STRONG TREE PROPERTIES FOR TWO SUCCESSIVE CARDINALS

LAURA FONTANELLA

Abstract. An inaccessible cardinal κ is supercompact when (κ, λ)-ITP holds for all $\lambda \geq \kappa$. We prove that if there is a model of ZFC with two supercompact cardinals, then there is a model of ZFC where simultaneously $(\aleph_2, \mu)$-ITP and $(\aleph_3, \mu')$-ITP hold, for all $\mu \geq \aleph_2$ and $\mu' \geq \aleph_3$.

1. Introduction

One of the most intriguing research axes in contemporary set theory is the investigation into those properties which are typically associated with large cardinals, though they can be satisfied by small cardinals as well. The tree property is a principle of that sort. Given a regular cardinal κ, we say that κ satisfies the *tree property* when every κ-tree has a cofinal branch. The result presented in the present paper concerns the so-called *strong tree property* and *super tree property*, which are two combinatorial principles that generalize the usual tree property. The definition of those properties will be presented in §3, for now let us just discuss some general facts about their connection with large cardinals. We know that an inaccessible cardinal is weakly compact if, and only if, it satisfies the tree property. The strong and the super tree properties provide a similar characterization for strongly compact and supercompact cardinals, indeed an inaccessible cardinal is strongly compact if, and only if, it satisfies the strong tree property, while it is supercompact if, and only if, it satisfies the super tree property (the former result follows from a theorem by Jech [3], the latter is due to Magidor [7]). In other words, when a cardinal satisfies one of the previous properties, it "behaves like a large cardinal".

While the previous characterizations date back to the early 1970s, a systematic study of the strong and the super tree properties has only recently been undertaken by Weiss (see [11] and [12]). He proved in [12] that for every $n \geq 2$, one can define a model of the super tree property for $\aleph_n$, starting from a model with a supercompact cardinal. By working on the super tree property at $\aleph_2$, Viale and Weiss (see [10] and [9]) obtained new results about the consistency strength of the Proper Forcing Axiom.

2010 *Mathematics Subject Classification.* 03E55 .

Key words and phrases. tree property, large cardinals, forcing.

They proved that if one forces a model of PFA using a forcing that collapse κ to ω_2 and satisfies the κ-covering and the κ-approximation properties, then κ has to be strongly compact; if the forcing is also proper, then κ is supercompact. Since every known forcing producing a model of PFA by collapsing κ to ω_2 satisfies those conditions, we can say that the consistency strength of PFA is, reasonably, a supercompact cardinal.

It is natural to ask whether two small cardinals can simultaneously have the strong or the super tree properties. Abraham defined in [1] a forcing construction producing a model of the tree property for $\aleph_2$ and $\aleph_3$, starting from a model of ZFC + GCH with a supercompact cardinal and a weakly compact cardinal above it. Cummings and Foreman [2] proved that if there is a model of set theory with infinitely many supercompact cardinals, then one can obtain a model in which every $\aleph_n$ with $n \geq 2$ satisfies the tree property. In the present paper, we construct a model of set theory in which $\aleph_2$ and $\aleph_3$ simultaneously satisfy the super tree property, starting from a model of ZFC with two supercompact cardinals $\kappa < \lambda$. We will collapse κ to $\aleph_2$ and λ to $\aleph_3$, in such a way that they will still satisfy the super tree property. The definition of the forcing construction required for that theorem is motivated by Abraham [1] and Cummings-Foreman [2]. We also conjecture that in the model defined by Cummings and Foreman, every $\aleph_n$ (with $n \geq 2$) satisfies the super tree property.

The paper is organized as follows. In §3 we introduce the strong and the super tree properties. §4 is devoted to the proof of two preservation theorems. In §5, §6 and §7 we define the forcing notion required for the final theorem and we discuss some properties of that forcing. Finally, the proof of the main theorem is developed in §8.

2. Preliminaries and Notation

Given a forcing $\mathbb{P}$ and conditions $p, q \in \mathbb{P}$, we use $p \leq q$ in the sense that p is stronger than q; we write $p||q$ when p and q are two compatible conditions (i.e. there is a condition $r \in \mathbb{P}$ such that $r \leq p$ and $r \leq q$). A poset $\mathbb{P}$ is *separative* if whenever $q \not\leq p$, then some extension of q in $\mathbb{P}$ is incompatible with p. Every partial order can be turned into a separative poset. Indeed, one can define $p \prec q$ iff all extensions of p are compatible with q, then the resulting equivalence relation, given by $p \sim q$ iff $p \prec q$ and $q \prec p$, provides a separative poset; we denote by $[p]$ the equivalence class of p.

A forcing $\mathbb{P}$ is *κ-closed* if, and only if, every descending sequence of conditions of $\mathbb{P}$ of size less than κ has a lower bound; $\mathbb{P}$ is *κ-directed closed* if, and only if, for every set of less than κ pairwise compatible conditions of $\mathbb{P}$ has a lower bound. We say that $\mathbb{P}$ is $< \kappa$-distributive if, and only if, no sequence of ordinals of length less than κ is added by $\mathbb{P}$. $\mathbb{P}$ is κ-c.c. when every antichain of $\mathbb{P}$ has size less than κ; $\mathbb{P}$ is κ-Knaster if, and only if, for all sequence of conditions $\langle p_\alpha;\ \alpha < \kappa\rangle$, there is $X \subseteq \kappa$ cofinal such that the conditions of the sequence $\langle p_\alpha;\ \alpha \in X\rangle$ are pairwise compatible.

Given two forcings $\mathbb{P}$ and $\mathbb{Q}$, we will write $\mathbb{P} \equiv \mathbb{Q}$ when $\mathbb{P}$ and $\mathbb{Q}$ are equivalent, namely:

1. for every filter $G_{\mathbb{P}} \subseteq \mathbb{P}$ which is generic over V, there exists a filter $G_{\mathbb{Q}} \subseteq \mathbb{Q}$ which is generic over V and $V[G_{\mathbb{P}}] = V[G_{\mathbb{Q}}]$;
2. for every filter $G_{\mathbb{Q}} \subseteq \mathbb{Q}$ which is generic over V, there exists a filter $G_{\mathbb{P}} \subseteq \mathbb{P}$ which is generic over V and $V[G_{\mathbb{P}}] = V[G_{\mathbb{Q}}]$.

If $\mathbb{P}$ is any forcing and $\dot{\mathbb{Q}}$ is a $\mathbb{P}$-name for a forcing, then we denote by $\mathbb{P} * \dot{\mathbb{Q}}$ the poset $\{(p, q);\ p \in \mathbb{P}, q \in V^{\mathbb{P}} \text{ and } p \Vdash q \in \dot{\mathbb{Q}}\}$, where for every $(p, q), (p', q') \in \mathbb{P} * \dot{\mathbb{Q}}$, $(p, q) \leq (p', q')$ if, and only if, $p \leq p'$ and $p \Vdash q \leq q'$.

If $\mathbb{P}$ and $\mathbb{Q}$ are two posets, a *projection* $\pi : \mathbb{Q} \to \mathbb{P}$ is a function such that:

1. for all $q, q' \in \mathbb{Q}$, if $q \leq q'$, then $\pi(q) \leq \pi(q')$;
2. $\pi(1_{\mathbb{Q}}) = 1_{\mathbb{P}}$;
3. for all $q \in \mathbb{Q}$, if $p \leq \pi(q)$, then there is $q' \leq q$ such that $\pi(q') \leq p$.

We say that $\mathbb{P}$ *is a projection of* $\mathbb{Q}$ when there is a projection $\pi : \mathbb{Q} \to \mathbb{P}$.
If $\pi : \mathbb{Q} \to \mathbb{P}$ is a projection and $G_{\mathbb{P}} \subseteq \mathbb{P}$ is a generic filter over V, define

$$\mathbb{Q}/G_{\mathbb{P}} := \{q \in \mathbb{Q};\ \pi(q) \in G_{\mathbb{P}}\},$$

$\mathbb{Q}/G_{\mathbb{P}}$ is ordered as a subposet of $\mathbb{Q}$. The following hold:

1. If $G_{\mathbb{Q}} \subseteq \mathbb{Q}$ is a generic filter over V and $H := \{p \in \mathbb{P};\ \exists q \in G_{\mathbb{Q}}(\pi(q) \leq p)\}$, then H is $\mathbb{P}$-generic over V;
2. if $\mathbb{G}_{\mathbb{P}} \subseteq \mathbb{P}$ is a generic filter over V, and if $G \subseteq \mathbb{Q}/G_{\mathbb{P}}$ is a generic filter over $V[\mathbb{G}_{\mathbb{P}}]$, then G is $\mathbb{Q}$-generic over V, and $\pi[G]$ generates $G_{\mathbb{P}}$;
3. if $G_{\mathbb{Q}} \subseteq \mathbb{Q}$ is a generic filter over V, and $H := \{p \in \mathbb{P};\ \exists q \in G_{\mathbb{Q}}(\pi(q) \leq p)\}$, then $\mathbb{G}_{\mathbb{Q}}$ is $\mathbb{Q}/G_{\mathbb{P}}$-generic over $V[H]$. That is, we can factor forcing with $\mathbb{Q}$ as forcing with $\mathbb{P}$ followed by forcing with $\mathbb{Q}/G_{\mathbb{P}}$ over $V[\mathbb{G}_{\mathbb{P}}]$.

Some of our projections $\pi : \mathbb{Q} \to \mathbb{P}$ will also have the following property: for all $p \leq \pi(q)$, there is $q' \leq q$ such that

1. $\pi(q') = p$,
2. for every $q^* \leq q$, if $\pi(q^*) \leq p$, then $q^* \leq q'$.

We denote by $\mathrm{ext}(q, p)$ any condition like q' above (if a condition q'' satisfies the previous properties, then $q' \leq q'' \leq q'$). In this case, if $G_{\mathbb{P}} \subseteq \mathbb{P}$ is a generic filter over V, we can define an ordering on $\mathbb{Q}/G_{\mathbb{P}}$ as follows: $p \leq^* q$ if, and only if, there is $r \leq \pi(p)$ such that $r \in G_{\mathbb{P}}$ and $\mathrm{ext}(p, r) \leq q$. Then, forcing over $V[G_{\mathbb{P}}]$ with $\mathbb{Q}/G_{\mathbb{P}}$ ordered as a subposet of $\mathbb{Q}$, is equivalent to forcing over $V[G_{\mathbb{P}}]$ with $(\mathbb{Q}/G_{\mathbb{P}}, \leq^*)$.

Let κ be a regular cardinal and λ an ordinal, we denote by $\mathrm{Add}(\kappa, \lambda)$ the poset of all partial functions $f : \lambda \to 2$ of size less than κ, ordered by reverse inclusion. We use $\mathrm{Add}(\kappa)$ to denote $\mathrm{Add}(\kappa, \kappa)$.

If $V \subseteq W$ are two models of set theory with the same ordinals and η is a cardinal in W, we say that (V, W) has the η-covering property if, and only if, every set $X \subseteq V$ in W of cardinality less than η in W, is contained in a set $Y \in V$ of cardinality less than η in V.

Assume that $\mathbb{P}$ is a forcing notion, we will use $\langle\mathbb{P}\rangle$ to denote the canonical $\mathbb{P}$-name for a $\mathbb{P}$-generic filter over V.

Lemma 2.1. *(Easton's Lemma) Let κ be regular. If $\mathbb{P}$ has the κ-chain condition and $\mathbb{Q}$ is κ-closed, then*

(1) $\Vdash_{\mathbb{Q}} \mathbb{P}$ *has the κ-chain condition*;
(2) $\Vdash_{\mathbb{P}} \mathbb{Q}$ *is a $< \kappa$-distributive*;
(3) *If G is $\mathbb{P}$-generic over V and H is $\mathbb{Q}$-generic over V, then G and H are mutually generic;*
(4) *If G is $\mathbb{P}$-generic over V and H is $\mathbb{Q}$-generic over V, then $(V, V[G][H])$ has the κ-covering property;*
(5) *If $\mathbb{R}$ is κ-closed, then $\Vdash_{\mathbb{P}\times\mathbb{Q}}$ $\mathbb{R}$ is $< \kappa$-distributive.*

For a proof of that lemma see [2, Lemma 2.11].

Let η be a regular cardinal, $\theta > \eta$ be large enough and $M \prec H_\theta$ of size η. We say that M is *internally approachable of length* η if it can be written as the union of an increasing continuous chain $\langle M_\xi : \xi < \eta\rangle$ of elementary submodels of $H(\theta)$ of size less than η, such that $\langle M_\xi : \xi < \eta'\rangle \in M_{\eta'+1}$, for every ordinal $\eta' < \eta$.

Lemma 2.2. *(Δ-system Lemma) Assume that λ is a regular cardinal and $\kappa < \lambda$ is such that $\alpha^{<\kappa} < \lambda$, for every $\alpha < \lambda$. Let $\mathscr{F}$ be a family of sets of cardinality less than κ such that $|\mathscr{F}| = \lambda$. There exists a family $\mathscr{F}' \subseteq \mathscr{F}$ of size λ and a set R such that $X \cap Y = R$, for any two distinct $X, Y \in \mathscr{F}'$.*

For a proof of that lemma see [5].

Lemma 2.3. *(Pressing Down Lemma) If f is a regressive function on a stationary set $S \subseteq [A]^{<\kappa}$ (i.e. $f(x) \in x$, for every non empty $x \in S$), then there exists a stationary set $T \subseteq S$ such that f is constant on T.*

For a proof of that lemma see [5].

We will assume familiarity with the theory of large cardinals and elementary embeddings, as developed for example in [4].

Lemma 2.4. *(Laver)* [6] *If κ is a supercompact cardinal, then there exists $L : \kappa \to V_\kappa$ such that: for all λ, for all $x \in H_{\lambda^+}$, there is $j : V \to M$ such that $j(\kappa) > \lambda$, ${}^\lambda M \subseteq M$ and $j(L)(\kappa) = x$.*

Lemma 2.5. *(Silver) Let $j : M \to N$ be an elementary embedding between inner models of* ZFC. *Let $\mathbb{P} \in M$ be a forcing and suppose that G is $\mathbb{P}$-generic over M, H is $j(\mathbb{P})$-generic over N, and $j[G] \subseteq H$. Then, there is a unique $j^* : M[G] \to N[H]$ such that $j^* \upharpoonright M = j$ and $j^*(G) = H$.*

Proof. If $j[G] \subseteq H$, then the map $j^*(\dot{x}^G) = j(\dot{x})^H$ is well defined and satisfies the required properties. □

3. The Strong and the Super Tree Properties

We start by recalling the definition of the tree property, for a regular cardinal κ.

Definition 3.1. *Let κ be a regular cardinal,*

1. *a κ-tree is a tree of height κ with levels of size less than κ;*
2. *we say that κ has the* tree property *if, and only if, every κ-tree has a cofinal branch (i.e. a branch of size κ).*

In order to define the strong tree property and the super tree property for a regular cardinal $\kappa \geq \aleph_2$, we need to define the notion of (κ, λ)-*tree*, for an ordinal $\lambda \geq \kappa$.

Definition 3.2. *Given $\kappa \geq \omega_2$ a regular cardinal and $\lambda \geq \kappa$, a (κ, λ)-*tree *is a set F satisfying the following properties:*

1. *for every $f \in F$, $f : X \to 2$, for some $X \in [\lambda]^{<\kappa}$*
2. *for all $f \in F$, if $X \subseteq \mathrm{dom}(f)$, then $f \upharpoonright X \in F$;*
3. *the set $\mathrm{Lev}_X(F) := \{f \in F;\ \mathrm{dom}(f) = X\}$ is non empty, for all $X \in [\lambda]^{<\kappa}$;*
4. *$|\mathrm{Lev}_X(F)| < \kappa$, for all $X \in [\lambda]^{<\kappa}$.*

When there is no ambiguity, we will simply write Lev_X instead of $\mathrm{Lev}_X(F)$. The main difference between κ-trees and (κ, λ)-trees is the following: in the former, levels are indexed by ordinals, while in the latter, levels are indexed by *sets of ordinals*, which are not totally ordered.

Definition 3.3. *Given $\kappa \geq \omega_2$ a regular cardinal, $\lambda \geq \kappa$, and F a (κ, λ)-tree,*

1. *a* cofinal branch *for F is a function $b : \lambda \to 2$ such that $b \upharpoonright X \in \mathrm{Lev}_X(F)$, for all $X \in [\lambda]^{<\kappa}$;*
2. *an F-*level sequence *is a function $D : [\lambda]^{<\kappa} \to F$ such that for every $X \in [\lambda]^{<\kappa}$, $D(X) \in \mathrm{Lev}_X(F)$;*
3. *given an F-level sequence D, an* ineffable branch *for D is a cofinal branch $b : \lambda \to 2$ such that $\{X \in [\lambda]^{<\kappa};\ b \upharpoonright X = D(X)\}$ is stationary.*

Definition 3.4. *Given $\kappa \geq \omega_2$ a regular cardinal and $\lambda \geq \kappa$,*

1. (κ, λ)-TP *holds if every (κ, λ)-tree has a cofinal branch;*

(2) (κ, λ)-ITP *holds if for every* (κ, λ)*-tree* F *and for every* F*-level sequence* D*, there is an an ineffable branch for* D;
(3) *we say that* κ *satisfies the* strong tree property *if* (κ, μ)-TP *holds, for all* $\mu \geq \kappa$;
(4) *we say that* κ *satisfies the* super tree property *if* (κ, μ)-ITP *holds, for all* $\mu \geq \kappa$;

4. The Preservation Theorems

It will be important in what follows that certain forcings cannot add ineffable branches. The following proposition is due to Silver (see [5, chap. VIII, Lemma 3.4] or [11, Proposition 2.1.12]), we include the proof for completeness.

Theorem 4.1. *(First Preservation Theorem) Let* θ *be a regular cardinal and* $\mu \geq \theta$ *be any ordinal. Assume that* F *is a* (θ, μ)*-tree and* $\mathbb{Q}$ *is an* η^+*-closed forcing with* $\eta < \theta \leq 2^\eta$*. For every filter* $G_{\mathbb{Q}} \subseteq \mathbb{Q}$ *generic over* V*, every cofinal branch for* F *in* $V[G_{\mathbb{Q}}]$ *is already in* V*.*

Proof. We can assume, without loss of generality, that η is minimal such that $2^\eta \geq \theta$. Assume towards a contradiction that $\mathbb{Q}$ adds a cofinal branch to F, let $\dot{b}$ be a $\mathbb{Q}$-name for such a function. For all $\alpha \leq \eta$ and all $s \in {}^{\alpha}2$, we are going to define by induction three objects $a_\alpha \in [\mu]^{<\theta}$, $f_s \in \mathrm{Lev}_{a_\alpha}$ and $p_s \in \mathbb{Q}$ such that:

(1) $p_s \Vdash \dot{b} \upharpoonright a_\alpha = f_s$;
(2) $f_{s^\frown 0}(\beta) \neq f_{s^\frown 1}(\beta)$, for some $\beta < \mu$;
(3) if $s \subseteq t$, then $p_t \leq p_s$;
(4) if $\alpha < \beta$, then $a_\alpha \subset a_\beta$.

Let $\alpha < \eta$, assume that a_α, f_s and p_s have been defined for all $s \in {}^{\alpha}2$. We define $a_{\alpha+1}$, f_s, and p_s, for all $s \in {}^{\alpha+1}2$. Let t be in ${}^{\alpha}2$, we can find an ordinal $\beta_t \in \mu$ and two conditions $p_{t^\frown 0}, p_{t^\frown 1} \leq p_t$ such that $p_{t^\frown 0} \Vdash \dot{b}(\beta_t) = 0$ and $p_{t^\frown 1} \Vdash \dot{b}(\beta_t) = 1$. (otherwise, $\dot{b}$ would be a name for a cofinal branch which is already in V). Let $a_{\alpha+1} := a_\alpha \cup \{\beta_t;\ t \in {}^{\alpha}2\}$, then $|a_{\alpha+1}| < \theta$, because $2^\alpha < \theta$. We just defined, for every $s \in {}^{\alpha+1}2$, a condition p_s. Now, by strengthening p_s if necessary, we can find $f_s \in \mathrm{Lev}_{a_{\alpha+1}}$ such that

$$p_s \Vdash \dot{b} \upharpoonright a_{\alpha+1} = f_s.$$

Finally, $f_{t^\frown 0}(\beta_t) \neq f_{t^\frown 1}(\beta_t)$, for all $t \in {}^{\alpha}2$: because $p_{t^\frown 0} \Vdash f_{t^\frown 0}(\beta_t) = \dot{b}(\beta_t) = 0$, while $p_{t^\frown 1} \Vdash f_{t^\frown 1}(\beta_t) = \dot{b}(\beta_t) = 1$.

If α is a limit ordinal $\leq \eta$, let t be any function in ${}^{\alpha}2$. Since $\mathbb{Q}$ is η^+-closed, there is a condition p_t such that $p_t \leq p_{t \upharpoonright \beta}$, for all $\beta < \alpha$. Define $a_\alpha := \bigcup_{\beta < \alpha} a_\beta$. By strengthening p_t if necessary, we can find $f_t \in \mathrm{Lev}_{a_\alpha}$ such that $p_t \Vdash \dot{b} \upharpoonright a_\alpha = f_t$. That completes the construction.

We show that $|\mathrm{Lev}_{a_\eta}| \geq {}^{\eta}2 \geq \theta$, thus a contradiction is obtained. Let $s \neq t$ be two functions in ${}^{\eta}2$, we are going to prove that $f_s \neq f_t$. Let α be the minimum ordinal less than η such that $s(\alpha) \neq t(\alpha)$, without loss of generality $r \frown 0 \sqsubset s$ and $r \frown 1 \sqsubset t$, for some $r \in {}^{\alpha}2$. By construction,

$$p_s \leq p_{r\frown 0} \Vdash \dot{b} \restriction a_{\alpha+1} = f_{r\frown 0} \text{ and } p_t \leq p_{r\frown 1} \Vdash \dot{b} \restriction a_{\alpha+1} = f_{r\frown 1},$$

where $f_{r\frown 0}(\beta) \neq f_{r\frown 1}(\beta)$, for some β. Moreover, $p_s \Vdash \dot{b} \restriction a_\eta = f_s$ and $p_t \Vdash \dot{b} \restriction a_\eta = f_t$, hence $f_s \restriction a_{\alpha+1}(\beta) = f_{r\frown 0}(\beta) \neq f_{r\frown 1}(\beta) = f_t \restriction a_{\alpha+1}(\beta)$, thus $f_s \neq f_t$. That completes the proof. □

The following proposition is rather ad hoc. It will be used several times in the final theorem.

Theorem 4.2. *(Second Preservation Theorem) Let $V \subseteq W$ be two models of set theory with the same ordinals and let $\mathbb{P} \in V$ be a forcing notion and κ a cardinal in V such that:*

1. $\mathbb{P} \subseteq \mathrm{Add}(\aleph_n, \tau)^V$, *for some* $\tau > \aleph_n$, *and for every* $p \in \mathbb{P}$, *if* $X \subseteq \mathrm{dom}(p)$, *then* $p \restriction X \in \mathbb{P}$;
2. $\aleph_m^V = \aleph_m^W$, *for every* $m \leq n$, *and* $W \models \kappa = \aleph_{n+1}$;
3. *for every set* $X \subseteq V$ *in* W *of size* $< \aleph_{n+1}$ *in* W, *there is* $Y \in V$ *of size* $< \kappa$ *in* V, *such that* $X \subseteq Y$;
4. *in* V, *we have* $\gamma^{<\aleph_n} < \kappa$, *for every cardinal* $\gamma < \kappa$.

Let $F \in W$ be a $(\aleph_{n+1}, \mu)$-tree with $\mu \geq \aleph_{n+1}$, then for every filter $G_{\mathbb{P}} \subseteq \mathbb{P}$ generic over W, every cofinal branch for F in $W[G_{\mathbb{P}}]$ is already in W.

Proof. Work in W. Let $\dot{b} \in W^{\mathbb{P}}$ and let $p \in \mathbb{P}$ such that

$$p \Vdash \dot{b} \text{ is a cofinal branch for } F.$$

We are going to find a condition $q \in \mathbb{P}$ such that $q || p$ and for some $b \in W$, we have $q \Vdash \dot{b} = b$. Let χ be large enough, for all $X \prec H_\chi$ of size $\aleph_n$, we fix a condition $p_X \leq p$ and a function $f_X \in Lev_{X\cap\mu}$ such that

$$p_X \Vdash \dot{b} \restriction X = f_X.$$

Let S be the set of all the structures $X \prec H_\chi$, such that X is internally approachable of length $\aleph_n$. Since every condition of $\mathbb{P}$ has size less than $\aleph_n$, there is, for all $X \in S$, a set $M_X \in X$ of size less than $\aleph_n$ such that

$$p_X \restriction X \subseteq M_X.$$

By the Pressing Down Lemma, there exists M^* and a stationary set $E^* \subseteq S$ such that $M^* = M_X$, for all $X \in E^*$. The set M^* has size less than $\aleph_n$ in W, hence $A := (\bigcup_{X\in E^*} p_X) \restriction M^*$ has size less than $\aleph_n$ in W. By the assumption, A is covered by some $N \in V$ of size $\gamma < \kappa$ in V. In V, we have $|[N]^{<\aleph_n}| \leq \gamma^{<\aleph_n} < \kappa$. It follows that in W there are less than $\aleph_{n+1}$ possible values for $p_X \restriction M^*$. Therefore, we can find in W a cofinal $E \subseteq E^*$ and a condition $q \in \mathbb{P}$, such that $p_X \restriction X = q$, for all $X \in E$.

Claim 4.3. *$f_X \restriction Y = f_Y \restriction X$, for all $X, Y \in E$.*

Proof. Let $X, Y \in E$, there is $Z \in E$ with $X, Y, \mathrm{dom}(p_X), \mathrm{dom}(p_Y) \subseteq Z$. Then, we have $p_X \cap p_Z = p_X \cap (p_Z \restriction Z) = p_X \cap q = q$, thus $p_X || p_Z$ and similarly $p_Y || p_Z$. Let $r \leq p_X, p_Z$ and $s \leq p_Y, p_Z$, then $r \Vdash f_Z \restriction X = \dot{b} \restriction X = f_X$ and $s \Vdash f_Z \restriction Y = \dot{b} \restriction Y = f_Y$. It follows that $f_X \restriction Y = f_Z \restriction (X \cap Y) = f_Y \restriction X$. □

Let b be $\bigcup_{X \in E} f_X$. The previous claim implies that b is a function and

$$b \restriction X = f_X, \text{ for all } X \in E.$$

Claim 4.4. $q \Vdash \dot{b} = b$.

Proof. We show that for every $X \in E$, the set $B_X := \{s \in \mathbb{P};\ s \Vdash \dot{b} \restriction X = b \restriction X\}$ is dense below q. Let $r \leq q$, there is $Y \in E$ such that $\mathrm{dom}(r), X \subseteq Y$. It follows that $p_Y \cap r = p_Y \restriction Y \cap r = q \cap r = q$, thus $p_Y || r$. Let $s \leq p_Y, r$, then $s \in B_X$, because $s \Vdash \dot{b} \restriction X = f_Y \restriction X = f_X = b \restriction X$. Since $\bigcup \{X \cap \mu; X \in E\} = \mu$, we have $q \Vdash \dot{b} = b$. □

That completes the proof. □

5. The Main Forcing

Definition 5.1. *Let η be a regular cardinal and let θ be any ordinal, we define*

$$\mathbb{P}(\eta, \theta) := \{p \in \mathrm{Add}(\eta, \theta);\ \textit{for every } \alpha \in \mathrm{dom}(p),\ \alpha \textit{ is a successor ordinal }\},$$

$\mathbb{P}(\eta, \theta)$ is ordered by reverse inclusion.

For $E \subseteq \theta$, we denote by $P(\eta, \theta) \restriction E$ the set of all functions in $P(\eta, \theta)$ with domain a subset of E. The following definition is due to Abraham [1].

Definition 5.2. *Assume that $V \subseteq W$ are two models of set theory with the same ordinals, let η be a regular cardinal in W and let $\mathbb{P} := \mathbb{P}(\eta, \theta)^V$, where θ is any ordinal. We define in W the poset $\mathbb{M}(\eta, \theta, V, W)$ as follows:*

$(p, q) \in \mathbb{M}(\eta, \theta, V, W)$ if, and only if,

1. *$p \in \mathbb{P}(\eta, \theta)^V$;*
2. *$q \in W$ is a partial function on θ of size $\leq \eta$ such that for every $\alpha \in \mathrm{dom}(q)$, α is a successor ordinal, $q(\alpha) \in W^{\mathbb{P} \restriction \alpha}$, and $\Vdash^W_{\mathbb{P} \restriction \alpha} q(\alpha) \in \mathrm{Add}(\eta^+)^{V[\langle \mathbb{P} \restriction \alpha \rangle]}$.*

$\mathbb{M}(\eta, \theta, V, W)$ is partially ordered by $(p, q) \leq (p', q')$ if, and only if,

1. *$p \leq p'$;*
2. *$\mathrm{dom}(q') \subseteq \mathrm{dom}(q)$;*
3. *$p \restriction \alpha \Vdash^W_{P \restriction \alpha} q(\alpha) \leq q'(\alpha)$, for all $\alpha \in \mathrm{dom}(q')$.*

If θ is a weakly compact cardinal, then $\mathbb{M}(\aleph_n, \theta, V, V)$ corresponds to the forcing defined by Mitchell for a model of the tree property at $\aleph_{n+2}$ (see [8]). Weiss proved that a variation of that forcing with θ supercompact, produces a model of the super tree property for $\aleph_{n+2}$. Let us discuss a naive attempt to build a model of the super tree property for two successive cardinals $\aleph_n$, $\aleph_{n+1}$ (with $n \geq 2$). We start with two supercompact cardinals $\kappa < \lambda$ in a model V, then we force with $\mathbb{M}(\aleph_{n-2}, \kappa, V, V)$ over V obtaining a model W; finally, we force over W with $\mathbb{M}(\aleph_{n-1}, \lambda, W, W)$. The problem with this approach is that the second stage might introduce an $(\aleph_n, \mu)$-tree F with no cofinal branches. Therefore, we have to define the first stage of the iteration so that it will make the super tree property at $\aleph_n$ "indestructible". The forcing notion required for that will "anticipate a fragment" of the forcing at the second stage, namely $\mathbb{M}(\aleph_{n-1}, \lambda, W, W)$.

Definition 5.3. *For V, W and η, θ like in Definition 5.2, we define*

$$\mathbb{Q}^*(\eta, \theta, V, W) := \{(\emptyset, q);\ (\emptyset, q) \in \mathbb{M}(\eta, \theta, V, W)\}.$$

The poset defined hereafter is a variation of the forcing construction defined by Abraham in [1, Definition 2.14].

Definition 5.4. *Let V be a model of set theory, and suppose that $\theta > \aleph_n$ is an inaccessible cardinal. Let $\mathbb{P} := \mathbb{P}(\aleph_n, \theta)^V$ and let $L : \theta \to V_\theta$ be any function. Define*

$$\mathbb{R} := \mathbb{R}(\aleph_n, \theta, L)$$

as follows. For each $\beta \leq \theta$, we define by induction $\mathbb{R} \restriction \beta$ and then we set $\mathbb{R} = \mathbb{R} \restriction \theta$.

$\mathbb{R} \restriction 0$ *is the trivial forcing.*

$(p, q, f) \in \mathbb{R} \restriction \beta$ *if, and only if*

1. $p \in \mathbb{P} \restriction \beta (= \mathbb{P}(\aleph_n, \beta)^V)$;
2. *q is a partial function on β of size $\leq \aleph_n$, such that for every $\alpha \in \mathrm{dom}(q)$, α is a successor ordinal, $q(\alpha) \in V^{\mathbb{P}\restriction\alpha}$ and $\Vdash_{\mathbb{P}\restriction\alpha} q(\alpha) \in \mathrm{Add}(\aleph_{n+1})^{V[\langle \mathbb{P}\restriction\alpha\rangle]}$*;
3. *f is a partial function on β of size $\leq \aleph_n$ such that for all $\alpha \in \mathrm{dom}(f)$, α is a limit ordinal, $f(\alpha) \in V^{\mathbb{R}\restriction\alpha}$ and*

$$\Vdash_{\mathbb{R}\restriction\alpha} L(\alpha) \text{ is an ordinal such that } f(\alpha) \in \mathbb{Q}^*(\aleph_{n+1}^{V[\langle \mathbb{R}\restriction\alpha\rangle]}, L(\alpha), V, V[\langle \mathbb{R} \restriction \alpha\rangle]).$$

$\mathbb{R} \restriction \beta$ *is partially ordered by $(p, q, f) \leq (p', q', f')$ if, and only if:*

1. $p \leq p'$;
2. $\mathrm{dom}(q') \subseteq \mathrm{dom}(q)$;
3. $p \restriction \alpha \Vdash_{P\restriction\alpha} q(\alpha) \leq q'(\alpha)$, *for all $\alpha \in \mathrm{dom}(q')$.*
4. $\mathrm{dom}(f') \subseteq \mathrm{dom}(f)$;
5. *for all $\alpha \in \mathrm{dom}(f')$, if $(p, q, f) \restriction \alpha := (p \restriction \alpha, q \restriction \alpha, f \restriction \alpha)$, then*

$$(p, q, f) \restriction \alpha \Vdash_{\mathbb{R}\restriction\alpha} f(\alpha) \leq f'(\alpha)$$

Assume that V is a model of ZFC with two supercompact cardinals $\kappa < \lambda$, and $L : \kappa \to V_\kappa$ is the Laver function. Let $\mathbb{R} := \mathbb{R}(\aleph_0, \kappa, L)$ and let $G_{\mathbb{R}} \subseteq \mathbb{R}$ be any generic filter over V. Assume that $G_{\mathbb{M}}$ is an $\mathbb{M}(\aleph_1, \lambda, V, V[G_{\mathbb{R}}])$-generic filter over $V[G_{\mathbb{R}}]$, we will prove in §8 that both $\aleph_2$ and $\aleph_3$ satisfy the super tree property in $V[G_{\mathbb{R}}][G_{\mathbb{M}}]$.

6. Factoring Mitchell's Forcing

In this section, V, W, η, θ are like in Definition 5.2. None of the result of this section are due to the author. For more details see [1].

Remark 6.1. *The function* $\pi : \mathbb{M}(\eta,\theta,V,W) \to \mathbb{P}(\eta,\theta)^V$ *defined by* $\pi(p,q) := p$ *is a projection. If* $\mathbb{P} := \mathbb{P}(\eta,\theta)^V$ *and if* $G_{\mathbb{P}}$ *is a* $\mathbb{P}$*-generic filter over* W*, then we define in* $W[G_{\mathbb{P}}]$ *the poset*

$$\mathbb{Q}(\eta,\theta,V,W,G_{\mathbb{P}}) := \mathbb{M}(\eta,\theta,V,W)/G_{\mathbb{P}}.$$

Lemma 6.2. *The function* $\sigma : \mathbb{P}(\eta,\theta)^V \times \mathbb{Q}^*(\eta,\theta,V,W) \to \mathbb{M}(\eta,\theta,V,W)$ *defined by* $\sigma(p,(\emptyset,q)) := (p,q)$ *is a projection. If* $G_{\mathbb{M}}$ *is a* W*-generic filter over* $\mathbb{M}(\eta,\theta,V,W)$*, then we define in* $W[G_{\mathbb{M}}]$ *the poset:*

$$\mathbb{S}(\eta,\theta,V,W,G_{\mathbb{M}}) := (\mathbb{P}(\eta,\theta)^V \times \mathbb{Q}^*(\eta,\theta,V,W))/G_{\mathbb{M}}.$$

Proof. Let $\mathbb{P} := \mathbb{P}(\eta,\theta)^V$ and $\mathbb{Q}^* := \mathbb{Q}^*(\eta,\theta,V,W)$. It is clear that σ preserves the identity and respect the ordering relation. Let $(p',q') \leq \sigma(p,(\emptyset,q))$. Define q^* as follows: $\mathrm{dom}(q^*) = \mathrm{dom}(q')$ and for $\alpha \in \mathrm{dom}(q')$, if $\alpha \notin \mathrm{dom}(q)$, then $q^*(\alpha) := q'(\alpha)$; if $\alpha \in \mathrm{dom}(q)$, we define $q^*(\alpha) \in W^{\mathbb{P}\restriction\alpha}$ such that the following hold:

(1) $p' \restriction \alpha \Vdash q^*(\alpha) = q'(\alpha)$,
(2) if $r \in \mathbb{P} \restriction \alpha$ is incompatible with $p' \restriction \alpha$, then $r \Vdash q^*(\alpha) = q(\alpha)$.

So $\Vdash^W_{\mathbb{P}\restriction\alpha} q^*(\alpha) \leq q(\alpha)$, hence $(p',(\emptyset,q^*)) \leq (p,(\emptyset,q))$ in $\mathbb{P}\times\mathbb{Q}^*$ and $\sigma(p',(\emptyset,q^*)) = (p',q^*)$. Moreover $[(p',q^*)] = [(p',q')]$, that completes the proof. ☐

Lemma 6.3. $\mathbb{Q}^*(\eta,\theta,V,W)$ *is* η^+*-directed closed in* W*.*

Proof. See [1] for a proof of that lemma. ☐

Lemma 6.4. *Assume that* $\mathbb{P} := \mathbb{P}(\eta,\theta)^V$ *is* η^+*-cc in* W*, for every filter* $G_{\mathbb{M}} \subseteq \mathbb{M}(\eta,\theta,V,W)$ *generic over* W*, if* $G_{\mathbb{P}} \subseteq \mathbb{P}$ *is the projection of* $G_{\mathbb{M}}$ *to* $\mathbb{P}$*, then all sets of ordinals in* $W[G_{\mathbb{M}}]$ *of size* η *are in* $W[G_{\mathbb{P}}]$*.*

Proof. By Lemma 6.2 it is enough to prove that if $G_{\mathbb{P}} \times G_{\mathbb{Q}} \subseteq \mathbb{P}\times\mathbb{Q}^*(\eta,\theta,V,W)$ is a generic filter over W, then every set of ordinals in $W[G_{\mathbb{P}} \times G_{\mathbb{Q}}]$ of size η is already in $W[G_{\mathbb{P}}]$. This is an easy consequence of Easton's Lemma. ☐

Proposition 6.5. *Assume that* θ *is inaccessible in* W *and let* $\mathbb{M} := \mathbb{M}(\eta,\theta,V,W)$*. The following hold:*

(i) $|\mathbb{M}| = \theta$ *and* $\mathbb{M}$ *is* θ*-c.c.;*
(ii) *If* $\mathbb{P}(\eta,\theta)^V$ *is* η^+*-cc in* W*, then* $\mathbb{M}$ *preserves* η^+*;*
(iii) *If* $\mathbb{P}(\eta,\theta)^V$ *is* η^+*-c.c. in* W*, then* $\mathbb{M}$ *makes* $\theta = \eta^{++} = 2^\eta$*.*

Proof. *(i)* The proof that $|\mathbb{M}| = \theta$ is omitted. The key point is that κ is inaccessible, so $\mathbb{P}(\eta,\theta)$ has size θ and for every $(p,q) \in \mathbb{M}$, there are fewer than θ possibilities for $q(\alpha)$. The proof that $\mathbb{M}$ is θ-c.c. is a standard application of the Δ-system Lemma.

(ii) It follows from Lemma 6.4.

(*iii*) For every cardinal $\alpha \in]\eta, \theta[$, $\mathbb{M}$ projects to $\mathbb{P}(\eta, \alpha)^V$ which makes $2^\eta \geq \alpha$ and then adds a Cohen subset of η^+. That forcing will collapse α to η^+. By the previous claims, η^+ is preserved and θ remains a cardinal after forcing with $\mathbb{M}$. So, $\mathbb{M}$ makes $\theta = \eta^{++}$. □

Lemma 6.6. *The following hold:*

1. *Assume that* $\mathbb{P} := \mathbb{P}(\eta, \theta)^V$. *If* $\mathbb{P}$ *adds no new* $< \eta$ *sequences to* W, *then*
$$\Vdash_{\mathbb{P}}^W \mathbb{Q}(\eta, \theta, V, W, \langle \mathbb{P} \rangle) \text{ is } \eta\text{-closed};$$
2. *Assume that* $\mathbb{P} := \mathbb{P}(\eta, \theta)^V$ *and* $\mathbb{M} := \mathbb{M}(\eta, \theta, V, W)$. *If* $\mathbb{P}$ *adds no new* $< \eta$ *sequences to* W, *then* $\Vdash_{\mathbb{M}}^W \mathbb{S}(\eta, \theta, V, W, \langle \mathbb{M} \rangle)$ *is* η*-closed.*

Proof. See [1]. □

For any ordinal $\alpha \in]\eta, \theta[$, the function $(p, q) \mapsto (p \upharpoonright \alpha, q \upharpoonright \alpha)$ is a projection from $\mathbb{M}(\eta, \theta, V, W)$ to $\mathbb{M}_\alpha := \mathbb{M}(\eta, \alpha, V, W)$. We want to analyse
$$\mathbb{M}(\eta, \theta, V, W)/G_{\mathbb{M}_\alpha},$$
where $G_{\mathbb{M}_\alpha} \subseteq \mathbb{M}_\alpha$ is any generic filter over W. Consider the following definition.

Definition 6.7. *Let* $\theta' \in]\eta, \theta[$ *be any ordinal and let* $\mathbb{P} := \mathbb{P}(\eta, \theta)^V$. *Let* $\mathbb{M}_{\theta'} := \mathbb{M}(\eta, \theta', V, W)$ *and assume that* $G_{\mathbb{M}_{\theta'}} \subseteq \mathbb{M}_{\theta'}$ *is any generic filter over* W, *then we define in* $W' := W[G_{\mathbb{M}_{\theta'}}]$, *the following poset* $\mathbb{M}(\eta, \theta - \theta', V, W')$.

$(p, q) \in \mathbb{M}(\eta, \theta - \theta', V, W')$ *if, and only if,*

1. $p \in \mathbb{P} \upharpoonright (\theta - \theta')$;
2. $q \in W'$ *is a partial function on* $]\theta', \theta[$ *of size* $\leq \eta$ *such that for every* $\alpha \in \mathrm{dom}(q)$, α *is a successor ordinal,* $q(\alpha) \in (W')^{\mathbb{P} \upharpoonright (\alpha - \theta')}$, *and*
$$\Vdash_{\mathbb{P} \upharpoonright (\alpha - \theta')}^{W'} q(\alpha) \in \mathrm{Add}(\eta^+)^{W'[\langle \mathbb{P} \upharpoonright (\alpha - \theta') \rangle]}.$$

$\mathbb{M}(\eta, \theta - \theta', V, W)$ *is partially ordered as in Definition 5.2.*

Lemma 6.8. [1, Lemma 2.12] *Let* $\theta' \in]\eta, \theta[$ *be any ordinal and let* $\mathbb{M}_{\theta'} := \mathbb{M}(\eta, \theta', V, W)$ *with* $G_{\mathbb{M}_{\theta'}} \subseteq \mathbb{M}_{\theta'}$ *a generic filter over* W. *Assume that* $\mathbb{P}(\eta, \theta)$ *is* η^+*-cc in* W *and in* $W[G_{\mathbb{M}_{\theta'}}]$, *then*
$$\mathbb{M}(\eta, \theta, V, W) \equiv \mathbb{M}_{\theta'} * \mathbb{M}(\eta, \theta - \theta', V, W[\langle \mathbb{M}_{\theta'} \rangle]).$$

Proof. One can prove that $\mathbb{M}_{\theta'} * \mathbb{M}(\eta, \theta - \theta', V, W[\langle \mathbb{M}_{\theta'} \rangle])$ contains a dense set isomorphic to $\mathbb{M}(\eta, \theta, V, W)$. The proof is omitted, for more details see [1] Lemma 2.12. □

Remark 6.9. *Lemma 6.2 and Lemma 6.3, can be generalized in the following way. Assume that* $\theta' < \theta$, $\mathbb{P} := \mathbb{P}(\eta, \theta)^V \upharpoonright (\theta - \theta')$, $\mathbb{M}_{\theta'} := \mathbb{M}(\eta, \theta', V, W)$ *and* $G_{\mathbb{M}_{\theta'}} \subseteq \mathbb{M}_{\theta'}$ *is a generic filter over* W, *define*
$$\mathbb{Q}^*(\eta, \theta - \theta', V, W[G_{\mathbb{M}_{\theta'}}]) := \{(\emptyset, q);\ (\emptyset, q) \in \mathbb{M}(\eta, \theta - \theta', V, W[G_{\mathbb{M}_{\theta'}}])\}.$$

Then, $\mathbb{M}(\eta, \theta - \theta', V, W[G_{\mathbb{M}_{\theta'}}])$ *is a projection of* $\mathbb{P} \times \mathbb{Q}^*(\eta, \theta - \theta', V, W[G_{\mathbb{M}_{\theta'}}])$ *and* $\mathbb{Q}^*(\eta, \theta - \theta', V, W[G_{\mathbb{M}_{\theta'}}])$ *is* η^+*-directed closed in* $W[G_{\mathbb{M}_{\theta'}}]$*.*

7. Factoring the Main Forcing

In this section θ, V, L are like in Definition 5.4. We want to analyse the forcing $\mathbb{R}(\aleph_0, \theta, L)$. As we said, that poset is a variation of the forcing defined by Abraham in [1, Definition 2.14], we have just to deal with the function L. The proofs of the lemmas presented in this section are very similar to the proofs of the corresponding lemmas in [1].

Remark 7.1. $(p, q, f) \mapsto (p, q)$ *is a projection of* $\mathbb{R}(\aleph_0, \theta, L)$ *to* $\mathbb{M}(\aleph_0, \theta, V, V)$ *and for every limit ordinal* $\alpha < \theta$*, if* $L(\alpha)$ *is an* $\mathbb{R} \restriction \alpha$*-name for an ordinal and* $\mathbb{Q}^*$ *is the canonical* $\mathbb{R} \restriction \alpha$*-name for* $\mathbb{Q}^*(\aleph_1, L(\alpha), V, V[\langle R \restriction \alpha \rangle])$*, then*

$$\mathbb{R} \restriction \alpha + 1 = \mathbb{R} \restriction \alpha * \mathbb{Q}^*.$$

Indeed, the functions in $\mathbb{M}(\aleph_0, \theta, V, V)$ *are not defined on limit ordinals.*

Lemma 7.2. *Let* $\mathbb{U}(\aleph_0, \theta, L) := \{(\emptyset, q, f);\ (\emptyset, q, f) \in \mathbb{R}\}$ *ordered as a subposet of* $\mathbb{R}$*. The following hold:*

(i) *the function* $\pi : \mathbb{P}(\aleph_0, \theta) \times \mathbb{U}(\aleph_0, \theta, L) \to \mathbb{R}$ *defined by* $\pi(p, (\emptyset, q, f)) = (p, q, f)$ *is a projection;*

(ii) $\mathbb{U}(\aleph_0, \theta, L)$ *is* σ*-closed.*

Proof. (*i*) Let $(p', q', f') \leq \pi(p, (\emptyset, q, f))$. By Lemma 6.2, the function $(p, (\emptyset, q)) \mapsto (p, q)$ is a projection and we can find $(\emptyset, q^*) \leq (\emptyset, q)$ such that $[(p', q^*)] = [(p', q')]$. We define a function f^* as follows: $\mathrm{dom}(f^*) = \mathrm{dom}(f')$ and for all $\alpha \in \mathrm{dom}(f')$, if $\alpha \notin \mathrm{dom}(f)$, then $f^*(\alpha) := f'(\alpha)$; if $\alpha \in \mathrm{dom}(f)$, we define $f^*(\alpha) \in V^{\mathbb{R}\restriction\alpha}$ such that the following hold:

(1) $(p', q', f') \restriction \alpha \Vdash_{\mathbb{R}\restriction\alpha} f^*(\alpha) = f'(\alpha)$,

(2) if $r \in \mathbb{R} \restriction \alpha$ is incompatible with $(p', q', f') \restriction \alpha$, then $r \Vdash_{\mathbb{R}\restriction\alpha} f^*(\alpha) = f(\alpha)$.

Since $(p', q', f') \restriction \alpha \Vdash_{\mathbb{R}\restriction\alpha} f'(\alpha) \leq f(\alpha)$, we have $\Vdash_{\mathbb{R}\restriction\alpha} f^*(\alpha) \leq f(\alpha)$. One can prove by induction on α that $[(p^*, q^*, f^*) \restriction \alpha] = [(p', q', f') \restriction \alpha]$, and we have $(\emptyset, q^*, f^*) \leq (\emptyset, q, f)$.

(*ii*) Let $\langle(\emptyset, q_n, f_n);\ n < \omega\rangle$ be a decreasing sequence of conditions in $\mathbb{U}(\aleph_0, \theta, L)$. By definition, $\langle(\emptyset, q_n);\ n < \omega\rangle$ is a decreasing sequence of conditions in $\mathbb{Q}^*(\aleph_0, \theta, V, V)$ which is σ-closed by Lemma 6.3. So there is $(\emptyset, q)$ such that $(\emptyset, q) \leq (\emptyset, q_n)$, for every $n < \omega$. We define a function f with $\mathrm{dom}(f) = \bigcup_{n<\omega} \mathrm{dom}(f_n)$ as follows. We define $f \restriction \alpha + 1$ by induction on α, so that

$$(\emptyset, q \restriction \alpha + 1, f \restriction \alpha + 1) \leq (\emptyset, q_n, f_n) \restriction \alpha + 1,$$

for all $n < \omega$. Assume $f \restriction \alpha$ has been defined. For every $m > n$, we have

$$(\emptyset, q_m, f_m) \restriction \alpha \Vdash_{\mathbb{R}\restriction\alpha} f_m(\alpha) \leq f_n(\alpha),$$

so by the inductive hypothesis we have $(\emptyset, q \restriction \alpha, f \restriction \alpha) \Vdash f_m(\alpha) \leq f_n(\alpha)$. By Lemma 6.3, if $G_\alpha \subseteq \mathbb{R} \restriction \alpha$ is a generic filter over V, then $\mathbb{Q}^*(\aleph_1, L(\alpha), V, V[G_\alpha])$ is $\aleph_2$-closed in $V[G_\alpha]$. It follows that for some $f(\alpha) \in V^{\mathbb{R}\restriction\alpha}$, we have

$$(\emptyset, q \restriction \alpha, f \restriction \alpha) \Vdash f(\alpha) \leq f_m(\alpha), \text{ for every } m < \omega.$$

Finally, the condition $(\emptyset, q, f)$ is a lower bound for $\langle(\emptyset, q_n, f_n);\ n < \omega\rangle$. □

Lemma 7.3. *Assume that V is a model of* ZFC *with two supercompact cardinals $\kappa < \lambda$, and $L : \kappa \to V_\kappa$ is the Laver function. Let $\mathbb{R} := \mathbb{R}(\aleph_0, \kappa, L)$, and let $\dot{\mathbb{M}}$ be the canonical $\mathbb{R}$-name for $\mathbb{M}(\aleph_1, \lambda, V, V[\langle\mathbb{R}\rangle])$. The following hold:*

1. *$\mathbb{R}$ has size κ and it is κ-c.c.;*
2. *$\Vdash_{\mathbb{R}} \lambda$ is inaccessible;*
3. *For every filter $G_{\mathbb{R}} \subseteq \mathbb{R}$ generic over V, if G_0 is the projection of $G_{\mathbb{R}}$ to $\mathbb{P}_0 := \mathbb{P}(\aleph_0, \kappa)$, then all countable sets of ordinals in $V[G_{\mathbb{R}}]$ are in $V[G_0]$;*
4. *$\mathbb{R}$ preserves $\aleph_1$ and makes $\kappa = \aleph_2 = 2^{\aleph_0}$;*
5. *If $G_{\mathbb{R}} \subseteq \mathbb{R}$ is a generic filter over V, then $\mathbb{P}_1 := \mathbb{P}(\aleph_1, \lambda)^V$ does not introduce new countable subsets to $V[G_{\mathbb{R}}]$;*
6. *$\Vdash_{\mathbb{R}} \mathbb{P}(\aleph_1, \lambda)^V$ is κ-c.c. (and even κ-Knaster).*

Proof. (1) The proof is similar to the proof of Lemma 6.5 (i) and it is omitted.

(2) It follows from the previous claim.

(3) By Lemma 7.2, it is enough to prove that if $G_0 \times H \subseteq \mathbb{P}_0 \times \mathbb{U}(\aleph_0, \kappa, L)$ is any generic filter over V, then every countable set of ordinals in $V[G_0 \times H]$ is already in $V[G_0]$. This is an easy consequence of Easton's Lemma.

(4) Since $\mathbb{P}(\aleph_0, \kappa)$ is c.c.c., Claim 3 implies that $\aleph_1$ is preserved. The forcing $\mathbb{R}$ is κ-c.c., hence κ remains a cardinal after forcing with $\mathbb{R}$. Moreover, $\mathbb{R}$ projects on $\mathbb{M}(\aleph_0, \kappa, V, V)$ which, by Proposition 6.5, collapses all the cardinals between $\aleph_1$ and κ and adds κ many Cohen reals. Therefore $\mathbb{R}$ makes $\kappa = \aleph_2 = 2^{\aleph_0}$.

(5) By Lemma 7.2, $\mathbb{R}$ is a projection of $\mathbb{P}_0 \times \mathbb{U}_0$, where $\mathbb{P}_0 := \mathbb{P}(\aleph_0, \kappa)$ and $\mathbb{U} := \mathbb{U}(\aleph_0, \kappa, L)$. By Easton's Lemma $\Vdash_{\mathbb{P}_0 \times \mathbb{U}} \mathbb{P}_1$ is $< \aleph_1$-distributive, so no countable sequence of ordinals is added by $\mathbb{P}_1$ to $V[G_0 \times H]$, where $G_0 \subseteq \mathbb{P}_0$ and $H \subseteq \mathbb{U}$ are generic filters over V such that $G_{\mathbb{R}}$ is the projection of $G_0 \times H$ to $\mathbb{R}$. Moreover, we proved in Claim 3, that every countable sequence of ordinals in $V[G_0 \times H]$ is already in $V[G_0]$. Since $V[G_0] \subseteq V[G_{\mathbb{R}}]$, this completes the proof.

(6) Let $G_{\mathbb{R}} \subseteq \mathbb{R}$ be a generic filter over V. Work in $V[G_{\mathbb{R}}]$. Assume that $\langle f_\alpha;\ \alpha < \kappa\rangle$ is a sequence of conditions in $\mathbb{P}_1 := \mathbb{P}(\aleph_1, \lambda)^V$. Let $D := \bigcup_{\alpha<\kappa} \mathrm{dom}(f_\alpha)$, then there is a bijection $h : D \to \kappa$. Since every condition of the sequence is a countable function we have, for every $\alpha < \kappa$ of uncountable cofinality $\sup(h[\mathrm{dom}(f_\alpha)] \cap \alpha) < \alpha$. So the function $\alpha \mapsto \sup(h[\mathrm{dom}(f_\alpha)] \cap \alpha)$ is regressive. By Fodor's Theorem, there is an ordinal τ and a stationary set $S \subseteq \kappa$ such that $\sup(h[\mathrm{dom}(f_\alpha)] \cap \alpha) = \tau$, for every $\alpha \in S$. The set $h^{-1}(\tau)$ has size $< \kappa$ in $V[G_{\mathbb{R}}]$ and $\mathbb{R}$ is κ-c.c., so there is a set $E \in V$ of size $< \kappa$ in V such that $h^{-1}(\tau) \subseteq E$. Since κ is inaccessible in V, we can find in $V[G_{\mathbb{R}}]$

a stationary set $S' \subseteq S$ such that $f_\alpha \upharpoonright E$ has a fixed value, for every $\alpha \in S'$. Then the sets in $\{\mathrm{dom}(f_\alpha) \setminus E;\ \alpha \in S'\}$ can be assumed to be pairwise disjoint, hence $f_\alpha \cup f_\beta$ is a function for every $\alpha, \beta \in S'$. $\square$

Lemma 7.4. [1, Lemma 2.18] *Assume that* $\alpha < \theta$ *is a limit ordinal, let* $\mathbb{P} := \mathbb{P}(\aleph_0, \theta) \upharpoonright (\theta - \alpha)$, $\mathbb{R} := \mathbb{R}(\aleph_0, \theta, L)$ *and let* $G_\alpha \subseteq \mathbb{R} \upharpoonright \alpha + 1$ *be a generic filter over* V. *We define in* $V[G_{\alpha+1}]$ *the following set:*

$$\mathbb{U}_{\alpha+1}(\aleph_0, \theta, L, G_{\alpha+1}) := \{(0, q, f) \in \mathbb{R}(\aleph_0, \theta, L);\ (0, q, f) \upharpoonright \alpha + 1 \in G_{\alpha+1}\}.$$

Then $\mathbb{R}/G_{\alpha+1}$ *is a projection of* $\mathbb{P} \times \mathbb{U}_{\alpha+1}(\aleph_0, \theta, L, G_{\alpha+1})$, *and* $\mathbb{U}_{\alpha+1}(\aleph_0, \theta, L, G_{\alpha+1})$ *is* σ*-closed in* $V[G_{\alpha+1}]$.

Proof. The proof is very similar to the proof of Lemma 2.18 in [1] and it is omitted. $\square$

8. The Main Theorem

Theorem 8.1. *Assume that* V *is a model of* ZFC *with two supercompact cardinals* $\kappa < \lambda$, *and suppose that* $L : \kappa \to V_\kappa$ *is the Laver function. If* $\mathbb{R} := \mathbb{R}(\aleph_0, \kappa, L)$, *and* $\dot{\mathbb{M}}$ *is the canonical* $\mathbb{R}$*-name for* $\mathbb{M}(\aleph_1, \lambda, V, V[\langle\mathbb{R}\rangle])$, *then for every filter* $G \subseteq \mathbb{R} * \dot{\mathbb{M}}$ *generic over* V, *both* $\aleph_2$ *and* $\aleph_3$ *satisfy the super tree property in* $V[G]$.

The proof that the model obtained is as required consists of three parts:

(1) $V[G] \models \aleph_1^V = \aleph_1,\ \kappa = \aleph_2$ and $\lambda = \aleph_3$;
(2) $\aleph_3$ satisfies the super tree property.
(3) $\aleph_2$ satisfies the super tree property;

Proof of (1)

First we show that $\aleph_1$ is preserved. Let $G_\mathbb{R}$ be the projection of G to $\mathbb{R}$ and let $G_\mathbb{M}$ be the projection of G to $\mathbb{M} := \dot{\mathbb{M}}^{G_\mathbb{R}}$. By Lemma 7.3, $\aleph_1$ is preserved by $\mathbb{R}$. Moreover, $\mathbb{P}(\aleph_1, \lambda)^V$ does not introduce new countable subsets to $V[G_\mathbb{R}]$ (see Lemma 7.3 (5)). So, by Lemma 6.6 (1) $\mathbb{M}$ does not introduce new countable sequences, hence $\aleph_1$ remains a cardinal in $V[G]$. Now, we show that κ remains a cardinal in $V[G]$. By Lemma 7.3, we know that κ remains a cardinal in $V[G_\mathbb{R}]$ and becomes $\aleph_2$. By Lemma 7.3 (6), $\mathbb{P}(\aleph_1, \lambda)^V$ is κ-c.c. in $V[G_\mathbb{R}]$, so κ remains a cardinal after forcing with $\mathbb{P}(\aleph_1, \lambda)^V$ over $V[G_\mathbb{R}]$ and it is equal to $\aleph_2$. By applying Lemma 6.4, we get that all sets of ordinals in $V[G]$ of cardinality $\aleph_1$ are in $V[G_\mathbb{R}][G_\mathbb{P}]$, where $G_\mathbb{P}$ is the projection of $G_\mathbb{M}$ to $\mathbb{P}(\aleph_1, \lambda)^V$. Therefore, κ remains a cardinal in $V[G]$. Finally, λ remains a cardinal because $\mathbb{R} * \dot{\mathbb{M}}$ is λ-c.c., and it becomes $\aleph_3$.

Proof of 2

By (1), we know that $\lambda = \aleph_3$ in $V[G]$, so we want to prove that λ has the super tree property in that model. Let $\mu \geq \lambda$ be any ordinal, we fix, in $V[G]$, a (λ, μ)-tree F and an F-level sequence D. Fix an elementary embedding $j : V \to N$ with critical point λ such that:

(1) if $\sigma := |\mu|^{<\lambda}$, then $j(\lambda) > \sigma$,
(2) ${}^{\sigma}N \subseteq N$.

The structure of the proof is the following. First, we find $H \subseteq j(\mathbb{R} * \dot{\mathbb{M}})$ generic over N such that j lifts to an elementary embedding $j^* : V[G] \to N[H]$. Then, we prove that $N[H]$ has an ineffable branch b for D. Finally, we show that $b \in V[G]$.

Claim 8.2. *We can lift j to an elementary embedding $j^* : V[G] \to N[H]$, with $H \subseteq j(\mathbb{R} * \dot{\mathbb{M}})$ generic over N.*

Proof. To simplify the notation we will denote all the extensions of j by "j" also. We let $G_{\mathbb{R}}$ be the projection of G to $\mathbb{R}$ and let $G_{\mathbb{M}}$ be the projection of G to $\mathbb{M} := \dot{\mathbb{M}}^{G_{\mathbb{R}}}$.

As $\lambda > \kappa$ and $|\mathbb{R}| = \kappa$, we have $j(\mathbb{R}) = \mathbb{R}$, so we can lift j to an elementary embedding

$$j : V[G_{\mathbb{R}}] \to N[G_{\mathbb{R}}].$$

Observe that $j(\mathbb{M}) \restriction \lambda = \mathbb{M}(\aleph_1, \lambda, N, N[G_{\mathbb{R}}]) = \mathbb{M}(\aleph_1, \lambda, V, V[G_{\mathbb{R}}]) = \mathbb{M}$. Force over $V[G_{\mathbb{R}}]$ to get a $j(\mathbb{M})$-generic filter $H_{j(\mathbb{M})}$ such that $H_{j(\mathbb{M})} \restriction \lambda = G_{\mathbb{M}}$. By Lemma 6.5 and Lemma 7.3 (2), $\mathbb{M}$ is λ-c.c. in $V[G_{\mathbb{R}}]$, so $j \restriction \mathbb{M}$ is a complete embedding from $\mathbb{M}$ into $j(\mathbb{M})$, hence we can lift j to an elementary embedding

$$j : V[G_{\mathbb{R}}][G_{\mathbb{M}}] \to N[G_{\mathbb{R}}][H_{j(\mathbb{M})}].$$

□

Rename j^* by j. We define $\mathscr{N}_1 := N[G]$ and $\mathscr{N}_2 := N[G_{\mathbb{R}}][H_{j(\mathbb{M})}]$. In $\mathscr{N}_2$, $j(F)$ is a $(j(\lambda), j(\mu))$-tree and $j(D)$ is a $j(F)$-level sequence. By the closure of N, the tree F and the F-level sequence D are in $\mathscr{N}_1$.

Claim 8.3. *In $\mathscr{N}_2$, there is an ineffable branch b for D.*

Proof. Let $a := j[\mu]$, clearly $a \in [j(\mu)]^{<j(\lambda)}$. Consider $f := j(D)(a)$, let $b : \mu \to 2$ be the function defined by $b(\alpha) := f(j(\alpha))$, we show that b is an ineffable branch for D. Assume for a contradiction that in $\mathscr{N}_2$ there is a club $C \subseteq [\mu]^{<|\lambda|} \cap \mathscr{N}_1$ such that $b \restriction X \neq D(X)$, for all $X \in C$. Then by elementarity,

$$j(b) \restriction X \neq j(D)(X),$$

for all $X \in j(C)$. But $a \in j(C)$ and $j(b) \restriction a = f = j(D)(a)$, so we have a contradiction.

□

We have found an ineffable branch b for D in the model $\mathscr{N}_2$. We conclude the proof by proving that b is already in $N[G]$ (if $b \in N[G]$, then b is ineffable since $\{X \in [\mu]^{<|\lambda|} \cap N[G];\ b \restriction X = D(X)\}$ is stationary in $\mathscr{N}_2$, hence it is stationary in $N[G]$), hence in $V[G]$. We assume, towards a contradiction, that $b \notin N[G]$. Step by step, we want to prove that $b \notin \mathscr{N}_2$, that will lead us to a contradiction.

By Remark 6.9, $\mathbb{M}(\aleph_1, j(\lambda) - \lambda, N, N[G])$ is a projection of

$$\mathbb{P} \times \mathbb{Q}^*(\aleph_1, j(\lambda) - \lambda, N, N[G]),$$

where $\mathbb{P} := \mathbb{P}(\aleph_1, j(\lambda))^N \upharpoonright (j(\lambda)-\lambda)$, and $\mathbb{Q}^* := \mathbb{Q}^*(\aleph_1, j(\lambda)-\lambda, N, N[G])$ is $\aleph_2$-closed in $N[G]$. In $N[G]$, we have $\lambda = \aleph_3 = 2^{\aleph_1}$ and F is an $(\aleph_3, \mu)$-tree, so we can apply the First Preservation Theorem, thus

$$b \notin N[G][H_{\mathbb{Q}^*}],$$

where $H_{\mathbb{Q}^*}$ is the projection of $H_{j(\mathbb{M})}$ to $\mathbb{Q}^*$. The filter $H_{\mathbb{Q}^*}$ collapses λ (which is $\aleph_3^{N[G]}$) to have size $\aleph_2$, so now F is an $(\aleph_2, \mu)$-tree. The model $\mathscr{N}_2 = N[G_{\mathbb{R}}][H_{j(\mathbb{M})}]$ is the result of forcing with $\mathbb{P}$ over $N[G][H_{\mathbb{Q}^*}]$; we want to apply the Second Preservation Theorem. Every set $X \subseteq N$ in $N[G][H_{\mathbb{Q}^*}]$ which has size $< \aleph_2$ in $N[G][H_{\mathbb{Q}^*}]$ is covered by a set $Y \in N$ which has size $< \lambda$ in N, so the hypothesis of the Second Preservation Theorem are satisfied, hence $b \notin N[G_{\mathbb{R}}][H_{\mathbb{Q}^*}] = \mathscr{N}_2$, a contradiction.

This completes the proof of (2).

Proof of 3

By (1), we know that $\kappa = \aleph_2$ in $V[G]$, so we want to prove that κ has the super tree property in that model. Let $\mu \geq \kappa$ be any ordinal, we fix, in $V[G]$, a (κ, μ)-tree F and an F-level sequence D. Since L is the Laver function, there is an elementary embedding $j : V \to N$ with critical point κ such that:

(1) if $\sigma := \max(\lambda, |\mu|^{<\kappa})$, then $j(\kappa) > \sigma$,
(2) ${}^{\sigma}N \subseteq N$,
(3) $j(L)(\kappa) = \lambda$.

Claim 8.4. *We can lift j to an elementary embedding $j^* : V[G] \to N[H]$, with $H \subseteq j(\mathbb{R} * \dot{\mathbb{M}})$ generic over N.*

Proof. To simplify the notation we will denote all the extensions of j by "j" also. Let $G_{\mathbb{R}}$ be the projection of G to $\mathbb{R}$ and let $G_{\mathbb{M}}$ be the projection of G to $\mathbb{M} := \dot{\mathbb{M}}^{G_{\mathbb{R}}}$. Observe that $j(\mathbb{R}) = \mathbb{R}(\aleph_0, j(\kappa), j(L))^N = \mathbb{R}(\aleph_0, j(\kappa), j(L))^V$, and $j(\mathbb{R}) \upharpoonright \kappa = \mathbb{R}$. Force over V to get a $j(\mathbb{R})$-generic filter $H_{j(R)}$ such that $H_{j(R)} \upharpoonright \kappa = G_{\mathbb{R}}$. By Lemma 7.3 (1) $\mathbb{R}$ is κ-c.c. So $j \upharpoonright \mathbb{R}$ is a complete embedding from $\mathbb{R}$ into $j(\mathbb{R})$, hence we can lift j to get an elementary embedding

$$j : V[G_{\mathbb{R}}] \to N[H_{j(R)}].$$

By Lemma 6.2, in $V[G_{\mathbb{R}}]$, the forcing $\mathbb{M}$ is a projection of

$$\mathbb{P}(\aleph_1, \lambda)^V \times \mathbb{Q}^*(\aleph_1, \lambda, V, V[G_{\mathbb{R}}])$$

(moreover, $\mathbb{P}(\aleph_1, \lambda)^V = \mathbb{P}(\aleph_1, \lambda)^N$ and $\mathbb{Q}^*(\aleph_1, \lambda, V, V[G_{\mathbb{R}}]) = \mathbb{Q}^*(\aleph_1, \lambda, N, N[G_{\mathbb{R}}])$). Recall that

$$\mathbb{S}(\aleph_1, \lambda, V, V[G_{\mathbb{R}}], G_{\mathbb{M}}) = (\mathbb{P}(\aleph_1, \lambda)^V \times \mathbb{Q}^*(\aleph_1, \lambda, V, V[G_{\mathbb{R}}]))/G_{\mathbb{M}},$$

so by forcing with $\mathbb{S}(\aleph_1, \lambda, V, V[G_{\mathbb{R}}], G_{\mathbb{M}})$ over $V[G]$ we obtain a model $V[G_{\mathbb{R}}][G_{\mathbb{P}} \times G_{\mathbb{Q}^*}]$ with $G_{\mathbb{P}} \times G_{\mathbb{Q}^*}$ generic for $\mathbb{P}(\aleph_1, \lambda)^V \times \mathbb{Q}^*(\aleph_1, \lambda, V, V[G_{\mathbb{R}}])$ over $V[G_{\mathbb{R}}]$ and such that

$G_{\mathbb{M}}$ is the projection of $G_{\mathbb{P}} \times G_{\mathbb{Q}^*}$ to $\mathbb{M}$.

If $\mathbb{P} := \mathbb{P}(\aleph_1, \lambda)^V$, then $\mathbb{P}$ is κ-c.c. in $V[G_{\mathbb{R}}]$ (Lemma 7.3 (6)), hence $j \restriction \mathbb{P}$ is a complete embedding of $\mathbb{P}$ into $j(\mathbb{P})$. Moreover, $\mathbb{P}$ is isomorphic via $j \restriction \mathbb{P}$ to $\mathbb{P}(\aleph_1, j[\lambda])^N = \mathbb{P}(\aleph_1, j[\lambda])^V$. By forcing with $\mathbb{P}(\aleph_1, j(\lambda))^V \restriction (j(\lambda) - j[\lambda])$ over $V[H_{j(\mathbb{R})}]$ we get a $j(\mathbb{P})$-generic filter $H_{j(\mathbb{P})}$ such that $j[G_{\mathbb{P}}] \subseteq H_{j(\mathbb{P})}$. Then j lifts to an elementary embedding

$$j : V[G_{\mathbb{R}}][G_{\mathbb{P}}] \to N[H_{j(\mathbb{R})}][H_{j(\mathbb{P})}].$$

Let $\mathbb{Q}^* := \mathbb{Q}^*(\aleph_1, \lambda, V, V[G_{\mathbb{R}}])$. By Remark 7.1 and since $j(\mathbb{R}) \restriction \kappa = \mathbb{R}$, we have $j(\mathbb{R}) \restriction \kappa + 1 = \mathbb{R} * \dot{\mathbb{Q}}^*$ where $\dot{\mathbb{Q}}^*$ is an $\mathbb{R}$-name for $\mathbb{Q}^*(\aleph_1, j(L)(\kappa), V, V[G_{\mathbb{R}}])$. We chose j so that $j(L)(\kappa) = \lambda$, therefore forcing with $j(\mathbb{R}) \restriction \kappa+1$ over V is the same as forcing with $\mathbb{R}$ followed by forcing with $\mathbb{Q}^*$ over $V[G_{\mathbb{R}}]$. It follows that, by the closure of N, we have $j[G_{\mathbb{Q}^*}] \in N[H_{j(\mathbb{R})}]$. By Lemma 6.3, $\mathbb{Q}^*$ is $\aleph_2$-directed closed in $V[G_{\mathbb{R}}]$, hence $j(\mathbb{Q}^*)$ is $\aleph_2$-directed closed in $N[H_{j(\mathbb{R})}]$. Moreover, the filter $H_{j(\mathbb{R})}$ collapses λ to have size $\aleph_1$, thus $j[G_{\mathbb{Q}^*}]$ has size $\aleph_1$ in $V[H_{j(\mathbb{R})}]$. Therefore, we can find $t \leq j(q)$, for all $q \in G_{\mathbb{Q}^*}$. We force over $V[G_{j(\mathbb{R})}]$ with $j(\mathbb{Q}^*)$ below t to get a $j(\mathbb{Q}^*)$-generic filter $H_{j(\mathbb{Q}^*)}$ containing $j[G_{\mathbb{Q}^*}]$. The filter $H_{j(\mathbb{P})} \times H_{j(\mathbb{Q}^*)}$ generates a filter $H_{j(\mathbb{M})}$ generic for $j(\mathbb{M})$ over $N[H_{j(\mathbb{R})}]$.

It remains to prove that $j[G_{\mathbb{M}}] \subseteq H_{j(\mathbb{M})}$: let (p, q) be a condition of $G_{\mathbb{M}}$, there are $\bar{p} \in G_{\mathbb{P}}$ and $(0, \bar{q}) \in G_{\mathbb{Q}^*}$ such that $(\bar{p}, \bar{q}) \leq (p, q)$. We have $j(\bar{p}) \in H_{j(\mathbb{P})}$ and $(0, j(\bar{q})) \in H_{j(\mathbb{Q}^*)}$, hence $(j(\bar{p}), 0)$ and $(0, j(\bar{q}))$ are both in $H_{j(\mathbb{M})}$. The condition $j(\bar{p}, \bar{q})$ is the greatest lower bound[1] of $(j(\bar{p}), 0)$ and $(0, j(\bar{q}))$; it follows that $j(\bar{p}, \bar{q}) \in H_{j(\mathbb{M})}$. We also have $j(\bar{p}, \bar{q}) \leq j(p, q)$, hence $j(p, q) \in H_{j(\mathbb{M})}$ as required. Therefore, j lifts to an elementary embedding

$$j : V[G_{\mathbb{R}}][G_{\mathbb{M}}] \to N[H_{j(\mathbb{R})}][H_{j(\mathbb{M})}].$$

□

Rename j^* by j. We define $\mathscr{N}_1 := N[G]$ and $\mathscr{N}_2 := N[H_{j(\mathbb{R})}][H_{j(\mathbb{M})}]$. In $\mathscr{N}_2$, $j(F)$ is a $(j(\kappa), j(\mu))$-tree and $j(D)$ is a $j(F)$-level sequence. By the closure of N, the tree F and the F-level sequence D are in $\mathscr{N}_1$.

Claim 8.5. *In $\mathscr{N}_2$, there is an ineffable branch b for D.*

Proof. Let $a := j[\mu]$, clearly $a \in [j(\mu)]^{<j(\kappa)}$. Consider $f := j(D)(a)$, let $b : \mu \to 2$ be the function defined by $b(\alpha) := f(j(\alpha))$, we show that b is an ineffable branch for D. Assume for a contradiction that for some club $C \subseteq [\mu]^{<|\kappa|} \cap \mathscr{N}_1$ we have $b \restriction X \neq D(X)$, for all $X \in C$. Then by elementarity,

$$j(b) \restriction X \neq j(D)(X),$$

[1] $j(\bar{p}, \bar{q}) = (j(\bar{p}), j(\bar{q}))$ is clearly a lower bound. Suppose that (p_1, q_1) is also a lower bound, then by definition $p_1 \leq j(\bar{p})$ and $p_1 \restriction \alpha \Vdash q_1(\alpha) \leq j(\bar{q})(\alpha)$, for every α. That is $(p_1, q_1) \leq (j(\bar{p}), j(\bar{q}))$.

for all $X \in j(C)$. But $a \in j(C)$ and $j(b) \upharpoonright a = f = j(D)(a)$, so we have a contradiction. □

We conclude the proof by showing that b is already in $\mathscr{N}_1$.

Claim 8.6. $b \in \mathscr{N}_1$.

Proof. Assume towards a contradiction that $b \notin \mathscr{N}_1$. Step by step, we are going to prove that $b \notin \mathscr{N}_2$, that will lead us to a contradiction. By Lemma 7.3 (5) and Lemma 6.6 (2), the poset $\mathbb{S} := \mathbb{S}(\aleph_1, \lambda, N, N[G_{\mathbb{R}}], G_{\mathbb{M}})$ is σ-closed in $\mathscr{N}_1$. In $\mathscr{N}_1$, we have $\kappa = \aleph_2 = 2^{\aleph_0}$, hence F is a $(\aleph_2, \mu)$-tree and we can apply the First Preservation Theorem to $\mathbb{S}$, thus

$$b \notin N[G_{\mathbb{R}}][G_{\mathbb{P}} \times G_{\mathbb{Q}^*}]$$

(we defined $G_{\mathbb{P}} \times G_{\mathbb{Q}^*}$ in Claim 8.4 as an $\mathbb{S}$-generic filter). $\mathbb{S}$ is $< \aleph_2$-distributive in $\mathscr{N}_1$ (this is a standard application of Easton's Lemma, see [2, Lemma 3.20] for more details) so F is still an $(\aleph_2, \mu)$-tree after forcing with $\mathbb{S}$. Now, the forcing that takes us from $\mathbb{P}$ to $j(\mathbb{P})$ is

$$\mathbb{P}_{tail} := \mathbb{P}(\aleph_1, j(\lambda))^N \upharpoonright (j(\lambda) - \lambda)).$$

The pair $(N, N[G_{\mathbb{R}}][G_{\mathbb{P}} \times G_{\mathbb{Q}^*}])$ has the κ-covering property, because $\mathbb{S}$ is $< \aleph_2$-distributive and $\mathbb{R}$ is κ-c.c. Since κ is inaccessible in N, we can apply the Second Preservation Theorem to $\mathbb{P}_{tail}$, so

$$b \notin N[G_{\mathbb{R}}][G_{\mathbb{Q}^*}][H_{j(\mathbb{P})}].$$

We already observed in the proof of the first claim that forcing with $j(\mathbb{R}) \upharpoonright \kappa + 1$ over V is the same as forcing with $\mathbb{R}$ followed by forcing with $\mathbb{Q}^*$ over $V[G_{\mathbb{R}}]$. So, if $H_{\kappa+1}$ is the projection of $H_{j(\mathbb{R})}$ to $j(\mathbb{R}) \upharpoonright \kappa + 1$, then $N[G_{\mathbb{R}}][G_{\mathbb{Q}^*}] = N[H_{\kappa+1}]$. This means that we proved

$$b \notin N[H_{\kappa+1}][H_{j(\mathbb{P})}].$$

Consider $\mathbb{R}_{tail} := j(\mathbb{R})/H_{\kappa+1}$, by Lemma 7.4, $\mathbb{R}_{tail}$ is a projection of $\mathbb{P}_0 \times \mathbb{U}_0$, where $\mathbb{P}_0 := \mathbb{P}(\aleph_0, j(\kappa))^N \upharpoonright (j(\kappa) - \kappa)$ and $\mathbb{U}_0 := \mathbb{U}_{\kappa+1}(\aleph_0, j(\kappa), j(L), H_{\kappa+1})$, moreover, $\mathbb{U}_0$ is σ-closed in $N[H_{\kappa+1}]$. Forcing with $j(\mathbb{P})$ does not add countable sequences to $N[H_{\kappa+1}]$ (the proof is analogous to the proof of Lemma 7.3 (5)) hence $\mathbb{U}_0$ is still σ-closed in $N[H_{\kappa+1}][H_{j(\mathbb{P})}]$.

We want to apply the First Preservation Theorem to $\mathbb{U}_0$, so consider the following facts. In $N[H_{\kappa+1}][H_{j(\mathbb{P})}]$, we have $2^{\aleph_0} = j(\kappa) > \kappa = \aleph_2$ but now F is not exactly an $(\aleph_2, \mu)$-tree because $N[H_{\kappa+1}][H_{j(\mathbb{P})}]$ was obtained by forcing with $\mathbb{P}_{tail}$ over $N[G_{\mathbb{R}}][G_{\mathbb{P}} \times G_{\mathbb{Q}^*}]$ and $\mathbb{P}_{tail}$ is not $< \aleph_2$-distributive. Neverthless, $\mathbb{P}_{tail}$ is $\aleph_2$-c.c. in $N[G_{\mathbb{R}}][G_{\mathbb{P}} \times G_{\mathbb{Q}^*}]$ (the proof is analogous to the proof of Lemma 7.3 (6), see [2] for more details), so F "covers" an $(\aleph_2, \mu)$-tree, namely there is in $N[H_{\kappa+1}][H_{j(\mathbb{P})}]$ a $(\aleph_2, \mu)$-tree F^* such that for cofinally many $X \in [\mu]^{<\aleph_2}$, $\mathrm{Lev}_X(F) \subseteq \mathrm{Lev}_X(F^*)$. Let $N[H_{\mathbb{U}_0}][H_{j(\mathbb{P})}]$ be the generic extension obtained by forcing with $\mathbb{U}_0$ over $N[H_{\kappa+1}][H_{j(\mathbb{P})}]$. If $b \in N[H_{\mathbb{U}_0}][H_{j(\mathbb{P})}]$, then b provides a cofinal branch for F^* in that model, hence by the First Preservation

Theorem, $b \in N[H_{\kappa+1}][H_{j(\mathbb{P})}]$. But we already proved that b does not belong to that model, so we must have

$$b \notin N[H_{\mathbb{U}_0}][H_{j(\mathbb{P})}].$$

The filter $H_{\mathbb{U}_0}$ collapses κ (hence $\aleph_2$) to have size $\aleph_1$, so now F^* is an $(\aleph_1, \mu)$-tree in $W := N[H_{\mathbb{U}_0}][H_{j(\mathbb{P})}]$. The model $N[H_{j(\mathbb{R})}][H_{j(\mathbb{P})}]$ is the result of forcing with $\mathbb{P}_0$ over W. Observe that $\mathbb{P}_0$ and (W, W) satisfy all the hypothesis of the Second Preservation Theorem: indeed, $\mathbb{P}_0 \subseteq \mathrm{Add}(\aleph_0, j(\kappa))^W$ and in W, we have $\gamma^{<\omega} < \omega_1$ for every cardinal $\gamma < \omega_1$. Therefore,

$$b \notin N[H_{j(\mathbb{R})}][H_{j(\mathbb{P})}].$$

$\mathbb{P}_0$ is c.c.c. in W (the proof is analogous to the proof of Lemma 7.3 (6)), so F^* covers an $(\aleph_1, \mu)$-tree in $N[H_{j(\mathbb{R})}][H_{j(\mathbb{P})}]$, we rename it F^*. $\mathscr{N}_2$ is the result of forcing with

$$\mathbb{Q}(\aleph_1, j(\lambda), N, N[H_{j(\mathbb{R})}], H_{j(\mathbb{P})}) = \mathbb{M}(\aleph_1, j(\lambda), N, N[H_{j(\mathbb{R})}])/H_{j(\mathbb{P})}$$

over $N[H_{j(\mathbb{R})}][H_{j(\mathbb{P})}]$ and by Lemma 6.6 (1), that poset is σ-closed in $N[H_{j(\mathbb{R})}][H_{j(\mathbb{P})}]$. The function b, which is in $\mathscr{N}_2$, provides a cofinal branch for F^* in $\mathscr{N}_2$. It follows, from the First Preservation Theorem, that $b \notin \mathscr{N}_2$, a contradiction. □

This completes the proof of (3).

9. Conclusion

Cummings and Foreman [2] defined a model of the tree property for every $\aleph_n$ $(n \geq 2)$, starting with an infinite sequence of supercompact cardinals $\langle \kappa_n \rangle_{n<\omega}$. Their forcing $\mathbb{R}_\omega$ is basically an iteration with length ω of our main forcing. We conjecture that $\mathbb{R}_\omega$ produces a model in which every $\aleph_n$ $(n \geq 2)$ satisfies even the super tree property. If we want to prove that stronger result, we have to deal with the following fact: every κ_n-tree in the Cummings-Foreman model appears in some intermediate stage, that is after forcing with $\mathbb{R}_\omega \restriction m$ for some m; in the case of a (κ_n, μ)-tree, that is not necessarily true.

10. Acknowledgements

I would like to thank my advisor Boban Veličković for suggesting to investigate the strong and the super tree property at small cardinals. I owe to him the idea to reformulate the definition of these properties in terms of (κ, λ)-trees, a very useful indication that helped me to prove the results presented in this paper. He gave me many other suggestions for which I am very grateful. Finally, I would like to thank the referee for carefully reading my manuscript and for giving such constructive comments which helped improving the quality of the present paper.

References

[1] U. Abraham. Aronszajn Trees on $\aleph_2$ and $\aleph_3$, *Annals of Pure and Applied Logic* 24: 213-230 (1983).
[2] J. Cummings and M. Foreman. The Tree Property, *Advances in Mathematics* 133: 1-32 (1998).
[3] T. Jech. Some Combinatorial Problems Concerning Uncountable Cardinals, *Annals of Mathematical Logic* 5: 165-198 (1972/73).
[4] A. Kanamori. *The Higher Infinite. Large Cardinals in Set Theory from their Beginnings*, Perspectives in Mathematical Logic. Springer-Verlag, Berlin, (1994).
[5] K. Kunen, *Set Theory. An introduction to independence proofs*, North-Holland, (1980).
[6] R. Laver. Making the supercompactness of κ indestructible under κ-directed closed forcing, *Israel Journal of Mathematics* 29: 385-388 (1978).
[7] M. Magidor. Combinatorial Characterization of Supercompact Cardinals, *Proc. American Mathematical Society* 42: 279-285 (1974).
[8] W. J. Mitchell, Aronszajn Trees and the Independence of the Transfer Property, *Annals of Mathematical Logic* 5: 21-46 (1972).
[9] M. Viale. On the Notion of Guessing Model, to appear in the *Annals of Pure and Applied Logic.*
[10] M. Viale and C. Weiss. On the Consistency Strength of the Proper Forcing Axiom, *Advances in Mathematics* 228: 2672-2687 (2011).
[11] C. Weiss. Subtle and Ineffable Tree Properties, Phd thesis, *Ludwig Maximilians Universitat Munchen* (2010)
[12] C. Weiss. The Combinatorial Essence of Supercompactness, submitted to the *Annals of Pure and Applied Logic.*

E-mail address: `fontanella@logique.jussieu.fr`

Equipe de Logique Mathématique, Université Paris Diderot Paris 7, UFR de mathématiques case 7012, site Chevaleret, 75205 Paris Cedex 13, France