filter $G_\omega \subseteq \mathbb{R}_\omega$ *generic over* V*, and for every* $n > 0$*, we denote by* $K_n := G_0 * ... * G_{n-1}$ *the initial segment of* G_ω *generic for* $\mathbb{R}_n = \mathbb{Q}_0 * ... * \dot{\mathbb{Q}}_{n-1}$ *over* V.

It is not clear that this definition is legitimate, because we can only define $\mathbb{R}(\tau, \kappa, V, W, L)$ when we know that certain things are true in W; namely τ must be regular, κ must be inaccessible etc. The following lemmas will prove that the previous definition is legitimate. In the statement of the lemmas, when we refer to "$\aleph_i$" we mean $\aleph_i$ in the sense of $V[K_n]$.

Lemma 3.3.2. *Let* $\mathbb{P}_0 := \mathrm{Add}(\aleph_0, \kappa_0)$ *and* $\mathbb{U}_0 := \mathbb{U}(\aleph_0, \kappa_0, V, V, L_0)$*. The following hold.*

(1) $\mathbb{Q}_0$ *has size* κ_0 *and it is* κ_0*-Knaster. In particular*

(a) all V*-cardinals* $\geq \kappa_0$ *are preserved in* $V[G_0]$

(b) all V*-inaccessibles* $> \kappa$ *remain inaccessible in* $V[G_0]$

(c) all sets of size less than κ_0 *in* $V[G_0]$ *are covered by sets of size less than* κ_0 *in* V.

(2) $\mathbb{Q}_0$ *is a projection of* $\mathbb{P}_0 \times \mathbb{U}_0$ *and* $V[g_0] \subseteq V[G_0] \subseteq V[g_0 \times u_0]$ *where* u_0 *is generic for* $\mathbb{U}_0$ *over* V.

(3) All ω*-sequences of ordinals from* $V[G_0]$ *are in* $V[g_0]$.

(4) $\mathbb{Q}_0$ *preserves* $\aleph_1$ *and* $2^{\aleph_0} = \kappa_0 = \aleph_2$ *in* $V[G_0]$.

(5) $\mathrm{Add}(\aleph_0, \eta)^V$ *has the* $\aleph_1$*-Knaster property in* $V[G_0]$.

(6) For all η*,* $V[G_0] \models \mathrm{Add}(\aleph_1, \eta)^V$ *is* $\aleph_1$*-distributive and* κ_0*-Knaster.*

Proof. We take the claims in turn.

(1) It follows from Lemma 3.1.2.

(2) That corresponds to Lemma 3.2.6.

(3) This follows from Lemma 3.2.7.

(4) It is immediate by Lemma 3.2.8.

(5) $\mathrm{Add}(\aleph_0, \eta)^V = \mathrm{Add}(\aleph_0, \eta)^V[G_0]$.

(6) It follows from Easton's Lemma.

$\square$

Lemma 3.3.3. *For $n \geq 1$, in $V[K_n]$ we let $\mathbb{Q}_n := \dot{\mathbb{Q}}_n^{K_n}$ and we define*

$$\mathbb{P}_n := \mathrm{Add}(\aleph_n, \kappa_n)^{V[K_{n-1}]} \text{ and } \mathbb{U}_n := \mathbb{U}(\aleph_n, \kappa_n, V[K_{n-1}], V[K_n], L_n^*)$$

(ordered as a subset of $\mathbb{Q}_n$). The following hold:

(1) $V[K_n] \models 2^{\aleph_i} = \aleph_{i+2} = \kappa_i$, *for $i < n$, and κ_j is inaccessible for every $j \geq n$.*

(2) $V[K_n] \models \mathbb{Q}_n$ *is $\aleph_n$-distributive, κ_n-Knaster, $\aleph_{n-1}$-directed closed and has size κ_n.*

(3) All $\aleph_{n-1}$-sequences of ordinals from $V[K_n]$ are in $V[K_{n-1} * g_{n-1}]$.

(4) All cardinals up to $\aleph_n$ are preserved in $V[K_n]$.

(5) $V[K_n] \models (\mathbb{Q}_n$ *is a projection of $\mathbb{P}_n \times \mathbb{U}_n$), hence there are filters $g_n \subseteq \mathbb{P}_n$ and $u_n \subseteq \mathbb{U}_n$ which are generic over $V[K_n]$ and satisfy $V[K_n * g_n] \subseteq V[K_n * G_n] \subseteq V[K_n * (g_n \times u_n)]$.*

(6) All $\aleph_n$-sequences of ordinals from $V[K_{n+1}]$ are in $V[K_n * g_n]$.

(7) $\aleph_{n+1}$ which is κ_{n-1} is preserved in $V[K_{n+1}]$. Cardinal arithmetic in $V[K_{n+1}]$ follows the pattern $2^{\aleph_i} = \aleph_{i+2} = \kappa_i$ for $i \leq n$.

(8) $\mathrm{Add}(\aleph_n, \eta)^{V[K_{n-1}]}$ *is $\aleph_{n+1}$-Knaster in $V[K_{n+1}]$ for any ordinal η.*

(9) $V[K_{n+1}] \models \mathrm{Add}(\aleph_{n+1}, \eta)^{V[K_n]}$ *is $\aleph_{n+1}$-distributive and κ_n-Knaster for any ordinal η.*

Proof. We prove the lemma by induction on $n \geq 1$. The previous lemma already gives us some information in the case $n = 1$.

(1) is immediate by induction.

(2) From the analysis of the main forcing that we have done in the previous sections, it follows that $\mathbb{Q}_n$ has size κ_n, it is κ_n-Knaster and $\aleph_n$-distributive in $V[K_n]$. For the closure observe that $\mathbb{P}_n$ is $\aleph_n$-closed in $V[K_{n-1}]$ and by induction $\mathbb{Q}_{n-1}$ is $\aleph_{n-1}$-distributive in $V[K_{n-1}]$, so that $\mathbb{P}_n$ is $\aleph_{n-1}$-closed in $V[K_n]$. By Lemma 3.1.6 the forcing $\mathbb{Q}_n$ is $\aleph_{n-1}$-directed closed in $V[K_n]$.

(3) Since $\mathbb{Q}_n$ is $\aleph_n$-distributive, every $\aleph_{n-1}$-sequence of ordinals from $V[K_{n+1}]$ is in $V[K_n]$. By induction every $\aleph_{n-1}$-sequence of ordinals from $V[K_n]$ is in $V[K_{n-2} * g_{n-1}]$.

(4) This follows immediately from the last claim.

(5) Apply Lemma 3.2.6 in $V[K_n]$.

(6) Apply Lemma 3.2.7.

(7) By Lemma 3.2.8 $\aleph_{n+1}$ is preserved in $V[K_{n+1}]$. Since $\mathbb{Q}_n$ is $\aleph_n$-distributive in $V[K_n]$ it follows that all cardinals up to $\aleph_n$ are preserved and that we still have $2^{\aleph_i} = \kappa_i = \aleph_{i+2}$ for $i < n$ in $V[K_{n+1}]$. By Lemma 3.2.8 again $2^{\aleph_n} = \kappa_n = \aleph_{n+2}$ in $V[K_{n+1}]$.

(8) Use Easton's Lemma.

(9) Use Easton's Lemma.

□

In the following chapter, we will use the previous lemma repeatedly and without comments.

Cummings and Foreman [2] also proved the following lemma.

Lemma 3.3.4. *For every $n < \omega$, let $X \in V[G_\omega]$ be a κ_n-sequence of ordinals, then $X \in V[K_{n+2} * g_{n+2}]$*

Proof. For every $m < \omega$, the poset $\mathbb{R}_\omega / K_{m+3}$ is κ_m-closed. Therefore $X \in V[K_{n+4}]$. Since $\mathbb{Q}_{n+3}$ is κ_{n+1}-distributive in $V[K_{n+3}]$, we have $X \in V[K_{n+3}]$. Finally every κ_n-sequence of ordinals in $V[K_{n+3}]$ is in $V[K_{n+2} * g_{n+2}]$, that completes the proof. □

In [2], this was used to show that if T is a κ_n-tree in $V[G_\omega]$, then $T \in V[K_{n+2} * g_{n+2}]$; we cannot prove the same for (κ_n, μ)-trees.

4

The Super Tree Property at Small Cardinals

In Chapter 2 we showed that for every $n \geq 2$, if there is a model of GCH with a supercompact cardinal, then one can construct a model where $\aleph_{n+2}$ has the super tree property. In this chapter we prove that starting from infinitely many supercompact cardinals, one can obtain a model where all the cardinals of the form $\aleph_{n+2}$ *simultaneously* satisfy the super tree property. This is the main result of [4]. To prove such theorem we will use Cummings-Foreman iteration discussed in Chapter 3.

4.1 Expanding Cummings-Foreman's Model

To prove the main theorem, we need to expand Cummings and Foreman's model. In the previous section we introduced several objects. We recall that G_ω is a generic filter for $\mathbb{R}_\omega$ over V and $K_n = G_0 * ... * G_{n-1}$ is the initial segment of G_ω generic for $\mathbb{R}_n = \mathbb{Q}_0 * ... * \dot{\mathbb{Q}}_{n-1}$ over V. We defined in $V[K_n]$ two posets $\mathbb{P}_n := \mathrm{Add}(\aleph_n, \kappa_n)^{V[K_{n-1}]}$ and $\mathbb{U}_n := \{(0, q, f);\ (0, q, f) \in \mathbb{Q}_n\}$, and we fixed $g_n \subseteq \mathbb{P}_n$ and $u_n \subseteq \mathbb{U}_n$ which are two generic filters over $V[K_n]$ such that $V[K_n * g_n] \subseteq V[K_n * G_n] \subseteq V[K_n * (g_n \times u_n)]$. For every $n > 0$, the poset $\mathbb{S}_n$ is a forcing notion in $V[K_n * G_n]$ and it denotes $(\mathbb{P}_n \times \mathbb{U}_n)/G_n$ (see Lemma 3.3.3). In this short section we observe what happens when we force over $V[G_\omega]$ with $\mathbb{S}_{n+1}$ and then with $\mathbb{S}_{n+2}$.

Definition 4.1.1. *For every $n < \omega$, we define in $V[K_{n+1}]$ the forcing*

$$\mathrm{Tail}_{n+1} := \mathbb{R}_\omega / K_{n+1}.$$

Tail_{n+3} is a κ_n-directed closed forcing in $V[K_{n+3}]$.

Definition 4.1.2. *For every $n < \omega$, we denote $V_n := V[G_0 * ... * G_n] = V[K_{n+1}]$ and we let $G_{tail(n+1)} \subseteq \mathrm{Tail}_{n+1}$ be the generic filter over V_n such that $V_n[G_{tail(n+1)}] = V[G_\omega]$.*

Definition 4.1.3. *We let $s_{n+1} \subseteq \mathbb{S}_{n+1}$ be the generic filter over V_{n+1} such that $V_{n+1}[s_{n+1}] = V_n[g_{n+1} \times u_{n+1}]$.*

By Theorem I.0.8, $G_{tail(n+1)}$ and s_{n+1} are mutually generic thus

$$V_n[g_{n+1} \times u_{n+1}][G_{tail(n+2)}] = V[G_\omega][s_{n+1}].$$

For the same reason,

$$V_n[g_{n+1} \times u_{n+1}][g_{n+2} \times u_{n+2}][G_{tail(n+3)}] = V[G_\omega][s_{n+1}][s_{n+2}].$$

So the model $V_n[g_{n+1} \times u_{n+1}][g_{n+2} \times u_{n+2}][G_{tail(n+3)}]$ is the result of forcing over $V[G_\omega]$ first with $\mathbb{S}_{n+1}$ and then with $\mathbb{S}_{n+2}$. Now, we want to show that this model can be seen as being obtained by forcing over V_n with a cartesian product that satisfies particular properties.

In order to define that product, first we need to introduce the notion of "term forcing" (that notion is due to Mitchell [16]).

Definition 4.1.4. *Let $\mathbb{P}$ be a forcing notion and let $\dot{\mathbb{Q}}$ be a $\mathbb{P}$-name for a poset. For every $\dot{q}, \dot{r}$ such that $\Vdash_{\mathbb{P}} \dot{q}, \dot{r} \in \dot{\mathbb{Q}}$, we let $\dot{q} \leq^* \dot{r}$ if and only if $\Vdash_{\mathbb{P}} \dot{q} \leq \dot{r}$. The* $\mathbb{P}$-term-forcing for $\dot{\mathbb{Q}}$ *is the set of all equivalence classes (corresponding to $\leq^*$) of minimal rank.*

Lemma 4.1.5. *In the situation of Definition 4.1.4, assume $\mathbb{T}$ is the $\mathbb{P}$-term-forcing for $\dot{\mathbb{Q}}$, then the following hold:*

(i) $\mathbb{P} * \dot{\mathbb{Q}}$ is a projection of $\mathbb{P} \times \mathbb{T}$;

(ii) if $\Vdash_{\mathbb{P}} \dot{\mathbb{Q}}$ is κ-directed closed, then $\mathbb{T}$ is κ-directed closed as well.

Proof.

(1) Let $\pi : \mathbb{P} \times \mathbb{T} \to \mathbb{P} * \dot{\mathbb{Q}}$ be the map $(p, \dot{q}) \mapsto (p, \dot{q})$, we prove that π is a projection. It is clear that π respects the ordering relation and $\pi(1_{\mathbb{P}\times\mathbb{T}}) = (1_{\mathbb{P}*\dot{\mathbb{Q}}})$. In $\mathbb{P}*\dot{\mathbb{Q}}$, let $(p_0, \dot{q}_0) \leq (p_1, \dot{q}_1)$, then $p_0 \leq p_1$ and $p_0 \Vdash \dot{q}_0 \leq \dot{q}_1$. Define $\dot{q}$ as a $\mathbb{P}$-name for an element of $\dot{Q}$ such that for every $\mathbb{P}$-generic filter G, we have $\dot{q}^G = \dot{q}_0^G$ if $p_0 \in G$, and $\dot{q}^G = \dot{q}_1^G$ otherwise. Then $(p_0, \dot{q}) = (p_0, \dot{q}_0)$.

(2) Assume that $\langle \dot{q}_\alpha;\ \alpha < \gamma \rangle$ is a sequence of less than κ pairwise compatible conditions in $\mathbb{T}$. Then,

$$\Vdash_{\mathbb{P}} \text{“}\langle \dot{q}_\alpha;\ \alpha < \gamma\rangle \text{ are pairwise compatible conditions in } \dot{\mathbb{Q}}\text{”},$$

hence there exists a $\mathbb{P}$-name $\dot{q}$ such that $\Vdash_{\mathbb{P}} \dot{q} \leq \dot{q}_\alpha$, for every $\alpha < \gamma$. This means that $\dot{q} \leq^* \dot{q}_\alpha$, for every $\alpha < \gamma$. □

Posets like $\mathbb{P}_n$, $\mathbb{U}_n$ and Tail_{n+1} can be defined in any generic extension of V by $\mathbb{R}_n$. We introduce names for such forcings.

Notation 4.1.6. *Let $\dot{K}_n$ be the canonical name for a generic filter for $\mathbb{R}_n$. We let $\dot{\mathbb{P}}_n, \dot{\mathbb{U}}_n \in V^{\mathbb{R}_n}$ and $\dot{\text{Tail}}_{n+1} \in V^{\mathbb{R}_{n+1}}$ be such that*

(i) $\Vdash_{\mathbb{R}_n} \dot{\mathbb{P}}_n = \text{Add}(\aleph_n, \kappa_n)^{V[\dot{K}_{n-1}]}$;

(ii) $\Vdash_{\mathbb{R}_n} \dot{\mathbb{U}}_n = \{(0,q,f);\ (0,q,f) \in \dot{\mathbb{Q}}_n\}$;

(iii) $\Vdash_{\mathbb{R}_{n+1}} \dot{\mathrm{Tail}}_{n+1} = \mathbb{R}_\omega / \dot{K}_{n+1}$.

Definition 4.1.7. *For every $n < \omega$, we let $\dot{\mathbb{T}}_{n+3} \in V^{\mathbb{R}_{n+2}}$ be such that*

$$\Vdash_{\mathbb{R}_{n+2}} \dot{\mathbb{T}}_{n+3} \textit{ is the } (\dot{\mathbb{P}}_{n+2} \times \dot{\mathbb{U}}_{n+2})\textit{-term-forcing for } \dot{\mathrm{Tail}}_{n+3}.$$

We also let $\dot{\mathbb{Z}}_{n+2} \in V^{\mathbb{R}_{n+1}}$ be such that

$$\Vdash_{\mathbb{R}_{n+1}} \dot{\mathbb{Z}} \textit{ is the } (\dot{\mathbb{P}}_{n+1} \times \dot{\mathbb{U}}_{n+1} \times \dot{\mathbb{P}}_{n+2})\textit{-term-forcing for the poset } \dot{\mathbb{U}}_{n+2} \times \dot{\mathbb{T}}_{n+3}.$$

Finally we define $\mathbb{T}_{n+3} := \dot{\mathbb{T}}_{n+3}^{K_{n+2}}$ and $\mathbb{Z}_{n+2} := \dot{\mathbb{Z}}_{n+2}^{K_{n+1}}$.

Remark 4.1.8. *In other words,*

(i) $(\mathbb{P}_{n+2} \times \mathbb{U}_{n+2}) * \mathrm{Tail}_{n+3}$ *is a projection of* $\mathbb{P}_{n+2} \times \mathbb{U}_{n+2} \times \mathbb{T}_{n+3}$;

(ii) $(\mathbb{P}_{n+1} \times \mathbb{U}_{n+1} \times \mathbb{P}_{n+2}) * (\mathbb{U}_{n+2} \times \mathbb{T}_{n+3})$ *is a projection of* $\mathbb{P}_{n+1} \times \mathbb{U}_{n+1} \times \mathbb{P}_{n+2} \times \mathbb{Z}_{n+2}$.

Lemma 4.1.9. *The following hold:*

(i) $\mathbb{T}_{n+3}$ *is κ_n-directed closed in V_{n+1};*

(ii) $\mathbb{Z}_{n+2}$ *is κ_n-directed closed in V_n.*

Proof.

(1) Tail_{n+3} is κ_n-directed closed in V_{n+2} and in $V_{n+1}[g_{n+2} \times u_{n+2}]$. By Lemma 4.1.5, then, $\mathbb{T}_{n+3}$ is κ_n-directed closed in V_{n+1}.

(2) By the previous claim, the product $\mathbb{U}_{n+2} \times \mathbb{T}_{n+3}$ is κ_n-directed closed in V_{n+1}. The poset $\mathbb{S}_{n+1}$ is κ_n-distributive in V_{n+1}, so $\mathbb{U}_{n+2} \times \mathbb{T}_{n+3}$ is κ_n-directed closed in $V_n[g_{n+1} \times u_{n+1}]$ as well. Now $\mathbb{P}_{n+2}$ is κ_n-distributive in $V_n[g_{n+1} \times u_{n+1}]$, so $\mathbb{U}_{n+2} \times \mathbb{T}_{n+3}$ is κ_n-directed closed even in $V_n[g_{n+1} \times u_{n+1}][g_{n+2}] = V_n[g_{n+1} \times u_{n+1} \times g_{n+2}]$. By Lemma 4.1.5, the poset $\mathbb{Z}_{n+2}$ is κ_n-directed closed in V_n. □

Remark 4.1.8 justifies the following definition.

Definition 4.1.10. *We let $t_{n+3} \subseteq \mathbb{T}_{n+3}$ be generic over $V_n[g_{n+1} \times u_{n+1}]$ such that*

$$V_n[g_{n+1} \times u_{n+1}][g_{n+2} \times u_{n+2}][G_{tail(n+3)}] \subseteq V_n[g_{n+1} \times u_{n+1}][g_{n+2} \times u_{n+2} \times t_{n+3}].$$

We also let $z_{n+2} \subseteq \mathbb{Z}_{n+2}$ be generic over V_n such that

$$V_n[g_{n+1} \times u_{n+1}][g_{n+2} \times u_{n+2} \times t_{n+3}] \subseteq V_n[g_{n+1} \times u_{n+1} \times g_{n+2} \times z_{n+2}].$$

Lemma 4.1.11. *The following hold:*

(i) $(\mathbb{P}_{n+2}\times\mathbb{U}_{n+2}\times\mathbb{T}_{n+3})/(g_{n+2}\times u_{n+2})*G_{tail(n+3)}$ *is* κ_n*-closed in* $V_{n+1}[g_{n+2}\times u_{n+2}][G_{tail(n+3)}]$;

(ii) $(\mathbb{P}_{n+1}\times\mathbb{U}_{n+1}\times\mathbb{P}_{n+2}\times\mathbb{Z}_{n+2})/(g_{n+1}\times u_{n+1}\times g_{n+2})*(u_{n+2}\times t_{n+3})$ *is* $\aleph_{n+1}$*-closed in* $V_n[g_{n+1}\times u_{n+1}\times g_{n+2}][u_{n+2}\times t_{n+3}]$.

Proof. The proof is standard: it follows from Lemma 4.1.9 and from the fact that $\mathbb{P}_{n+2}\times\mathbb{U}_{n+2}*\dot{\text{Tail}}_{n+3}$ is κ_n-distributive and $(\mathbb{P}_{n+1}\times\mathbb{U}_{n+1}\times\mathbb{P}_{n+2})*(\dot{\mathbb{U}}_{n+2}\times\dot{\mathbb{T}}_{n+3})$ is $\aleph_{n+1}$-distributive. □

Remark 4.1.12. *Summing up, we have:*

(i) $V[G_\omega]\subseteq V_n[g_{n+1}\times u_{n+1}][G_{tail(n+2)}]$*, the latter model has been obtained by forcing with* $\mathbb{S}_{n+1}$ *over* $V[G_\omega]$*;*

(ii) $V_n[g_{n+1}\times u_{n+1}][G_{tail(n+2)}]\subseteq V_n[g_{n+1}\times u_{n+1}][g_{n+2}\times u_{n+2}][G_{tail(n+3)}]$*, the latter model has been obtained by forcing with* $\mathbb{S}_{n+2}$ *over the former;*

(iii) $V_n[g_{n+1}\times u_{n+1}][g_{n+2}\times u_{n+2}][G_{tail(n+3)}]\subseteq V_n[g_{n+1}\times u_{n+1}][g_{n+2}\times u_{n+2}\times t_{n+3}]$*, the latter model has been obtained by forcing over the former with a* κ_n*-closed forcing, namely* $(\mathbb{P}_{n+2}\times\mathbb{U}_{n+2}\times\mathbb{T}_{n+3})/(g_{n+2}\times u_{n+2})*G_{tail(n+3)}$*;*

(iv) $V_n[g_{n+1}\times u_{n+1}][g_{n+2}\times u_{n+2}\times t_{n+3}]\subseteq V_n[g_{n+1}\times u_{n+1}\times g_{n+2}\times z_{n+2}]$*, the latter model has been obtained by forcing over the former with an* $\aleph_{n+1}$*-closed forcing, namely* $(\mathbb{P}_{n+1}\times\mathbb{U}_{n+1}\times\mathbb{P}_{n+2}\times\mathbb{Z}_{n+2})/(g_{n+1}\times u_{n+1}\times g_{n+2})*(u_{n+2}\times t_{n+3})$.

4.2 More Preservation Results

In the proof of the main theorem we will use repeatedly the Second Preservation Lemma discussed in Chapter 2. For that we will need the following two lemmas.

Lemma 4.2.1. *Let* $\tau<\kappa$ *be two regular cardinals in a model* V *and assume* κ *is inaccessible. Let* $W\supseteq V$ *be a generic extension such that:*

(i) κ *and* τ *are still cardinals in* W*;*

(ii) (V,W) *has the* κ*-covering property.*

Then $\text{Add}(\tau,\kappa)^V$ *has the* κ*-sunflower property in* W*.*

Proof. We mimic the usual Δ-system argument. Work in W. Let $\langle p_X;\ X\in[\lambda]^{<\kappa}\rangle$ be a sequence of conditions with $\lambda\geq\kappa$. For χ large enough we consider the set S of all the substructures $X\prec H_\chi$ internally approachable of length τ. For every $X\in S$ there is $M_X\in X$ of size τ such that the condition $p_{X\cap\lambda}\restriction X\subseteq M_X$. By the pressing down lemma there is a set M of size τ and a stationary subset $S'\subseteq S$ such that $M_X=M$ for every $X\in S'$. It follows that the set $A:=\bigcup_{X\in S'}p_{X\cap\lambda}\restriction M$ has size less than κ

(it has size $\leq \tau$) in W. By the κ-covering property there exists $A' \in V$ of size less than κ in V such that $A \subseteq A'$. Since κ is inaccessible in V we have $|[A']^{<\tau}| < \kappa$ in V. Moreover κ is still a cardinal in W, so $[A']^{<\tau}$ has size less than κ even in W. It follows that there are less than κ possible values for $p_{X\cap\lambda} \upharpoonright M$, hence there is a condition q and a stationary subset $S^* \subseteq S$ such that $p_{X\cap\lambda} \upharpoonright M = q$ for every $X \in S^*$. The sequence $\langle p_{X\cap\lambda};\ X \in S^*\rangle$ forms a sunflower with root q. Indeed, for every $X, Y \in S^*$, pick $Z \in S^*$ such that $X, Y, \mathrm{dom}(p_{X\cap\lambda}), \mathrm{dom}(p_{Y\cap\lambda}) \subseteq Z$, then $p_{X\cap\lambda} \cap p_{Z\cap\lambda} = p_{X\cap\lambda} \cap p_{Z\cap\lambda} \upharpoonright Z = p_{X\cap\lambda} \cap q = q$ and similarly $p_{Y\cap\lambda} \cap p_{Z\cap\lambda} = q$. □

Lemma 4.2.2. *Let $\kappa > \aleph_n$ be a regular cardinal in a model V and assume W is a generic extension of V such that*

(i) $\aleph_m^V = \aleph_m^W$ for every $m \leq n+1$;

(ii) if $A \subseteq V$ is a set in W of size less than $\aleph_{n+1}$, then $[A]^{<\aleph_n}$ has size less than $\aleph_{n+1}$ in W.

Then $\mathrm{Add}(\aleph_n, \kappa)^V$ *has the $\aleph_{n+1}$-sunflower property in W.*

Proof. Work in W. Let $\langle p_X;\ X \in [\lambda]^{<\kappa}\rangle$ be a sequence of conditions with $\lambda \geq \kappa$. As usual, for χ large enough we consider the set S of all the substructures $X \prec H_\chi$ internally approachable of length $\aleph_n$ and by using the Pressing Down Lemma we find a set M of size $\aleph_n$ in W such that $p_{X\cap\lambda} \upharpoonright X \subseteq M$ for every X in a stationary subset $S' \subseteq S$. The set $A := \bigcup_{X \in S'} p_{X\cap\lambda} \upharpoonright M$ has size less than $\aleph_{n+1}$ in W. By hypothesis $[A]^{<\aleph_n}$ has size less than $\aleph_{n+1}$ in W. It follows that there are less than $\aleph_{n+1}$ possible values for $p_{X\cap\lambda} \upharpoonright M$, hence there is a condition q and a stationary subset $S^* \subseteq S$ such that $p_{X\cap\lambda} \upharpoonright M = q$ for every $X \in S^*$. The sequence $\langle p_{X\cap\lambda};\ X \in S^*\rangle$ forms a sunflower with root q. □

It will be important, in what follows, that the forcing that takes us from G_ω to the model $V_n[g_{n+1} \times u_{n+1} \times g_{n+2} \times z_{n+2}]$ defined in the previous section, cannot add cofinal branches to an $(\aleph_{n+2}, \mu)$-tree.

Lemma 4.2.3. *Let $F \in V[G_\omega]$ be an $(\aleph_{n+2}, \mu)$-tree, where $\mu \geq \aleph_{n+2}$ is an ordinal. If b is a cofinal branch for F in $V_n[g_{n+1} \times u_{n+1} \times g_{n+2} \times z_{n+2}]$, then $b \in V[G_\omega]$.*

Proof. Assume towards a contradiction that $b \notin V[G_\omega]$. The forcing $\mathbb{S}_{n+1}$ is κ_{n-1}-closed in V_{n+1} and, since Tail_{n+2} is κ_{n-1}-closed, $\mathbb{S}_{n+1}$ remains κ_{n-1}-closed (that is $\aleph_{n+1}$-closed) in $V[G_\omega]$, where $\kappa_n = \aleph_{n+2} = 2^{\aleph_n}$. By the First Preservation Lemma, we have

$$b \notin V_n[g_{n+1} \times u_{n+1}][G_{tail(n+2)}].$$

Now $\mathbb{S}_{n+2}$ is κ_n-closed in V_{n+2} and, since $\mathbb{S}_{n+1}$ is κ_n-distributive and Tail_{n+3} is κ_n-closed, the poset $\mathbb{S}_{n+2}$ remains κ_n-closed (that is $\aleph_{n+2}$-closed) in the model $V_n[g_{n+1} \times u_{n+1}][G_{tail(n+2)}]$. Another application of the First Preservation Lemma gives

$$b \notin V_n[g_{n+1} \times u_{n+1}][g_{n+2} \times u_{n+2}][G_{tail(n+3)}].$$

The passage from $V_n[g_{n+1} \times u_{n+1}][g_{n+2} \times u_{n+2}][G_{tail(n+3)}]$ to $V_n[g_{n+1} \times u_{n+1}][g_{n+2} \times u_{n+2} \times t_{n+3}]$ is done by a κ_n-closed forcing (see Remark 4.1.12), hence by the First Preservation Lemma, we get $b \notin V_n[g_{n+1} \times u_{n+1}][g_{n+2} \times u_{n+2} \times t_{n+3}] = V_n[g_{n+1} \times u_{n+1} \times g_{n+2}][u_{n+2} \times t_{n+3}]$. The forcing that takes us from $V_n[g_{n+1} \times u_{n+1} \times g_{n+2}][u_{n+2} \times t_{n+3}]$ to $V_n[g_{n+1} \times u_{n+1} \times g_{n+2} \times z_{n+2}]$ is $\aleph_{n+1}$-closed (see Remark 4.1.12), hence by the First Preservation Lemma, we have

$$b \notin V_n[g_{n+1} \times u_{n+1} \times g_{n+2} \times z_{n+2}],$$

that leads to a contradiction. □

4.3 The Final Theorem

Theorem 4.3.1. *In $V[G_\omega]$ every cardinal $\aleph_{n+2}$ has the super tree property.*

Proof. Let $F \in V[G_\omega]$ be an $(\aleph_{n+2}, \mu)$-tree, where $\mu \geq \aleph_{n+2}$ is an ordinal, and let D be an F-level sequence. In $V[G_\omega]$, we have $\kappa_n = \aleph_{n+2}$, so F is a (κ_n, μ)-tree. We start working in V. Let $\lambda := \sup_{n<\omega} \kappa_n$ and fix ν grater than both $\mu^{<\kappa_n}$ and λ^ω. There is an elementary embedding $j : V \to M$ with critical point κ_n such that:

(i) $j(\kappa_n) > \nu$ and ${}^{<\nu}M \subseteq M$;

(ii) $j(L_n)(\kappa_n)$ is an $\mathbb{R}_{n+1}$-name for the product

$$\dot{\mathbb{U}}_{n+1} \times \dot{\mathbb{P}}_{n+2} \times \dot{\mathbb{Z}}_{n+2}$$

($\dot{\mathbb{U}}_{n+1}$, $\dot{\mathbb{P}}_{n+2}$ and $\dot{\mathbb{Z}}_{n+2}$ were defined in Notation 4.1.6 and Definition 4.1.7).

Note that $j(L_n)(\kappa_n)$ is a name for a κ_n-directed closed forcing in V_n.

The proof of the theorem consists of three parts:

(1) we show that we can lift j to get an elementary embedding

$$j^* : V[G_\omega] \to M[H_\omega],$$

where $H_\omega \subseteq j(\mathbb{R}_\omega)$ is generic over V;

(2) we prove that there is in $M[H_\omega]$ an ineffable branch b for D;

(3) we show that $b \in V[G_\omega]$.

Part 1

We prove Claim 1. To simplify the notation we will denote all the extensions of j by "j" also. Recall that

$$V[G_\omega] \subseteq V_n[g_{n+1}\times u_{n+1}][G_{tail(n+2)}] \subseteq V_n[g_{n+1}\times u_{n+1}][g_{n+2}\times u_{n+2}][G_{tail(n+3)}]$$

$$\subseteq V_n[g_{n+1} \times u_{n+1}][g_{n+2} \times u_{n+2} \times t_{n+3}] \subseteq V_n[g_{n+1} \times u_{n+1} \times g_{n+2} \times z_{n+2}]$$

(see Remark 4.1.12). The forcing $\mathbb{R}_n$ has size less than κ_n, so we can lift j to get an elementary embedding

$$j : V_{n-1} \to M_{n-1}.$$

For every $i < \omega$, we denote by M_i the model $M[G_0]...[G_i]$. We will use repeatedly and without comments the resemblance between V and M. In M_{n-1}, we have

$$j(\mathbb{Q}_n) \restriction \kappa_n = \mathbb{Q}_n,$$

and at stage κ_n, the forcing at the third coordinate will be $j(L_n)(\kappa_n)$ (see Lemma 3.1.3). By our choice of $j(L_n)(\kappa_n)$, this means that we can look at the model $M_n[u_{n+1} \times g_{n+2} \times z_{n+2}]$ as a generic extension of M_{n-1} by $j(\mathbb{Q}_n) \restriction \kappa_n+1$. Force with $j(\mathbb{Q}_n)$ over W to get a generic filter H_n such that $H_n \restriction \kappa_n+1 = G_n * (u_{n+2} \times g_{n+2} \times z_{n+2})$. The forcing $\mathbb{Q}_n$ is κ_n-c.c. in M_{n-1}, so $j \restriction \mathbb{Q}_n$ is a complete embedding from $\mathbb{Q}_n$ into $j(\mathbb{Q}_n)$. Consequently, we can lift j to get an elementary embedding

$$j : V_n \to M_{n-1}[H_n].$$

We know that $\mathbb{P}_{n+1}$ is κ_n-c.c. in V_n, hence $j \restriction \mathbb{P}_{n+1}$ is a complete embedding from $\mathbb{P}_{n+1}$ into $j(\mathbb{P}_{n+1}) = \mathrm{Add}(\aleph_{n+1}, j(\kappa_{n+1}))^{M_{n-1}}$. $\mathbb{P}_{n+1}$ is even isomorphic via j to $\mathrm{Add}(\aleph_{n+1}, j[\kappa_{n+1}])^{M_{n-1}}$. Force with $\mathrm{Add}(\aleph_{n+1}, j(\kappa_{n+1})- j[\kappa_{n+1}])^{M_{n-1}}$ over $V_n[H_n][g_{n+1}]$ to get a generic filter $h_{n+1} \subseteq j(\mathbb{P}_{n+1})$ such that $j[g_{n+1}] \subseteq h_{n+1}$. We can lift j to get an elementary embedding

$$j : V_n[g_{n+1}] \to M_{n-1}[H_n][h_{n+1}].$$

By the previous observations on $j(\mathbb{Q}_n) \restriction \kappa_n + 1$ and by the closure of M, we have $j[u_{n+1} \times g_{n+2} \times z_{n+2}] \in M_{n-1}[H_n]$. The filter H_n collapses every cardinal below $j(\kappa_n)$ to have size $\aleph_{n+1}$ in $M_{n-1}[H_n]$, therefore the set $j[u_{n+1} \times g_{n+2} \times z_{n+2}]$ has size $\aleph_1$ in that model. Moreover, $j(\mathbb{U}_{n+1}) \times j(\mathbb{P}_{n+2}) \times j(\mathbb{Z}_{n+2})$ is a $j(\kappa_n)$-directed closed forcing and $j(\kappa_n) = \aleph_{n+2}^{M_{n-1}[H_n]}$. So, we can find a condition t^* stronger than every condition $j(q) \in j[u_{n+1} \times g_{n+2} \times z_{n+2}]$. By forcing over $V_{n-1}[H_n][h_{n+1}]$ with $j(\mathbb{U}_{n+1}) \times j(\mathbb{P}_{n+2}) \times j(\mathbb{Z}_{n+2})$ below t^* we get a generic filter $x_{n+1} \times h_{n+2} \times l_{n+2}$ such that $j[u_{n+1}] \subseteq x_{n+1}$, $j[g_{n+2}] \subseteq h_{n+2}$ and $j[z_{n+2}] \subseteq l_{n+2}$. The filters h_{n+1} and $x_{n+1} \times h_{n+2} \times l_{n+2}$ are mutually generic over $M_{n-1}[H_n]$, and $h_{n+1} \times x_{n+1}$ generates a filter

H_{n+1} generic for $j(\mathbb{Q}_{n+1})$ over $M_{n-1}[H_n]$. By the properties of projections, we have $j[G_{n+1}] \subseteq H_{n+1}$. Therefore the embedding j lifts to an elementary embedding

$$j : V_{n+1} \to M_{n-1}[H_n][H_{n+1}].$$

By definition of $\mathbb{Z}_{n+2}$, the filter $h_{n+2} \times l_{n+2}$ which is generic for $j(\mathbb{P}_{n+2}) \times j(\mathbb{Z}_{n+2})$ determines a generic filter $(h_{n+2} \times x_{n+2}) * H_{tail(n+3)}$ for $(j(\mathbb{P}_{n+2}) \times j(\mathbb{U}_{n+2})) * j(\mathrm{Tail}_{n+3})$. On the other hand $h_{n+2} \times x_{n+2}$ determines a filter H_{n+2} generic for $j(\mathbb{Q}_{n+2})$ over $M_{n-1}[H_n][H_{n+1}]$. By the properties of projections, we have $j[G_{n+2}] \subseteq H_{n+2}$. Therefore, j lifts to an elementary embedding

$$j : V_{n+2} \to M_{n-1}[H_n][H_{n+1}][H_{n+2}].$$

It remains to prove that $j[G_{tail(n+3)}] \subseteq H_{tail(n+3)}$, but this is an immediate consequence of $j[z_{n+2}] \subseteq l_{n+2}$. Finally j lifts to an elementary embedding

$$j : V[G_\omega] \to M_{n-1}[H_n][H_{n+1}][H_{n+2}][H_{tail(n+3)}].$$

This completes the proof of Claim 1.

Part 2

Let $\mathscr{M}_1 := M[G_\omega]$ and $\mathscr{M}_2 := M_{n-1}[H_n][H_{n+1}][H_{n+2}][H_{tail(n+3)}]$. In $\mathscr{M}_2$, $j(F)$ is a $(j(\kappa_n), j(\mu))$-tree and $j(D)$ is a $j(F)$-level sequence. By the closure of M, the tree F and the F-level sequence D are in $\mathscr{M}_1$. We want to find in $\mathscr{M}_2$ an ineffable branch for D. Let $a := j[\mu]$, clearly $a \in [j(\mu)]^{<j(\kappa_n)}$. Consider $f := j(D)(a)$ and define $b : \mu \to 2$ be the function defined by $b(\alpha) := f(j(\alpha))$. We show that b is an ineffable branch for D. Let $S := \{X \in [\mu]^{<|\kappa_n|};\ b \restriction X = D(X)\}$, then $j[\mu] = a \in j(S)$, hence S is stationary by Lemma 1.1.7.

Part 3

We proved that an ineffable branch b for D exists in $\mathscr{M}_2$. Now we show that $b \in \mathscr{M}_1$, thereby proving that $\mathscr{M}_1$ (hence $V[G_\omega]$) has an ineffable [1] branche for D. We will use repeatedly and without comments the resemblance between V and M. Assume towards a contradiction that $b \notin \mathscr{M}_1$. Step by step, we are going to prove that $b \notin \mathscr{M}_2$. By Lemma 4.2.3, we have $b \notin M_n[g_{n+1} \times u_{n+1} \times g_{n+2} \times z_{n+2}]$. Consider $\mathrm{Add}(\aleph_{n+1}, j(\kappa_{n+1}) - j[\kappa_{n+1}])^{M_{n-1}}$, by forcing with this poset over $M_n[g_{n+1} \times u_{n+1} \times g_{n+2} \times z_{n+2}]$ we obtained the generic extension $M_n[h_{n+1} \times u_{n+1} \times g_{n+2} \times z_{n+2}]$; we want to prove that

[1] If $b \in \mathscr{M}_1$, then b is ineffable since $\{X \in [\mu]^{<|\kappa_n|} \cap \mathscr{M}_1;\ b \restriction X = D(X)\}$ is stationary in $\mathscr{M}_2$, hence it is stationary in $\mathscr{M}_1$.

b does not belong to that model. To that purpose observe that κ_n was inaccessible in M_{n-1} and it is still a cardinal in $M_n[g_{n+1} \times u_{n+1} \times g_{n+2} \times z_{n+2}]$, so the latter model satisfies $\forall \gamma < \aleph_n \ (\gamma^{<\aleph_n} < \kappa_n)$. By Lemma 2.2.4 the poset $\mathrm{Add}(\aleph_{n+1}, j(\kappa_{n+1}) - j[\kappa_{n+1}])^{M_{n-1}}$ has the $\aleph_{n+2}$-sunflower property (recall that $\kappa = \aleph_{n+2}$). So we can apply the Second Preservation Lemma and we have

$$b \notin M_n[h_{n+1} \times u_{n+1} \times g_{n+2} \times z_{n+2}].$$

As we said in Part 1, we have $j(\mathbb{Q}_n) \restriction \kappa_n = \mathbb{Q}_n$, and at stage κ_n, the forcing at the third coordinate is $\mathbb{U}_{n+1} \times \mathbb{P}_{n+2} \times \mathbb{Z}_{n+2}$. It follows that for $H^* = H_n \restriction \kappa_n + 1$ we have just proved

$$b \notin M_{n-1}[H^*][h_{n+1}] = M_{n-1}[h_{n+1}][H^*].$$

Now we want to show that $\mathbb{R}^* := j(\mathbb{Q}_n)/H^*$ cannot add cofinal branches to F, hence b does not belong to the model $M_{n-1}[h_{n+1}][H_n]$. This part is quite technical. First observe that F is not exactly a (κ_n, μ)-tree in $M_{n-1}[h_{n+1}][H^*]$ because the filter h_{n+1} may add sets in $[\mu]^{<\kappa_n}$. However, the poset $j(\mathbb{P}_{n+1})$ is κ_n-c.c. in $M_{n-1}[H^*]$, so we can say that F covers a (κ_n, μ)-tree F^* in $M_{n-1}[h_{n+1}][H^*]$. If $b \in M_{n-1}[h_{n+1}][H_n]$, then b is a cofinal branch for F^*. By Lemma 3.2.16, the forcing $\mathbb{R}^*$ is a projection of $\mathbb{P}^* \times \mathbb{U}^*$, where $\mathbb{P}^* = \mathrm{Add}(\aleph_n, j(\kappa_n) - \kappa_n)^{M_{n-2}}$ and $\mathbb{U}^*$ is $\aleph_{n+1}$-closed in $M_{n-1}[h_{n+1}][H^*]$. Let $g^* \times u^* \subseteq \mathbb{P}^* \times \mathbb{U}^*$ be generic over $M_{n-1}[h_{n+1}][H^*]$ such that $M_{n-1}[h_{n+1}][H_n] \subseteq M_{n-1}[h_{n+1}][H^*][g^* \times u^*]$. In $M_{n-1}[h_{n+1}][H^*]$ we have $\kappa_n = \aleph_{n+1} = 2^{\aleph_n}$ and F is a (κ_n, μ)-tree. By the First Preservation Lemma $b \notin M_{n-1}[h_{n+1}][H^*][u^*]$. The filter u^* collapses κ to $\aleph_{n+1}$. We can assume that F has become a $(\aleph_{n+1}, \mu)$-tree in $M_{n-1}[h_{n+1}][H^*][u^*]$, then we prove that $\mathbb{P}^*$ cannot add cofinal branches to that model by using the Second Preservation Lemma[2] and by showing that $\mathbb{P}^*$ has the $\aleph_{n+1}$-sunflower property in the model $M_{n-1}[h_{n+1}][H^*][u^*]$. We check that M_{n-2} and $M_{n-1}[h_{n+1}][H^*][u^*]$ satisfy the hypothesis of Lemma 4.2.2. If $A \subseteq M_{n-2}$ is a set of size less than $\aleph_{n+1}$ in $M_{n-1}[h_{n+1}][H^*][u^*]$, then A is covered by a set $A' \in M_{n-2}$ of size less than κ_{n-1} in M_{n-2} and κ_{n-1} is inaccessible in that model. It follows that $|[A']^{<\aleph_n}| < \kappa_{n-1}$ in M_{n-2}. Since κ_{n-1} remains a cardinal in $M_{n-1}[H^*]$, the set $[A']^{<\aleph_n}$ has size less than κ_{n-1} even in this model. $M_{n-1}[H^*] \models \kappa_{n-1} = \aleph_{n+1}$ and the filters h_{n+1} and u^* preserve $\aleph_{n+1}$ so $[A']^{<\aleph_n}$ (hence $[A]^{<\aleph_n}$) has size less than $\aleph_{n+1}$ in $M_{n-1}[H^*][h_{n+1}][u^*]$ as well. Therefore $\mathbb{P}^*$ has the $\aleph_{n+1}$-sunflower property in $M_{n-1}[H^*][h_{n+1}][u^*]$; by the Second Preservation Lemma $b \notin M_{n-1}[H^*][h_{n+1}][u^*][g^*]$ and in particular

$$b \notin M_{n-1}[h_{n+1}][H_n] = M_{n-1}[H_n][h_{n+1}].$$

[2] Strictly speaking F is not exactly an $(\aleph_{n+1}, \mu)$-tree in $M_{n-1}[h_{n+1}][H^*][u^*]$. However, after proving that $\mathbb{P}^*$ has the $\aleph_{n+1}$-sunflower property, we can argue as at the end of the proof of Theorem 2.3.1 and check that $\mathbb{P}^*$ cannot add the branch b.

F^* is no longer an $(\aleph_{n+1}, \mu)$-tree in $M_{n-1}[H_n][h_{n+1}]$. However, we obtained this model by forcing with $j(\mathbb{Q}_n)/H^*$ which is $\aleph_{n+1}$-c.c. in the model $M_{n-1}[h_{n+1}][H^*]$, this means that F^* covers an $(\aleph_{n+1}, \mu)$-tree that we can rename F. Consider $j(\mathbb{Q}_{n+1})/h_{n+1}$, by Lemma 3.1.6, this is an $\aleph_{n+1}$-closed forcing in the model $M_{n-1}[H_n][h_{n+1}]$, where $2^{\aleph_n} \geq j(\kappa_n) = \aleph_{n+2}$. By the First Preservation Lemma, we have

$$b \notin M_{n-1}[H_n][H_{n+1}].$$

We continue our analysis by working with $j(\mathbb{Q}_{n+2})$ which is a projection of $j(\mathbb{P}_{n+2}) \times j(\mathbb{U}_{n+2})$. This poset is $\aleph_{n+2}$-closed in $M_{n-1}[H_n][H_{n+1}]$ and F is an $(\aleph_{n+1}, \mu)$-tree. By the First Preservation Lemma, we have $b \notin M_{n-1}[H_n][H_{n+1}][h_{n+1} \times u_{n+1}]$, in particular

$$b \notin M_{n-1}[H_n][H_{n+1}][H_{n+2}].$$

Finally $j(\mathrm{Tail}_{n+3})$ is $\aleph_{n+2}$-closed in $M_{n-1}[H_n][H_{n+1}]$, where F is still an $(\aleph_{n+1}, \mu)$-tree. By applying again the First Preservation Lemma, we get that

$$b \notin M_{n-1}[H_n][H_{n+1}][H_{n+2}][H_{tail(n+3)}] = \mathscr{M}_2,$$

that leads to a contradiction and completes the proof of the theorem.

□

5

The Strong Tree Property at Successors of Singular Cardinals

In this chapter we prove that starting from infinitely many supercompact cardinals, one can obtain a model where $\aleph_{\omega+1}$ has the strong tree property. This is the main result of [6]. We also prove that if ν is a singular limit of strongly compact cardinals, then the strong tree property holds at ν^+.

5.1 The Tree Property at $\aleph_{\omega+1}$

In [14] Magidor and Shelah proved the following result.

Theorem 5.1.1. *(Magidor and Shelah) Assume there is a model of ZFC with an increasing sequence $\langle\lambda_n\rangle_{n<\omega}$ such that*

(i) if $\lambda = \sup_{n\geq 0}\lambda_n$, then λ_n is λ^+-supercompact, for all $n > 0$;

(ii) λ_0 is the critical point of an embedding $j : V \to M$ where $j(\lambda_0) = \lambda_1$ and ${}^{\lambda^+}M \subseteq M$.

Then there is a model of ZFC where $\aleph_{\omega+1}$ has the tree property.

The hypotheses on λ_0 imply that λ_0 is between a huge and a 2-huge. Such result has been recently improved by Sinapova [19] who proved the consistency of the tree property at $\aleph_{\omega+1}$ under weaker assumptions.

Theorem 5.1.2. *(Sinapova) Assume there is a model of ZFC with infinitely many supercompact cardinals, then there is a model of ZFC where $\aleph_{\omega+1}$ has the tree property.*

By combining Cummings-Foreman's result and tools from Sinapova's paper, Neeman [17] constructed a model where the tree property holds "up" to $\aleph_{\omega+1}$ – i.e. at every regular cardinal $\leq \aleph_{\omega+1}$ – starting from infinitely many supercompact cardinals.

Theorem 5.1.3. *(Neeman) Assume there is a model of ZFC with infinitely many supercompact cardinals, then there is a model of ZFC where the tree property holds at every $\aleph_n$ with $n \geq 2$ and at $\aleph_{\omega+1}$.*

We are going to prove that $\aleph_{\omega+1}$ can consistently satisfy even the strong tree property. Our proof of such result is very closed to Neeman's proof of Theorem 5.1.3.

5.2 A Partition Property for Strongly Compact Cardinals

Let μ be a regular cardinal and let $\lambda \geq \mu$ be any ordinal. For every cofinal set $I \subseteq [\lambda]^{<\mu}$ we denote by $[[\ I\ ]]^2$ the set of all pairs $(X, Y) \in I \times I$ such that $X \subseteq Y$.

Definition 5.2.1. *Let $\mu > \kappa$ be two regular cardinals and let $S \subseteq [\lambda]^{<\mu}$ be a cofinal set and $c : [[\ S\ ]]^2 \to \gamma$ a function such that $\gamma < \kappa$. We say that a cofinal set $H \subseteq S$ is a* quasi homogenous *set of color $i < \gamma$ iff for every $X, Y \in H$ there is $W \supseteq X, Y$ in H such that $c(X, W) = i = c(Y, W)$.*

We focus on the following partition property.

Definition 5.2.2. *Given a regular cardinal κ and $\nu \geq \kappa$, we define the principle $\varphi(\kappa, \nu^+)$ establishing that for every $\lambda \geq \nu^+$ if $S \subseteq [\lambda]^{<\nu^+}$ is a stationary set, then every function $c : [[\ S\ ]]^2 \to \gamma$ with $\gamma < \kappa$ has a quasi homogenous set H which is also stationary.*

We now prove that strongly compact cardinals satisfy that property for every $\nu \geq \kappa$.

Theorem 5.2.3. *Let κ be a strongly compact cardinal, then $\varphi(\kappa, \nu^+)$ holds for every $\nu \geq \kappa$.*

Proof. Fix $\lambda \geq \nu^+$ and a function $c : [[\ S\ ]]^2 \to \gamma$ where $\gamma < \kappa$ and $S \subseteq [\lambda]^{<\nu^+}$ is a stationary set. Consider all the sets of the form $C \cap S$ where $C \subseteq [\lambda]^{<\nu^+}$ is a club; they form a κ-complete family. Since κ is strongly compact, there exists a κ-complete ultrafilter U that contains all these sets. Note that every set of U is stationary. First we show that for every $X \in S$, there is $i_X < \gamma$ and a set $H_X \subseteq S$ in U such that for every Y in H_X we have $c(X, Y) = i_X$. Assume by contradiction that for every $i < \gamma$, the set $K_i := \{Y \in S;\ Y \supseteq X \text{ and } c(X, Y) \neq i\} \in U$ then, from the κ-completeness of U, we have $\bigcap_{i<\gamma} C_i$ is in U and it is empty, a contradiction. With a similar argument we can use the κ-completeness of U to get that the function $X \mapsto i_X$ is constant on a set $H \in U$; let i be such that $i = i_X$, for every $X \in H$. Now, it is easy to see that H is quasi-homogenous of color i. Indeed, if $X, Y \in H$, then $H_X \cap H_Y \cap H$ belongs to U and it is, therefore, non empty. Let $Z \in H_X \cap H_Y \cap H$, then we have $c(X, Z) = i = c(Y, Z)$ as required. □

The following result show the connection between this new partition property and the strong tree property.

Proposition 5.2.4. *Given a regular cardinal κ, if $\varphi(\kappa^+, \kappa^+)$ holds, then κ^+ has the strong tree property.*

Proof. Let F be a (κ^+, λ)-tree where $\lambda \geq \kappa^+$. Assume that for every $X \in [\lambda]^{<\kappa^+}$, we have $\mathrm{Lev}_X(F) := \{f_i^X;\ i < \gamma_X\}$. We define a function $c : [[\ [\lambda]^{<\kappa^+}\ ]]^2 \to \kappa \times \kappa$ by letting $c(X, Y) = (i, j)$ if and only if $f_j^Y \restriction X = f_i^X$. c can be seen as a function from $[[\ [\lambda]^{<\kappa^+}\ ]]^2$ into κ, so there exists a quasi-homogenous and stationary set $H \subseteq [\lambda]^{<\kappa^+}$. Assume H has color $(i, j) \in \kappa \times \kappa$, we define $b := \bigcup_{X \in H} f_i^X$ and we prove that b is a cofinal branch. As usual, it is enough to prove that b is a function. Let $X, Y \in H$, there is $Z \in H$ such that $X, Y \subseteq Z$ and $c(X, Z) = (i, j) = c(Y, Z)$. By definition of c, we have

(i) $f_j^Z \restriction X = f_i^X$;

(ii) $f_j^Z \restriction Y = f_i^Y$.

It follows that f_i^X and f_i^Y are comparable and b is a function. □

5.3 The Strong Tree Property at Successors of Singular Cardinals

In the proof of Theorem 5.1.1 an important theorem plays a crucial role. The theorem establishes the following.

Theorem 5.3.1. *(Magidor and Shelah) Assume ν is a singular limit of strongly compact cardinals, then ν^+ has the tree property.*

We prove that every successor of a singular limit of strongly compact cardinals satisfies even the strong tree property.

Theorem 5.3.2. *Let ν be a singular cardinal such that $\nu = \lim_{i<cof(\nu)} \kappa_i$ and $\varphi(\kappa_i, \nu^+)$ holds for every i. Then ν^+ has the strong tree property.*

Proof. To simplify the proof we will assume that ν has countable cofinality, so $\nu = \lim_{n<\omega} \kappa_n$. Suppose without loss of generality that $\langle \kappa_n \rangle_{n<\omega}$ is increasing. Let $\mu \geq \nu^+$ and let F be a (ν^+, μ)-tree. For every $X \in [\mu]^{<\nu^+}$, we assume that $\mathrm{Lev}_X(F) = \{f_i^X;\ i < |\mathrm{Lev}_X(F)|\}$.

Lemma 5.3.3. *(Spine Lemma) There exists $n < \omega$ and a stationary set $S \subseteq [\mu]^{<\nu^+}$, such that for all $X, Y \in S$, there are $\zeta, \eta < \kappa_n$ with $f_\zeta^X \restriction (X \cap Y) = f_\eta^Y \restriction (X \cap Y)$.*

Proof. Given a function $f \in Lev_X$, we write $\#f = i$ for $i < \nu$, when $f = f_i^X$. Define $c : [[\ [\mu]^{<\nu^+}]]^2 \to \omega$ by $c(X,Y) = \min\{i;\ \#(f_0^Y \upharpoonright X) < \kappa_i\}$. By hypothesis $\varphi(\kappa_0, \nu^+)$ holds, hence there is $n < \omega$ and a stationary quasi homogenous set $S \subseteq [\mu]^{<\nu^+}$ of color n. Then, for every $X, Y \in S$, there is $Z \supseteq X, Y$ in S such that $c(X,Z) = n = c(Y,Z)$. This means that $\#(f_0^Z \upharpoonright X)$, $\#(f_0^Z \upharpoonright Y) < \kappa_n$. So, let $\zeta, \eta < \kappa_n$ be such that $f_0^Z \upharpoonright X = f_\zeta^X$ and $f_0^Z \upharpoonright Y = f_\eta^Y$, then $f_\zeta^X \upharpoonright (X \cap Y) = f_0^Z \upharpoonright (X \cap Y) = f_\eta^Y \upharpoonright (X \cap Y)$, as required. □

Let n and S be like in the Spine Lemma, we prove the following fact.

Lemma 5.3.4. *There is a cofinal $S' \subseteq S$ and an ordinal $\zeta < \kappa_n$ such that for all $X, Y \in S'$, we have $f_\zeta^X \upharpoonright (X \cap Y) = f_\zeta^Y \upharpoonright (X \cap Y)$ (the set S' is even stationary).*

Proof. For every $(X,Y) \in [[\ S\ ]]^2$, we define $\bar{c}(X,Y)$ as the minimum couple $(\zeta, \eta) \in \kappa_n \times \kappa_n$, in the lexicografical order, such that $f_\eta^Y \upharpoonright X = f_\zeta^X$; the function is well defined by definition of n and S. We can apply $\varphi(\kappa_{n+1}, \nu^+)$ to the function $\bar{c} : [[\ S\ ]]^2 \to \kappa_n \times \kappa_n$. So there exists a quasi homogenous set S' of color $(\zeta, \eta) \in \kappa_n \times \kappa_n$. It follows that for every $X, Y \in S'$, there is $Z \supseteq X, Y$ in S' such that $\bar{c}(X,Z) = (\zeta, \eta) = \bar{c}(Y,Z)$. This means that $f_\eta^Z \upharpoonright X = f_\zeta^X$ and $f_\eta^Z \upharpoonright Y = f_\zeta^Y$, hence $f_\zeta^X \upharpoonright (X \cap Y) = f_\eta^Z \upharpoonright (X \cap Y) = f_\zeta^Y \upharpoonright (X \cap Y)$. □

Set $b = \bigcup_{X \in S'} f_\zeta^X$, by the previous lemma b is a function; b is a cofinal branch for F. □

Corollary 5.3.5. *Let ν be a singular limit of strongly compact cardinals, then ν^+ has the strong tree property.*

Proof. Apply Theorem 5.3.2 and Theorem 5.2.3. □

Whether such result can be generalized to the super tree property is still an open problem — we do not know whether under some hypotheses the successor of a singular cardinal can satisfy the super tree property.

5.4 Systems

Definition 5.4.1. *Given an ordinal $\lambda \geq \nu^+$, a cofinal set $D \subseteq [\lambda]^{<\nu^+}$ and a family $\mathscr{S} := \{S_i\}_{i \in I}$ of transitive, reflexive binary relations over $D \times \nu$, we say that $\mathscr{S}$ is a* system *if the following hold:*

(i) if $(X, \zeta)\ S_i\ (Y, \eta)$ and $(X, \zeta) \neq (Y, \eta)$, then $X \subsetneq Y$;

(ii) for every $X \subseteq Y$, if both $(X, \zeta)\ S_i\ (Z, \theta)$ and $(Y, \eta)\ S_i\ (Z, \theta)$, then $(X, \zeta)\ S_i\ (Y, \eta)$;

(iii) for every $X, Y \in D$, there is $Z \supseteq X, Y$ and $\zeta_X, \zeta_Y, \eta \in \nu$ such that for some $i \in I$ we have $(X, \zeta_X)\ S_i\ (Z, \eta)$ and $(Y, \zeta_Y)\ S_i\ (Z, \eta)$ (in particular, if $X \subseteq Y$, then $(X, \zeta)\ S_i\ (Y, \zeta_Y)$).

In the previous definition, the elements of $D \times \nu$ are called *nodes.* Given two nodes u and v, we say that they are *S_i-incompatible,* for some $i \in I$, if there is no $w \in D \times \nu$ such that $u\ S_i\ w$ and $v\ S_i\ w$. Sometimes we will say that a node u belongs to the X'th level if the first coordinate of u is X (i.e. $u = (X, \zeta)$, for some $\zeta \in \nu$).

Definition 5.4.2. *Let $\{S_i\}_{i \in I}$ be a system on $D \times \nu$ and let $i \in I$, a partial function $b : D \to \nu$ is an S_i-*branch *if it satisfies the following conditions. For every $X \in \mathrm{dom}(b)$ and for every $Y \in D$ with $Y \subseteq X$, we have:*

(i) $Y \in \mathrm{dom}(b)$ iff there exists $\zeta < \nu$ such that $(Y, \zeta)\ S_i\ (X, b(X))$,

(ii) $b(Y)$ is the unique ζ witnessing this.

In the situation of the previous definition, we say that a branch b is *cofinal* if $X \in \mathrm{dom}(X)$ for cofinally many X's.

Definition 5.4.3. *Let $\{S_i\}_{i \in I}$ be a system on $D \times \nu$, a* system of branches *is a family $\{b_j\}_{j \in J}$ such that*

(i) every b_j is an S_i-branch for some $i \in I$;

(ii) for every $X \in D$, there is $j \in J$ such that $X \in \mathrm{dom}(b_j)$.

The following two results (Lemma 5.4.4 and Lemma 5.4.5) generalize a theorem by Sinapova (see [19] Preservation Lemma). The proofs of such results are very similar to the proof of Sinapova's Preservation Lemma, we have just to deal with *sets of ordinals* instead of ordinals.

Lemma 5.4.4. *Let ν be a singular cardinal of countable cofinality, $\lambda \geq \nu^+$, and let $\{R_i\}_{i \in I}$ be a system on $D \times \tau$ with $D \subseteq [\lambda]^{<\nu^+}$ cofinal and $\max(|I|, \tau) < \nu$. Suppose that $\mathbb{P}$ is a κ-closed forcing, for a regular $\kappa > \max(|I|, \tau)^+$, and for some $p \in \mathbb{P}$, $\dot{b} \in V^{\mathbb{P}}$ and $i \in I$, we have*

$$p \Vdash \dot{b} \text{ is a cofinal } R\text{-branch},$$

where $R = R_i$. If V has no cofinal branches for the system, then for all $\eta < \kappa$, we can find a sequence $\langle v_\zeta;\ \zeta < \eta \rangle$ of pairwise R-incompatible elements of $D \times \tau$ such that for every $\zeta < \eta$, there exists $q \leq p$ forcing $v_\zeta \in \dot{b}$.

Proof. Let $E := \{u \in D \times \tau;\ \exists q \leq p (q \Vdash u \in \dot{b})\}$. First remark that, since p forces that $\dot{b}$ is cofinal, the set $\{X \in D;\ \exists \zeta \in \tau\ (X, \zeta) \in E\}$ is cofinal. Since V has no cofinal branches for the system, we can find, for all $v \in E$ two R-incompatible nodes $w_1, w_2 \in E$ such that $v\ R\ w_1$, $v\ R\ w_2$.

We inductively define for all $\zeta < \eta$ two nodes $u_\zeta, v_\zeta \in E$ and a condition $p_\zeta \leq p$ such that:

(i) $p_\zeta \Vdash u_\zeta \in \dot{b}$;

(ii) u_ζ and v_ζ are pairwise R-incompatible;

(iii) for all $\varepsilon < \zeta$, $u_\varepsilon \ R \ u_\zeta$ and $u_\varepsilon \ R \ v_\zeta$;

(iv) the sequence $\langle p_\varepsilon;\ \varepsilon \leq \zeta \rangle$ is decreasing;

Let u be any node in E. From the remark above, there are $u_0, v_0 \in E$ which are R-incompatible and both $u \ R \ u_0$ and $u \ R \ v_0$. By definition of E, there is a condition $p_0 \leq p$ such that $p_0 \Vdash u_0 \in \dot{b}$.

Let $\zeta > 0$ and assume that $u_\varepsilon, v_\varepsilon, p_\varepsilon$ are defined for every $\varepsilon < \zeta$. Let q be stronger than every condition in $\{p_\varepsilon;\ \varepsilon < \zeta\}$. By the inductive hypothesis (claim 3.), the nodes $\langle u_\varepsilon;\ \varepsilon < \zeta \rangle$ form an R-chain, so we can find a node h and a condition $q^* \leq q$ such that $u_\varepsilon \ R \ h$, for all $\varepsilon < \zeta$, and $q^* \Vdash h \in \dot{b}$. Since p force that $\dot{b}$ is cofinal and there is no cofinal branch in V for the system, we can find two R-incompatible nodes $u_\zeta, v_\zeta \in E$ and a condition $p_\zeta \leq q^*$ such that $h \ R \ u_\zeta$, $h \ R \ v_\zeta$ and $p_\zeta \Vdash u_\zeta \in \dot{b}$. That completes the construction.

The sequence $\langle v_\zeta;\ \zeta < \eta \rangle$ is as required: for if $\zeta' < \zeta < \eta$, then by definition $u_{\zeta'}$ and $v_{\zeta'}$ are R-incompatible, and $u_{\zeta'} \ R \ v_\zeta$, hence $v_{\zeta'}$ and v_ζ are R-incompatible. □

Lemma 5.4.5. *(Third Preservation Lemma) Let ν be a singular cardinal of countable cofinality, $\lambda \geq \nu^+$, and let $\{R_i\}_{i \in I}$ be a system on $D \times \tau$ with $D \subseteq [\lambda]^{<\nu^+}$ cofinal and $\max(|I|, \tau) < \nu$. Suppose that $\mathbb{P}$ is a κ-closed forcing, for a regular $\kappa > \max(|I|, \tau)^+$, and assume that $\mathbb{P}$ forces a system of branches $\{\dot{b}_j\}_{j \in J}$ through $\{R_i\}_{i \in I}$ with $|J|^+ < \kappa$ and such that for some $j \in J$, the branch b_j is cofinal. Then, there exists in V a cofinal R_i-branch, for some $i \in I$.*

Proof. Suppose for contradiction that V has no cofinal branches for the system. We start working in $V[\mathbb{R}]$. Let $B := \{j \in J;\ b_j \text{ is not cofinal}\}$. By the closure of $\mathbb{P}$ (we have $|J|^+ < \kappa$), we can find a condition p deciding, for every $j \in J$ whether or not b_j is cofinal, hence $B \in V$. For every $j \in B$, fix $X_j \in [\lambda]^{<\nu^+}$ such that $\mathrm{dom}(\dot{b}_j)$ has empty intersection with every $Y \supseteq X_j$. Since B has size less than ν, the set $X^* \bigcup_{j \in B} X_j$ is in $[\lambda]^{<\nu^+}$. Let $C^* := \{Z \in D;\ X^* \subseteq Z\}$. Define $A := \{j \in J;\ p \Vdash \dot{b}_j \text{ is cofinal}\}$, then $A \in V$ and by hypothesis of the theorem A is non empty. Moreover, by strengthening p if necessary, we have $p \Vdash \forall X \in C^* \exists a \in A (X \in \mathrm{dom}(\dot{b}_j))$ (use condition (2) and the definition of C^*). Let η be a regular cardinal with $\max(|D|, \tau) < \eta < \kappa$.

Claim 4. *Let $\lhd$ be a well ordering of A. For every $a \in A$, we can define $\langle q^a_\gamma;\ \gamma < \eta \rangle$ and $\langle u^a_\gamma;\ \gamma < \eta \rangle$ such that*

(i) *for all* $\gamma < \eta$, $q^a_\gamma \leq p$ *and* $q^a_\gamma \Vdash u^a_\gamma \in \dot{b}_a$,

(ii) *the nodes* $\langle u^a_\gamma;\ \gamma < \eta \rangle$ *are pairwise* R*-incompatible,*

in such a way that for all $\gamma < \eta$, $\langle q^a_\gamma;\ a \in A \rangle$ *is* $\lhd$*-decreasing.*

Proof. We proceed by induction on $a \in A$. Assume that the sequences have been defined up to $a \in A$. For every $\gamma < \eta$, let r_γ be stronger than every condition in the set $\{q^c_\gamma;\ c \lhd a\}$, and let $E_\gamma := \{u \in D \times \tau;\ \exists q \leq r_\gamma (q \Vdash u \in \dot{b}_a)\}$. For all $\gamma < \eta$, let $\langle v^\gamma_\zeta;\ \zeta < \eta \rangle$ be as in the conclusion of Lemma 5.4.4 applied to r_γ and $\dot{b}_a$, and let $X_\gamma \in [\chi]^{<\mu^+}$ be such that the level of each v^γ_ζ is below X_γ. Let $X^* \supsetneq \bigcup_{\gamma<\eta} X_\gamma$ in D. We want to define the sequence $\langle u^a_\gamma;\ \gamma < \eta \rangle$ with each $u^a_\gamma \in E_\gamma$ belonging to a level above X^*. We proceed by induction: suppose we have defined $\langle u^a_\gamma;\ \gamma < \delta \rangle$ for some $\delta < \eta$. For every $\gamma < \delta$, there is at most one $\zeta < \eta$ such that $v^\delta_\zeta \ R\ u^a_\gamma$ (because the v^δ_ζ's are pairwise R-incompatible). For all $\gamma < \delta$, let ζ_γ be that unique index if it exists and let ζ_γ be 0 otherwise. Choose $\zeta \in \eta - \{\zeta^\delta_\gamma;\ \gamma < \delta\}$. Then, for all $\gamma < \delta$, the nodes v^δ_ζ and u_γ are R-incompatible. Let $u^a_\delta \in E_\delta$ be such that $v^\delta_\zeta \ R\ u^a_\delta$. Then, for all $\gamma < \delta$, the nodes u^a_γ and u^a_δ are R-incompatible. Since for every $\gamma < \eta$, we have $u^a_\gamma \in E_\gamma$, we can find a condition $q^a_\gamma \leq r_\gamma$ such that $q^a_\gamma \Vdash u^a_\gamma \in \dot{b}_a$. That completes the construction. □

We return to the proof of the theorem. For every $\gamma < \eta$, let p_γ be stronger than all the conditions $\langle q^a_\gamma;\ a \in A \rangle$, and let $Y_\gamma \in D$ be such that the nodes in $\{u^a_\gamma;\ a \in A\}$ belong to levels below Y_γ. Fix $Y^* \in C^*$ such that $Y^* \supseteq \bigcup_\gamma Y_\gamma$. For all $\gamma < \eta$, let $p^*_\gamma \leq p_\gamma$, let w_γ of level Y^* and $a_\gamma \in A$ such that $p^*_\gamma \Vdash w_\gamma \in \dot{b}_{a_\gamma}$. Since A has size less than η, there is w^* on level Y^* and $a^* \in A$ such that $w_\gamma = w^*$, $a_\gamma = a^*$, for almost all $\gamma < \eta$. Let $b^* := \dot{b}_{a^*}$. Given two distinct $\gamma, \delta < \eta$ large enough, if $u := u^{a^*}_\gamma$ and $v := u^{a^*}_\delta$, then the following hold:

(i) $p^*_\gamma \Vdash u \in b^*$, $p^*_\gamma \Vdash w^* \in b^*$;

(ii) $p^*_\delta \Vdash v \in b^*$, $p^*_\delta \Vdash w^* \in b^*$;

(iii) both the level of u and the level of v are subsets of Y^*;

(iv) u and v are R_{a^*}-incompatible,

that leads to a contradiction. □

5.5 The Strong Tree Property at $\aleph_{\omega+1}$

Now we are ready to prove the main theorem of this chapter.

Theorem 5.5.1. *Let $\langle\kappa_n\rangle_n < \omega$ be an increasing sequence of indestructibly supercompact cardinals. There is a strong limit cardinal $\mu < \kappa_0$ of cofinality ω such that by forcing over V with the poset*

$$\mathrm{Coll}(\omega, \mu) \times \mathrm{Coll}(\mu^+, <\kappa_0) \times \Pi_{n<\omega}\mathrm{Coll}(\kappa_n, <\kappa_{n+1}),$$

one gets a model where the strong tree property holds at $\aleph_{\omega+1}$.

Proof. Let κ denote κ_0, for every $\mu < \kappa$ we let

(i) $\mathbb{R}(\mu) := \mathrm{Coll}(\omega, \mu) \times \mathrm{Coll}(\mu^+, <\kappa_0) \times \Pi_{n<\omega}\mathrm{Coll}(\kappa_n, <\kappa_{n+1})$,

(ii) $\mathbb{L}(\mu) := \mathrm{Coll}(\omega, \mu) \times \mathrm{Coll}(\mu^+, <\kappa_0)$,

(iii) $\mathbb{C} := \Pi_{n<\omega}\mathrm{Coll}(\kappa_n, <\kappa_{n+1})$.

Assume that $\nu = \sup_n \kappa_n$, then the forcing $\mathbb{R}(\mu)$ produces a model where $\aleph_{\omega+1} = \nu^+$. We fix $H := \Pi_{n<\omega}H_n \subseteq \mathbb{C}$ generic over V. We work in $W := V[H]$. Assume for contradiction that in every extension of W by $\mathbb{L}(\mu)$ with $\mu < \kappa$ strong limit of cofinality ω, the strong tree property fails at ν^+. For every such μ, let λ_μ and $\dot{F}(\mu) \in W^{\mathbb{L}(\mu)}$ be a name for a (ν^+, λ_μ)-tree with no cofinal branches. Let $\lambda = \sup_{\mu<\kappa} \lambda_\mu$, without loss of generality we can assume that $\lambda_\mu = \lambda$ for every μ, since a (ν^+, λ_μ)-tree with no cofinal branches can be extended to a (ν^+, λ)-tree with no cofinal branches. Given $X, Y \in [\lambda]^{<\nu^+}$ and $\zeta, \eta < \nu$, we will write $\Vdash_{\mathbb{L}(\mu)} (X, \zeta) <_{\dot{F}_\mu} (Y, \eta)$ when

$$\Vdash_{\mathbb{L}(\mu)} \text{ the } \eta\text{'th function on level } Y \text{ extends the } \zeta\text{'th function on level } X$$

(formally, for every μ and X, we fix an $\mathbb{L}(\mu)$-name $\dot{e}^\mu_X$ for an enumeration of the level of X into at most ν elements, then we write $\Vdash_{\mathbb{L}(\mu)} (X, \zeta) <_{\dot{F}_\mu} (Y, \eta)$ when $\Vdash_{\mathbb{L}(\mu)} \dot{e}^\mu_X(\zeta) = \dot{e}^\mu_Y(\eta) \restriction X$). Consider the following set

$$I := \{(a, b, \mu);\ \mu < \kappa \text{ is strong limit of cof } \omega \text{ and } (a, b) \in \mathbb{L}(\mu)\}.$$

We define a system $\mathscr{S} = \{S_i\}_{i\in I}$ on $[\lambda]^{<\nu^+} \times \nu$ as follows. Given $i = (a, b, \mu) \in I$, for every $X, Y \in [\lambda]^{<\nu^+}$ and for every $\zeta, \eta < \nu$, we let

$$(X, \zeta)\ S_i\ (Y, \eta) \iff (a, b) \Vdash (X, \zeta) <_{\dot{F}_\mu} (Y, \eta).$$

Lemma 5.5.2. *There is, in W, an integer $n < \omega$ and a cofinal set $D \subseteq [\lambda]^{<\nu^+}$ such that $\{\ S_i \restriction D \times \kappa_n\}_{i\in I}$ is a system.*

Proof. κ is indestructible supercompact, so we can fix $j : W \to W^*$ a σ-supercompact elementary embedding with critical point κ, where σ is large enough for the argument that follows. We have $a^* := j[\lambda] \in W^* \cap [j(\lambda)]^{<j(\nu^+)}$. We denote by F^* the name $j(\dot{F})(\nu)$, where $\dot{F}$ is the map $\mu \mapsto \dot{F}(\mu)$. We denote by $\ll \lambda \gg^{<\nu^+}$ the set of all the strictly increasing sequences X from an ordinal $\alpha < \nu^+$. into λ. For every $X \in \ll \lambda \gg^{<\nu^+}$, the image of X is a subset of $[\lambda]^{<\nu^+}$. Now, we define a sequence $\langle(p_X, q_X, \zeta_X, n_X);\ X \in \ll \lambda \gg^{<\nu^+}\rangle$ such that

(i) $(p_X, q_X) \in \mathrm{Coll}(\omega, \nu) \times \mathrm{Coll}(\nu^+, < j(\kappa))$, $n_X < \omega$, and $\zeta_X < j(\kappa_{n_X})$;

(ii) $(p_X, q_X) \Vdash (j[Im(X)], \zeta_X) <_{F^*} (a^*, 0)$;

(iii) for every $X \sqsubseteq Y$ in $\ll \lambda \gg^{<\nu^+}$, we have $q_Y \leq q_X$.

The sequence is inductively defined as follows. Let $X : \alpha \to \lambda$ be a strictly increasing sequence, assume by inductive hypothesis that

$$\langle (p_X, q_X, \zeta_X, n_X);\ X \in \ll \lambda \gg^{<\alpha} \rangle$$

is defined. By condition (3), the sequence $\langle q_{X \restriction \beta};\ \beta < \alpha \rangle$ is decreasing and $\mathrm{Coll}(\nu^+, < j(\kappa))$ is ν^+-closed, so there exists a lower bound $\bar{q}_X$. The set $j[Im(X)]$ is in $[\lambda]^{<\nu^+}$, so there exists $p_X \in \mathrm{Coll}(\omega, \nu)$, $q_X \leq \bar{q}_X$ in $\mathrm{Coll}(\nu^+, < j(\kappa))$ and $\zeta_X < j(\nu)$ such that

$$(p_X, q_X) \Vdash (j[Im(X)], \zeta_X) <_{F^*} (a^*, 0).$$

If we let n_X be the minimum integer such that $\zeta_X < j(\kappa_{n_X})$, then p_X, q_X, ζ_X and n_X satisfy conditions (1) and (2) for the sequence X. That completes the definition.

For every $X \in [\lambda]^{<\nu^+}$ we denote by s_X the unique increasing sequence whose image is X. The poset $\mathrm{Coll}(\omega, \nu)$ has size less than $\lambda^{<\nu^+}$, hence there is a condition p and a cofinal set $D \subseteq [\lambda]^{<\nu^+}$ such that for every $X \in D$, we have $p = p_{s_X}$. By shrinking D, we can also assume that there exists $n < \omega$ such that $n = n_{s_X}$, for every $X \in D$.

Claim 5. *$\{ S_i \restriction D \times \kappa_n \}_{i \in I}$ is a system.*

Proof. We just have to prove that it satisfies condition (3) of Definition 5.4.1. Fix $X, Y \in D$, by construction we have $(p, q_{s_X}) \Vdash (j[X], \zeta_X) <_{F^*} (a^*, 0)$ and $(p, q_{s_Y}) \Vdash (j[Y], \zeta_Y) <_{F^*} (a^*, 0)$. Take any set Z in D such that $s_Z \sqsupseteq s_X, s_Y$ (in particular $Z \supseteq X, Y$), then q_Z is stronger than both q_X and q_Y. Therefore, condition (p, q_Z) forces that:

(i) $(j[X], \zeta_X) <_{F^*} (a^*, 0)$;

(ii) $(j[Z], \zeta_Z) <_{F^*} (a^*, 0)$;

(iii) $(j[Z], \zeta_Z) <_{F^*} (a^*, 0)$;

(iv) $(j[Y], \zeta_Y) <_{F^*} (a^*, 0)$.

From (i) and (ii) follows that $(p, q_Z) \Vdash (j[X], \zeta_X) <_{F^*} (j[Z], \zeta_Z)$; from (iii) and (iv) follows that $(p, q_Z) \Vdash (j[Y], \zeta_Y) <_{F^*} (j[Z], \zeta_Z)$. Then, by elementarity, there exists $\mu < \kappa$ and $(\bar{p}, \bar{q}) \in \mathbb{L}(\mu)$ and $\bar{\zeta}_X, \bar{\zeta}_Y, \bar{\zeta}_Z < \kappa_n$ such that $(\bar{p}, \bar{q}) \Vdash (X, \bar{\zeta}_X) <_{\dot{F}_\mu} (Z, \bar{\zeta}_Z)$ and $(Y, \bar{\zeta}_Y) <_{\dot{F}_\mu} (Z, \bar{\zeta}_Z)$. If we let $i = (\bar{p}, \bar{q}, \mu)$, then we just proved $(X, \bar{\zeta}_X)\ S_i\ (Z, \bar{\zeta}_Z)$ and $(Y, \bar{\zeta}_Y)\ S_i\ (Z, \bar{\zeta}_Z)$. □

That completes the proof of the lemma. □

To simplify the notation, we define $R_i := S_i \upharpoonright D \times \kappa_n$, for every $i \in I$. Let $m = n + 2$, the following holds by the indestructibility of κ_{m+1} : forcing over $W = V[H]$ with $\mathrm{Coll}(\kappa_m, \gamma)^V$ for sufficiently large γ, adds an elementary embedding $\pi : V[H] \to M[H^*]$ with critical point κ_{m+1} and $\pi(\kappa_m) > \sup \pi[\lambda]$ (use standard extension of embedding).

Lemma 5.5.3. *There is in $V[H^*]$ a system of branches $\{b_j\}_{j \in J}$ for the system $\{R_i\}_{i \in I}$ with $J = I \times \kappa_n$, such that for some $j \in J$, the branch b_j is cofinal.*

Proof. First note that since $\kappa_n, |I| < cr(\pi)$, we may assume that $\pi(I) = I$ and $\pi(\{R_i\}_{i \in I}) = \{\pi(R_i)\}_{i \in I}$. This is a system on $\pi(D) \times \kappa_n$. Let a^* be a set in $\pi(D)$ such that $\pi[\lambda] \subseteq a^*$. For every $(i, \delta) \in I \times \kappa_n$, let $b_{i,\delta}$ be the partial map sending $X \in D$ to the unique $\zeta < \kappa_n$ such that $(\pi[X], \zeta)\ \pi(R_i)\ (a^*, \delta)$ if such ζ exists. Every $b_{i,\delta}$ is an R_i-branch. Condition (2) of Definition 5.4.3 is satisfied as well: indeed, if $X \in D$, then by condition (3) of Definition 5.4.1, there exists $\zeta, \eta < \kappa_n$ and $i \in I$ such that $(\pi[X], \zeta)\ \pi(R_i)\ (a^*, \eta)$, hence $X \in \mathrm{dom}(b_{i,\eta})$. It remains to prove that for some $j \in J$, b_j is cofinal. For every $X \in D$, we fix i_X, δ_X such that $X \in \mathrm{dom}(b_{i_X,\delta_X})$. The set I has size less than κ_m in W, moreover $\mathrm{Coll}(\kappa_m, \gamma)^V$ is κ_m-closed in $V[H_m \times H_{m+1} \times \ldots]$ and $W = V[H]$ is a κ_m-c.c. forcing extension of $V[H_m \times H_{m+1} \times \ldots]$, so I has size $< \kappa_m$ even in $V[H]^*$. It follows that there exists a cofinal $D' \subseteq D$ and i, δ in $V[H^*]$ such that $i = i_X$ and $\delta = \delta_X$, for every $X \in D$. This means that $X \in \mathrm{dom}(b_{i,\delta})$ for every $X \in D$, namely $b_{i,\delta}$ is a cofinal branch. □

By the Third Preservation Lemma, a cofinal R_i-branch b exists in W, for some $i \in I$. Assume that $i = (a, b, \mu)$, for every $X \subseteq Y$ in $\mathrm{dom}(b)$, we have $(a, b) \Vdash (X, b(X)) <_{\dot{F}_\mu} (Y, b(Y))$. If $G_0 \times G_1 \subseteq \mathbb{L}(\mu)$ is any generic filter containing the condition (a, b), then the branch b determines a cofinal branch for $\dot{F}_\mu$ in $W[G_0 \times G_1]$ (note that $([\lambda]^{<\nu^+})^W$ is cofinal in $([\lambda]^{<\nu^+})^{W[G_0 \times G_1]}$) contradicting the fact that $\dot{F}_\mu$ is a name for a (ν^+, λ)-tree with no cofinal branches. This completes the proof of the theorem. □

We do not know whether a model of the super tree property at $\aleph_{\omega+1}$ can be found. As we said in §5.3 we do not even know whether a successor of a singular cardinal can satisfy the super tree property.

Conclusions

Three questions where asked in the Introduction.

- What cardinals can satisfy the strong or the super tree properties?
- How can we use the "strong compactness" or "supercompactness" of small cardinals satisfying the strong or the super tree properties?
- Is it possible to find analogous combinatorial characterizations of other large cardinals?

The results presented in this thesis provide a partial answer to the first of the above questions for regular cardinals less than or equal to $\aleph_{\omega+1}$. We proved that a model where all the $\aleph_n$'s simultaneously satisfy the super tree property can be found, and we showed that even $\aleph_{\omega+1}$ can consistently satisfy the strong tree property. It remains open whether we can combine all these results to get a model of the strong or the super tree property "up to" $\aleph_{\omega+1}$, and we do not know whether larger cardinals such as $\aleph_2$ can have the strong or the super tree property.

In addiction to these problems, it would be interesting to explore the consequences of these properties. An intriguing problem would be, for example, to determine whether one of these properties decide the singular cardinal hypothesis, SCH. Solovay showed that if κ is a strongly compact cardinal, then the singular cardinal hypothesis holds *above* κ. Since the strong tree property characterizes strongly compact cardinals, we may ask whether assuming the strong tree property at a cardinal κ is enough to get the singular cardinal hypothesis above κ. If it is the case, then in particular the strong tree property at $\aleph_2$ would imply SCH.

As we said, the epistemological issues associated to large cardinals axioms suggest we explore the possibility to replaced them by weaker assumptions. The strong and super tree properties are very good candidates for replacing strongly compact and supercompact cardinals in many contexts, because they capture the combinatorial essence of these large cardinals. We can also explore the possibility for other large cardinals to have analogous characterization. For example, we can investigate the combinatorics of stronger large cardinals such as extensible or huge cardinals, and ask the same questions for the corresponding properties. By characterizing the main large cardinal notions in terms of combinatorial properties, we would open the road to a new axiomatization. We can consider replacing the hierarchy of large cardinals by these more natural principles in tuning with Erdös tradition. All those problems promise a wide range of applications in many fields.

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Notation

$X - Y$	the set $\{x \in X;\ x \notin Y\}$
$\lvert X\rvert$	cardinality of X
$\text{o.t.}(X)$	the order type of X
$\text{t.c.}(X)$	transitive closure of X
H_κ	the set of all x with $\text{t.c.}(x) < \kappa$
V_α	the α-th set of the cumulative hierarchy of sets
$\mathscr{P}(X)$	powerset of X
$[X]^{<\kappa}$	the set of all subsets of X of size less than κ
$\ll \lambda \gg^{<\nu^+}$	the set of all the strictly increasing sequences $X : \alpha \to \lambda$ such that $\alpha < \nu^+$.
$[[I]]^2$	the set of all pairs $(X, Y) \in I \times I$ such that $X \subseteq Y$
$]\alpha, \beta[$	the set $\{\gamma > \alpha;\ \gamma < \beta\}$
$]\alpha, \beta]$	the set $\{\gamma > \alpha;\ \gamma \leq \beta\}$
$[\alpha, \beta[$	the set $\{\gamma \geq \alpha;\ \gamma < \beta\}$
$[\alpha, \beta]$	the set $\{\gamma \geq \alpha;\ \gamma \leq \beta\}$
$\text{dom}(f)$	domain of f
$f \restriction X$	the restriction of a function f to a set X
$f \frown x$	the extension of a sequence f by an element x
$f[X]$	the set $\{f(x);\ x \in X\}$
$g \sqsubseteq f$	the function f extends g
$f \circ g$	the composition of f and g
${}^{Y}X$	the set of all functions $f : Y \to X$
$\text{ht}(T)$	the height of a tree T
Lev_α, $\text{Lev}_\alpha(T)$	the α-th level of T
$\text{pred}_T(t)$	the set of all predecessors of t in the tree T
$t \restriction \alpha$	the predecessor of t in Lev_α
$\max(\alpha, \beta)$	the maximum between α and β
$\sup X$	the supremum of X
$\inf X$	the infimum of X
$\lim_{\gamma<\delta} \alpha_\gamma$	the limit of a sequence $\langle \alpha_\gamma;\ \gamma < \delta \rangle$
$\alpha^{<\beta}$	the set $\bigcup_{\gamma<\beta} \alpha^\gamma$
$\prod_{i \in I} X_i$	the product of sets X_i, $i \in I$
$\mathbb{P}$	a forcing notion
$p \vert\vert q$	the conditions p and q are compatible
$V^{\mathbb{P}}$	the set of $\mathbb{P}$-names
$\dot{a}$	a $\mathbb{P}$-name
$V[G]$	the generic extension of a model V by a generic filter G
$\dot{a}^G$	the interpretation of a $\mathbb{P}$-name $\dot{a}$ by a generic filter G

$\mathbb{P} \equiv \mathbb{Q}$	the two forcings $\mathbb{P}$ and $\mathbb{Q}$ are equivalent
$\mathbb{P} \times \mathbb{Q}$	the product forcing
$\mathbb{P} \times \dot{\mathbb{Q}}$	the two step iteration of forcing notions
$\mathbb{Q}/G_{\mathbb{P}}$	the set $\{q \in \mathbb{Q};\ \pi(q) \in G_{\mathbb{P}}\}$ where $\pi : \mathbb{Q} \to \mathbb{P}$ is a projection and $G_{\mathbb{P}} \subseteq \mathbb{P}$ is a generic filter
$\mathrm{Add}(\tau, \kappa)$	the set of all $p : \kappa \to 2$ of size $< \tau$, partially ordered by reverse inclusion
$\mathrm{Add}(\kappa)$	$\mathrm{Add}(\kappa, \kappa)$
$\mathrm{Coll}(\kappa, \lambda)$	the poset $\{p : \kappa \to \lambda;\ \lvert\mathrm{dom}(p)\rvert < \kappa\}$ ordered by reverse inclusion
$\mathrm{Coll}(\kappa, < \lambda)$	the product $\Pi_{\alpha<\lambda}\mathrm{Coll}(\kappa, \alpha)$
φ^M	the relativization of a formula φ to the model M

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