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A report on the thesis "Large Properties at Small Cardinals" by Ms. Fontanella

The thesis "Large Properties at Small Cardinals" by Ms. Fontanella discusses what kind of large cardinal properties small cardinals can have, which is one of the most important topics in modern set theory.

A classical result in this area is the consistency of the tree property of small uncountable regular cardinals. The tree property is a combinatorial essence of a weakly compact cardinal. It has been well-known from the first half of the 20th century, due to Aronszajn, that the first uncountable cardinal $\aleph_1$ does not have the tree property. On the other hand, in 1972 Mitchell proved the consistency of that $\aleph_n$ has the tree property for any (single) natural number $n \geq 2$. Extending his result, in 1998 Cummings and Foreman proved the consistency of that $\aleph_n$'s simultaneously have the tree property for all natural numbers $n \geq 2$. Furthermore in 1996 Magidor and Shelah proved the consistency of the tree property of $\aleph_{\omega+1}$.

In 2010, in his Ph.D thesis, Weiss introduced the strong tree property and the super tree property, which are combinatorial essence of a strongly compact cardinal and a supercompact cardinal respectively. (Supercompactness implies strong compactness, which in turn implies weak compactness. Similarly, the super tree property implies the strong tree property, which in turn implies the tree property.) Moreover, extending Mitchell's result, he proved the consistency of that $\aleph_n$ has the super tree property for any (single) natural number $n \geq 2$. Strongly compact cardinals and supercompact cardinals are among most important large cardinals, and many set theorists are interested in his result.

Here the following natural questions arise: Is it consistent that $\aleph_n$'s simultaneously have the strong or the super tree property for all natural numbers $n \geq 2$? Is it consistent that $\aleph_{\omega+1}$ has the strong or the super tree property?

Extending Cummings-Foreman's and Magidor-Shelah's results mentioned above, Fontanella's thesis gives an affirmative answer to the first question and a partially affirmative answer to the second one:

Theorem 1. *It is consistent that $\aleph_n$'s simultaneously have the super tree property for all natural numbers $n \geq 2$.*

Theorem 2. *It is consistent that $\aleph_{\omega+1}$ has the strong tree property.*

These results themselves are important and fundamental in the large cardinal theory. Moreover their proofs are technically interesting. The original proof of Cummings-Foreman's result is already involved, and there are many technical difficulties to prove Theorem 1 by generalizing their proof. But she has overcome all these difficulties. Also, to prove Theorem 2, a new method developed by Neeman and Sinapova recently is used.

Furthermore this thesis is quite well-written. In the introduction the historical background of this research is explained in details. In Chapter 1–3 results and tools by Mitchell, Weiss and Cummings-Foreman are put in order. In Chapter 4 and 5, Theorem 1 and 2 are proved using tools in Chapter 1–3. The proofs of theorems are somewhat involved, but they are written in an efficient and clear way.

It remains open whether $\aleph_{\omega+1}$ can consistently have the super tree property, which is an interesting and important problem. I expect that she would solve this problem in the future. Moreover tree properties are central notions in the large cardinal theory and infinite combinatorics. Her thesis allows me to expect that she would make a great contributions to these areas.

Conclusion. I consider that the thesis by Ms. Fontanella deserves to be a Ph.D thesis by international standard. I propose to award the degree to her after a successful defense of the thesis.

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