## Lowness of the piegeonhole principle

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## Section 1





It all started with this guy...

#### Theorem (Ramsey's theorem)

Let  $n \ge 1$ . For each coloration of  $[\omega]^n$  in a finite number of color, there exists a set  $X \in [\omega]^{\omega}$  such that each element of  $[X]^n$  has the same color (X is said to be monochromatic).

Splitting  $\omega$  in two

Iterating through the ordinals

## Motivation

## **Ramsey Theory**

#### A general question

Suppose we have some mathematical structure that is then cut into finitely many pieces. How big must the original structure be in order to ensure that at least one of the pieces has a given interesting property?

Examples :

- Van der Waerden's theorem
- Hindman's theorem

• ...

## Motivation

#### Example (Van der Waerden's theorem)

For any given c and n, there is a number w(c, n), such that if w(c, n) consecutive numbers are colored with c different colors, then it must contain an arithmetic progression of length n whose elements all have the same color.

We know that :

$$w(c,n) \leqslant 2^{2^{c^{2^{n+9}}}}$$

#### Example (Hindmam's theorem)

If we color the natural numbers with finitely many colors, there must exists a monochromatic infinite set closed by finite sums.

## Partition regularity

Theorems in Ramsey theory often assert, in their stronger form, that certain classes are *partition regular* :

#### Definition (Partition regularity)

A partition regular class is a non-empty collection of sets  $\mathcal{L}\subseteq 2^\omega$  such that :

- $\mathcal{L}$  is upward closed : If  $X \in \mathcal{L}$  and  $X \subseteq Y$ , then  $Y \in \mathcal{L}$
- If  $X \in \mathcal{L}$  and  $Y_0 \cup \cdots \cup Y_k \supseteq X$ , then there is  $i \leq k$  such that  $Y_i \in \mathcal{L}$

Proper partition regular classes are exactly the complements of proper set theoretic ideals :

#### Definition (Ideals)

An *ideal* class is a non-empty collection of sets  $\mathcal{I} \subseteq 2^{\omega}$  such that :

- $\mathcal{I}$  is downward closed : If  $X \in \mathcal{L}$  and  $X \supseteq Y$ , then  $Y \in \mathcal{I}$
- If  $Y_0, \ldots, Y_k \in \mathcal{I}$ , then  $Y_0 \cup \cdots \cup Y_k \in \mathcal{I}$

## Partition regularity

The following classes are partition regular :

Classical combinatorial results :

- The class of infinite sets
- The class of sets with positive upper density
- The class of sets X s.t.  $\sum_{n \in X} \frac{1}{n} = \infty$
- The class of sets containing arbitrarily long arithmetic progressions (Van der Waerden's theorem)
- The class of sets containing an infinite set closed by finite sum (Hindman's theorem)
- ... and *new* type of results involving computability :
  - Given X non-computable, the class of sets containing an infinite set which does not compute X (Dzhafarov and Jockusch)

## Seetapun's theorem

#### Theorem (Dzhafarov and Jockusch)

Given X non-computable, Given  $A^0 \cup A^1 = \omega$ , there exists  $G \in [A^0]^{\omega} \cup [A^1]^{\omega}$  such that G does not compute X.

This theorem comes from Reverse mathematics :

What is the computational strength of Ramsey's theorem?

that is, given a computable coloring of say  $[\omega]^2$ , must all monochromatic sets have a specific computational power?

Theorem (Seetapun)

For any non-computable set X and any computable coloring of  $[\omega]^2$ , there is an infinite monochromatic set which does not compute X.

#### Theorem (Jockusch)

There exists a computable coloring of  $[\omega]^3$ , every solution of which computes  $\emptyset'$ .

Iterating through the ordinals

## Modern approach of Seetapun's theorem

Modern approach of Seetapun's theorem (Cholak, Jockusch, Slaman) :

#### Definition

A set C is  $\{R_n\}_{n \in \omega}$ -cohesive if  $C \subseteq R_n$  or  $C \subseteq \overline{R_n}$  for every n.

#### Definition

A coloring  $c: \omega^2 \to \{0,1\}$  is *stable* if  $\forall x \lim_{y \in \omega} c(x,y)$  exists.

- Given a computable coloring  $c : \omega^2 \to \{0, 1\}$ , let  $R_n = \{y : c(n, y) = 0\}$ . Let C be  $\{R_n\}_{n \in \omega}$ -cohesive. Then c restricted to C is stable.
- Let c be a stable coloring. Let  $A_c$  be the  $\Delta_2^0(c)$  set defined as  $A_c(x) = \lim_y c(x, y)$ . An infinite subset of  $A_c$  or of  $\overline{A_c}$  can be used to compute a solution to c.

 $\rightarrow$  Find a cohesive set *C* (cohesive for the recursive sets) which does not compute *X* and use Dzhafarov and Jockusch relative to *C* with  $A_{c\uparrow_C}$ .

## The general question

The following version of Dzhafarov and Jockusch's is also true :

#### Theorem (Dzhafarov and Jockusch)

Let X be non-computable. The class of sets

 $\{A : There exists G \in [A]^{\omega} \text{ such that } G \text{ does not compute } X\}$  is partition regular.

Dzhafarov and Jockusch's theorem is sometimes called strong cone avoidance of  $RT_2^1$ : the instance of  $RT_2^1$  we consider does not need to be computable. We study here the following general question, that we derive from Dzhafarov and Jockusch's

What computational power can we encode inside every infinite subsets of both two halves of  $\omega$ ?

## Splitting $\omega$ in two



## Splitting $\omega$ in two

#### The question

What can we encode inside every infinite subsets of both two halves of  $\omega$  ?

A splitting :

Such that :

- Each infinite subset of the blue part has some comp. power
- Each infinite subset of the red part has some comp. power

Answer : Not much...

A precision

What if we drop the complement thing?

Consider any set X. Then we can encode X into every infinite subset of a set A the following way : We let A be all the integers which correspond to an encoding of the prefixes of X (using some computable bijection between  $2^{\omega}$  and  $\omega$ ).

$$\sigma_0 < \sigma_1 < \sigma_2 < \dots X$$

A(n) = 1 iff *n* encodes  $\sigma_n$  for some *n* 

Splitting  $\omega$  in two

Iterating through the ordinals

## Encoding Hyperimmunity

#### Definition (Hyperimmunity)

A set X is of hyperimmune degree if X computes a function  $f : \omega \to \omega$ , which is not dominated by any computable function.



#### Theorem

There exists a covering  $A^0 \cup A^1 \supseteq \omega$ , such that every  $X \in [A^0]^{\omega} \cup [A^1]^{\omega}$  is of hyperimmune degree.

## Encoding Hyperimmunity

#### Theorem

There exists a covering  $A^0 \cup A^1 \supseteq \omega$ , such that every  $X \in [A^0]^\omega \cup [A^1]^\omega$  is of hyperimmune degree.

We split  $\omega$  by alternating larger and larger blocks of consecutive integers in  $A^0$  and  $A^1$ .

For X infinite subset of  $A^0$  or  $A^1$ , the hyperimmune function is given by f(n) to be the *n*-th number which appears in X.

## Encoding DNC

#### Definition (Diagonally non-computable degree)

A set X is of DNC degree (diagonally non-computable) if X computes a function  $f: \omega \to \omega$ , such that  $f(n) \neq \Phi_n(n)$  for every n.

#### Theorem

The following are equivalent for a set X :

- X is of DNC degree.
- X computes a function which on input n can output a string of Kolmorogov complexity greater than n.
- X computes an infinite subset of a Martin-Löf random set.

## Encoding DNC

#### Definition (Informal definition of Kolmorogov complexity)

We say  $K(\sigma) \ge n$  if the size of the smallest program which outputs  $\sigma$  is at least n.

#### Definition (Informal definition of Martin Löf randomness)

We say X is Martin Löf random if the Kolmogorov complexity of each of its prefix  $\sigma$  is greater than  $|\sigma|$ .

#### Theorem

X is of DNC degree iff X computes an infinite subset of a Martin-Löf random set.

## 

## Encoding enumerating non-enumerable things

#### Theorem [Tennenbaum, Denisov]

There exists a computable order of  $\omega$ , of order type  $\omega + \omega^*$  which has no infinite ascending or descending c.e. sequence.

Consider  $A \subseteq \omega$  the initial segment of order-type  $\omega$ .

- Any infinite subset X ⊆ A enumerates A (by enumerating things smaller than elements of X)
- Any infinite subset of X ⊆ A
   enumerates A
   (by enumerating things larger than elements of X)

#### Corollary [Tennenbaum, Denisov]

There exists a set A such that every set  $G \in [A]^{\omega} \cup [\overline{A}]^{\omega}$  can make c.e. something which is not c.e.

## Cone avoidance

#### Theorem [Dzhafarov and Jockusch]

Let  $X \subseteq \omega$  be non-computable. For every covering  $A^0 \cup A^1 \supseteq \omega$ , we have some  $G \in [A^0]^{\omega} \cup [A^1]^{\omega}$  such that  $G \ngeq_T X$ .

The proof uses computable Mathias Forcing : Dzhafarov and Jockusch's technic have then been enhanced an reused in various manner by multiple authors to show other results of the same type, that we shall now expose.

#### Theorem [Strong form of Dzhafarov and Jockusch]

Let  $X \subseteq \omega$  be non-computable. The class of sets

 $\{A : \text{ There exists } G \in [A]^{\omega} \text{ such that } G \text{ does not compute } X\}$ 

is partition regular.

## More on cone avoidance

Theorem [Dzhafarov and Jockusch]

Let  $X \subseteq \omega$  be non-c.e. The class of sets

 $\{A : \text{ There exists } G \in [A]^{\omega} \text{ such that } X \text{ is not c.e. in } G\}$ 

is partition regular

But we cannot avoid more than one c.e. set. On the other hand :

#### Theorem [Dzhafarov and Jockusch]

Let  $\{X_n\}_{n\in\omega}$  be all non-computable. The class of sets

 $\{A : \text{There exists } G \in [A]^{\omega} \text{ such that } G \text{ computes no } X_n\}$ 

is partition regular

## PA degrees

#### Definition

A set X is of P.A. degree if X computes a complete and consistent extension of Peano arithmetic.

#### Theorem

The following are equivalent :

- X is of P.A. degree.
- X is diagonally non-computable with a  $\{0,1\}$ -valued function.
- X computes an infinite path in any non-empty  $\Pi_1^0$  class.

#### Theorem (Liu)

The class of sets

 $\{A : There \ exists \ G \in [A]^{\omega} \ which \ is \ not \ of \ PA \ degree \ \}$  is partition regular

## Non high

#### Definition

A set X is high if it computes a function which eventually grows faster than any computable function.



#### Theorem (M., Patey)

The class of sets

 $\{A : \text{ There exists } G \in [A]^{\omega} \text{ such that } G \text{ is not high } \}$ 

is partition regular

## Non high

#### Theorem (Martin)

The following are equivalent for a set X :

- X is high
- $X' \ge_T \emptyset''$

#### Theorem (M., Patey)

Let  $X \subseteq \omega$  be non- $\emptyset'$ -computable. The class of sets

 $\{A : \text{ There exists } G \in [A]^{\omega} \text{ such that } G' \text{ does not compute } X\}$ 

is partition regular

The proof uses of new forcing technique that builds upon Mathias forcing to control the second jump.

Partition regularity is in particular a key concept of the used forcing.

## Computing random sets

#### Theorem (Liu)

Let f be a computabe order function. The class of sets

 $\{A : \text{ There exists } G \in [A]^{\omega} \text{ which is not of } DNC_f \text{ degree } \}$ 

is partition regular

#### Fact

Every Martin-Löf random Z is  $DNC_{n\mapsto 2^n}$ , that is, Z computes a DNC function bounded by  $n \mapsto 2^n$ .

#### Corollary [Liu]

The class of sets

 $\{A : \text{ There exists } G \in [A]^{\omega} \text{ which compute no MLR set } \}$ 

is partition regular

## Computing generic sets

#### Definition

A set is *weakly-n-generic* if it is in every  $\Sigma_1^0(\emptyset^{(n-1)})$  dense open set. It is 1-generic if for every  $\Sigma_1^0(\emptyset^{(n-1)})$  open set U, it is in U or in the interior of the complement of U.

#### Theorem

There exists a covering  $A^0 \cup A^1 \supseteq \omega$ , such that for every  $G \in [A^0]^{\omega} \cup [A^1]^{\omega}$  we have that G computes a 2-generic.

This is because any function which is not bounded by any  $\Delta_3^0$  function can compute a 2-generic. This does not work anymore with weakly-3-genericity and above.

## Computing generic sets

#### Theorem (Andrews, Gerdes, Miller)

There is a function bounded by no  $\Delta_3^0$  function which computes no weakly-3-generic set.

The previous theorem gives us material for the following conjecture :

#### Conjecture

The class of sets

 $\{A : \text{ There is } G \in [A]^{\omega} \text{ which computes no weakly-3-generic set } \}$ 

is partition regular

## Computing cohesive sets

#### Definition (Cohesiveness)

A set X if p-cohesive if for any primitive recursive set  $R_e$  we have  $X \subseteq^* R_e$  or  $X \subseteq^* \overline{R_e}$ 

#### Theorem (Folklore)

A set X computes a p-cohesive set iff X' is  $PA(\emptyset')$ , that is, iff X' computes a function  $f : \omega \to \{0, 1\}$  such that  $f(n) \neq \Phi_e^{\emptyset'}(e)$ .

#### Theorem (M., Patey)

For every  $\Delta_2^0$  set A, there is an element  $G \in [A]^{\omega} \cup [\overline{A}]^{\omega}$  such that G' is not  $PA(\emptyset')$ .

#### Question

Is the former true for any set A?

## Iterating through the ordinals



# lterating through the ordinals

## The goal

#### Theorem (M., Patey)

Let  $X \subseteq \omega$  be non- $\emptyset'$ -computable. The class of sets

 $\{A : \text{ There exists } G \in [A]^{\omega} \text{ such that } G' \text{ does not compute } X\}$ 

is partition regular

Does this generalizes to any jump?

#### Fact

The first "second jump control" forcing did not generalize to the third jump control.

## Largeness and partition regularity

#### Definition (Largeness)

A largeness class is a collection of sets  $\mathcal{L}\subseteq 2^\omega$  such that :

- $\mathcal{L}$  is upward closed : If  $X \in \mathcal{L}$  and  $X \subseteq Y$ , then  $Y \in \mathcal{L}$
- If  $Y_0 \cup \cdots \cup Y_k \supseteq \omega$ , then there is  $i \leq k$  such that  $Y_i \in \mathcal{L}$
- If  $X \in \mathcal{L}$  then  $|X| \ge 2$

#### Definition (Partition regularity)

A partition regular class is a collection of sets  $\mathcal{L}\subseteq 2^\omega$  such that :

- $\mathcal{L}$  is a largeness class
- If  $X \in \mathcal{L}$  and  $Y_0 \cup \cdots \cup Y_k \supseteq X$ , then there is  $i \leq k$  such that  $Y_i \in \mathcal{L}$

We add the condition  $|X| \ge 2$  to ensure that  $\mathcal{L}$  contains only infinite elements.

## Generalities

#### Proposition

A partition regular class  $\mathcal L$  contains only infinite sets.

#### Proposition

Let  $\mathcal{L}$  be a partition regular class. Then  $\mathcal{L}$  is closed by finite change of its elements. Furthermore if  $\mathcal{L}$  is measurable it has measure 1.

#### Proof sketch :

- $\ensuremath{\mathcal{L}}$  contains only infinite set
- $\rightarrow \mathcal{L}$  is closed by finite change
- $\rightarrow$   ${\cal L}$  has measure 0 or 1
- $\rightarrow$  If  ${\cal L}$  has measure 0, sufficiently MLR Z and  $\omega-Z$  are not in  ${\cal L}$
- $\rightarrow$  But Z or  $\omega Z$  must be in  $\mathcal{L}$ . Contradiction.
- $\rightarrow \mathcal{L}$  has measure 1

## Generalities

#### Proposition (Compactness for largeness classes)

Suppose  $\{\mathcal{A}_n\}_{n\in\omega}$  is a collection of largeness classes with  $\mathcal{A}_{n+1} \subseteq \mathcal{A}_n$ . Thus  $\bigcap_{n\in\omega} \mathcal{A}_n$  is a largeness class.

#### Proposition (Compactness for partition regular classes)

Suppose  $\{\mathcal{L}_n\}_{n\in\omega}$  is a collection of partition regular classes with  $\mathcal{L}_{n+1} \subseteq \mathcal{L}_n$ . Thus  $\bigcap_{n\in\omega} \mathcal{L}_n$  is partition regular.

#### Proposition

Let  ${\mathcal A}$  be any set. Then  ${\mathcal A}$  is a largeness class iff the set

$$\mathcal{L}(\mathcal{A}) = \{ X \in 2^{\omega} : \forall k \; \forall X_0 \cup \cdots \cup X_k \supseteq X \; \exists i \leqslant k \; X_i \in \mathcal{A} \}$$

is a partition regular subclass of A (in which case it is the largest).

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## $\Pi_2^0$ partition regular classes

#### Proposition

If  $\mathcal{U}$  is a  $\Sigma_1^0$  large class. Then  $\mathcal{L}(\mathcal{U})$  is a  $\Pi_2^0$  partition regular class.

#### Proposition

If  ${\cal U}$  is a  $\Sigma^0_1$  upward closed class. Then predicate  ${\cal U} ~is~ large$  is  $\Pi^0_2.$ 

Fix k, the class of element :

 $\{Y_0 \oplus \cdots \oplus Y_k : X \subseteq Y_0 \oplus \cdots \oplus Y_k \land \forall i < k \; Y_i \notin \mathcal{U}\}$ 

is a  $\Pi_1^0(X)$  class uniformly in X.

## A glance at the forcing idea

Let  $(\sigma, X)$  be a condition.

•  $\sigma \mathrel{?}\vdash \exists n \ \Phi(G, n) \text{ iff}$ 

 $\{Y : \exists n \ \exists \tau \subseteq Y \ \Phi(\sigma \cup \tau, n)\}$  is a largeness class

• 
$$\sigma \mathrel{?}\vdash \exists n \; \forall m_0 \ldots Qm_k \Phi(G, n, m_0, \ldots, m_k)$$
 iff

 $\{Y : \exists n \exists \tau \subseteq Y \sigma \cup \tau ? \not\vdash \exists m_0 \dots \neg Qm_k \neg \Phi(G, n, m_0, \dots, m_k)\}$ 

is a largeness class

 $\rightarrow$  If yes, X is in the largeness class. Take an extention of  $\tau \geq \sigma$  with  $\tau \subseteq X$ 

 $\rightarrow$  If no, there is a cover  $Y_0 \cup \cdots \cup Y_k \supseteq \omega$ , such that for every extention  $\tau \geq \sigma$  in  $Y_i$  and every n, "something is satisfied". Take an extention  $Y_i \cap X \subseteq X$  for the "right"  $Y_i$ .

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## Canonical $\Pi_2^0$ partition regular classes

The following classes are  $\Pi_2^0$  partition regular classes.

Exemple	
For <i>X</i> c.e. :	
	$\mathcal{L}_X = \{Y :  X \cap Y  = \infty\}$

#### Exemple

The class :

$$\mathcal{L}_{1/n} = \left\{ X : \sum_{n \in X} 1/(1+n) = \infty \right\}$$

#### Exemple

The class :

 $\mathcal{L}_W = \{X : X \text{ contains arbitrarily long arithmetic progressions}\}$ 

## Minimal largeness classes

The challenge is to **fix in advance** all the possible largeness classes we want to work with, whithout being definitionally too complex.

#### Notation

For 
$$C \subseteq \omega$$
 we write  $\mathcal{U}_C = \bigcap_{e \in C} \mathcal{U}_e$ 

#### Definition (M., Patey)

Let  $\mathcal{M}$  be a countable set. A largeness class  $\mathcal{U}_C$  is  $\mathcal{M}$ -minimal if for every  $\Sigma_1^0(X)$  class  $\mathcal{U}$  for  $X \in \mathcal{M}$  we have :

• 
$$\mathcal{U}_C \subseteq \mathcal{U}$$

 $\bullet$  or  $\mathcal{U} \cap \mathcal{U}_C$  is not a largeness class

### Cohesive largeness classes

#### Definition (M., Patey)

Let  $\mathcal{M}$  be a countable set. A largeness class  $\mathcal{L}$  is  $\mathcal{M}$ -cohesive if for every  $X \in \mathcal{M}$  we have :

- $\mathcal{L} \subseteq \mathcal{L}_X$
- or  $\mathcal{L} \subseteq \mathcal{L}_{\overline{X}}$

#### Proposition (M., Patey)

Let  $\mathcal M$  be a Scott set. An  $\mathcal M\text{-cohesive}$  largeness class contains a unique  $\mathcal M\text{-minimal}$  largeness class.

#### Notation

Let  $\mathcal{M}$  be a Scott set and  $\mathcal{U}_C$  be an  $\mathcal{M}$ -cohesive class. Then  $\langle \mathcal{U}_C \rangle$  is the unique minimal largness subclass of  $\mathcal{U}_C$ .

## The forcing (1)

- Let  $\{M_{\alpha}\}_{\alpha < \omega_1^{ck}}$  be such that
  - $M_{lpha}$  codes for a countable Scott set  $\mathcal{M}_{lpha}$
  - $\emptyset^{(lpha)}$  is uniformly coded by an element of  $\mathcal{M}_{lpha}$
  - Each  $M'_{lpha}$  is uniformly computable in  $\emptyset^{(lpha+1)}$
- Let  $\{C_{\alpha}\}_{\alpha < \omega_1^{ck}}$  be such that :
  - $\mathcal{U}_{\mathcal{C}_{\alpha}}^{\mathcal{M}_{\alpha}}$  is an  $\mathcal{M}_{\alpha}$ -cohesive largeness class
  - $\beta < \alpha$  implies  $\mathcal{U}_{\mathcal{C}_{\alpha}}^{\mathcal{M}_{\alpha}} \subseteq \langle \mathcal{U}_{\mathcal{C}_{\beta}}^{\mathcal{M}_{\beta}} \rangle$
  - Each  $C_{\alpha}$  is coded by an element of  $\mathcal{M}_{\alpha+1}$  uniformly in  $\alpha$  and  $\mathcal{M}_{\alpha+1}$ .

Let 
$$S = \bigcap_{\alpha < \omega_1^{ck}} \mathcal{U}_{C_{\alpha}}^{\mathcal{M}_{\alpha}}$$
. At least one among  $A$  or  $\overline{A}$  belongs to  $S$ 

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Iterating through the ordinals

## The forcing (2)

Let A be such that  $A\in\mathcal{S}.$  Forcing conditions are Mathias conditions  $(\sigma,X)$  such that :

- $\sigma \subseteq A$
- $X \subseteq A$

• 
$$X \cap \{0, \ldots, |\sigma|\} = \emptyset$$

• 
$$X \in S$$

#### Theorem (M., Patey)

Let B be not  $\Delta_1^0(\emptyset^{(\alpha)})$  for  $\alpha < \omega_1^{ck}$ . If G is sufficiently generic then B is not  $\Delta_1^0(G^{(\alpha)})$ .

#### Theorem (M., Patey)

If B is not  $\Delta_1^1$ , for every covering  $A^0 \cup A^1 \supseteq \omega$ . If G is sufficiently generic then B is not  $\Delta_1^1(G)$  (with in particular  $\omega_1^G = \omega_1^{ck}$ ).

## Question

#### Theorem (Wang)

Let X be non-computable. Let  $c : \mathbb{N}^2 \to \{0, 1, 2\}$  be any coloring. Then there exists G and  $i \in \{0, 1, 2\}$  such that

- For all  $n, m \in G$  we have  $c(n, m) \neq i$ .
- G does not compute X

#### Theorem (Cholak, Patey)

Let X be non-computable. Let n and  $m > d_n$ . Let  $c : \mathbb{N}^n \to \{0, m-1\}$  be any coloring. Then there exists G such that

- $\#\{i : c(a, b) = i \text{ for } a, b \in G\} \leq d_n$ .
- G does not compute X

where  $d_0 = 1$  and  $d_n = \sum_{i=0}^n d_i d_{n-1}$  are the Catalan numbers

#### Question

Can we iterate this (with maybe different numbers) through the jumps?