STRONG TREE PROPERTIES FOR SMALL CARDINALS

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Abstract. An inaccessible cardinal κ is supercompact when (κ, λ)-ITP holds for all $\lambda \geq \kappa$. We prove that if there is a model of ZFC with infinitely many supercompact cardinals, then there is a model of ZFC where for every $n \geq 2$ and $\mu \geq \aleph_n$, we have $(\aleph_n, \mu)$-ITP.

1. Introduction

One of the most intriguing research axes in contemporary set theory is the investigation into those properties which are typically associated with large cardinals, though they can be satisfied by small cardinals as well. The tree property is a principle of that sort. Given a regular cardinal κ, we say that κ satisfies the *tree property* when every κ-tree has a cofinal branch. The result presented in the present paper concerns the so-called *strong tree property* and *super tree property*, which are two combinatorial principles that generalize the usual tree property. The definition of those properties will be presented in §**??**, for now let us just discuss some general facts about their connection with large cardinals. We know that an inaccessible cardinal is weakly compact if and only if it satisfies the tree property. The strong and the super tree properties provide a similar characterization of strongly compact and supercompact cardinals, indeed an inaccessible cardinal is strongly compact if and only if it satisfies the strong tree property, while it is supercompact if and only if it satisfies the super tree property (the former result follows from a theorem by Jech [**?**], the latter is due to Magidor [**?**]). In other words, when a cardinal satisfies one of the previous properties, it "behaves like a large cardinal".

While the previous characterizations date back to the early 1970s, a systematic study of the strong and the super tree properties has only recently been undertaken by Weiss [**?**]. He proved that for every $n \geq 2$, one can define a model of the super tree property for $\aleph_n$, starting from a model with a supercompact cardinal. It is natural to ask whether all small cardinals of the form $\aleph_n$ (with $n \geq 2$) can *simultaneously* have the strong or the super tree properties. Fontanella [**?**] proved that a forcing

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construction due to Abraham [?] generalizes to show that the super tree property can hold for two successive cardinals. Cummings and Foreman [?] proved that if there is a model of set theory with infinitely many supercompact cardinals, then one can obtain a model in which every $\aleph_n$ with $n \geq 2$ satisfies the tree property. In the present paper, we prove that in the Cummings and Foreman's model even the *super* tree property holds at every $\aleph_n$ with $n \geq 2$. The same result has been proved independently by Unger [?].

The paper is organized as follows. In §**??** we introduce the strong and the super tree properties. §**??** is devoted to the proof of two preservation theorems. In §**??** we define Cummings and Foreman's model. In §**??**, and §**??**, we expand that model and we analyze some properties of the new generic extension. Finally, we prove in §**??** that in Cummings and Foreman's model every cardinal $\aleph_n$ (with $n \geq 2$) has the super tree property.

2. Preliminaries and Notation

Given a forcing $\mathbb{P}$ and conditions $p, q \in \mathbb{P}$, we use $p \leq q$ in the sense that p is stronger than q; we write $p||q$ when p and q are two compatible conditions (i.e. there is a condition $r \in \mathbb{P}$ such that $r \leq p$ and $r \leq q$). A poset $\mathbb{P}$ is *separative* if whenever $q \not\leq p$, then some extension of q in $\mathbb{P}$ is incompatible with p. Every partial order can be turned into a separative poset. Indeed, one can define $p \prec q$ iff all extensions of p are compatible with q, and the resulting equivalence relation is given by $p \sim q$ iff $p \prec q$ and $q \prec p$, provides a separative poset. Then the set of all equivalence classes of $\mathbb{P}$ is separative.

Assume that $\mathbb{P}$ is a forcing notion in a model V, we will use $V^{\mathbb{P}}$ to denote the class of $\mathbb{P}$-names. If $a \in V^{\mathbb{P}}$ and $G \subseteq \mathbb{P}$ is generic over V, then a^G denotes the interpretation of a in $V[G]$. Every element x of the ground model V is represented in a canonical way by a name $\check{x}$. However, to simplify the notation, we will use just x instead of $\check{x}$ in forcing formulas. The set $\dot{G} := \{(\check{p}, p);\ p \in \mathbb{P}\} \in V^{\mathbb{P}}$ is called the *canonical name for a generic filter for* $\mathbb{P}$, thus for every filter $G \subseteq \mathbb{P}$ generic over V, the interpretation of $\dot{G}$ in V^G is precisely G.

A forcing $\mathbb{P}$ is κ-*closed* if and only if every descending sequence of conditions of $\mathbb{P}$ of size less than κ has a lower bound; $\mathbb{P}$ is κ-*directed closed* if and only if for every set of less than κ pairwise compatible conditions of $\mathbb{P}$ has a lower bound. We say that $\mathbb{P}$ is κ-*distributive* if and only if no sequence of ordinals of length less than κ is added by $\mathbb{P}$. $\mathbb{P}$ is κ-c.c. when every antichain of $\mathbb{P}$ has size less than κ; $\mathbb{P}$ is κ-*Knaster* if and only if for all sequence of conditions $\langle p_\alpha;\ \alpha < \kappa\rangle$, there is $X \subseteq \kappa$ cofinal such that the conditions of the sequence $\langle p_\alpha;\ \alpha \in X\rangle$ are pairwise compatible.

Given two forcings $\mathbb{P}$ and $\mathbb{Q}$, we will write $\mathbb{P} \equiv \mathbb{Q}$ when $\mathbb{P}$ and $\mathbb{Q}$ are equivalent, namely:

(1) for every filter $G_{\mathbb{P}} \subseteq \mathbb{P}$ which is generic over V, there exists a filter $G_{\mathbb{Q}} \subseteq \mathbb{Q}$ which is generic over V, and $V[G_{\mathbb{P}}] = V[G_{\mathbb{Q}}]$;
(2) for every filter $G_{\mathbb{Q}} \subseteq \mathbb{Q}$ which is generic over V, there exists a filter $G_{\mathbb{P}} \subseteq \mathbb{P}$ which is generic over V, and $V[G_{\mathbb{P}}] = V[G_{\mathbb{Q}}]$.

If $\mathbb{P}$ is any forcing and $\dot{\mathbb{Q}}$ is a $\mathbb{P}$-name for a forcing, consider the class of all $(p, q) \in \mathbb{P} \times V^{\mathbb{P}}$ such that $p \Vdash q \in \dot{\mathbb{Q}}$. We define an ordering on the elements of this class by setting $(p, q) \leq (p', q')$ if and only if $p \leq p'$ and $p \Vdash q \leq q'$. Then $\mathbb{P} * \dot{\mathbb{Q}}$ denotes the set of all equivalence classes (corresponding to this ordering) of minimal rank.

Theorem 2.1. *(folklore) Assume $\mathbb{P}$ and $\mathbb{Q}$ are two forcing notions in V. For every $G_P \subseteq \mathbb{P}$ and $G_Q \subseteq \mathbb{Q}$, the following are equivalent:*

(1) *$G_P \times G_Q$ is generic for $\mathbb{P} \times \mathbb{Q}$ over V;*
(2) *G_P is generic for $\mathbb{P}$ over V and G_Q is generic for $\mathbb{Q}$ over $V[G_P]$;*
(3) *G_Q is generic for $\mathbb{Q}$ over V and G_P is generic for $\mathbb{P}$ over $V[G_Q]$.*

Furthermore, if $(1) - (3)$ holds, then $V[G_P \times G_Q] = V[G_P][G_Q] = V[G_Q][G_P]$ and we say that G_P and G_Q are mutually generic.

For a proof of the previous theorem see for example [**?**, Theorem 1.4. Chapter VIII].

If $\mathbb{P}$ and $\mathbb{Q}$ are two posets, a *projection* $\pi : \mathbb{Q} \to \mathbb{P}$ is a function such that:

(1) for all $q, q' \in \mathbb{Q}$ if $q \leq q'$, then $\pi(q) \leq \pi(q')$;
(2) $\pi(1_{\mathbb{Q}}) = 1_{\mathbb{P}}$;
(3) for all $q \in \mathbb{Q}$ if $p \leq \pi(q)$, then there is $q' \leq q$ such that $\pi(q') \leq p$.

We say that $\mathbb{P}$ *is a projection of* $\mathbb{Q}$ when there is a projection $\pi : \mathbb{Q} \to \mathbb{P}$.

If $\pi : \mathbb{Q} \to \mathbb{P}$ is a projection and $G_{\mathbb{P}} \subseteq \mathbb{P}$ is a generic filter over V, define

$$\mathbb{Q}/G_{\mathbb{P}} := \{q \in \mathbb{Q};\ \pi(q) \in G_{\mathbb{P}}\},$$

$\mathbb{Q}/G_{\mathbb{P}}$ is ordered as a subposet of $\mathbb{Q}$. The following hold:

(1) If $G_{\mathbb{Q}} \subseteq \mathbb{Q}$ is a generic filter over V and $H := \{p \in \mathbb{P};\ \exists q \in G_{\mathbb{Q}}(\pi(q) \leq p)\}$, then H is $\mathbb{P}$-generic over V;
(2) if $G_{\mathbb{P}} \subseteq \mathbb{P}$ is a generic filter over V and if $G \subseteq \mathbb{Q}/G_{\mathbb{P}}$ is a generic filter over $V[\mathbb{G}_{\mathbb{P}}]$, then G is $\mathbb{Q}$-generic over V and $\pi[G]$ generates $G_{\mathbb{P}}$;
(3) if $G_{\mathbb{Q}} \subseteq \mathbb{Q}$ is a generic filter and $H := \{p \in \mathbb{P};\ \exists q \in G_{\mathbb{Q}}(\pi(q) \leq p)\}$, then $G_{\mathbb{Q}}$ is $\mathbb{Q}/G_{\mathbb{P}}$-generic over $V[H]$. In other words we can factor forcing with $\mathbb{Q}$ as forcing with $\mathbb{P}$ followed by forcing with $\mathbb{Q}/G_{\mathbb{P}}$ over $V[G_{\mathbb{P}}]$.

Some of our projections $\pi : \mathbb{Q} \to \mathbb{P}$ will also have the following property: for all $p \leq \pi(q)$, there is $q' \leq q$ such that

(1) $\pi(q') = p$,
(2) for every $q^* \leq q$, if $\pi(q^*) \leq p$, then $q^* \leq q'$.

Let κ be a regular cardinal and λ an ordinal, we denote by $\mathrm{Add}(\kappa, \lambda)$ the poset of all partial functions $f : \lambda \to 2$ of size less than κ which is ordered by reverse inclusion. We use $\mathrm{Add}(\kappa)$ to denote $\mathrm{Add}(\kappa, \kappa)$.

If $V \subseteq W$ are two models of set theory with the same ordinals and η is a cardinal in W, we say that (V, W) has the η-covering property if and only if every set $X \subseteq V$ in W of cardinality less than η in W, is contained in a set $Y \in V$ of cardinality less than η in V.

Lemma 2.2. *(Easton's Lemma) Let κ be regular. If $\mathbb{P}$ has the κ-chain condition and $\mathbb{Q}$ is κ-closed, then*
(1) $\Vdash_{\mathbb{Q}} \mathbb{P}$ *has the κ-chain condition*;
(2) $\Vdash_{\mathbb{P}} \mathbb{Q}$ *is a κ-distributive*;
(3) *If G is $\mathbb{P}$-generic over V and H is $\mathbb{Q}$-generic over V, then $(V, V[G][H])$ has the κ-covering property;*
(4) *If $\mathbb{R}$ is κ-closed, then $\Vdash_{\mathbb{P}\times\mathbb{Q}}$ $\mathbb{R}$ is κ-distributive.*

For a proof of that lemma see [**?**, Lemma 2.11].

Let η be a regular cardinal, $\theta > \eta$ be large enough and $M \prec H_\theta$ of size η. We say that M is *internally approachable of length η* if it can be written as the union of an increasing continuous chain $\langle M_\xi : \xi < \eta\rangle$ of elementary submodels of $H(\theta)$ of size less than η, such that $\langle M_\xi : \xi < \eta'\rangle \in M_{\eta'+1}$, for every ordinal $\eta' < \eta$.

We will assume familiarity with the theory of large cardinals and elementary embeddings, as developed for example in [**?**].

Lemma 2.3. *(Laver)* [**?**] *If κ is a supercompact cardinal, then there exists $L : \kappa \to V_\kappa$ such that: for all λ, for all $x \in H_{\lambda^+}$, there is an elementary embedding $j : V \to M$ with critical point κ such that $j(\kappa) > \lambda$, ${}^{\lambda}M \subseteq M$ and $j(L)(\kappa) = x$.*

Lemma 2.4. *(Silver) Let $j : M \to N$ be an elementary embedding between inner models of* ZFC. *Let $\mathbb{P} \in M$ be a forcing and suppose that G is $\mathbb{P}$-generic over M, H is $j(\mathbb{P})$-generic over N, and $j[G] \subseteq H$. Then there is a unique $j^* : M[G] \to N[H]$ such that $j^* \restriction M = j$ and $j^*(G) = H$.*

Proof. If $j[G] \subseteq H$, then the map $j^*(\dot{x}^G) = j(\dot{x})^H$ is well defined and satisfies the required properties. □

3. The Strong and the Super Tree Properties

We recall the definition of the tree property for a regular cardinal κ.

Definition 3.1. *Let κ be a regular cardinal,*

(1) *a κ-tree is a tree of height κ with levels of size less than κ;*
(2) *we say that κ has the* tree property *if and only if every κ-tree has a cofinal branch (i.e. a branch of size κ).*

The strong and the super tree property concern special objects that generalize the notion of κ-tree, for a regular cardinal κ.

Definition 3.2. *Given $\kappa \geq \omega_2$ a regular cardinal and $\lambda \geq \kappa$, a (κ, λ)*-tree *is a set F satisfying the following properties:*

(1) *every $f \in F$ is a function $f : X \to 2$, for some $X \in [\lambda]^{<\kappa}$*
(2) *for all $f \in F$, if $X \subseteq \mathrm{dom}(f)$, then $f \upharpoonright X \in F$;*
(3) *the set $\mathrm{Lev}_X(F) := \{f \in F;\ \mathrm{dom}(f) = X\}$ is non empty for all $X \in [\lambda]^{<\kappa}$;*
(4) *$|\mathrm{Lev}_X(F)| < \kappa$ for all $X \in [\lambda]^{<\kappa}$.*

When there is no ambiguity, we will simply write Lev_X instead of $\mathrm{Lev}_X(F)$. In a (κ, λ)-tree levels are not indexed by ordinals, but by *sets of ordinals.* Therefore, predecessors are not well ordered, hence a (κ, λ)-tree is not a tree.

Definition 3.3. *Given $\kappa \geq \omega_2$ a regular cardinal, $\lambda \geq \kappa$, and a (κ, λ)-tree F,*

(1) *a* cofinal branch *for F is a function $b : \lambda \to 2$ such that $b \upharpoonright X \in \mathrm{Lev}_X(F)$ for all $X \in [\lambda]^{<\kappa}$;*
(2) *an F*-level sequence *is a function $D : [\lambda]^{<\kappa} \to F$ such that for every $X \in [\lambda]^{<\kappa}$, $D(X) \in \mathrm{Lev}_X(F)$;*
(3) *given an F-level sequence D, an* ineffable branch *for D is a cofinal branch $b : \lambda \to 2$ such that $\{X \in [\lambda]^{<\kappa};\ b \upharpoonright X = D(X)\}$ is stationary.*

Definition 3.4. *Given $\kappa \geq \omega_2$ a regular cardinal and $\lambda \geq \kappa$,*

(1) (κ, λ)-TP *holds if every (κ, λ)-tree has a cofinal branch;*
(2) (κ, λ)-ITP *holds if for every (κ, λ)-tree F and for every F-level sequence D, there is an an ineffable branch for D;*
(3) *we say that κ satisfies the* strong tree property *if (κ, μ)*-TP *holds for all $\mu \geq \kappa$;*
(4) *we say that κ satisfies the* super tree property *if (κ, μ)*-ITP *holds for all $\mu \geq \kappa$;*

4. The Preservation Theorems

It will be important, in what follows, that certain forcings cannot add ineffable branches. The following proposition is due to Silver (see [**?**, chap. VIII, Lemma 3.4] or [**?**, Proposition 2.1.12]), we include the proof for completeness.

Theorem 4.1. *(First Preservation Theorem) Let θ be a regular cardinal and $\mu \geq \theta$ be any ordinal. Assume that F is a (θ, μ)-tree and $\mathbb{Q}$ is an η^+-closed forcing with $\eta < \theta \leq 2^\eta$. For every filter $G_{\mathbb{Q}} \subseteq \mathbb{Q}$ generic over V, every cofinal branch for F in $V[G_{\mathbb{Q}}]$ is already in V.*

Proof. We can assume without loss of generality that η is minimal such that $2^\eta \geq \theta$. Assume towards a contradiction that $\mathbb{Q}$ adds a cofinal branch to F, let $\dot{b}$ be a $\mathbb{Q}$-name for such a function. For all $\alpha \leq \eta$ and all $s \in {}^\alpha 2$, we are going to define by induction three objects $a_\alpha \in [\mu]^{<\theta}$, $f_s \in \mathrm{Lev}_{a_\alpha}$ and $p_s \in \mathbb{Q}$ such that:

(1) $p_s \Vdash \dot{b} \upharpoonright a_\alpha = f_s$;
(2) $f_{s^\frown 0}(\beta) \neq f_{s^\frown 1}(\beta)$, for some $\beta < \mu$;
(3) if $s \subseteq t$, then $p_t \leq p_s$;
(4) if $\alpha < \beta$, then $a_\alpha \subset a_\beta$.

Let $\alpha < \eta$, assume that a_α, f_s and p_s have been defined for all $s \in {}^\alpha 2$. We define $a_{\alpha+1}$, f_s, and p_s, for all $s \in {}^{\alpha+1}2$. Let t be in ${}^\alpha 2$, we can find an ordinal $\beta_t \in \mu$ and two conditions $p_{t^\frown 0}, p_{t^\frown 1} \leq p_t$ such that $p_{t^\frown 0} \Vdash \dot{b}(\beta_t) = 0$ and $p_{t^\frown 1} \Vdash \dot{b}(\beta_t) = 1$. (otherwise $\dot{b}$ would be a name for a cofinal branch which is already in V). Let $a_{\alpha+1} := a_\alpha \cup \{\beta_t;\ t \in {}^\alpha 2\}$, then $|a_{\alpha+1}| < \theta$, because $2^\alpha < \theta$. We just defined, for every $s \in {}^{\alpha+1}2$, a condition p_s. Now by strengthening p_s if necessary, we can find $f_s \in \mathrm{Lev}_{a_{\alpha+1}}$ such that

$$p_s \Vdash \dot{b} \upharpoonright a_{\alpha+1} = f_s.$$

Finally $f_{t^\frown 0}(\beta_t) \neq f_{t^\frown 1}(\beta_t)$, for all $t \in {}^\alpha 2$: because $p_{t^\frown 0} \Vdash f_{t^\frown 0}(\beta_t) = \dot{b}(\beta_t) = 0$, while $p_{t^\frown 1} \Vdash f_{t^\frown 1}(\beta_t) = \dot{b}(\beta_t) = 1$.

If α is a limit ordinal $\leq \eta$, let t be any function in ${}^\alpha 2$. Since $\mathbb{Q}$ is η^+-closed, there is a condition p_t such that $p_t \leq p_{t\upharpoonright\beta}$, for all $\beta < \alpha$. Define $a_\alpha := \bigcup\limits_{\beta<\alpha} a_\beta$. By strengthening p_t if necessary, we can find $f_t \in \mathrm{Lev}_{a_\alpha}$ such that $p_t \Vdash \dot{b} \upharpoonright a_\alpha = f_t$. That completes the construction.

We show that $|\mathrm{Lev}_{a_\eta}| \geq {}^\eta 2 \geq \theta$, thus a contradiction is obtained. Let $s \neq t$ be two functions in ${}^\eta 2$, we are going to prove that $f_s \neq f_t$. Let α be the minimum ordinal less than η such that $s(\alpha) \neq t(\alpha)$, without loss of generality $r \frown 0 \sqsubset s$ and $r \frown 1 \sqsubset t$, for some $r \in {}^\alpha 2$. By construction we have

$$p_s \leq p_{r^\frown 0} \Vdash \dot{b} \upharpoonright a_{\alpha+1} = f_{r^\frown 0} \text{ and } p_t \leq p_{r^\frown 1} \Vdash \dot{b} \upharpoonright a_{\alpha+1} = f_{r^\frown 1},$$

where $f_{r^\frown 0}(\beta) \neq f_{r^\frown 1}(\beta)$ for some β. Moreover $p_s \Vdash \dot{b} \upharpoonright a_\eta = f_s$ and $p_t \Vdash \dot{b} \upharpoonright a_\eta = f_t$, hence $f_s \upharpoonright a_{\alpha+1}(\beta) = f_{r^\frown 0}(\beta) \neq f_{r^\frown 1}(\beta) = f_t \upharpoonright a_{\alpha+1}(\beta)$, thus $f_s \neq f_t$. That completes the proof. $\square$

We will also use the following theorem from Fontanella [**?**], we include the proof for completeness.

Theorem 4.2. *(Second Preservation Theorem) Let $V \subseteq W$ be two models of set theory with the same ordinals and let $\mathbb{P} \in V$ be a forcing notion and κ a cardinal in V such that:*

(1) $\mathbb{P} \subseteq \mathrm{Add}(\aleph_n, \tau)^V$, *for some* $\tau > \aleph_n$,
and for every $p \in \mathbb{P}$, *if* $X \subseteq \mathrm{dom}(p)$, *then* $p \upharpoonright X \in \mathbb{P}$;
(2) $\aleph_m^V = \aleph_m^W$, *for every* $m \leq n$, *and* $W \models |\kappa| = \aleph_{n+1}$;
(3) *for every set* $X \subseteq V$ *in* W *of size* $< \aleph_{n+1}$ *in* W, *there is* $Y \in V$ *of size* $< \kappa$ *in* V, *such that* $X \subseteq Y$;
(4) *in* V, *we have* $\gamma^{<\aleph_n} < \kappa$ *for every cardinal* $\gamma < \kappa$.

Let $F \in W$ *be a* $(\aleph_{n+1}, \mu)$*-tree with* $\mu \geq \aleph_{n+1}$, *then for every filter* $G_{\mathbb{P}} \subseteq \mathbb{P}$ *generic over* W, *every cofinal branch for* F *in* $W[G_{\mathbb{P}}]$ *is already in* W.

Proof. Work in W. Let $\dot{b} \in W^{\mathbb{P}}$ and let $p \in \mathbb{P}$ such that

$$p \Vdash \dot{b} \text{ is a cofinal branch for } F.$$

We are going to find a condition $q \in \mathbb{P}$ such that $q||p$ and for some $b \in W$, we have $q \Vdash \dot{b} = b$. Let χ be large enough, for all $X \prec H_\chi$ of size $\aleph_n$, we fix a condition $p_X \leq p$ and a function $f_X \in Lev_{X \cap \mu}$ such that

$$p_X \Vdash \dot{b} \upharpoonright X = f_X.$$

Let S be the set of all the structures $X \prec H_\chi$ such that X is internally approachable of length $\aleph_n$. Since every condition of $\mathbb{P}$ has size less than $\aleph_n$, there is for all $X \in S$, a set $M_X \in X$ of size less than $\aleph_n$ such that

$$p_X \upharpoonright X \subseteq M_X.$$

By the Pressing Down Lemma, there exists M^* and a stationary set $E^* \subseteq S$ such that $M^* = M_X$, for all $X \in E^*$. The set M^* has size less than $\aleph_n$ in W, hence $A := (\bigcup_{X \in E^*} p_X) \upharpoonright M^*$ has size less than $\aleph_n$ in W. By the assumption, A is covered by some $N \in V$ of size $\gamma < \kappa$ in V. In V we have $|[N]^{<\aleph_n}| \leq \gamma^{<\aleph_n} < \kappa$. It follows that in W there are less than $\aleph_{n+1}$ possible values for $p_X \upharpoonright M^*$. Therefore we can find in W a cofinal $E \subseteq E^*$ and a condition $q \in \mathbb{P}$ such that $p_X \upharpoonright X = q$ for all $X \in E$.

Claim 4.3. $f_X \upharpoonright Y = f_Y \upharpoonright X$, *for all* $X, Y \in E$.

Proof. Let $X, Y \in E$, there is $Z \in E$ with $X, Y, \mathrm{dom}(p_X), \mathrm{dom}(p_Y) \subseteq Z$. Then we have $p_X \cap p_Z = p_X \cap (p_Z \upharpoonright Z) = p_X \cap q = q$, thus $p_X||p_Z$ and similarly $p_Y||p_Z$. Let $r \leq p_X, p_Z$ and $s \leq p_Y, p_Z$, then $r \Vdash f_Z \upharpoonright X = \dot{b} \upharpoonright X = f_X$ and $s \Vdash f_Z \upharpoonright Y = \dot{b} \upharpoonright Y = f_Y$. It follows that $f_X \upharpoonright Y = f_Z \upharpoonright (X \cap Y) = f_Y \upharpoonright X$. □

Let b be $\bigcup_{X \in E} f_X$. The previous claim implies that b is a function and

$$b \upharpoonright X = f_X, \text{ for all } X \in E.$$

Claim 4.4. $q \Vdash \dot{b} = b$.

Proof. We show that for every $X \in E$, the set $B_X := \{s \in \mathbb{P};\ s \Vdash \dot{b} \upharpoonright X = b \upharpoonright X\}$ is dense below q. Let $r \leq q$, there is $Y \in E$ such that $\mathrm{dom}(r), X \subseteq Y$. It follows that $p_Y \cap r = p_Y \upharpoonright Y \cap r = q \cap r = q$, thus $p_Y||r$. Let $s \leq p_Y, r$, then $s \in B_X$, because $s \Vdash \dot{b} \upharpoonright X = f_Y \upharpoonright X = f_X = b \upharpoonright X$. Since $\bigcup\{X \cap \mu; X \in E\} = \mu$, we have $q \Vdash \dot{b} = b$. $\square$

That completes the proof. $\square$

5. Cummings and Foreman's Iteration

In this section we discuss a forcing construction which is due to Cummings and Foreman [?]. We will prove in §?? that this iteration produces a model where every $\aleph_n$ (with $n \geq 2$) satisfies the super tree property. A few considerations will help the reader to understand the definition of this iteration. The standard way to produce a model of the super tree property for $\aleph_{n+2}$ (where $n < \omega$) is the following: we start with a supercompact cardinal κ – by Magidor's theorem it is inaccessible and it satisfies the super tree property –, then we turn κ into $\aleph_{n+2}$ by forcing with a poset that preserves the super tree property at κ. The forcing notion required for that, is a variation of an iteration due to Mitchell that we denote by $\mathbb{M}(\aleph_n, \kappa)$ (see [?]). A naive attempt to construct a model where the super tree property holds simultaneously for two cardinals $\aleph_{n+2}$ and $\aleph_{n+3}$, would be to start with two supercompact cardinals $\kappa < \lambda$, and force with $\mathbb{M}(\aleph_n, \kappa)$ first, and then with $\mathbb{M}(\aleph_{n+1}, \lambda)$. The problem with that approach is that, at the second step of this iteration, we could lose the super tree property at κ, that is at $\aleph_{n+2}$. For this reason, the first step of the iteration must be reformulated so that, not only it will turn κ into $\aleph_{n+2}$ and preserve the super tree property at κ, but it will also "anticipate a fragment" of $\mathbb{M}(\aleph_{n+1}, \lambda)$. We are going to define a forcing $\mathbb{R}(\tau, \kappa, V, W, L)$ that will constitute the main brick of Cummings and Foreman's iteration (Definition **??**). If κ is supercompact cardinal in the model V, then $\mathbb{R}(\tau, \kappa, V, W, L)$ turns κ into τ^{++} and it makes τ^{++} satisfy the super tree property in a larger model W. The parameter L refers to the Laver function for κ (which is in V), such function will be used to "guess the tail" of the iteration.

None of the results of this section are due to the author.

Definition 5.1. *Let $V \subseteq W$ be two models of set theory and suppose that for some τ, κ, we have $W \models (\tau < \kappa$ is regular and κ is inaccessible). Let $\mathbb{P} := \mathrm{Add}(\tau, \kappa)^V$ and suppose that $W \models \mathbb{P}$ is τ^+-c.c. and τ-distributive. Let $L \in W$ be a function with $L : \kappa \to (V_\kappa)^W$. Define in W a forcing*

$$\mathbb{R} := \mathbb{R}(\tau, \kappa, V, W, L)$$

as follows. The definition is by induction; for each $\beta \leq \kappa$ we will define a forcing $\mathbb{R} \upharpoonright \beta$ and we will finally set $\mathbb{R} := \mathbb{R} \upharpoonright \kappa$. $\mathbb{R} \upharpoonright 0$ is the trivial forcing.
(p, q, f) is a condition in $\mathbb{R} \upharpoonright \beta$ if and only if

(1) $p \in \mathbb{P} \upharpoonright \beta := \mathrm{Add}(\tau, \beta)^V$;
(2) *q is a partial function on β, $|q| \leq \tau$, $\mathrm{dom}(q)$ consists of successor ordinals, and if $\alpha \in \mathrm{dom}(q)$, then $q(\alpha) \in W^{\mathbb{P} \upharpoonright \alpha}$ and $\Vdash^W_{\mathbb{P} \upharpoonright \alpha} q(\alpha) \in \mathrm{Add}(\tau^+)$*

(3) *f is a partial function on β, $|f| \leq \tau$, $\mathrm{dom}(f)$ consists of limit ordinals and $\mathrm{dom}(f)$ is a subset of*

$$\{\alpha;\ \Vdash^W_{\mathbb{R}\restriction\alpha} L(\alpha) \textit{ is a } \tau^+\textit{-directed closed forcing } \}$$

(4) *If $\alpha \in \mathrm{dom}(f)$, then $f(\alpha) \in W^{\mathbb{R}\restriction\alpha}$ and $\Vdash^W_{\mathbb{R}\restriction\alpha} f(\alpha) \in L(\alpha)$.*

The conditions in $\mathbb{R} \restriction \beta$ are ordered by $(p', q', f') \leq (p, q, f)$ if and only if

(1) *$p' \leq p$;*
(2) *for all $\alpha \in \mathrm{dom}(q)$, $p' \restriction \alpha \Vdash q'(\alpha) \leq q(\alpha)$;*
(3) *for all $\alpha \in \mathrm{dom}(f)$, $(p', q', f') \restriction \alpha \Vdash^W_{\mathbb{R}\restriction\alpha} f'(\alpha) \leq f(\alpha)$.*

Let us discuss some easy properties of that forcing.

Lemma 5.2. *In the situation of Definition **??**, $\mathbb{R}$ can be projected to $\mathbb{P}$, $\mathbb{R} \restriction \alpha * L(\alpha)$, and $\mathbb{P} \restriction \alpha * \dot{\mathbb{A}}$ where $\dot{\mathbb{A}}$ is a $\mathbb{P} \restriction \alpha$-name for $Add(\tau^+)$.*

Proof. The projection maps are defined as follows:

(1) $\pi_0 : (p, q, f) \mapsto p$ is the projection to $\mathbb{P}$;
(2) $\pi_1 : (p, q, f) \mapsto ((p, q, f) \restriction \alpha, f(\alpha))$ is the projection to $\mathbb{R} \restriction \alpha * L(\alpha)$;
(3) $\pi_2 : (p, q, f) \mapsto (p \restriction \alpha, q(\alpha))$ is the projection to $\mathbb{P} \restriction \alpha * \dot{\mathbb{A}}$.

See also [**?**, Lemma 3.3]. □

The proof of the following lemma is analogous to the proof of [**?**, Lemma 3.6], we include it for completeness.

Lemma 5.3. *In the situation of Definition **??**, if $g \subseteq \mathbb{P}$ is a generic filter and if $\mathbb{P}$ is τ-distributive in W, then $\mathbb{R}/g$ is τ-directed closed in $W[g]$. In particular if $\mathbb{P}$ is τ-closed, then $\mathbb{R}$ is τ-closed.*

Proof. In $W[g]$, let $\langle (p_i, q_i, f_i);\ i < \gamma \rangle$ be a sequence of less than τ pairwise compatible conditions of $\mathbb{R}/g$. Since $\mathbb{P}$ is τ-distributive, the sequence belongs to W. By definition of $\mathbb{R}/g$, we have $p_i \in g$ for every g, so we can fix a condition p such that $p \leq p_i$ for every $i < \gamma$ (as $\mathbb{P}$ is separative, we can take for example $p \in g$ such that $p \Vdash p_i \in \dot{g}$ for all i, where $\dot{g}$ is the canonical name for a generic filter for $\mathbb{P}$). We define a function q with domain $\bigcup_{i<\gamma} \mathrm{dom} q_i$ as follows. For every $\alpha \in \mathrm{dom}(q)$, let $I_\alpha \subseteq \gamma$ such that $\alpha \in \mathrm{dom}(q_i)$ for every $i \in I_\alpha$. Then we have

$$p \restriction \alpha \Vdash \langle q_i(\alpha);\ i \in I_\alpha \rangle \text{ are pairwise compatible conditions in } \mathrm{Add}(\tau^+).$$

Therefore there is $q(\alpha) \in W^{\mathbb{P}\restriction\alpha}$ such that $p \restriction \alpha \Vdash q(\alpha) \leq q_i(\alpha)$ for every $i \in I_\alpha$. Now we define a function f with domain $\bigcup_{i<\gamma} \mathrm{dom}(f_i)$. By induction on α, we define $f(\alpha)$ so that $(p, q, f) \restriction \alpha$ is a lower bound for the sequence $\langle (p_i, q_i, f_i) \restriction \alpha;\ i < \gamma \rangle$. Assume that $f(\beta)$ has been defined for every $\beta < \alpha$, and let $J_\alpha \subseteq \gamma$ such that $\alpha \in \mathrm{dom}(f_i)$ for every $i \in J_\alpha$, then

$$(p, q, f) \restriction \alpha \Vdash \langle f_i(\alpha);\ i \in J_\alpha \rangle \text{ are pairwise compatible conditions in } L(\alpha).$$

By definition we have $\Vdash^W_{\mathbb{R}\restriction\alpha} L(\alpha)$ is τ^+-directed closed, so there is $f(\alpha) \in W^{\mathbb{R}\restriction\alpha}$ such that $(p,q,f)\restriction\alpha \Vdash f(\alpha) \leq f_i(\alpha)$, for every $i \in J_\alpha$. That completes the definition of f. Finally the condition (p,q,f) is a lower bound for the sequence $\langle (p_i,q_i,f_i);\ i<\gamma\rangle$. □

Definition 5.4. *(Cummings and Foreman's Iteration) We consider $\langle \kappa_n;\ n<\omega\rangle$ an increasing sequence of supercompact cardinals. For every $n<\omega$, let $L_n : \kappa_n \to V_{\kappa_n}$ be the Laver function for κ_n. We define a forcing iteration $\mathbb{R}_\omega$ of length ω as follows.*

1. *The first stage of the iteration $\mathbb{R}_1$ is the poset $\mathbb{Q}_0 := \mathbb{R}(\aleph_0, \kappa_0, V, V, L_0)$.*
2. *Assume $\dot{L}_1$ is an $\mathbb{R}_1$-name for a function so that $\Vdash_{\mathbb{R}_1} \dot{L}_1(\alpha) = L_1(\alpha)$, if $L_1(\alpha)$ is a $\mathbb{R}_1$-name, and $\Vdash_{\mathbb{R}_1} \dot{L}_1(\alpha) = 0$ otherwise. We let $\dot{\mathbb{Q}}_1$ be an $\mathbb{R}_1$-name for $\mathbb{R}(\aleph_1^V, \kappa_1, V, V[\dot{K}_1], \dot{L}_1)$, where $\dot{K}_1$ denotes the canonical name for a generic filter for $\mathbb{R}_1$. We define $\mathbb{R}_2 := \mathbb{Q}_0 * \dot{\mathbb{Q}}_1$.*
3. *Suppose $\mathbb{R}_n := \mathbb{Q}_0 * ... * \dot{\mathbb{Q}}_{n-1}$ has been defined and assume $\dot{L}_n$ is an $\mathbb{R}_n$-name for a function so that $\Vdash_{\mathbb{R}_n} \dot{L}_n(\alpha) = L_n(\alpha)$, if $L_n(\alpha)$ is a $\mathbb{R}_n$-name, and $\Vdash_{\mathbb{R}_n} L_n(\alpha) = 0$, otherwise. We let $\dot{\mathbb{Q}}_n$ be an $\mathbb{R}_n$-name for the poset $\mathbb{R}(\kappa_{n-2}, \kappa_n, V[\dot{K}_{n-1}], V[\dot{K}_n], \dot{L}_n)$, where $\dot{K}_n$ is the canonical name for a generic filter for $\mathbb{R}_n$. Finally, we define $\mathbb{R}_{n+1} := \mathbb{Q}_0 * ... * \dot{\mathbb{Q}}_n$.*

We also fix a filter $G_\omega \subseteq \mathbb{R}_\omega$ generic over V, and for every $n>0$, we denote by $K_n := G_0 * ... * G_{n-1}$ the initial segment of G_ω generic for $\mathbb{R}_n = \mathbb{Q}_0 * ... * \dot{\mathbb{Q}}_{n-1}$ over V.

The following lemma will prove that the previous definition is legitimate. In the statement of the lemma, when we refer to "$\aleph_i$" we mean $\aleph_i$ in the sense of $V[K_n]$.

Lemma 5.5. *For $n \geq 1$, in $V[K_n]$ we let $\mathbb{Q}_n := \dot{\mathbb{Q}}_n^{K_n}$ and we define $\mathbb{P}_n := \mathrm{Add}(\aleph_n, \kappa_n)^{V[K_{n-1}]}$ and $\mathbb{U}_n := \{(0,q,f);\ (0,q,f) \in \mathbb{Q}_n\}$ (ordered as a subset of $\mathbb{Q}_n$). The following hold:*

1. *$V[K_n] \models 2^{\aleph_i} = \aleph_{i+2} = \kappa_i$, for $i<n$, and κ_j is inaccessible for every $j \geq n$.*
2. *$V[K_n] \models \mathbb{Q}_n$ is $\aleph_n$-distributive, κ_n-Knaster, $\aleph_{n-1}$-directed closed and has size κ_n.*
3. *All cardinals up to $\aleph_{n+1}$ are preserved in $V[K_n * G_n]$.*
4. *$V[K_n] \models (\mathbb{Q}_n$ is a projection of $\mathbb{P}_n \times \mathbb{U}_n)$, hence there are filters $g_n \subseteq \mathbb{P}_n$ and $u_n \subseteq \mathbb{U}_n$ which are generic over $V[K_n]$ and satisfy $V[K_n * g_n] \subseteq V[K_n * G_n] \subseteq V[K_n * (g_n \times u_n)]$.*
5. *$V[K_n] \models \mathbb{P}_n \times \mathbb{U}_n$ is κ_n-c.c.*
6. *$V[K_n] \models \mathbb{U}_n$ is $\aleph_{n+1}$-directed closed and κ_n-c.c.*
7. *In $V[K_n * G_n]$ we let $\mathbb{S}_n := (\mathbb{P}_n \times \mathbb{U}_n)/G_n$, then*
$$V[K_n * G_n] \models \mathbb{S}_n \text{ is } \aleph_{n+1}\text{-distributive, } \aleph_n\text{-closed and } \kappa_n\text{-c.c..}$$
8. *$\mathrm{Add}(\aleph_n, \eta)^{V[K_{n-1}]}$ is $\aleph_{n+1}$-Knaster in $V[K_n * G_n]$ for any ordinal η.*
9. *$V[K_n * G_n] \models \mathrm{Add}(\aleph_{n+1}, \eta)^{V[K_n]}$ is $\aleph_{n+1}$-distributive and κ_n-Knaster for any ordinal η.*
10. *All $\aleph_n$-sequences of ordinals from $V[K_n * G_n]$ are in $V[K_n * g_n]$.*

Proof. See [?, Lemma 4.3] (for claim 5. and 10. see [?, Lemma 3.11], for claim 6. see [?, Lemma 3.8 and 3.9], finally, claim 7. corresponds to [?, Lemma 3.20]) □

In the following sections, we will use the previous lemma repeatedly and without comments.

Definition 5.6. *In the situation of Definition* **??***, let* $\beta < \kappa$ *and* X_β *be* $\mathbb{R} \restriction \beta$*-generic over* W*, we define* $\mathbb{R}^* := \mathbb{R}/X_\beta$ *(i.e.* $\mathbb{R}^* := \{r \in \mathbb{R};\ r \restriction \beta \in X_\beta\}$*).* $\mathbb{R}^*$ *is ordered as a subposet of* $\mathbb{R}$*. We also let* $\mathbb{U}^* := \{(0,q,f);\ (0,q,f) \in \mathbb{R}^*\}$ *which is ordered as a suborder of* $\mathbb{R}^*$*. Finally, we let* $\mathbb{P}^* := \{p \in \mathbb{P};\ (p,0,0) \in \mathbb{R}^*\}$ *be ordered as a suborder of* $\mathbb{P}$*.*

Lemma 5.7. *In the situation of Definition* **??***, the following hold:*

(1) *the function* $\pi : \mathbb{P}^* \times \mathbb{U}^* \to \mathbb{R}^*$ *defined by* $\pi(p,(0,q,f)) \mapsto (p,q,f)$ *is a projection;*
(2) $\mathbb{U}^*$ *is* τ^+*-closed in* $W[X_\beta]$*.*

Proof. See [**?**, Lemma 3.24 and 3.25]. □

Cummings and Foreman [**?**] also proved the following lemma.

Lemma 5.8. *For every* $n < \omega$*, let* $X \in V[G_\omega]$ *be a* κ_n*-sequence of ordinals, then* $X \in V[K_{n+2} * g_{n+2}]$

Proof. For every $m < \omega$, the poset $\mathbb{R}_\omega/K_{m+3}$ is κ_m-closed. Therefore $X \in V[K_{n+4}]$. Since $\mathbb{Q}_{n+3}$ is κ_{n+1}-distributive in $V[K_{n+3}]$, we have $X \in V[K_{n+3}]$. Finally every κ_n-sequence of ordinals in $V[K_{n+3}]$ is in $V[K_{n+2} * g_{n+2}]$, that completes the proof. □

In [**?**], this was used to show that if T is a κ_n-tree in $V[G_\omega]$, then $T \in V[K_{n+2} * g_{n+2}]$; we cannot prove the same for (κ_n, μ)-trees.

6. Expanding Cummings and Foreman's Model

To prove the main theorem, we need to expand Cummings and Foreman's model. In the previous section we introduced several objects. We recall that G_ω is a generic filter for $\mathbb{R}_\omega$ over V and $K_n = G_0 * ... * G_{n-1}$ is the initial segment of G_ω generic for $\mathbb{R}_n = \mathbb{Q}_0 * ... * \dot{\mathbb{Q}}_{n-1}$ over V. We defined in $V[K_n]$ two posets $\mathbb{P}_n := \mathrm{Add}(\aleph_n, \kappa_n)^{V[K_{n-1}]}$ and $\mathbb{U}_n := \{(0,q,f);\ (0,q,f) \in \mathbb{Q}_n\}$, and we fixed $g_n \subseteq \mathbb{P}_n$ and $u_n \subseteq \mathbb{U}_n$ which are two generic filters over $V[K_n]$ such that $V[K_n * g_n] \subseteq V[K_n * G_n] \subseteq V[K_n * (g_n \times u_n)]$. For every $n > 0$, the poset $\mathbb{S}_n$ is a forcing notion in $V[K_n * G_n]$ and it denotes $(\mathbb{P}_n \times \mathbb{U}_n)/G_n$ (see Lemma **??**). In this short section we observe what happens when we force over $V[G_\omega]$ with $\mathbb{S}_{n+1}$ and then with $\mathbb{S}_{n+2}$.

Definition 6.1. *For every* $n < \omega$*, we define in* $V[K_{n+1}]$ *the forcing*

$$\mathrm{Tail}_{n+1} := \mathbb{R}_\omega/K_{n+1}.$$

Tail_{n+3} is a κ_n-directed closed forcing in $V[K_{n+3}]$.

Definition 6.2. *For every* $n < \omega$*, we denote* $V_n := V[G_0 * ... * G_n] = V[K_{n+1}]$ *and we let* $G_{tail(n+1)} \subseteq \mathrm{Tail}_{n+1}$ *be the generic filter over* V_n *such that* $V_n[G_{tail(n+1)}] = V[G_\omega]$*.*

Definition 6.3. *We let $s_{n+1} \subseteq \mathbb{S}_{n+1}$ be the generic filter over V_{n+1} such that $V_{n+1}[s_{n+1}] = V_n[g_{n+1} \times u_{n+1}]$.*

By Theorem **??**, $G_{tail(n+1)}$ and s_{n+1} are mutually generic thus

$$V_n[g_{n+1} \times u_{n+1}][G_{tail(n+2)}] = V[G_\omega][s_{n+1}].$$

For the same reason,

$$V_n[g_{n+1} \times u_{n+1}][g_{n+2} \times u_{n+2}][G_{tail(n+3)}] = V[G_\omega][s_{n+1}][s_{n+2}].$$

So the model $V_n[g_{n+1} \times u_{n+1}][g_{n+2} \times u_{n+2}][G_{tail(n+3)}]$ is the result of forcing over $V[G_\omega]$ first with $\mathbb{S}_{n+1}$ and then with $\mathbb{S}_{n+2}$. Now, we want to show that this model can be seen as being obtained by forcing over V_n with a cartesian product that satisfies particular properties. In order to define that forcing notion, first we need to introduce the notion of "term forcing" (that notion is due to Mitchell [**?**]).

Definition 6.4. *Let $\mathbb{P}$ be a forcing notion and let $\dot{\mathbb{Q}}$ be a $\mathbb{P}$-name for a poset. For every $\dot{q}, \dot{r}$ such that $\Vdash_{\mathbb{P}} \dot{q}, \dot{r} \in \dot{\mathbb{Q}}$, we let $\dot{q} \leq^* \dot{r}$ if and only if $\Vdash_{\mathbb{P}} \dot{q} \leq \dot{r}$. The* $\mathbb{P}$-term-forcing for $\dot{\mathbb{Q}}$ *is the set of all equivalence classes (corresponding to $\leq^*$) of minimal rank.*

Lemma 6.5. *In the situation of Definition* **??**, *assume $\mathbb{T}$ is the $\mathbb{P}$-term-forcing for $\dot{\mathbb{Q}}$, then the following hold:*

(1) *$\mathbb{P} * \dot{\mathbb{Q}}$ is a projection of $\mathbb{P} \times \mathbb{T}$;*
(2) *if $\Vdash_{\mathbb{P}} \dot{\mathbb{Q}}$ is κ-directed closed, then $\mathbb{T}$ is κ-directed closed as well.*

Proof.

(1) Let $\pi : \mathbb{P} \times \mathbb{T} \to \mathbb{P} * \dot{\mathbb{Q}}$ be the map $(p, \dot{q}) \mapsto (p, \dot{q})$, we prove that π is a projection. It is clear that π respects the ordering relation and $\pi(1_{\mathbb{P}\times\mathbb{T}}) = (1_{\mathbb{P}*\dot{\mathbb{Q}}})$. In $\mathbb{P} * \dot{\mathbb{Q}}$, let $(p_0, \dot{q}_0) \leq (p_1, \dot{q}_1)$, then $p_0 \leq p_1$ and $p_0 \Vdash \dot{q}_0 \leq \dot{q}_1$. Define $\dot{q}$ as a $\mathbb{P}$-name for an element of $\dot{Q}$ such that for every $\mathbb{P}$-generic filter G, we have $\dot{q}^G = \dot{q}_0^G$ if $p_0 \in G$, and $\dot{q}^G = \dot{q}_1^G$ otherwise. Then $(p_0, \dot{q}) = (p_0, \dot{q}_0)$.

(2) Assume that $\langle \dot{q}_\alpha;\ \alpha < \gamma \rangle$ is a sequence of less than κ pairwise compatible conditions in $\mathbb{T}$. Then,

$$\Vdash_{\mathbb{P}} \text{"}\langle \dot{q}_\alpha;\ \alpha < \gamma \rangle \text{ are pairwise compatible conditions in } \dot{\mathbb{Q}}\text{"},$$

hence there exists a $\mathbb{P}$-name $\dot{q}$ such that $\Vdash_{\mathbb{P}} \dot{q} \leq \dot{q}_\alpha$, for every $\alpha < \gamma$. This means that $\dot{q} \leq^* \dot{q}_\alpha$, for every $\alpha < \gamma$. □

Posets like $\mathbb{P}_n$, $\mathbb{U}_n$ and Tail_{n+1} can be defined in any generic extension of V by $\mathbb{R}_n$. We introduce names for such forcings.

Notation 6.6. *Let $\dot{K}_n$ be the canonical name for a generic filter for $\mathbb{R}_n$. We let $\dot{\mathbb{P}}_n, \dot{\mathbb{U}}_n \in V^{\mathbb{R}_n}$ and $\dot{\mathrm{Tail}}_{n+1} \in V^{\mathbb{R}_{n+1}}$ be such that*

(1) $\Vdash_{\mathbb{R}_n} \dot{\mathbb{P}}_n = \mathrm{Add}(\aleph_n, \kappa_n)^{V[\dot{K}_{n-1}]}$;
(2) $\Vdash_{\mathbb{R}_n} \dot{\mathbb{U}}_n = \{(0, q, f);\ (0, q, f) \in \dot{\mathbb{Q}}_n\}$;

(3) $\Vdash_{\mathbb{R}_{n+1}} \dot{\text{Tail}}_{n+1} = \mathbb{R}_\omega / \dot{K}_{n+1}$.

Definition 6.7. *For every $n < \omega$, we let $\dot{\mathbb{T}}_{n+3} \in V^{\mathbb{R}_{n+2}}$ be such that*

$$\Vdash_{\mathbb{R}_{n+2}} \dot{\mathbb{T}}_{n+3} \textit{ is the } (\dot{\mathbb{P}}_{n+2} \times \dot{\mathbb{U}}_{n+2})\textit{-term-forcing for } \dot{\text{Tail}}_{n+3}.$$

We also let $\dot{\mathbb{Z}}_{n+2} \in V^{\mathbb{R}_{n+1}}$ be such that

$$\Vdash_{\mathbb{R}_{n+1}} \dot{\mathbb{Z}} \textit{ is the } (\dot{\mathbb{P}}_{n+1} \times \dot{\mathbb{U}}_{n+1} \times \dot{\mathbb{P}}_{n+2})\textit{-term-forcing for the poset } \dot{\mathbb{U}}_{n+2} \times \dot{\mathbb{T}}_{n+3}.$$

Finally we define $\mathbb{T}_{n+3} := \dot{\mathbb{T}}_{n+3}^{K_{n+2}}$ and $\mathbb{Z}_{n+2} := \dot{\mathbb{Z}}_{n+2}^{K_{n+1}}$.

Remark 6.8. *In other words,*

(1) $(\mathbb{P}_{n+2} \times \mathbb{U}_{n+2}) * \text{Tail}_{n+3}$ *is a projection of* $\mathbb{P}_{n+2} \times \mathbb{U}_{n+2} \times \mathbb{T}_{n+3}$;
(2) $(\mathbb{P}_{n+1} \times \mathbb{U}_{n+1} \times \mathbb{P}_{n+2}) * (\mathbb{U}_{n+2} \times \mathbb{T}_{n+3})$ *is a projection of* $\mathbb{P}_{n+1} \times \mathbb{U}_{n+1} \times \mathbb{P}_{n+2} \times \mathbb{Z}_{n+2}$.

Lemma 6.9. *The following hold:*

(1) $\mathbb{T}_{n+3}$ *is κ_n-directed closed in V_{n+1};*
(2) $\mathbb{Z}_{n+2}$ *is κ_n-directed closed in V_n.*

Proof.
(1) Tail_{n+3} is κ_n-directed closed in V_{n+2} and in $V_{n+1}[g_{n+2} \times u_{n+2}]$. By Lemma **??**, then, $\mathbb{T}_{n+3}$ is κ_n-directed closed in V_{n+1}.
(2) By the previous claim, the product $\mathbb{U}_{n+2} \times \mathbb{T}_{n+3}$ is κ_n-directed closed in V_{n+1}. The poset $\mathbb{S}_{n+1}$ is κ_n-distributive in V_{n+1}, so $\mathbb{U}_{n+2} \times \mathbb{T}_{n+3}$ is κ_n-directed closed in $V_n[g_{n+1} \times u_{n+1}]$ as well. Now $\mathbb{P}_{n+2}$ is κ_n-distributive in $V_n[g_{n+1} \times u_{n+1}]$, so $\mathbb{U}_{n+2} \times \mathbb{T}_{n+3}$ is κ_n-directed closed even in $V_n[g_{n+1} \times u_{n+1}][g_{n+2}] = V_n[g_{n+1} \times u_{n+1} \times g_{n+2}]$. By Lemma **??**, the poset $\mathbb{Z}_{n+2}$ is κ_n-directed closed in V_n. □

Remark **??** justifies the following definition.

Definition 6.10. *We let $t_{n+3} \subseteq \mathbb{T}_{n+3}$ be generic over $V_n[g_{n+1} \times u_{n+1}]$ such that*

$$V_n[g_{n+1} \times u_{n+1}][g_{n+2} \times u_{n+2}][G_{tail(n+3)}] \subseteq V_n[g_{n+1} \times u_{n+1}][g_{n+2} \times u_{n+2} \times t_{n+3}].$$

We also let $z_{n+2} \subseteq \mathbb{Z}_{n+2}$ be generic over V_n such that

$$V_n[g_{n+1} \times u_{n+1}][g_{n+2} \times u_{n+2} \times t_{n+3}] \subseteq V_n[g_{n+1} \times u_{n+1} \times g_{n+2} \times z_{n+2}].$$

Lemma 6.11. *The following hold:*

(1) $(\mathbb{P}_{n+2} \times \mathbb{U}_{n+2} \times \mathbb{T}_{n+3})/(g_{n+2} \times u_{n+2}) * G_{tail(n+3)}$ *is κ_n-closed in* $V_{n+1}[g_{n+2} \times u_{n+2}][G_{tail(n+3)}]$;
(2) $(\mathbb{P}_{n+1} \times \mathbb{U}_{n+1} \times \mathbb{P}_{n+2} \times \mathbb{Z}_{n+2})/(g_{n+1} \times u_{n+1} \times g_{n+2}) * (u_{n+2} \times t_{n+3})$ *is $\aleph_{n+1}$-closed in* $V_n[g_{n+1} \times u_{n+1} \times g_{n+2}][u_{n+2} \times t_{n+3}]$.

Proof. The proof is standard: it follows from Lemma **??** and from the fact that $\mathbb{P}_{n+2} \times \mathbb{U}_{n+2} * \dot{\text{Tail}}_{n+3}$ is κ_n-distributive and $(\mathbb{P}_{n+1} \times \mathbb{U}_{n+1} \times \mathbb{P}_{n+2}) * (\dot{\mathbb{U}}_{n+2} \times \dot{\mathbb{T}}_{n+3})$ is $\aleph_{n+1}$-distributive. □

Remark 6.12. *Summing up, we have:*

(1) $V[G_\omega] \subseteq V_n[g_{n+1} \times u_{n+1}][G_{tail(n+2)}]$, *the latter model has been obtained by forcing with* $\mathbb{S}_{n+1}$ *over* $V[G_\omega]$;
(2) $V_n[g_{n+1} \times u_{n+1}][G_{tail(n+2)}] \subseteq V_n[g_{n+1} \times u_{n+1}][g_{n+2} \times u_{n+2}][G_{tail(n+3)}]$, *the latter model has been obtained by forcing with* $\mathbb{S}_{n+2}$ *over the former;*
(3) $V_n[g_{n+1} \times u_{n+1}][g_{n+2} \times u_{n+2}][G_{tail(n+3)}] \subseteq V_n[g_{n+1} \times u_{n+1}][g_{n+2} \times u_{n+2} \times t_{n+3}]$, *the latter model has been obtained by forcing over the former with a* κ_n*-closed forcing, namely* $(\mathbb{P}_{n+2} \times \mathbb{U}_{n+2} \times \mathbb{T}_{n+3})/(g_{n+2} \times u_{n+2}) * G_{tail(n+3)}$;
(4) $V_n[g_{n+1} \times u_{n+1}][g_{n+2} \times u_{n+2} \times t_{n+3}] \subseteq V_n[g_{n+1} \times u_{n+1} \times g_{n+2} \times z_{n+2}]$, *the latter model has been obtained by forcing over the former with an* $\aleph_{n+1}$*-closed forcing, namely* $(\mathbb{P}_{n+1} \times \mathbb{U}_{n+1} \times \mathbb{P}_{n+2} \times \mathbb{Z}_{n+2})/(g_{n+1} \times u_{n+1} \times g_{n+2}) * (u_{n+2} \times t_{n+3})$.

7. More Preservation Results

It will be important, in what follows, that the forcing that takes us from G_ω to the model $V_n[g_{n+1} \times u_{n+1} \times g_{n+2} \times z_{n+2}]$ defined in the previous section, cannot add cofinal branches to an $(\aleph_{n+2}, \mu)$-tree.

Lemma 7.1. *Let* $F \in V[G_\omega]$ *be an* $(\aleph_{n+2}, \mu)$*-tree, where* $\mu \geq \aleph_{n+2}$ *is an ordinal. If* b *is a cofinal branch for* F *in* $V_n[g_{n+1} \times u_{n+1} \times g_{n+2} \times z_{n+2}]$, *then* $b \in V[G_\omega]$.

Proof. Assume towards a contradiction that $b \notin V[G_\omega]$. The forcing $\mathbb{S}_{n+1}$ is κ_{n-1}-closed in V_{n+1} and, since Tail_{n+2} is κ_{n-1}-closed, $\mathbb{S}_{n+1}$ remains κ_{n-1}-closed (that is $\aleph_{n+1}$-closed) in $V[G_\omega]$, where $\kappa_n = \aleph_{n+2} = 2^{\aleph_n}$. By the First Preservation Theorem, we have

$$b \notin V_n[g_{n+1} \times u_{n+1}][G_{tail(n+2)}].$$

Now $\mathbb{S}_{n+2}$ is κ_n-closed in V_{n+2} and, since $\mathbb{S}_{n+1}$ is κ_n-distributive and Tail_{n+3} is κ_n-closed, the poset $\mathbb{S}_{n+2}$ remains κ_n-closed (that is $\aleph_{n+2}$-closed) in the model $V_n[g_{n+1} \times u_{n+1}][G_{tail(n+2)}]$. Another application of the First Preservation Theorem gives

$$b \notin V_n[g_{n+1} \times u_{n+1}][g_{n+2} \times u_{n+2}][G_{tail(n+3)}].$$

The passage from $V_n[g_{n+1} \times u_{n+1}][g_{n+2} \times u_{n+2}][G_{tail(n+3)}]$ to $V_n[g_{n+1} \times u_{n+1}][g_{n+2} \times u_{n+2} \times t_{n+3}]$ is done by a κ_n-closed forcing (see Remark **??**), hence by the First Preservation Theorem, we get $b \notin V_n[g_{n+1} \times u_{n+1}][g_{n+2} \times u_{n+2} \times t_{n+3}] = V_n[g_{n+1} \times u_{n+1} \times g_{n+2}][u_{n+2} \times t_{n+3}]$. The forcing that takes us from $V_n[g_{n+1} \times u_{n+1} \times g_{n+2}][u_{n+2} \times t_{n+3}]$ to $V_n[g_{n+1} \times u_{n+1} \times g_{n+2} \times z_{n+2}]$ is $\aleph_{n+1}$-closed (see Remark **??**), hence by the First Preservation Theorem, we have

$$b \notin V_n[g_{n+1} \times u_{n+1} \times g_{n+2} \times z_{n+2}],$$

that leads to a contradiction. □

For the proof of the final theorem, we will also need the following lemma.

Lemma 7.2. *Let* $\mathbb{R} := \mathbb{R}(\tau, \kappa, V, W, L)$ *be like in Definition* **??** *and let* $\theta < \kappa$ *be such that:*

(1) *for some* $n < \omega$, $\tau = \aleph_n$ *and* $\aleph_m^V = \aleph_m^W$, *for every* $m \leq n$.
(2) *in* W *we have* $\gamma^{<\tau} < \theta$, *for every* $\gamma < \theta$,

Suppose $X \subseteq \mathbb{R}$ *is a generic filter over* W, *and* $X_\theta := X \upharpoonright \theta$. *Given a* (θ, μ)*-tree* F *in* $W[X_\theta]$ *with* $\mu \geq \theta$, *if* b *is a cofinal branch for* F *in* $W[X]$, *then* $b \in W[X_\theta]$.

Proof. Assume towards a contradiction that $b \notin W[X_\theta]$. By Lemma **??**, the forcing $\mathbb{R}^* := \mathbb{R}/X_\theta$ is a projection of $\mathbb{P}^* \times \mathbb{U}^*$, where $\mathbb{P}^* = \mathrm{Add}(\tau, \kappa - \theta)^V$ and $\mathbb{U}^*$ is τ^+-closed in $W[X_\theta]$. Let $g^* \times u^* \subseteq \mathbb{P}^* \times \mathbb{U}^*$ be any generic filter over W that projects on X. In $W[X_\theta]$ we have $\theta = \tau^{++} = 2^\tau$ and F is a (θ, μ)-tree. Therefore, we can apply the First Preservation Theorem, hence $b \notin W[X_\theta][u^*]$. The filter u^* collapses θ to τ^+, so now F is a (τ^+, μ)-tree in $W[X_\theta][u^*]$. We want to use the Second Preservation Theorem to prove that $\mathbb{P}^*$ cannot add cofinal branches to $W[X_\theta][u^*]$. We can see $\mathbb{P}^*$ as a subset of $\mathrm{Add}(\tau, \kappa)^{W[X_\theta]}$. By hypothesis, $W \models \gamma^{<\tau} < \theta$ for every $\gamma < \theta$. Moreover, $\mathbb{R} \upharpoonright \theta$ is τ-distributive and θ-c.c., so $W[X_\theta] \models \gamma^{<\tau} < \theta$, for every $\gamma < \theta$. Since $\mathbb{U}^*$ is τ^+-closed, the pair $(W[X_\theta], W[X_\theta][u^*])$ satisfies condition (3) of the Second Preservation Theorem. So all the hypothesis of the Second Preservation Theorem are satisfied, hence $b \notin W[X_\theta][u^*][g^*]$ and in particular $b \notin W[X]$. That leads to a contradiction. □

8. The Final Theorem

Theorem 8.1. *In* $V[G_\omega]$, *every cardinal* $\aleph_{n+2}$ *has the super tree property.*

Proof. Let $F \in V[G_\omega]$ be an $(\aleph_{n+2}, \mu)$-tree, where $\mu \geq \aleph_{n+2}$ is an ordinal, and let D be an F-level sequence. In $V[G_\omega]$, we have $\kappa_n = \aleph_{n+2}$, so F is a (κ_n, μ)-tree. We start working in V. Let $\lambda := \sup_{n<\omega} \kappa_n$ and fix ν grater than both $\mu^{<\kappa_n}$ and λ^ω. There is an elementary embedding $j : V \to M$ with critical point κ_n such that:

(i) $j(\kappa_n) > \nu$ and ${}^{<\nu}M \subseteq M$;
(ii) $j(L_n)(\kappa_n)$ is an $\mathbb{R}_{n+1}$-name for the product

$$\dot{\mathbb{U}}_{n+1} \times \dot{\mathbb{P}}_{n+2} \times \dot{\mathbb{Z}}_{n+2}$$

($\dot{\mathbb{U}}_{n+1}$, $\dot{\mathbb{P}}_{n+2}$ and $\dot{\mathbb{Z}}_{n+2}$ were defined in Notation **??** and Definition **??**).

Note that $j(L_n)(\kappa_n)$ is a name for a κ_n-directed closed forcing in V_n.
The proof of the theorem consists of three parts:

(1) we show that we can lift j to get an elementary embedding

$$j^* : V[G_\omega] \to M[H_\omega],$$

where $H_\omega \subseteq j(\mathbb{R}_\omega)$ is generic over V;
(2) we prove that there is in $M[H_\omega]$ an ineffable branch b for D;
(3) we show that $b \in V[G_\omega]$.

Part 1

We prove Claim 1. To simplify the notation we will denote all the extensions of j by "j" also. Recall that

$$V[G_\omega] \subseteq V_n[g_{n+1} \times u_{n+1}][G_{tail(n+2)}] \subseteq V_n[g_{n+1} \times u_{n+1}][g_{n+2} \times u_{n+2}][G_{tail(n+3)}]$$

$$\subseteq V_n[g_{n+1} \times u_{n+1}][g_{n+2} \times u_{n+2} \times t_{n+3}] \subseteq V_n[g_{n+1} \times u_{n+1} \times g_{n+2} \times z_{n+2}]$$

(see Remark **??**). The forcing $\mathbb{R}_n$ has size less than κ_n, so we can lift j to get an elementary embedding

$$j : V_{n-1} \to M_{n-1}.$$

For every $i < \omega$, we denote by M_i the model $M[G_0]...[G_i]$. We will use repeatedly and without comments the resemblance between V and M. In M_{n-1}, we have

$$j(\mathbb{Q}_n) \restriction \kappa_n = \mathbb{Q}_n,$$

and at stage κ_n, the forcing at the third coordinate will be $j(L_n)(\kappa_n)$ (see Lemma **??**). By our choice of $j(L_n)(\kappa_n)$, this means that we can look at the model $M_n[u_{n+1} \times g_{n+2} \times z_{n+2}]$ as a generic extension of M_{n-1} by $j(\mathbb{Q}_n) \restriction \kappa_n + 1$. Force with $j(\mathbb{Q}_n)$ over W to get a generic filter H_n such that $H_n \restriction \kappa_n + 1 = G_n * (u_{n+2} \times g_{n+2} \times z_{n+2})$. The forcing $\mathbb{Q}_n$ is κ_n-c.c. in M_{n-1}, so $j \restriction \mathbb{Q}_n$ is a complete embedding from $\mathbb{Q}_n$ into $j(\mathbb{Q}_n)$. Consequently, we can lift j to get an elementary embedding

$$j : V_n \to M_{n-1}[H_n].$$

We know that $\mathbb{P}_{n+1}$ is κ_n-c.c. in V_n, hence $j \restriction \mathbb{P}_{n+1}$ is a complete embedding from $\mathbb{P}_{n+1}$ into $j(\mathbb{P}_{n+1}) = \mathrm{Add}(\aleph_{n+1}, j(\kappa_{n+1}))^{M_{n-1}}$. $\mathbb{P}_{n+1}$ is even isomorphic via j to $\mathrm{Add}(\aleph_{n+1}, j[\kappa_{n+1}])^{M_{n-1}}$. Force with $Add(\aleph_{n+1}, j(\kappa_{n+1}) - j[\kappa_{n+1}])^{M_{n-1}}$ over $V_n[H_n][g_{n+1}]$ to get a generic filter $h_{n+1} \subseteq j(\mathbb{P}_{n+1})$ such that $j[g_{n+1}] \subseteq h_{n+1}$. We can lift j to get an elementary embedding

$$j : V_n[g_{n+1}] \to M_{n-1}[H_n][h_{n+1}].$$

By the previous observations on $j(\mathbb{Q}_n) \restriction \kappa_n + 1$ and by the closure of M, we have $j[u_{n+1} \times g_{n+2} \times z_{n+2}] \in M_{n-1}[H_n]$. The filter H_n collapses every cardinal below $j(\kappa_n)$ to have size $\aleph_{n+1}$ in $M_{n-1}[H_n]$, therefore the set $j[u_{n+1} \times g_{n+2} \times z_{n+2}]$ has size $\aleph_1$ in that model. Moreover, $j(\mathbb{U}_{n+1}) \times j(\mathbb{P}_{n+2}) \times j(\mathbb{Z}_{n+2})$ is a $j(\kappa_n)$-directed closed forcing and $j(\kappa_n) = \aleph_{n+2}^{M_{n-1}[H_n]}$. So, we can find a condition t^* stronger than every condition $j(q) \in j[u_{n+1} \times g_{n+2} \times z_{n+2}]$. By forcing over $V_{n-1}[H_n][h_{n+1}]$ with $j(\mathbb{U}_{n+1}) \times j(\mathbb{P}_{n+2}) \times j(\mathbb{Z}_{n+2})$ below t^* we get a generic filter $x_{n+1} \times h_{n+2} \times l_{n+2}$ such that $j[u_{n+1}] \subseteq x_{n+1}$, $j[g_{n+2}] \subseteq h_{n+2}$ and $j[z_{n+2}] \subseteq l_{n+2}$. The filters h_{n+1} and $x_{n+1} \times h_{n+2} \times l_{n+2}$ are mutually generic over $M_{n-1}[H_n]$, and $h_{n+1} \times x_{n+1}$ generates a filter H_{n+1} generic for $j(\mathbb{Q}_{n+1})$ over $M_{n-1}[H_n]$. By the properties of projections, we have $j[G_{n+1}] \subseteq H_{n+1}$. Therefore the embedding j lifts to an elementary embedding

$$j : V_{n+1} \to M_{n-1}[H_n][H_{n+1}].$$

By definition of $\mathbb{Z}_{n+2}$, the filter $h_{n+2} \times l_{n+2}$ which is generic for $j(\mathbb{P}_{n+2}) \times j(\mathbb{Z}_{n+2})$ determines a generic filter $(h_{n+2} \times x_{n+2}) * H_{tail(n+3)}$ for $(j(\mathbb{P}_{n+2}) \times j(\mathbb{U}_{n+2})) * j(\mathrm{Tail}_{n+3})$. On the other hand $h_{n+2} \times x_{n+2}$ determines a filter H_{n+2} generic for $j(\mathbb{Q}_{n+2})$ over $M_{n-1}[H_n][H_{n+1}]$. By the properties of projections, we have $j[G_{n+2}] \subseteq H_{n+2}$. Therefore, j lifts to an elementary embedding

$$j : V_{n+2} \to M_{n-1}[H_n][H_{n+1}][H_{n+2}].$$

It remains to prove that $j[G_{tail(n+3)}] \subseteq H_{tail(n+3)}$, but this is an immediate consequence of $j[z_{n+2}] \subseteq l_{n+2}$. Finally j lifts to an elementary embedding

$$j : V[G_\omega] \to M_{n-1}[H_n][H_{n+1}][H_{n+2}][H_{tail(n+3)}].$$

This completes the proof of Claim 1.

Part 2

Let $\mathscr{M}_1 := M[G_\omega]$ and $\mathscr{M}_2 := M_{n-1}[H_n][H_{n+1}][H_{n+2}][H_{tail(n+3)}]$. In $\mathscr{M}_2$, $j(F)$ is a $(j(\kappa_n), j(\mu))$-tree and $j(D)$ is a $j(F)$-level sequence. By the closure of M, the tree F and the F-level sequence D are in $\mathscr{M}_1$. We want to find in $\mathscr{M}_2$ an ineffable branch for D. Let $a := j[\mu]$, clearly $a \in [j(\mu)]^{<j(\kappa_n)}$. Consider $f := j(D)(a)$ and let $b : \mu \to 2$ be the function defined by $b(\alpha) := f(j(\alpha))$. We show that b is an ineffable branch for D. Assume towards a contradiction that for some club $C \subseteq [\mu]^{<|\kappa_n|}$ in $\mathscr{M}_2$ we have $b \restriction X \neq D(X)$, for all $X \in C$. By elementarity,

$$j(b) \restriction X \neq j(D)(X),$$

for all $X \in j(C)$. Observe that $a \in j(C)$ and $j(b) \restriction a = f = j(D)(a)$, that leads to a contradiction.

Part 3

We proved that an ineffable branch b for D exists in $\mathscr{M}_2$. Now we show that $b \in \mathscr{M}_1$, thereby proving that $\mathscr{M}_1$ (hence $V[G_\omega]$) has an ineffable [1] branche for D. We will use repeatedly and without comments the resemblance between V and M. Assume towards a contradiction that $b \notin \mathscr{M}_1$. Step by step, we are going to prove that $b \notin \mathscr{M}_2$. By Lemma **??**, we have $b \notin M_n[g_{n+1} \times u_{n+1} \times g_{n+2} \times z_{n+2}]$. Consider $\mathrm{Add}(\aleph_{n+1}, j(\kappa_{n+1}) - j[\kappa_{n+1}])^{M_{n-1}}$, by forcing with this poset over $M_n[g_{n+1} \times u_{n+1} \times g_{n+2} \times z_{n+2}]$ we obtained the generic extension $M_n[h_{n+1} \times u_{n+1} \times g_{n+2} \times z_{n+2}]$; we want to prove that b does not belong to that model. The pair $(M_{n-1}, M_n[g_{n+1} \times u_{n+1} \times g_{n+2} \times z_{n+2}])$ has the κ_n-covering property. Moreover, in V_{n-1}, the cardinal κ_n is inaccessible, therefore the hypothesis of the Second Preservation Theorem is satisfied and we have

$$b \notin M_n[h_{n+1} \times u_{n+1} \times g_{n+2} \times z_{n+2}].$$

[1] If $b \in \mathscr{M}_1$, then b is ineffable since $\{X \in [\mu]^{<|\kappa_n|} \cap \mathscr{M}_1;\ b \restriction X = D(X)\}$ is stationary in $\mathscr{M}_2$, hence it is stationary in $\mathscr{M}_1$.

As we said in Part 1, we have $j(\mathbb{Q}_n) \upharpoonright \kappa_n = \mathbb{Q}_n$, and at stage κ_n, the forcing at the third coordinate is $\mathbb{U}_{n+1} \times \mathbb{P}_{n+2} \times \mathbb{Z}_{n+2}$. It follows that for $H^* = H_n \upharpoonright \kappa_n + 1$ we have just proved

$$b \notin M_{n-1}[H^*][h_{n+1}] = M_{n-1}[h_{n+1}][H^*].$$

Now we want to show that $\mathbb{R}^* := j(\mathbb{Q}_n)/H^*$ cannot add cofinal branches to F, hence b does not belong to the model $M_{n-1}[h_{n+1}][H_n]$. So we check that the hypothesis of Lemma **??** are satisfied. The cardinal κ_n was inaccessible in M_{n-1}, and h_{n+1} is a generic filter for an $\aleph_{n+1}$-closed forcing, so $M_{n-1}[h_{n+1}] \models \gamma^{<\aleph_{n+1}} < \kappa_n$, for every $\gamma < \kappa_n$. Then all the hypothesis of Lemma **??** are satisfied except for the fact that F is not exactly a (κ_n, μ)-tree in $M_{n-1}[h_{n+1}][H^*]$ because the filter h_{n+1} may add sets in $[\mu]^{<\kappa_n}$. However, the poset $j(\mathbb{P}_{n+1})$ is κ_n-c.c. in $M_{n-1}[H^*]$, so we can say that F covers a (κ_n, μ)-tree F^* in $M_{n-1}[h_{n+1}][H^*]$. If $b \in M_{n-1}[h_{n+1}][H_n]$, then b is a cofinal branch for F^*. Then, by Lemma **??**, we have

$$b \notin M_{n-1}[h_{n+1}][H_n] = M_{n-1}[H_n][h_{n+1}].$$

F^* is no longer an $(\aleph_{n+1}, \mu)$-tree in $M_{n-1}[H_n][h_{n+1}]$. However, we obtained this model by forcing with $j(\mathbb{Q}_n)/H^*$ which is $\aleph_{n+1}$-c.c. in $M_{n-1}[h_{n+1}][H^*]$, this means that F^* covers an $(\aleph_{n+1}, \mu)$-tree that we can rename F. Consider $j(\mathbb{Q}_{n+1})/h_{n+1}$, by Lemma **??**, this is an $\aleph_{n+1}$-closed forcing in $M_{n-1}[H_n][h_{n+1}]$, where $2^{\aleph_n} \geq j(\kappa_n) = \aleph_{n+2}$. By the First Preservation Theorem, we have

$$b \notin M_{n-1}[H_n][H_{n+1}].$$

We continue our analysis by working with $j(\mathbb{Q}_{n+2})$ which is a projection of $j(\mathbb{P}_{n+2}) \times j(\mathbb{U}_{n+2})$. This poset is $\aleph_{n+2}$-closed in $M_{n-1}[H_n][H_{n+1}]$ and F is an $(\aleph_{n+1}, \mu)$-tree. By the First Preservation Theorem, we have $b \notin M_{n-1}[H_n][H_{n+1}][h_{n+1} \times u_{n+1}]$, in particular

$$b \notin M_{n-1}[H_n][H_{n+1}][H_{n+2}].$$

Finally $j(\mathrm{Tail}_{n+3})$ is $\aleph_{n+2}$-closed in $M_{n-1}[H_n][H_{n+1}]$, where F is still an $(\aleph_{n+1}, \mu)$-tree. By applying again the First Preservation Theorem, we get that

$$b \notin M_{n-1}[H_n][H_{n+1}][H_{n+2}][H_{tail(n+3)}] = \mathscr{M}_2,$$

that leads to a contradiction and completes the proof of the theorem. □

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